Generating Shared Interpretive Resources in the Mathematics Classroom: Using Philosophy of Mathematics to Teach Mathematics Better

A thesis submitted in partial fulfilment of the

requirements for the degree of

MASTERS IN PHILOSOPHY
of

RHODES UNIVERSITY

by

LAURA DE LANGE

June 2016

Abstract

Every student has a unique mathematical lived experience: a unique amalgamation of ideas about mathematics, exposure to mathematical concepts and feelings about mathematics. A student's unique set of circumstances means that not every explanatory account of mathematics will cohere with her previous experiences. For an explanation to have explanatory potential, it must provide an account which coheres with the other beliefs a student has about mathematics. If an explanation has no such coherence, it will not be recognisable as an explanation of the phenomenon of mathematics for the student.

Our explanatory accounts of mathematics and mathematical knowledge are our *philosophies of mathematics.* Different philosophies of mathematics will better explain different sets of mathematical lived experiences. In this thesis I will argue that students should be exposed to a multiplicity of philosophies of mathematics so that they can endorse the philosophy of mathematics which has the most explanatory potential for their particular set of mathematical lived experiences. I argue that this will improve student understanding of mathematics.

The claims inherent in any given philosophy of mathematics, when combined with other stereotypes or prejudices, can work to unjustly exclude members of subordinated groups, such as poor, black or female students, from mathematical participation. If we want to avoid reinforcing and reinscribing prejudicial claims about people in the mathematics classroom, we need to be aware of how a certain philosophy of mathematics can exclude certain students. In this thesis I will be defending the idea that, as mathematics educators, we should diversify the way we see mathematics so that we decrease this exclusion from mathematics. In order to diversify the way in which we see mathematics so as to decrease

unjust exclusion, members of subordinated groups should be encouraged to share their mathematical experiences in a space sensitive to the power dynamics present in the mathematics classroom. These accounts can then be combined with existing philosophies of mathematics to create new ways of making sense of mathematics which do not unjustly exclude members of subordinated groups.

The financial assistance from the Rhodes University Prestigious Scholarship Fund towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the author and are not necessarily to be attributed to Rhodes University or the donor.

Education is a moral activity, and epistemology and ethics are intertwined.

(Haynes & Murris, 'Picturebooks, Pedagogy and Philosophy')

Values and morality give meaning to our individual and social relationships. They are the common currencies that help make life more meaningful than might otherwise have been. An education system does not exist to simply serve a market, important as that may be for economic growth and material prosperity. Its primary purpose must be to enrich the individual and, by extension, the broader society. The kind of learner that is envisaged is one who will be imbued with the values and acts in the interests of a society based on respect for democracy, equality, human dignity and social justice as promoted in the Constitution.

(South African Department of Education, Curriculum Statement on Mathematics)

Table of Content

Introduction

I chose mathematics education as the subject of this thesis because I want others to gain as much value from mathematics as I have. Mathematics is weird and wonderful, concerned as it is with the interface between abstract theory and the world around us. In trying to explain my love of mathematics to the students I tutored – students who were performing badly at school mathematics and students who considered themselves to be stupid because they were performing badly at mathematics – I found that they responded well, and gained interest in mathematics, when I spoke about some of the weirdness inherent to mathematics: What does the existence of a non-Euclidean geometry mean for how we should see mathematics? Why, when I put two pairs of socks into the drier and only three socks come out, do I not think that 2+2=3? Why is zero divided by zero not equal to one, or equal to zero? How do mathematical rules work?

These are not areas of mathematics that typically get covered in the South African classroom. Mathematical teaching and learning aids focus mainly on enabling students to pass standardised tests. The type of mathematical resources I am interested in in this thesis are those that allow students to explore what the value of mathematics is, and to realise – that is, to make real – this value in their own lives.

Of course, there are many other factors which influence a student's ability to realise the value of mathematics in their own lives. In South Africa, there are students who come to school hungry, students who do not have access to textbooks or workbooks, students who do not have access to qualified and knowledgeable teachers, students who cannot enter their school grounds due to trade union politics. These students are not able to access what is valuable about mathematics. In order to empower these students, we have to continue implementing feeding schemes, textbook roll-outs, teacher training etc.

But I believe that these factors do not tell the whole story behind the poor mathematical performance in South African schools. There are students who have all of the above-mentioned resources but still do not realise any of the value of mathematics in their lives. Furthermore, knowing the value of mathematics and how to realise it in her life will give a student access to resources that will augment her understanding of any other resources she already has access to. It is therefore most important that those who do not have many resources are given access to ways of making the most of what little resources they do have.

During one of my dialogues about some of the puzzling aspects of mathematics with a Grade 9 class, I found that those students who claimed to enjoy mathematics also had a nuanced picture of what mathematics was: so for instance one of the students said that "maths is putting the universe on paper". Conversely, those who said that they disliked mathematics defined mathematics as "numbers, +, -, ÷, x". Part of the problem of poor mathematical performance is that students do not know much about the ontological or epistemological nature of mathematics. This problem is not necessarily addressed by access to all the mathematics worksheets and dedicated teachers in the world. The ontological and epistemological nature of mathematics is, of course, under the purview of the philosophy of mathematics. Students are not exposed to coherent (philosophical) accounts of the nature of mathematics and, I will argue, are worse off for it. Without access to an explanatory account of what mathematics is, students are less able to know how to make sense of events (relating to mathematics) which happen to them. For example, knowing that mathematics is more than "numbers, +, -, ÷, x" might help a student reject the claim that she is bad at mathematics merely because she struggles with mental arithmetic.

When making sense of an event, we also come to realise what the appropriate attitude is towards the event, e.g. if we interpret an event as being scary, the appropriate attitude is one of avoidance or fear. How a student interprets mathematical events will therefore influence the attitude she has towards mathematics. A student who believes herself to be bad at mathematics can easily hold attitudes of fear or avoidance. My claim is that students do not have access to resources which will help them figure out what attitude to have towards mathematics (both the mathematics they encounter in the classroom and the mathematics which they encounter in their everyday lives) and what attitude to assume towards themselves *qua* students of mathematics. I argue that the absence of a coherent philosophy of mathematics in the sphere of mathematics education in South Africa (and elsewhere, although my focus will be on South Africa), as well as the accompanying absence of interpretative resources for students of mathematics, constitutes a specific type of injustice: a hermeneutical injustice.

Outline

The first chapter of this thesis starts off by giving the reader an example of a dialogue about the puzzling aspects of mathematics that I have had with students. This example (let us call it the A4 Paper example) aims to give the reader an entry point into this thesis. It also shows how philosophical discussions about mathematics can, given the right environment for dialogue, arise from our encounters with everyday objects; in this example, the starting point for the philosophical discussion is a piece of A4 paper. The A4 Paper example shows that the particular combination of experiences, observations and beliefs about mathematics any student has, is unique to her. This includes her experiences, observations and beliefs about the mathematical aspects of her encounters with the world, *as well as* the experiences she has, observations she makes and the beliefs she forms about mathematics during her time in the mathematics classroom. These things taken together are what I refer to as a student's mathematical lived experiences.

In Chapter 1 I aim to show the reader what I mean in this thesis by the term coherent philosophy of mathematics. Using the A4 Paper example, I show that the answers to the ontological and epistemological questions pertaining to mathematics have to cohere with each other in order to provide an account of mathematics that can be used by the student to make sense of her mathematical lived experiences.

Finally, Chapter 1 engages with theorists who are doing work in academic fields adjacent to mine: authors writing about hermeneutical injustice, injustice in the classroom, philosophy of mathematics education, the value of philosophy to the philosophy outsider, and philosophy for children. I show how these authors affected the course of my own research, and highlight how my work differs from the work they have done.

Chapter 2 is concerned with the value of mathematics, and concludes that students need to be able to make sense of mathematical ideas and concepts in their everyday lives, in addition to being able to perform in the mathematics classroom, in order to be realising the value of mathematics. This gain seems to be thwarted by students' inability to make sense of mathematics, and by their dislike of mathematics in the classroom, expressed in the form of absenteeism, special needs, disruptive behaviour, disengagement and invisibility. This dislike of mathematics, I argue, should be seen as an expression of an emotion which is judged by society at large to be the wrong type of response towards something a valuable as mathematics and this should point us towards an injustice happening in the mathematics classroom. The conclusion reached in Chapter 1 that the way in which we see mathematics affects how we teach and learn mathematics gives me reason to think that the dislike of mathematics should be viewed as a sign of a hermeneutical injustice in the mathematics classroom.

In Chapter 3, I show that the concept of hermeneutical injustice applies to South African mathematics education. In this chapter I give evidence that indicates that in South Africa, black and female students tend not to be part of the 'chosen few' who are privileged to have easy and straightforward access to mathematics. This is because what constitutes mathematics and what counts as valued mathematical knowledge is determined by the norms, values and discourses of the dominant group. These discourses about mathematics track the lived experience of the dominant group and produce accounts that save the phenomenon for members of the dominant group – in other words, members of the dominant group have access to a mathematical account which is consistent with all of their observations and beliefs about individual instances of mathematics. But because there is no equivalent explanatory account for the lived experiences of members of subordinated groups, they experience a hermeneutical lacuna.

In consequence, in order to encourage diversity in academic fields like mathematics, we have to change, or at least diversify, our conception of what it means to be good at mathematics. This diversification depends however upon a diversification of our understandings of what mathematics is. Hence, in order to be self-reflexive about the concepts, norms, values, and framing perspectives that are assumed in mathematics, more philosophy of mathematics needs to be done by students and teachers of mathematics.

Chapter 4 delves further into how practising philosophy of mathematics can produce the interpretative resources needed by students to overcome the hermeneutical lacuna in the mathematics classroom. The main claim of the chapter is that when students talk about their lived mathematical experiences and discuss the explanatory accounts of mathematics developed by philosophers of mathematics – all this taking place within the safe space of a community of inquiry – the shared interpretative resources required to make sense of mathematics can be generated. Through such inquiries, a student is presented with many options for making sense of any given mathematical concept, experience or belief in terms of how it fits into her view of what mathematics is, how it fits in with her other experiences and beliefs, how it fits into the system of what she values and, more broadly, how mathematics fits into her life.

I also provide an appendix with short descriptions of explanatory accounts of mathematics. These are the explanatory accounts developed by philosophers of mathematics which I recommend in Chapter 4 for use in a community of inquiry in the mathematics classroom. I also spell out some of the pedagogical consequences of endorsing a given philosophy of mathematics in the hope that any teacher or student reading this will have the option of choosing the philosophy of mathematics she wishes to endorse based on the type of teaching or learning she wishes to do, while realising that others may choose differently.

Chapter 1

1.1 The A4 Example: Doing Philosophy of Mathematics with High School Students

The case I will proceed to describe exemplifies what occurs when I facilitate discussions about the philosophy of mathematics with high school students.

One lesson I particularly enjoy giving starts with me reading out an excerpt from *The Secret Lives of Numbers: The Curious Truth Behind Everyday Digits* (Millar, 2012). One of my favourite chapters concerns the A4 piece of paper. In Millar's words:

> 'So what?' you'll say. 'It's only a piece of paper, right?'
> Wrong. It's actually an impressive piece of mathematics.

Miller shows how the ratio of the shorter side to the longer side of the A4 is $1{:}\sqrt{2}$. This ratio is special because if you cut a sheet of A4 paper in half through its longer side, the two sheets you are left with will always maintain the same ratio (i.e. they will be exactly half the size but stay in proportion at the same time).

I then ask some of the following questions:

- Do you think that the author is right to say that this A4 paper is not just a piece of paper, but also an "impressive piece of mathematics"? What do you think it means for an object to *be* a piece of mathematics? Is this apple a piece of mathematics?
- The design of this piece of paper is based on a mathematical idea. Does this mean that mathematics is everywhere in the world around us? Is this type of mathematics the same or different to the mathematics we do in the classroom?
- The ratio of the sides of a square is 1:1. What happens to the ratio if we cut a square in half? Can you think of any time when it would be useful for the ratio to stay the same? Can you think of any time when it would be useful for the ratio to change? (If the students have just studied *ratio* and *proportion,* or geometric *similarity* and *congruency* in class, then it might be useful to get students to incorporate these concepts.)
- In the story, one of the physics professors said that this special ratio was "aesthetically pleasing". What is aesthetics? In philosophy, aesthetics is the study of art and beauty, so the professor thought that the special ration was beautiful. Who

agrees with the professor? Who disagrees? Why? Have you heard people say that mathematics is beautiful? What do you think they mean? Do you think they are right?

- Is this piece of paper a piece of art? Is art subjective? Scientists have studied what makes people attractive, and they found that we think people with symmetrical bodies and faces are more attractive. What does this mean? Does it mean that mathematics is beautiful? Is mathematics subjective?
- Leonardo da Vinci's "Vitruvian Man" emphasizes the proportions in the human body, e.g. (foot to navel) : (navel to head). These distances follow the Golden Ration, based on Fibonacci Numbers. If students are interested, I recommend doing Andrew Day's activity entitled *Beauty Secrets* (Day, 2012, Chapter 17) with them.

Any one of these points should be enough to get a healthy discussion going in the classroom. I try not to impose my own thoughts on the group, rather allowing the students to practise sharing their ideas with a group and encouraging students to engage with the positions other students take.

Some of the philosophical questions raised by an activity like this are:

- What is the nature of mathematics? What counts as a mathematical entity? What counts as a mathematical practice?
- What is the relationship between mathematics and the world? What is the relationship between the mathematics classroom and the world?
- What is the purpose of mathematics? Is mathematics purpose driven? Is mathematics driven by *human* purposes?
- Does mathematics have an aesthetic quality? Why is it a common intuition that the secret of beauty lies in mathematics?
- Are mathematical truths universally and necessarily true? What would it mean to say that mathematics is not objective?

1.2 Philosophy of Mathematics

The answers to these and other questions make up the content of any philosophy of mathematics. For the remainder of this thesis, I define a 'philosophy of mathematics' as an account which provides coherent answers to the ontological and epistemological questions pertaining to mathematics.

Ontological questions concerning mathematics are questions about the nature of mathematical objects and about what counts as practising or doing mathematics, since the aim of these questions is to figure out what mathematics is. Epistemological questions concerning mathematics are questions about what counts as mathematical knowledge and how we come to acquire it, since the aim of these questions is to figure out what mathematical knowledge is.

The answers to ontological and epistemological questions often influence each other. Let us imagine a hypothetical student, Abongile, who in response to the questions raised in the dialogue above argues that an A4 piece of paper is *not* a piece of mathematics. Instead, she thinks that the piece of paper is one instantiation of the abstract entity of a rectangle, and that this abstract rectangle (let us assume that it is the form of a rectangle and exists in the realm of the forms) is what we mean when we say something is a piece of mathematics or a mathematical entity. In other words, Abongile is a realist about mathematical entities. This is an attractive position to her because it explains mathematical *certainty*: we know for certain that when we cut an A4 piece of paper in half its ratio remains 1:√2 not because we empirically tested it, but because it is a mathematical property of the A4 piece of paper. For Abongile, this certainty comes from the fact that mathematics is objective and that mathematical truths are necessary truths. In order for this view of the nature of mathematical entities to serve an explanatory role in her life, her account needs to include an epistemology of mathematics that explains how humans come to know about these abstract, but real, entities. Without an account of how we gain access to mathematical knowledge, Abongile's account of mathematics will fall prey to Benacerraf's (1973) concern that an account of mathematics which makes it clear that mathematical truths are indeed *true* (i.e. what allows us to say with certainty that an A5 piece of paper has the ratio $1{:}\sqrt{2}$ since it is half of an A4 piece of paper without having to measure any given piece of A5 paper) will not be able to explain how we have the mathematical knowledge that we do have (i.e. how do we come to have knowledge of the abstract, idealised form of a rectangle which provides us with mathematical certainty). An account of this nature lacks explanatory potential, since it cannot explain all of the observations and beliefs about mathematics, nor can it encapsulate all of the practices which we generally accept as being mathematical.

Now imagine a different student, Buhle, who argues that the 1: √2 ratio was made up by humans to allow us, when we are using the photocopier, to change the size of a picture or diagram, from A4 to A5 for example, without distorting it or having to trim the edges to remove excess paper. In other words, Buhle denies the existence of objective mathematical truth, instead supporting a pragmatist-type account where mathematical concepts are true if they are successful in serving the interests of the humans who created them. Given Buhle's account, mathematical truths are easy to come by – they can be found by testing whether the theory works as explanation of what is happening around us. Buhle therefore has a strong mathematical epistemology, but her account does not include a clear picture of mathematical ontology. An account of this nature lacks explanatory potential, since it has no way of pointing to uniquely *mathematical* practices or entities. Buhle might want to say that humans made up the Higgs-Boson particle because it was useful for science, but she would not want to say that the Higgs-Boson particle is mathematical in the same way that a ratio or a shape is.

Abongile could not simply employ Buhle's epistemology since Buhle's epistemology rests on the fact that humans made up the 1: √2 ratio; similarly, Buhle could not simply help herself to Abongile's ontology because she denies the type of objective mathematical truth that Abongile's account rests on.

For the remainder of this thesis, a coherent philosophy of mathematics is an account where the epistemological and ontological claims contained in a particular philosophy do not conflict, but instead hang together (or cohere) to create a coherent and (relatively complete) picture of mathematics.

1.3 Saving the Phenomenon

In the previous section I mentioned that if a philosophy of mathematics is not coherent, it loses its explanatory potential. In addition, I assume that we come up with accounts of how the world works in order to better understand it. Therefore, we need an account of mathematics and how mathematics works so that we can better understand the mathematical aspects of our world.

Any old explanation will not do; for an explanation to be useful to an agent, it has to 'save the phenomenon'. A general explanatory account is said to save the phenomena if it is (1) about those phenomena, while (2) being consistent with all of our observations and beliefs about individual instances of those phenomena (Jones, 2014). For a mathematical explanatory account (such as a philosophy of mathematics) to save the phenomena it has to be consistent with all of an individual's observations and beliefs about individual instances of mathematics.

In this context, the individual instances of mathematics that I am talking about is the sum of an individual's mathematical lived experiences. Every agent will have a particular combination of experiences, observations and beliefs about mathematics unique to her. These include:

(1) her experiences, observations and beliefs about the mathematical aspects of her encounters with the world, e.g. the amount of change she gets from the cashier in the shop, the fact that even though she might put two pairs of socks into the tumble drier and later only be able to find three socks, 2+2 still equals 4, etc.
(2) the experiences she has, observations she makes and the beliefs she forms about mathematics during her time in the mathematics classroom and how she applies these to the world beyond the classroom
(3) the mathematical ideas or concepts she has been exposed to, e.g. whether she knows about the existence of imaginary numbers or non-Euclidean geometries, whether she has been exposed to the idea that mathematical concepts are foundational in human thought, etc.
(4) the ways in which she hears people speak about mathematics, which may include, but are not limited to, the following statements: women aren't good at mathematics, mathematics is beautiful, you have to be very smart to do mathematics, mathematics is very difficult, mathematics is important in order to

get a job, mathematics is only really useful if you want to be an engineer or scientist.

Any explanatory account of mathematics has to be such that it is consistent with all, or at least most, of these experiences. Some factors which may influence a student's mathematical experiences and beliefs might be: their social, economic and political context (e.g. the type of mathematics education they have access to), the stereotypes about their social groups (e.g. the stereotype that women are not good at mathematics), their future hopes and aspirations (e.g. wanting to become an engineer), and so on.

Not all philosophies of mathematics will be able to explain all of these experiences. Consider the example of Buhle above. Buhle might want to become an engineer and her parents, teachers and peers have a positive view of female engineers. She might also be exposed to the idea of non-Euclidean geometries and have taken from this to mean that human interests can influence mathematics, e.g. by denying Euclid's parallel postulate because it is sometimes useful to do so. Buhle's desire to go on to work in a job which requires mathematical problem-solving together with her exposure to non-Euclidean geometries, cohere best with a philosophy of mathematics which holds that certain mathematical concepts are created to solve human problems (as described in the previous section). Abongile, the mathematical realist, might, on the other hand, never have been exposed to some of the big ideas in mathematics and believes that mathematics "is just numbers, +, -, ÷, x". She has never been encouraged to consider science, engineering or finance as career options since all of the women in her family become housewives. She is not exposed much to numbers or calculations in her day-to-day life and therefore does not see mathematics in the world around her. Some form of mathematical Platonism is therefore the explanatory account which best coheres with her previous mathematical experiences, and is hence the explanation Abongile would be drawn towards to save the phenomena of mathematics in her life.

Is Abongile's way of saving the phenomena better than Buhle's way? Firstly, it is important to note that because their experiences of mathematics are different, it is unlikely that the same account of mathematics *could* save the phenomena of mathematics for both Abongile and Buhle. But it is important that we want both accounts to be accounts about the phenomena that we call *mathematics.* For this to happen, the idea of mathematical communities might be useful. Members of different mathematical communities do not disagree with each other about the particular content of mathematical theories, but instead disagree on issues such as “what are the questions most worth asking, appropriate ways of tackling them and adequate criteria for appraising success” (Glas, 2007, p. 26). The questions that are most worth asking are determined by the ontology of mathematics embraced by an individual: Abongile might think that the only practices that count as doing mathematics are abstract, proof-based practices, while Buhle might think that only the mathematics which can be used to solve real-world problems counts as mathematics in the proper sense. The appropriate ways of tackling mathematical problems will be defined by one’s mathematical epistemology (deductive logic for Abongile and mathematical practices whose efficacy can be tested in the real-world for Buhle), as will the criteria for appraising success (mathematical certainty for Abongile and empirical verification for Buhle). Therefore, the type of mathematics that Abongile values should not be seen as generating mathematical work that is of a lesser standard than the type of mathematics endorsed by Buhle, but they should rather be seen to be orientated towards different cognitive aims (Glas, 2007, p. 36).

To sum up: the mathematical questions which Abongile thinks are most worth asking, what she considers to be appropriate ways of tackling them and what in her opinion counts as criteria for appraising mathematical success, are based on her philosophy of mathematics. The philosophy of mathematics which she endorses depends on which philosophy of mathematics best saves the phenomena of mathematics for her, and this depends on the mathematical lived experiences which she has had in her life so far. The same chain can be followed for Buhle. Therefore, how we see mathematics, the attitudes we have towards mathematics and the type of mathematics which we value as being the best type of mathematics, are all determined by our lived experiences of mathematics.

This means that the type of mathematics which we value can be influenced by many things: the mathematical quality of our everyday interactions, the type of mathematics education we have had, our social or economic status, our gender, the mathematical ideas and concepts that we have been exposed to, our future career aspirations and the stereotypes about mathematics we come into contact with, to name but a few.

For each individual student above, finding a philosophy of mathematics that saves the phenomenon for her (i.e. an account which explains and helps her to make sense of her mathematical experiences, observations and beliefs) makes her part of a mathematical community. Being a member of a recognised mathematical community gives her legitimacy and allows her to engage in mathematical activity which resonates with her lived experiences. Mathematical knowledge ceases to be an externally imposed knowledge from which students feel alienated and instead becomes something which students can appropriate for themselves.

1.4 The Role of Philosophy of Mathematics

Mathematics is not usually seen as a field which is subject to an individual's perception or experience. This is because, on the standard view of mathematics, mathematics is seen as universal, objective and power neutral (see for example Brown, 2007). It is useful to remember Charles Mills's claim that all theorising takes place in an intellectual realm "dominated by concepts, assumptions, norms, values, and framing perspectives that reflect the experience and group interests of the privileged group (whether the bourgeoisie, or men, or whites)" (Mills, 2005, p. 175). This means that even the conceptions underlying how we do mathematics reflect the experience and group interests of the privileged group. Modern mathematicians, and teachers of mathematics, living in a world where gender, race and economic inequalities are so obviously visible, have to be self-conscious about the concepts that "spontaneously" occur to them. Many of these concepts "will not arise naturally but as the result of social structures and hegemonic ideational patterns" (Mills, 2005, p. 175). Whatever one's field, it is necessary to scrutinize the dominant conceptual tools and the way the boundaries are drawn.

Morris Kline, in *Mathematics in Western Culture,* provides us with one such example of how power dynamics influenced the field of mathematics.

Kline identifies the ancient Greeks as being the source of the focus on proof by deduction and abstraction that characterises modern mathematics. He points to the structure of Greek society to explain this movement away from proof by induction or the use of experience and analogy, which characterised mathematics before the Greeks.

In Greek society "the philosophers, mathematicians and artists were members of the highest social class" and disdained commercial pursuits and manual work as "taking time away from intellectual and social activities and the duties of citizenship" (Kline, 1964, p. 29). The running of businesses, households, industries as well as the pursuit of professions and also technical and unskilled work, were passed on to the slave class. The slave basis of classical Greek society "fostered a divorce of theory from practice and the development of the speculative and abstract side of science and mathematics with a consequent neglect of experimentation and practical applications" since if a person, like the Greek mathematician, does not 'live' in the world about him, "experience teaches him very little" (Kline, 1964, p. 29).

This political fact about Greek society "removed mathematics from the carpenter's tool box, the farmer's shed and the surveyor's kit, and installed it as a system of thought in man's mind. Man's reason, not his senses, was to decide thenceforth what was correct. … and thus the Greeks revealed more clearly than in any other manner the supreme importance they attached to the rational powers of man" (Kline, 1964, p. 30). Hence, the nature of mathematics, the epistemology of mathematics, what mathematics could be used for and who would be good at mathematics all stemmed from concepts, assumptions, norms, values, and framing perspectives that reflect the experience and group interests of the privileged group.

There tends to be a blindness to the social consequences of these deep-seated conceptions of an academic discipline precisely because those in the discipline already are exactly that cross-section of the population which is not affected by these issues. Philosophy of mathematics, then, has to be done while keeping in mind what Mills calls the 'social ontology' of the place of study. Not only the history and power relations of South Africa should be kept in mind, for example, but also the power relations throughout the history of mathematics.

1.5 The Impact of Philosophies of Mathematics on Pedagogy

In his book *Philosophy of Mathematics Education* Paul Ernest claims that "the philosophy of mathematics, or philosophies of mathematics underpin all mathematics curricula and teaching" (Ernest, 1991, p. 230). Using the example of Abongile and Buhle, we can make sense of this claim in the following way:

Imagine that Abongile became a teacher of mathematics. Since Abongile is a realist about mathematics, whether or not she ever consciously reflects on her endorsed philosophy of mathematics, she will teach in a particular way. She will stress to her students that mathematics is an activity which is independent of human concerns and involves only objective content. The motivation for studying mathematics, therefore, is to get closer to the objective human-independent truth. Mathematics should be studied for its intrinsic value.

Abongile's pedagogical aims would be to teach students the axioms and deductive skills the need to be able to do mathematics. This means that some mathematical facts have to be accepted by students as true axioms, and that the teacher demonstrates mathematical skills to students who are then expected to practise until they know all of the axioms and are able to swiftly and accurately apply the mathematical skills.

If Buhle were to become a mathematics teacher, on the other hand, her instrumentalist view of mathematics would affect the particular way in which she teaches. For Buhle, the motivation for studying mathematics is to solve problems related to science, finance or engineering. Buhle would aim to teach students the 'dry bones' of mathematics so that they could apply this to real-world problems. In Buhle's class, mathematics would therefore be assessed by means of 'word sums' or other real-world scenarios where students need to choose between different approaches and uses of knowledge in order to solve the problem. Only students who can successfully do this are considered to be good at mathematics. (This example is adapted from Ernest, 1991.)

When examining pedagogy, it is important to keep in mind that knowledge, and its transferral, is not power-neutral. The way in which we teach is influenced by certain (often unexamined) concepts, assumptions, norms, values and framing perspectives – our philosophy of mathematics provides us, as individual teachers, with the background conditions for teaching mathematics (Mills, 2005, p. 175). This is true for every teacher, not only for Abongile and Buhle. The philosophy of mathematics which we embrace, no matter what it may be, affects which community of mathematics we belong to, and hence influences the mathematical questions which we think are most worth asking, what we consider to be appropriate ways of tackling them and what we think counts as criteria for appraising mathematical success. In other words, our philosophy of mathematics influences what we teach, how we teach it and who we think has succeeded in our classrooms.

However, student philosophies of mathematics also affect their learning. Imagine that Abongile has a daughter, Aviwe, who is also an (unexamined) realist about mathematics. If Aviwe were to be taught mathematics by Buhle, she would in all likelihood find that she did not find Buhle's mathematical questions worth asking; she would think that there were better ways of answering mathematical questions and she would consider that Buhle's criteria for appraising mathematical success were foreign to her. Since Aviwe has no interest in doing science, engineering or finance-related jobs, she comes to view mathematics as boring, useless and "increasingly incomprehensible" (Haglund, 2009, p. 4). Additionally, if she does badly at Buhle's word sums, Aviwe might become convinced that she is bad at mathematics, or that mathematics is just not for her. Therefore, the ways in which we make sense of mathematics affect both mathematical teaching and learning.

1.6 Academic Study of Philosophy of Mathematics Education

I included the above example of a philosophical inquiry about mathematics in order to help the reader to understand what counts as philosophical questions about mathematics and how they can arise from our varied experiences, observations and beliefs about mathematics.

The reason why I am interested in philosophical questions about mathematics is because I believe that the effect of philosophies of mathematics on teaching and learning is very important, and yet not widely acknowledged. The academic field of the philosophy of mathematics education has not yet reached mainstream status, and additionally, the philosophy of mathematics education community tends to be associated more strongly with the teachers than the philosophers (Bart van Kerkhove, 2007, p. xii). In this thesis, I aim to take seriously the role of philosophy in the field of philosophy of mathematics education.

My interest in this thesis is to explain the phenomenon of, and provide a new type of solution to the problem of poor mathematics performance in South African schools. I believe that a special type of hermeneutical injustice is taking place in most South African mathematics classrooms and that this injustice negatively affects performance.

As I mentioned in the introduction, the lack of hermeneutical resources in the mathematics classroom is not the only cause of poor mathematical performance. In order to address this problem, we also need to ask questions such as: "How many of these underperforming students live in poverty or do not have access to educational resources and good teachers?" "Do students perform worse in some areas of mathematics than in others?" "Do students who dislike mathematics perform worse than students who like mathematics?" "On average, do female students perform worse than male students? How about black students?" Answers to questions such as these are no doubt important, but they are not questions that interest me here. What I am discussing here is an absence of hermeneutical resources which interacts with the absence of physical resources. This absence of hermeneutical resources may look relatively harmless, but when it interacts with other unjust practices in our society it produces a hermeneutical injustice. Injustice in the mathematics classroom is not taken into account. This thesis will be concerned with how looking at the problem of mathematical performance in South Africa through the lens of justice gives us an alternative – and I think valuable, even if incomplete – explanation for why it is so that mathematical performance in schools in South Africa is really poor.

Sometimes all the physical resources will be in place, but the students will still only gain "at best a fragmented sense of the subject matter and understand few if any of the connections that tie together the procedures that they had studied" (Schoenfeld, 1988, p. 145). In both well and poorly resourced schools, students can develop "perspectives regarding mathematics that are not only inaccurate, but are likely to impede their acquisition and use of other mathematical knowledge" (Schoenfeld, 1988, p. 145).

There are two important aspects to an explanatory account: firstly, it has to be able to give a good explanation of the phenomenon. I believe that my identification of a special type of injustice taking place in most South African mathematics classrooms to the detriment of performance is able to explain parts of the phenomenon which other accounts are less equipped to answer. Secondly, different ways of explaining a phenomenon produce different solutions to the problem. Choosing the wrong explanation can mean the implementation of an ineffective or even harmful solution to the problem. I believe that my account also provides a new way of solving (in part) the problem of poor mathematics performance: in order to overcome the hermeneutical injustice, we should engage mathematics students in philosophical dialogue.

Thinkers such as Paulo Freire and Henry Giroux have considered the role of justice in teaching and learning. Freire in particular stresses the importance of dialogue in overcoming injustice in pedagogy. My approach to the general problem of teaching in an unjust world, and what authentic and inauthentic teaching looks like, is heavily influenced by Freire's *Pedagogy of the Oppressed* and the dialogical approach he advocates there. (See Freire, 2005; Giroux, 1997). Research about, for example, gender equality in mathematics education does not generally provide us with an account that addresses the injustice in the mathematics classroom (see Hanna, 2002).

Alison Jaggar gives us an account of how feeling certain emotions towards an object can serve as a warning sign that injustice is taking place. I apply this to mathematics by saying that the disaffection students feel towards mathematics should alert us to the existence of a hermeneutical injustice. The study of epistemic and hermeneutical injustice (how injustice can occur in our processes of interpreting and coming to know about phenomena) is pioneered by Miranda Fricker. Part of the aim of this thesis is to apply Fricker's concept of hermeneutical injustice (which she uses to address gender inequality) to a different phenomenon – mathematics education. I believe that Fricker's work is especially useful when we are trying to account for an epistemic practice such as education. Using a socially situated account of a human practice allows us to address injustice and inequality since it does not perceive the student in abstraction from relations of social power. I hope that more scholars will apply her concepts to other epistemic practices. Ward Jones, in his published and forthcoming work, is one of the few scholars who does take Fricker's concept of hermeneutical injustice seriously, and has greatly influenced my views on how philosophy can possibly be the type of activity that can help us to overcome a hermeneutical injustice. (See Fricker, 2007; Jaggar, 1989; Jones, 2010, 2014)

Although he does not explicitly engage questions of justice, Paul Ernest's (1991) seminal book, *Philosophy of Mathematics Education* has been a formative influence on this thesis. This book, by setting out, in chronological order, different philosophies of mathematics and how these have been implemented in the UK schooling system, gives a clear picture of how important our endorsed philosophy of mathematics is for how we teach mathematics. His focus on the political aspects of these choices also influenced my thinking on who does, or does not, benefit from the way we currently teach mathematics in South Africa.

Work on ‘personal epistemologies’ gives an account of the influence that our beliefs and knowledge have on teaching and learning (see Brownlee, 2011; Cao, 2015; Elby, 2009; Hammer & Elby, 2002; Hofer, 2001). These studies focus mainly on science education, but show how students’ understandings of the nature of science and the learning of science can be categorised, and that different epistemologies affect how students approach scientific problems and concepts. This thesis examines explicitly how students view mathematics and mathematical learning, but goes further to examine the historical development of South Africa’s endorsed philosophy of mathematics, how it can negatively affect subordinated students, and what solution could be implemented to decrease injustice in the classroom.

The Philosophy for Children (P4C) movement does not explicitly engage with issues of injustice, nor, in general, with mathematics education. But the movement’s implicit philosophical views of children and childhood, philosophy of child and its metaphilosophical claims, especially the claims about the value of philosophy to the philosophy outsider, have been valuable in constructing the solution being proposed here. (See Hannam, Echeverria, 2009; Haynes & Murris, 2012; Murris, 2000; Stanley, 2012)

As far as I know, there is only one P4C practitioner who has published work specifically concerning philosophy with children on mathematical topics. Andrew Day (2012), in his book *The Numberverse,* provides the reader with a collection of mathematical starting points from which philosophical conversations can develop, with the intent of giving people who study mathematical topics “both pleasure and purpose”. His motivation for employing philosophy in mathematics lessons is that “we are always in a better position if children are happy to be doing maths, even occasionally. Whereas if they know there is no chance of them ever enjoying a maths lesson, they will switch off before it even starts. … Critically, [students learn] to judge themselves as bad at (or bored by) maths very early on, and, as adults, the moment they get something wrong, or can't solve something, they are not intrigued but repelled" (Day, 2012, Chapter 1). He also sees philosophy as being important, because if we get caught up in the traditional approach of only wanting to know whether students 'get the answers right' then "many thoughtful children turn their backs on maths, emotionally speaking, because their approach and their needs are seen as inappropriate to maths, but that is nonsense. You can come at it from different angles and be good at it in different ways. That is not the same as saying that anyone can be good at maths or that anyone can reach any level" (Day, 2012, Chapter 1).

My project differs from what Day is doing in important ways. Day backs up his suggestions by using his intuition and experience. Without an explanatory account to go with his solution, the reader is left wondering why his philosophical interventions work in the ways he says they do. In this thesis, I aim to provide just such a (theoretical) explanatory account. Which explanatory account is chosen to justify the value of P4C affects the motivation of the P4C practitioner: there is a big difference between doing philosophy of mathematics with students because you think that it will improve their maths marks, and doing philosophy of mathematics with students because you want them to be able to access the value of mathematics; between wanting them to be able to make sense of their mathematical experience, and wanting to address injustice. The theoretical background provided by this thesis would not, I think, drastically change the content of the philosophical inquiries that educators like Day recommend. It might, however, change the motivation and attitude of the facilitators of the sessions and the manner in which inquiries are run, and alert facilitators to the importance of being sensitive to injustice.

My thesis is not a piece of philosophy of mathematics. However, it is obviously influenced by philosophy of mathematics. Philosophy of mathematics is used to identify different ways of understanding mathematics, and as a source for starting philosophical discussions with students. I point interested readers to Bostock, 2009; Brown, 2007; Irvine, 2009; Bart van Kerkhove, 2007.

Therefore, although the field of hermeneutical lacunae in the mathematics classroom is quite a niche field, my approach to the topic is indebted to the work of many different philosophical thinkers. As a philosopher I have a rich tradition of thinking about justice, epistemology, mathematics and education to draw upon.

1.7 Concluding Remarks

This chapter has provided the lens through which the reader is invited to view this thesis. It provided many examples so that the reader can get a sense of the real-life applications of the theoretical account I provide in the remainder of the thesis.

This thesis aims to give an explanatory account of why mathematics performance in South African schools is so poor (a special type of hermeneutical injustice exists in most South African mathematics classrooms which negatively affects performance) and to suggest a solution to the problem (that in order to overcome the hermeneutical injustice, we should engage mathematics students in philosophical dialogue).

In order to show that there is a case for hermeneutical injustice being present in the South African mathematics classroom, I first have to establish what a philosophy of mathematics is, how this is related to mathematical lived experiences and mathematical communities, and how an endorsed philosophy of mathematics affects teaching and learning.

The basic line of argumentation goes like this:

(i) Every student (and teacher) has a unique set of mathematical lived experiences.

(ii) These experiences influence which explanatory account of mathematics (which philosophy of mathematics) saves the phenomenon of mathematics for each individual.

(iii) The philosophy of mathematics endorsed by each individual determines which mathematical community they belong to.

(iv) Different mathematical communities have different views about which mathematical questions are most worth asking, how to tackle these mathematical questions, and what counts as adequate criteria for appraising mathematical success.

(v) These differences change how we teach and learn mathematics.

In the next chapter, I examine how being exposed to a variety of endorsed philosophies of mathematics can benefit students, and how aspects of certain philosophies of mathematics can lead to injustice.

Chapter 2

This chapter is concerned with the particular value of mathematics and why it should concern us that mathematics students do not like mathematics, perform badly in mathematics and do not choose to continue with post-compulsory mathematics. I also show how this disaffection, poor performance and lack of further study are affected by the philosophies of mathematics which we endorse.

The fact that students do not like mathematics is strange because there is widespread consensus that mathematics is valuable, so we would expect a positive affective response to mathematics from students. That this is not the case should lead us to investigate whether any injustice is taking place in the mathematics classroom. Such an investigation leads us to see that black and female students are systematically disadvantaged by the philosophies and representations of mathematics currently at work in our schooling system. This not only reproduces the harmful prejudice which they encounter in other areas of their lives, but also prevents subordinated groups from generating the interpretative resources needed to address the problem, and leaves a subordinated individual "unable to make sense of her ongoing mistreatment, and this in turn prevents her from protesting it, let alone securing effective measures to stop it" (Fricker, 2007, p. 151).

2.1 Mathematical performance

It is an uncontested fact that mathematics performance amongst school children in South Africa is really, *really* bad. It might be an exaggeration to say that most of South Africa is mathematically illiterate, but only a slight exaggeration. For example, the World Economic Forum report ranks the quality of South Africa's mathematics and science education as the worst in the world, while the Trends in Mathematics and Science Studies of 1995, 1999 and 2002 rank South African students last or second to last amongst the other participating countries. In the 2011 Trends in Mathematics and Science Study, South African Grade 9 students were taking the Grade 8 tests and still fared worse than the Grade 8s in most countries. In the Southern and Eastern Africa Consortium for Monitoring Educational Quality study, South Africa came 8th out of 15 amongst a “more comparable group of counties in the region” (Muller, 2014).[1]

If mathematics is truly valuable, then this lack of performance should be troubling. Students who perform so poorly cannot be mathematically literate.

Mathematical literacy is important to individual agents. To be mathematically literate is to have the resources necessary to engage with the mathematical aspects of our world. Additionally, the mathematical skills or knowledge which we consider to be necessary and sufficient conditions for mathematical literacy, reflect what we think is valuable about mathematics.

Mathematics forms a very foundational part of our everyday lives in order, say, to know when we are outnumbered by predators or how many offspring we have so that we can notice when one goes missing. But we also need mathematical ideas in order to make sense of most of the perceptions our senses relay to our brains. Moving around requires concepts of shape (geometry) and distance (discrete mathematics) as well as concepts such as bigger than or smaller than (> or <). Ethics is dependent on the concept that I am me and you are you (1 or unity) and the concept that you and I are not the same person (x≠y). To say here that mathematics describes the world seems to be missing the point. Mathematics precedes description: mathematics seems to be a condition for experiencing the world as we do. Regarded in this way, mathematics can be seen as a set of ‘master symbols’ that participate in the fundamental organisation of our ways of looking at the world, ourselves and others (Jaggar, 1989). Like language, mathematics seems to be a precondition of living a life that is identifiably human.

Additionally, it is indisputable that in today’s society, the ability to deal with numbers and to interpret quantitative information is an important component of literacy (Jablonka, 2003, p. 76). Mathematics is a part of everyday life and we use it in a variety of basic calculations, including when we go shopping or driving or when we make sure that we get to work on time.

[1] There is considerable debate as to whether or not these tests are an accurate reflection of mathematics performance in South Africa. Critics often cite the difference in context and the gendered/racialised language as a barrier to the performance of South African students. While I am sympathetic to this point, as such differences are a result of holding different views about the purpose of mathematics and how to measure mathematical success, I will not be engaging with this issue here. Readers interested in the debates around this issue are encouraged to refer to Fleisch, 2008; Gustafsson, 2006; Howie & Plomp, 2005; Reddy, 2006; van der Berg, 2008

This also explains why out mathematical lived experiences are so varied – because so many of our everyday interactions involve mathematical concepts, we have lots of varied experiences, observations and beliefs about the mathematical aspects of our encounters with the world.

The resources we require to deal with the basic mathematical parts of our lives and to make sense of our perceptions of the world include basic mathematical practices such as basic arithmetic, interacting with the duodecimal system[1] needed for calculations relating to time, using the decimal system needed for calculations relating to money, and basic geometry. These skills make up what I call basic mathematical literacy.

Basic mathematical literacy is what is lacking when school mathematics performance is so poor that students do not know how to divide or multiply by factors of ten, assume that a third is smaller than a quarter because 3 is smaller than 4, or when students cannot calculate their percentage in a test if they got 3 out of 75. In order for students to be mathematically literate, and in order for students to get value out of mathematics, they should be able to see the basic role that mathematics plays in our lives and be able to do the basic calculations needed to thrive in today's society.

There does seem to be a second level of mathematical literacy which involves having a body of mathematical knowledge and the ability to apply this knowledge to other situations. The precise nature of this secondary mathematical literacy depends on how we define mathematics and mathematical practices, in other words, the characterisation of secondary mathematical literacy depends on our philosophy of mathematics.

Let us return to Abongile and Buhle to try to understand the concept of secondary literacy. According to Abongile's realist philosophy of mathematics, in order to reach true mathematical literacy one has to be able to work deductively from axioms to mathematically certain conclusions. On Buhle's instrumentalist account of mathematics, to be mathematically literate is to apply mathematical concepts and mathematical thinking to real-world problems, in the field of engineering, for example. Note that secondary mathematical literacy is often enacted in tertiary education or the working environment. If

[1] Duodecimal system: the base 12 system of counting used by Babylonians; contrasts with the decimal (base 10) system on which the SI units are based. Exercises regarding different base systems make good starting points for mathematical discussions since it usually brings us face-to-face with the fact that our mathematical system seems to have been arbitrarily made up by us, instead of being a real (if abstract) thing which we discover.

Buhle performed very poorly at school mathematics, for example, she might never realise the value of mathematics in her everyday life since she is basically mathematically illiterate. But her poor mathematical performance also prevents her from studying mathematics further (at tertiary level) – she cannot get into engineering or science or accounting degree programmes – and hence she cannot realise the instrumentalist value of mathematics which sees mathematics as useful for real-world problem-solving either.

Classroom mathematics is meant to be a formalisation of everyday mathematics in the same way that literature/grammar and musical theory are meant to be formalisations of our everyday interactions with language and music respectively. Classroom mathematics should, therefore, help students to make sense of their lived experiences of mathematics and to become better at doing mathematics, in the same way that we think studying language or music at school should improve our everyday understanding and use of language and music. In the same way that we think studying language or music should help students to see the value of the subject and to appreciate the real-life instantiations of the subject, students should be able to come to a conclusion about the value of mathematics and to a greater appreciation of the different instantiations of mathematics that they might come into contact with.

Additionally, the classroom determines which students get to enter tertiary mathematics education or a mathematically-orientated workplace. Whatever the value of mathematics, it is deemed important by society. High school mathematics scores are often taken as an indicator by universities of whether an applicant would be able to succeed in tertiary education, while most big companies screen applicants using basic mathematical competency tests. It is in the interest of all students to be seen as mathematically competent by society.

It is therefore important to study how mathematics is experienced in the mathematics classroom and how students come to view mathematics during the pedagogical process.

When students perform poorly at school mathematics, they do not have the resources needed to be able to make sense of, and usefully engage with, either their day-today mathematical experiences or their mathematical experiences at high school. Additionally, poor high school performance means that students cannot continue studying mathematics at higher levels. We can then say that these students do not receive the value of mathematics in their everyday lives, nor any of the value of mathematics as described by the endorsed philosophy of mathematics of the system they find themselves in.

2.2 Continuing with Mathematics and Shared Interpretative Resources

In the previous section, I pointed out that poor mathematics performance also affects students' ability to continue with non-compulsory mathematics. In many countries, an increasingly small percentage of students appear to be pursuing the study of mathematics at upper secondary level and beyond. Researchers like Nardi & Steward (2003) show that the student's choice is seriously influenced not only by her mathematical performance, but also by her attitudes towards mathematics. Mathematical performance and attitude are deeply shaped by students' mathematical experiences at school.

Furthermore, a disproportionately small percentage of black and female students continues with post-compulsory mathematics or go on to work in mathematics-heavy positions. If we take Nardi's claim seriously that these choices are shaped by the school mathematical experiences, we have to consider the aspects of teaching and learning which disproportionately affect black and female students.

In 2001, Leslie et al. (2001) examined the underrepresentation of women amongst those who attain Ph.D.'s in the fields of science, technology, engineering and mathematics (STEM) in the United States. They were investigating 'field specific ability beliefs', i.e. the way in which these academic fields were viewed by professional academics, and how these affected the representation of women in those fields. They did this by looking at what faculty, postdoctoral fellows and graduate students from different disciplines considered to be abilities necessary for success in their given field. They found that "the extent to which practitioners of a discipline believe that success depends on sheer brilliance is a strong predictor of women's and African Americans' representation in that discipline" (Leslie, 2001, p. 265). The more a field is thought to require raw intellectual talent, innate skills or inherent aptitudes that just can't be taught, the fewer female and black high-level practitioners there are. Mathematics in particular was thought to require a lot of sheer brilliance, and as such was one of the least representative fields in the Leslie et al. study.

In other words, how a discipline is represented had a big effect on who was allowed to be a part of that discipline and/or who was considered to be good at that discipline. This goes back to the conclusion of the first chapter: our account of mathematics and the reason for it being valuable, structure how students engage with mathematics.

Another way of seeing how the representation of a subject is important comes from the idea of self-to-prototype matching which comes from Hannover & Kessels (2004). Here self-to-prototype matching is seen as a way of making choices: in order to decide between alternative situation options, an individual imagines the prototypical person who would be found in each of the available options, and then compares the defining characteristics of these prototypes with the characteristics that she possesses or wishes to possess. She would then choose the option where there is the most overlap between who she is or wants to be and the prototypical person in that situation. Hannover & Kessels (2004) suggest that this is how students make subject choices at school. In order to decide, for example, whether to do mathematics, a student would imagine the prototypical mathematics student and see whether that is the type of person she is or wants to be. Therefore 'school students' avoidance of the sciences is due to (a) the specific image this subject domain has … and (b) the image students …. have of themselves or want to have of themselves" (Hannover & Kessels, 2004). The claim, more strongly put, is that the general image of science subjects and student self-image are "highly incompatible, consequently school students do not like science subjects (Hannover & Kessels, 2004). What this prototypical mathematics student looks like is a consequence of what we believe mathematics to be, how we think mathematics should be done, what counts as mathematical success and who gets to be good at mathematics.

In previous sections, I have already spoken about philosophies of mathematics as the coherent answers to ontological and epistemological questions pertaining to mathematics; and therefore as different ways of viewing and understanding mathematics. In this section, I add to that account, with the claim that our philosophies of mathematics should be understood as our *shared interpretative resources about mathematics*.

Our shared mathematical interpretative resources are the concepts and ways of thinking and talking from which individuals in a particular society can draw in order to make sense of mathematical practices, ideas and discourses when they encounter these. A coherent philosophy of mathematics can be seen as providing interpretative resources for mathematical lived experiences. In other words, a philosophy of mathematics allows students to see the value of mathematical practices, to make sense of the relationship between mathematical concepts and other beliefs they hold, to build a picture of what a successful student of mathematics looks like, etc. Generally, a philosophy of mathematics, seen as an interpretative resource, is a lens which colours how students interpret their interactions with all things mathematical. It allows students to explain some features of mathematics as they are experiencing it.

Let us go back to the examples of Abongile and Buhle. In Section 1.3. I hypothesised that Buhle had been exposed to the idea of non-Euclidean geometries. The instrumentalist philosophy of mathematics which Buhle endorses gives her a certain set of 'tools' that she can use to make sense of this mathematical concept. Buhle will use the instrumentalist answers to the ontological and epistemological mathematical questions, and in order to make sense of non-Euclidian geometry, she will conclude that by denying Euclid's parallel postulate, humans have created geometries which are useful for solving real-world problems e.g. in the field of special relativity. Her philosophy of mathematics provides her with the resources needed to interpret the mathematical phenomenon and to make sense of it. In the same way, if Buhle came into contact with the field-specific ability belief that to be successful in mathematics one requires raw brilliance, she would interpret this using the instrumentalist philosophy of mathematics. The instrumentalist belief holds that any mathematical method is acceptable as long as it produces real-world results, and if Buhle truly believes this, she would evaluate the claim that mathematics requires raw brilliance as false.

Abongile, with her realist philosophy of mathematics, on the other hand, would battle to make sense of non-Euclidean geometries since it is difficult to think of mutually exclusive geometries being instantiations of the same form of geometry. Additionally, since she thinks that mathematical knowledge is obtained through deductive reasoning, the claim that mathematics requires raw brilliance might strike a chord with her. If Abongile additionally believed herself to lack this raw brilliance, she would probably perform poorly at mathematics, or at the very least, not continue with post-compulsory mathematics. This is because her endorsed philosophy of mathematics creates a prototypical mathematics student which Abongile feels that she cannot hope to live up to.

Our shared mathematical interpretative resources, i.e. our endorsed philosophy of mathematics, provides the tools needed to make sense of our unique mathematical lived experiences. But as we saw in Section 1.3., our mathematical lived experiences also influence which philosophy of mathematics saves the phenomenon for us, and hence which philosophy of mathematics we endorse. However, we cannot just 'make up' ways of understanding mathematics, since we will not be understood by our epistemic community. Therefore, interpretative resources have to be shared in the sense that they are agreed upon by the community we find ourselves in.

According to Fricker, shared interpretative resources are generated by the public practices of journalists, politicians and academics. In terms of mathematics, shared interpretative resources are therefore generated by: how journalists write about mathematics and issues regarding mathematics, e.g. mathematical literacy; how politicians think about mathematics, particularly when setting up the mathematics curriculum or giving funding towards projects involving mathematics; how academics write about, think about and teach mathematics, especially in the sense that this has a trickle-down effect on how mathematics teachers are trained, how policy is influenced, how funding is allocated, and so on.

A natural consequence of the underrepresentation of black and female mathematicians in South Africa (and globally), is that the high-level positions in mathematical fields of all kinds, including mathematical journalism, mathematical policy making and academic mathematics, are dominated by white males. If black people and women never occupy the high-level positions in mathematical journalism, do not sit on policy-making bodies concerning mathematics education and are not represented in the academy which takes a position on mathematics and mathematics education, then the voices of these subordinated groups are prevented from participating on equal terms in the shaping of collective interpretative resources for mathematics. This means that the voice of subordinated groups is not heard when society creates concepts, words, ideas, norms and values that structure how we interpret our lived experiences of mathematics.

It is important that subordinated voices be heard when shared interpretative resources are generated because members of subordinated groups have different mathematical lived experiences. A white male typically does not have the experience of being stereotyped to lack raw brilliance and therefore stereotyped to be bad at mathematics. A white male typically does not have the experience of choosing to study mathematics-heavy fields to bolster his social identity in the context of a South Africa in transformation, nor does he know what it is like to be a black woman motivated to study mathematics so that she can open up opportunities to serve her community, nor does he know what it feels like to be a woman wanting to prove herself in a career traditionally dominated by white males (Jawitz, Case, & Tshabalala, 2000; Jawitz & Case, 1998). In order to create shared interpretative resources which help students to explain and make sense of their unique lived experiences, people with different lived experiences need to be involved in mathematical journalism, politics and academia.

In the absence of these types of shared interpretative resources, black and female students often do not see any value in studying mathematics; nor do they see how mathematics fits into other parts of their world and lived experiences. This would, predictably, decrease motivation and student performance would drop even further, which would lead to fewer black and female students continuing with post-compulsory mathematics.

This creates a negative feedback loop. If it is true that:

(i) poor performance in mathematics by subordinated groups is at least partly a result of a lack of interpretative resources regarding mathematics (that is, philosophies of mathematics) which track the lived experiences of these students and which these students can use to make sense of mathematics, and

(ii) members from subordinated groups are prevented from getting high-level positions in mathematical fields, and

(iii) interpretative resources are generated by people in the high-level positions in mathematical fields, and

(iv) without the creation of new interpretative resources subordinated students will still underperform and not make it to high-level mathematical positions, then

(v) it seems clear that the problem of unequal access to mathematics cannot be solved without specifically addressing this injustice resulting from how mathematics is viewed.

2.3 Affective Evaluation

In Section 2.2. we saw that a very small percentage of students continue with non-compulsory mathematics. In a study done by Howie & Hughes (1998, p. 38) a positive relationship has been observed between liking mathematics and achievement in mathematics. In this section I examine the relationship between attitudes and emotions towards mathematics and mathematical performance.

A peculiar thing about mathematics is that, although we agree that mathematics is valuable (as demonstrated in the Section 2.1.), we do not react to mathematics in a way which reflects the view that mathematics is valuable. Particularly, the layperson (and more importantly, most students), *does not feel positive emotion towards mathematics.*

Ask most people whether they like mathematics and the resounding answer will be 'no'. In general, people tend to feel what is known as 'disaffection' towards mathematics. Research often focuses on disaffection in the mathematics classroom as evidenced in disruptive behaviour, absenteeism or special needs. Nardi & Steward (2003) add disengagement and invisibility to this list. Disengaged or invisible students are students who attend classes and do the work but feel disaffection towards mathematics: they do not like mathematics because they find it tedious and something you can like only if you follow the exact rules and cues of the classroom. Depersonalised and isolating, mathematics is only something you can like if you are good at it.

Feminist literature on the emotions has shown us that emotions and values are closely related (see Jaggar, 1989; Mendus, 2000; Meyers, 1997). Emotions presuppose value; expressing emotion is partly a way of expressing the value attached to the object of that emotion. The object of an emotion, "that is, the object of fear, grief, pride, and so on – is a complex state of affairs that is appraised or evaluated by the individual. … Emotions and evaluation are logically or conceptually connected" (Jaggar, 1989). Not liking mathematics, therefore, is an evaluation of mathematics. Students who feel disaffection towards mathematics are making a negative appraisal or evaluation of mathematics as they experience and conceptualise it.

In the mathematics classroom in particular, we have many students who are expressing negative emotions about mathematics and in doing so are expressing that they attach disvalue to mathematics. But in the previous section we established that the societal consensus is that mathematics *is* valuable. Given that this is so, it would not unreasonable to expect that most students value mathematics, and to expect them to attach value to mathematics by liking mathematics.

But students do not feel these societally acceptable, positive emotions towards mathematics; instead, they feel disaffection which is a societally unacceptable emotion towards something deemed valuable. Societally unacceptable emotions are what Jaggar (1989) calls *outlaw emotions*. An outlaw emotion is an affective evaluation of an aspect of the world that does not match up with the affective evaluation prescribed by society-atlarge. The example Jaggar gives is of a black person who experiences anger rather than amusement when a racist joke is recounted. The societally prescribed emotion (in a racist society) is amusement – the anecdote is told in the form of a joke – but the values, norms and experiences of the black person clashes with those of the joke-teller, resulting in a societally unacceptable, or outlaw emotion, from the black person.

The object of an emotion, in this case mathematics, is 'a complex state of affairs that is appraised or evaluated by the individual'. To appraise mathematics is to try to make sense of mathematics, to give a description of the 'complex state of affairs' that makes up mathematics. After mathematics has been appraised, the appropriate emotion is meant to follow. These descriptions of mathematics are what I refer to as philosophies of mathematics and include ways of talking about mathematics, as well as ways of delineating mathematical concepts and ideas. Different philosophies of mathematics therefore offer different evaluations of mathematics.

Since emotions reflect value, societally acceptable emotions reflect the values, norms and experiences of members of the dominant social group. These values, norms and experiences are also reflected in how the dominant social group makes sense of mathematics, i.e. which philosophy of mathematics they endorse.

Mathematics students who experience outlaw emotions are individuals for whom the descriptions, concepts, ideas and ways of talking about mathematics employed by the dominant social group makes it more difficult, not easier, to make sense of mathematics. In other words, the philosophy of mathematics endorsed by the dominant social group, upon appraisal from these individuals, does not merit a positive valuation and therefore these mathematics students feel disaffection instead of the more socially acceptable positive attitudes towards mathematics.

Jaggar (1989) suggests that when we encounter an example of people feeling outlaw emotions it should motivate us to investigate the prevailing conceptions of the world. Outlaw emotions may provide the first indications that something is wrong with socially accepted or standard understandings of how things are. Conventionally unexpected or inappropriate emotions may precede our conscious recognition that the way in which we are describing or justifying a part of the world (in this case, mathematics) is making it difficult or impossible for members of subordinated groups to make sense of this part of the world. Outlaw emotions are useful because they invite us to "reflect on our initially puzzling irritability, revulsion, anger, or fear [so that] we may bring to consciousness our 'gut-level' awareness that we are in situations of coercion, cruelty, injustice or danger" (Jaggar, 1989, p. 167).

I apply the idea of outlaw emotions to mathematics in the following way: the fact that students do not like mathematics should lead us to investigate our standard ways of making sense of mathematics. It may be the case that how we describe mathematics or justify mathematics education exemplify certain norms and values which are exclusionary. Therefore, the very resources which should help students understand the nature and value of mathematics makes mathematics harder to understand and appreciate for students who are members of subordinated groups.

When we are examining the standard definitions, concepts, ideas and ways of talking about mathematics, outlaw emotions can alert us to two types of injustice. Firstly, individuals with mathematical lived experiences which cause them to appraise and evaluate mathematics in different ways to the dominant social group, are living in a world where mathematics is structured by the dominant group for their purposes. These purposes have not been determined by the subordinated individual and are "in various degrees inimical to their development and even existence" (Fricker, 2007, p. 147). The effect of this is that the subordinated individual does not have the definitions, concepts, ideas and ways of talking about mathematics which *do* serve her purposes. The second type of injustice is that the "prejudicial flaws in shared interpretive resources" (Fricker, 2007, p. 147), that is, an exclusionary philosophy of mathematics, prevents the subject from making sense of an experience which it is strongly in her interests to render intelligible. This links up to the value of mathematical performance examined in the previous section. If a student cannot make sense of mathematics because the resources she has access to in order to do so are prejudicially flawed or skewed, then her inability to realise any of the value of mathematics is a matter of injustice.

Since in this thesis we are concerned with the influence of shared interpretative resources on mathematical teaching and learning, the specific type of injustice we would want to investigate is that of *hermeneutical injustice.*

2.4 Hermeneutical Injustice

As we give up on sexist and racist beliefs, we also endorse the belief that black or female students are not bad at mathematics because of their race or gender. The poor mathematical performance, dislike of mathematics and poor representation of black and female students in post-compulsory mathematics is therefore related not to a feature of being black or female, but instead strongly suggests that some form of structural injustice is stifling certain students' ability to learn mathematics. One facet of this structural injustice is that the resources available to us to make sense of our mathematical experiences are skewed or contain prejudicial flaws. In this thesis I am concerned with this facet of the injustice – hermeneutical injustice – which is present in the mathematics classroom.

When we are investigating a hermeneutical injustice, we are concerned with assessing shared interpretative resources to see whether they are exclusionary, particularly whether they prevent individuals or groups from making sense of a phenomenon which it is strongly in their interests to render intelligible. Specifically, a group's unequal hermeneutical participation will tend to show up in a localized manner in hermeneutical hotspots — locations in social life where the powerful have no interest in achieving a proper interpretation, perhaps indeed where they have a positive interest in sustaining the extant misinterpretation. (Fricker, 2007, p. 152)

The school environment, considered broadly, is one such hotspot, since it is, as a social epistemic practice, particularly vulnerable to injustice concerning coming to know and the interpretation and transfer of knowledge.

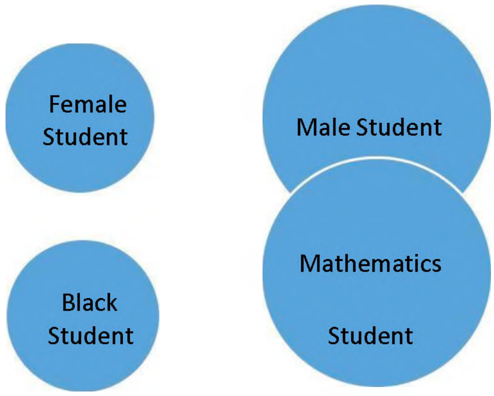
Because mathematics-heavy jobs tend to be lucrative and influential, mathematics education has particularly close ties to societal power structures. Due to the systemic nature of the inequality, only certain people can be members of the powerful group, and it tends to be these people who currently hold lucrative and influential mathematics-heavy jobs. The members of the powerful group are, due to their social positioning, often unable to recognise inequity in the system they find themselves in. Even when they do recognise inequity, they rarely know how to redress it (see for instance Mills, 2005).

In previous sections of this chapter, I have discussed how poor mathematical performance, a lack of black and female representation in post-compulsory mathematics and the affective evaluation of mathematics, are influenced by how we talk about mathematics. These shared interpretative resources regarding mathematics do not exist in isolation; they work together with other social interpretative resources. Particularly important are other discriminatory social discourses, especially racist and sexist discourse, as well as other stereotypes and identity prejudices. These discriminatory discourses change the context of mathematical shared interpretative resources.

For example, the belief that mathematics requires innate talent is not always a harmful shared interpretative resource. But combined with the fact that "laboratory, observational, and historical evidence reveals pervasive cultural associations linking men but not women with raw intellectual talent" (Leslie, 2001, p. 262) and the ambient stereotypes produced by this perception, the belief that mathematics requires innate talent in effect excludes women. There are several mechanisms by which field-specific ability beliefs might influence women's participation. Firstly, the practitioners themselves might doubt that women possess the correct innate abilities needed to succeed in their field and therefore exhibit biases against them. Secondly, the emphasis on raw aptitude may activate the negative stereotypes in women's own minds, making them vulnerable to stereotype threat. As a result of these processes, women may be less represented in 'brilliance required' fields.

Leslie et al. also show that since the same stereotypes often refer to black people, the underrepresentation of black people in mathematics-heavy fields can also be explained by referring to field-specific ability beliefs.

The belief that mathematics requires innate talent is part of a system of shared interpretative resources which is flawed and unjustly skewed in favour of already powerful groups. Leslie et al. suggests that practitioners who want to diversify their fields "might want to downplay talk of innate intellectual giftedness and instead highlight the importance of sustained effort for top-level success in their field" (Leslie, 2001, p. 265). Using the vocabulary used in this thesis, the solution to the lack of diversity is to diversify the shared interpretative resources to include a variety of different philosophies of mathematics.



Additionally, if the prototypical student of mathematics is determined by our philosophy of mathematics, as described in Section 2.2., then the prototypical female or black student is influenced by the social gender and race stereotypes. White male students, for example, are stereotypically held to be intelligent, innately brilliant, practical, good at abstract and deductive thinking and to have good spatial and visual perception. Consequently, they are socialised by their culture to enact this stereotype. White male students might therefore more easily (than members of other groups) match who they take themselves to be with who they take an ideal mathematics student to be (Hannover & Kessels, 2004).

If, due to prejudice and stereotyped socialisation, the self-image of a female or black student does not include any characteristics which would usually be attributed to the prototypical mathematics student, then there will be no overlap between her idea of herself and the prototype.

When a philosophy of mathematics becomes problematic due to discriminatory beliefs about subordinated groups held by a given society, then members of subordinated groups do not have adequate resources to make sense of mathematics in the context of stereotypes about their groups. In Section 2.2. I gave the hypothetical example of Abongile and Buhle responding to the idea that mathematics requires raw brilliance. Buhle's instrumentalist philosophy of mathematics allows her to dismiss the raw brilliance claim as false, so that even if she was stereotyped to lack raw brilliance, she could still continue with mathematics. Abongile's realist philosophy of mathematics does not allow her this recourse. When faced with the stereotype that women lack raw brilliance of the mathematical kind, Abongile has to either agree that she cannot be good at mathematics, or she has to deny that women lack raw brilliance.

Leslie et al. mentions the possibility that women like Abongile in Britain internalise the stereotypes that women lack raw brilliance, and therefore decide that STEM fields are not for them. Jawitz et al. give us reason to believe that the same thing is happening with South African students. Female students in their study found mathematics "a total turnoff" and while students consider themselves "logical and… talent[ed] in that area [STEM fields]", when they think of mathematics they say "I just wanna run" (Jawitz et al., 2000). Furthermore, they choose not to continue with mathematics or mathematics-heavy careers because "everybody was saying that it's too male-dominated. It's not a career for females" (Jawitz et al., 2000). A student's philosophy of mathematics therefore also influences how she deals with gender or race or class stereotypes as they affect her mathematical lived experiences. Graven & Heyd-Metzuyanim (2014) discuss the descriptions of a successful mathematics student held by younger South African students. They found that 22% of learners think that a good student must have an innate talent for mathematics, while only 2% of students believe that a good mathematics student thinks hard about mathematics, and only 1% believe that the good mathematics student is the one who works hard (Graven & Heyd-Metzuyanim, 2014, pp. 47–8).

From the discussion above, we can see that how a subject is represented (and consequently how a subject is taught) can reinscribe or reflect structural injustice. Part of the harm that is done is that subordinated students are stuck with a self-image highly incompatible with their image of a mathematics student, and therefore they will be unable to learn more about mathematics in their everyday lives; nor will they be able to continue to high-level mathematics classes or careers. All of the value of mathematics is lost on these students.

The problem here seems to be that subordinated groups do not have the necessary collective forms of social understanding that would enable them to interpret mathematics and the mathematics classroom in such a way that it seems relevant to their self-image, to their interests and hopes, to their economic position, to what they find valuable, to their mathematical lived experiences. There are no hermeneutical resources available to subordinated students that allow them to see mathematics as something which relates to them and the lives that they live.

Instead, hermeneutical resources have been skewed by unequal power relations so that the powerful tend to have access to an appropriate understanding of their experiences, ready to draw on as they make sense of their lived experiences, whereas the subordinated are more likely to "find themselves having some experiences through a glass darkly, with at best ill-fitting meanings to draw on in the effort to render them intelligible" (Fricker, 2007, p. 148).

Knowing that hermeneutical injustice is present in mathematics education is important because if we know that shared interpretative resources that subordinated groups need to make sense of mathematics is missing, then we can start to put processes in place to allow subordinated groups to create these interpretative resources.

2.6 Concluding Remarks

Mathematics is valuable as a precondition for making sense of the world. It has further value in its application to everyday activities. Mathematics opens doors for further study and career options. It is because of the value of mathematics that we teach mathematics at school. However, many students from subordinated groups do not get any value from their mathematics education because the way in which mathematics is represented does not give them any resources to make sense of what mathematics is, how it fits into their lives, why it is valuable or why they should care. The creation of the needed interpretative resources is hindered by the fact that those in positions which allow for the creation of interpretative resources – journalists, policy makers and academics – are not members of the subordinated groups which are in need of interpretative resources.

Seeing a dislike of mathematics, in the form of absenteeism, special needs, disruptive behaviour, disengagement and invisibility, as an expression of an outlaw emotion, allows us to analyse the ill-fitting interpretative resources discussed above using the lens of injustice.

In Chapter 1 I showed that our hermeneutical resources like our philosophies of mathematics have a significant impact on how we teach and learn mathematics. When considering injustice in the mathematics classroom, as Chapter 2 gives us reason to think that we should, it is therefore reasonable to investigate the existence of a hermeneutical injustice.

The next chapter of this thesis is devoted to applying the concept of hermeneutical injustice to South African mathematics education.

Chapter 3

3.1 Hermeneutical Injustice in the South African Mathematics Classroom

This chapter applies the conclusions from the last chapter to the South African context. First, I examine whether South African students also experience outlaw emotions regarding mathematics. Then I examine ways in which we talk about mathematics in South Africa. I focus particularly on the *shared* interpretative resources available to mathematics teachers in the form of the South African mathematics curriculum statement. I show that this resource, due to its fragmented nature, is unable to provide a *coherent* philosophy of mathematics which has explanatory potential (or manages to save the phenomenon) for students or teachers.

3.2 Outlaw Emotions in the South African Mathematics Classroom

It is clear that South African mathematics students feel disaffection towards mathematics. According to Howie & Hughes (1998), 14% of South African mathematics students who participated in the TIMSS claimed to dislike mathematics, while 8% disliked mathematics a lot. The same report states that "academic performance is supported by student perceptions" of mathematics and that "liking mathematics is positively related to higher achievement" (Howie & Hughes, 1998, p. 147). The poor mathematical performance is therefore also an indicator of wide-scale disaffection with mathematics.

Mathematics is also valued in South Africa. Former Education Minister Naledi Pandor said that "Maths, science and technology are now more important than they have been in our recorded history" (Unknown, 2007). We also value mathematics enough to make it compulsory for every school child to study mathematics for the duration of their school career. South African disaffection with mathematics is unexpected and socially unacceptable.

Jawitz, Case and Tshabalala (Jawitz et al., 2000) also acknowledge the importance of liking mathematics and science for continuing on to mathematics-heavy fields. They note that female students at South African universities, despite having performed well in these subjects at school, were either indifferent or had a strong dislike for mathematics and that this disaffection is what caused them not to pursue ‘maths-heavy’ careers. However, in this study there is also an investigation of student perceptions through investigating students’ beliefs about themselves. These are formed “from their interpretation of past experiences and perceptions of the attitudes and expectations of others, such as teachers and parents” (Jawitz et al., 2000). In other words, students use the interpretative resources provided by their teachers and parents to form beliefs about themselves *qua* mathematicians. Jawitz et al. note that, based on these interpretative resources, students find mathematics “a total turnoff”; additionally, students think of themselves as students who are “logical and [have] got the talent in [STEM subject areas]” but when “I just think of [mathematics] I just wanna run” (Jawitz et al., 2000).

This matches my experiences with students. In my conversations with students both from well-resourced private schools and disadvantaged township schools, in groups or oneon-one, most students display some disaffection towards mathematics. Some students consider themselves to be too stupid to be good at mathematics; others feel that they have no affinity for mathematics, or lack innate mathematical talent; some just shake their heads at me and refuse to discuss mathematics at all; some classes talk amongst themselves or ask disruptive questions whenever I ask mathematical questions; other students do not see the value of mathematics since even with a matric pass in their peers still end up doing menial jobs in the retail sector.

Indications are, therefore, that South African mathematics students feel disaffection with mathematics, a disaffection which is largely reinforced in the classroom. Furthermore, this disaffection arises from the view of mathematics the students hold themselves, but also from the views of the parents and teachers who influence them. I examine how mathematics is viewed in the next section.

3.3 Shared Interpretative Resources in the South African Mathematics Classroom

The ways in which mathematics is commonly conceived of in South Africa are varied. Mathematical success is considered a measure of intelligence, and a mathematics-heavy career is seen as a mark of status (see Jawitz et al., 2000; Jawitz & Case, 1998).

Mathematics is also punted as a necessary skill for students to get jobs, with Communications Deputy Minister Ndabeni-Abrahams saying to students: “Believe me when I say maths and science gets you everywhere and every sector of our economy has been transformed by technology and there is no job where maths is not useful. For you to succeed in the modern world, you need maths and science and this message should resonate with you when choosing subjects of choice” (Unknown, 2015b). The Deputy Minister also stressed the importance of scientists, engineers and mathematicians in ‘moving South Africa forward’. This includes the normative claim that social identity in the context of a South Africa in transformation is important to how we talk about mathematics. This way of viewing mathematics is also reflected in the black female students interviewed by Jawitz et al. (2000; 1998) who were motivated to study mathematics and mathematics-heavy fields in order to open up opportunities to serve their community. The Department of Education said that improved performance in mathematics and science, especially at high level, is of “great strategic importance both for economic growth and in order to empower individuals with globally competitive skills” (Unknown, 2015a). This is a variation on the often cited idea that literacy in mathematics and science affects economic productivity. Particularly, that “world-class competence in these subject domains is essential to compete successfully in today’s global marketplace” (Hannover, 2004:52). But it also contains the normative claims that mathematics and mathematics education should empower individuals, which is to say that it must be accessible to most students and relevant to their economic, social and political activities (Mathume, 2012).

Since hermeneutical injustice is concerned with the prejudicial flaws in, and skewing of, shared interpretive resources (Fricker, 2007, p. 147), we need a way of assessing mathematical shared interpretative resources in South African classrooms to see whether they are exclusionary; specifically whether they prevent individuals or groups from making sense of a phenomenon which it is strongly in their interests to render intelligible. I will be doing this by examining the mathematics curriculum and curriculum statements in South Africa. The mathematics curriculum is the piece of policy which most affects how mathematics is taught. It provides the outcomes and aims of mathematics education; classroom mathematics is assessed according to the curriculum, and mathematics teachers are trained in order to be able to teach the curriculum. As a shared interpretative resource, the mathematics curriculum has clearly developed over time in response to a clearer understanding of the social ramifications of the way we think about mathematics, come to know about mathematics, and therefore also of how we teach and learn mathematics. What follows is a compilation of quotations from the Curriculum Statement and Curriculum and Assessment Policy Statement provided by the South African Department of Education:

The curriculum for mathematics is based on the following view of the nature of the discipline (D. of E. DoE, 2003):

> Mathematics enables creative and logical reasoning about problems in the physical and social world and in the context of Mathematics itself. It is a distinctly human activity practised by all cultures. Knowledge in the mathematical sciences is constructed through the establishment of descriptive, numerical and symbolic relationships. Mathematics is based on observing patterns; with rigorous logical thinking, this leads to theories of abstract relations. Mathematical problem solving enables us to understand the world and make use of that understanding in our daily lives. Mathematics is developed and contested over time through both language and symbols by social interaction and is thus open to change.

This definition is built upon in the Curriculum and Assessment Policy Statement (D. of B. E. DoE, 2011):

> Mathematics is a language that makes use of symbols and notations for describing numerical, geometric and graphical relationships. It is a human activity that involves observing, representing and investigating patterns and qualitative relationships in physical and social phenomena and between mathematical objects themselves. It helps to develop mental processes that enhance logical and critical thinking, accuracy and problem solving that will contribute in decision-making.

The purpose of mathematics is defined in the curriculum statements as follows (D. of E. DoE, 2003):

> In an ever-changing society, it is essential that [students] … acquire a functioning knowledge of the Mathematics that empowers them to make sense of society. A suitable range of mathematical process skills and knowledge enables an appreciation of the discipline itself. It also ensures access to an extended study of mathematical sciences and a variety of career paths.
> The study of Mathematics contributes to personal development through a deeper understanding and successful application of its knowledge and skills, while maintaining appropriate values and attitudes. Mathematics is a discipline in its

own right and pursues the establishment of knowledge without necessarily requiring applications in real life. Competence in mathematical process skills such as investigating, generalising and proving is more important than the acquisition of content knowledge for its own sake.

Mathematical competence provides access to rewarding activity and contributes to personal, social, scientific and economic development. … Individual and collective engagement with Mathematics will provide valuable opportunities for the development of a variety of values, as well as personal and interpersonal skills.

While the CAPS adds that (D. of B. E. DoE, 2011):

The teaching and learning of Mathematics aims to develop the following in the learner:

- a critical awareness of how mathematical relationships are used in social, environmental, cultural and economic relations;
- confidence and competence to deal with any mathematical situation without being hindered by a fear of Mathematics;
- an appreciation for the beauty and elegance of Mathematics;
- a spirit of curiosity and a love for Mathematics;
- recognition that Mathematics is a creative part of human activity;
- deep conceptual understandings in order to make sense of Mathematics; and
- acquisition of specific knowledge and skills necessary for: o the application of Mathematics to physical, social and mathematical problems,
    - the study of related subject matter (e.g. other subjects), and
    - further study in Mathematics.

The description of mathematics listed above does not match any accepted philosophy of mathematics or philosophy of mathematics education (see the Appendix for a description of existing and possible philosophies of mathematics, or the Definitions section for a brief overview of any particular philosophy of mathematics). In other words, the South African mathematics curriculum does not contain a single, coherent philosophy of mathematics. Instead, the ontological account of mathematics given above seems to incorporate ideas from lots of different philosophies of mathematics. Using the quotes from the curriculum statements above, I perform an exegesis of the definitions of mathematics. Each definition has component parts that fit best with a variety of philosophies of mathematics. The different philosophies of mathematics drawn upon by the curriculum statements can be seen below:

From the Standard View of Mathematics:

- Mathematics is a discipline in its own right and pursues the establishment of knowledge without necessarily requiring applications in real life.
    - o E.g. being able to appreciate the beauty and elegance of mathematics
- A suitable range of mathematical skills and knowledge enables an appreciation of the discipline itself. Here are two examples:
    - deep conceptual understanding needed in order to make sense of mathematics
    - the knowledge and skills necessary for further study in mathematics
- Mathematics is the act of observing patterns which are already there.
- Rigorous logical thinking, together with observation, leads to theories of abstract relations.
- These numerical, geometric and graphical relationships are described by the language of mathematics.

From the Instrumentalist View of Mathematics:

- Mathematics is a distinctively (creative) human activity, that is, mathematics is created by people rather than being discovered.
- Mathematical problem solving enables us to understand the world.
- It also ensures access to continued study of mathematical sciences and a variety of career paths.
- The acquisition of specific mathematical knowledge and skills are necessary for the application of mathematics to physical, social and mathematical problem, and for the study of related subject matter (e.g. other subjects like science, finance, engineering, medicine, aeronautics etc.).

- Mathematical competence provides access to rewarding activity and contributes to personal, social, scientific and economic development.

From the view of Mathematics as Process and Practice:

- Competence in mathematical process skills such as investigating, generalising and proving is more important than the acquisition of content knowledge for its own sake.

From the Social Constructivist View of Mathematics:

- Mathematics is a distinctively (creative) human activity i.e. mathematics is created by people, not discovered, and is not power neutral. Links to a humanistic mathematics or ethno-mathematics.
- Mathematical problem solving enables us to understand the world and make use of that understanding in our daily lives. Students should acquire a functioning knowledge of the mathematics that empowers them to make sense of society and gives them a critical awareness of how mathematical relationships are used in social, environmental, cultural and economic relations.
- Mathematics is developed and contested over time through both language and symbols by social interaction and is thus open to change.
- Mathematical competence provides access to rewarding activity and contributes to personal, social, scientific and economic development. … Individual and collective engagement with mathematics will provide valuable opportunities for the development of a variety of values, as well as personal and interpersonal skills.
- Mathematics education should give students the confidence and competence to deal with any mathematical situation without being hindered by a fear of mathematics.

From the Virtue Epistemology View of Mathematics:

- Mathematics is the type of subject which enables creative and logical reasoning.

- It helps to develop mental processes that enhance logical and critical thinking, accuracy and problem solving that will contribute in decision-making.
- The study of mathematics contributes to personal development through a deeper understanding and successful application of its knowledge and skills, while maintaining appropriate values and attitudes.

The Wonder Account of Mathematics

- Mathematics education should develop in students a spirit of curiosity as well as an appreciation for the beauty and elegance of mathematics, and a love for mathematics.

One may be inclined to think that the inclusive nature of this curriculum is a step in the right direction. Perhaps both Abongile's and Buhle's view of mathematics can be accommodated in this curriculum statement?[1]

I am inclined to think that neither Abongile nor Buhle (nor their mathematics teacher) would find reading through the curriculum statement particularly helpful in the process of trying to make sense of what they think mathematics is. These ideas are not properly combined in the curriculum document to form one or more complete philosophies of mathematics; instead, the curriculum as an interpretative resource only gives bits and pieces of the explanatory account necessary to make sense of mathematics. Some accounts provide answers to the ontological questions regarding mathematics (for instance, what *is* mathematics, what makes an entity mathematical, what counts as doing mathematics, what content should we teach in the mathematics classroom, and so forth) without providing coherent answers to the epistemological questions regarding mathematics (for instance, how do we gain mathematical knowledge, how should I assess mathematics in the classroom), and vice versa. The standard view of mathematics is quite clear on *what* mathematics is, for example, but not clear on how exactly we should go about discovering these abstract relationships; the instrumentalist view of mathematics on the other hand is clear on the fact that mathematics has to have a specific *result* (real-world problem solving)

[1] My thanks to Megan Laverty for pointing this out.

but is unclear about what makes mathematics different from science or engineering or any other field that solves problems in the world.

But within the curriculum there are also philosophical claims which are incompatible with each other, notably clashing answers to the ontological and epistemological questions regarding mathematics. For example, the value of mathematics as described by the standard view of mathematics and as described by the instrumentalist view of mathematics are incompatible. If something is seen as an instrument to achieve an end, it is not obvious at first glance how it is possible to also see it as something valuable in itself. For example, money is an instrumental good, so by definition it is only valuable because we can get ultimately valuable goods by spending money. In the same way, in an instrumentalist view of mathematics, mathematics is seen as a tool to solve problems. Mathematics is not valuable in itself, it is only valuable in so far as it achieves the end of solving problems. On the standard view of mathematics, mathematics is a beautiful system of patterns which we discover through logical thought and which should be valued in itself. One cannot straightforwardly value mathematics 'in itself' while also valuing it as a tool to achieve other ends, unless one agrees with the instrumentalists that mathematics is essentially and solely a tool. The same clash applies to accounts which see mathematics as a tool to acquire jobs, or to the country's economic success, or even a tool to develop critical thinking.

There is also a clash between instrumentalist and standard views of mathematics on the one hand, and the view of mathematics as practice and process. Instrumentalist and standard views of mathematics recommend the acquisition of specific knowledge necessary for the application of mathematics to physical, social and mathematical problems; to the study of related subject matter (e.g. other subjects); and to further study in mathematics. Mathematics as practice and process, on the other hand, stresses that competence in mathematical process skills such as investigating, generalising and proving is more important than the acquisition of content knowledge for its own sake. These are just two examples that reveal the inconsistencies in the conception of mathematics contained in the syllabus and endorsed by the department of education. More examples of incompatible mathematical claims can also be found within the curriculum and other supporting documents.

After the fact, then, we may be able to find a place for both Abongile and Buhle in different fragments of the philosophies included in the curriculum statement. But the curriculum statement is not much help in providing educators and learners with different (viable and complete) philosophies of mathematics which can serve as the basis of their mathematical teaching and learning.

The combination of different philosophies of mathematics in the mathematics Curriculum is a result of South Africa's political history. Vithal & Volmink identify the "mathematics and mathematics education knowledge foundations (theories and practices) that have informed particular [curriculum] reforms" (Vithal, 2005), giving special attention to the socio-political and economic imperatives that have necessitated particular reforms. Powerful groups, as they go about their lives, create hermeneutical resources that suit their purposes. However, these hermeneutical resources often overlook subordinated groups, or actively disadvantage them. Because less powerful groups do not occupy important positions in a society, the opportunities to challenge or differently engage with these interpretations do not easily arise or are purposefully blocked. Theory and practice are always intertwined with historical, socio-political and economic factors. As Charles Mills points out, all theorizing takes place in an intellectual realm "dominated by concepts, assumptions, norms, values, and framing perspectives that reflect the experience and group interests of the privileged group (whether the bourgeoisie, or men, or whites)" (Mills, 2005, p. 175).

In the South African mathematics education context, the Apartheid government created a way of seeing people that actively disadvantaged subordinate groups. The official stance of the Apartheid government, in the words of Dr HF Verwoerd regarding mathematics in Bantu Education, was as follows:

> When I have control over native education I will reform it so that the Natives will be taught from childhood to realise that equality with Europeans is not for them … What is the use of teaching the Bantu mathematics when he cannot use it in practice? The idea is quite *absurd.* (*House of Assembly Debates* Vol. 78, August/September 1953:3585)

Expressed in this quote is a racist ideology, but also implied philosophical claims about what mathematics is, in the ontological sense. However, these claims only state that mathematics is the type of thing which black students cannot put into practice because of their racial inferiority, or because of their role in society (like the slaves in Rome). Vithal & Volmink

(2005) identify Fundamental Pedagogics as the pervasive "ideological practice masquerading as theoretical practice" in Apartheid mathematics education which, together with behaviourism, determined how mathematics was taught in Apartheid South Africa.

Furthermore, with only 12% of black secondary school teachers having a degree, mathematics teaching by and large was tackled by teachers barely one step ahead of their students. As a result authoritarianism and rote-learning methods predominated. In many schools, this is still the case. Such approaches nurtured the view that mathematics was an unalterable body of truths.

During the late Apartheid years, two developments in mathematics education were introduced into the South African mathematics education sphere. Firstly, there was a world-wide move towards a constructivist approach towards mathematics. Secondly, from anti-Apartheid resistance 'People's Mathematics for People's Power' arose.

Constructivism focuses on mathematics as a human construction used to address, describe and solve problems facing society at any particular moment. Constructivism can broadly be called a problem-centred approach and provides a prescriptive methodology for how mathematics should be taught and learnt – it therefore gives good answers to the epistemological questions about mathematics.

It is mathematics' responsiveness to the environment which was suppressed and obscured during the Bantu education regime. Vithal & Volmink worry that constructivism "takes for granted that mathematics is an endeavour for self-empowerment in which the issues of broader social responsiveness remain un- or underdeveloped" (Vithal, 2005), therefore failing to take the socio-economic and political dimensions of the mathematics curricula of Apartheid education seriously enough. The question that still remains to be answered is "Whose interest is this suppression serving?"

People's Mathematics shows that, by using mathematics as a way of keeping certain people out of power, the Apartheid way of thinking about mathematics serves the interest of the Apartheid government and the white populace. Vithal & Volmink sees people's mathematics not so much as a mathematics education theory or philosophy but rather as a political programme intended to bring an awareness of the injustices of Apartheid ideologies into the mathematics curriculum. It is important to note, however, that there are still implicit beginnings of answers to the epistemological and ontological questions about mathematics in the programme of People's Mathematics. For instance, it implies that mathematics is the type of topic that can be learnt (and used) by all students if the structural obstacles put up by the Apartheid state are removed from the mathematics classroom.

In the new South Africa, the Mathematics Curriculum had to determine the direction in which mathematics education would move in South Africa. The creators of postApartheid shared mathematical interpretative resources were faced with (1) the Apartheid regime's racist philosophy of mathematics and the rote learning tradition which the Bantu education system created, (2) constructivism which provides a clear outlook on *how* to teach mathematics, but neglects *why* or *what* we should teach, and (3) People's

Mathematics which is unclear both on the ontology and epistemology of mathematics, but states that *everyone* should learn mathematics for their social, political, economic and democratic development. Their solution was outcomes-based education (OBE). While OBE is set to be phased out and replaced with Schooling 2025 (content as yet unspecified), it still dominates the current educational climate in South Africa.

OBE was adopted in South Africa from the mid-1990s onward, and is concerned with the implications of a rapidly globalising economy, where a larger range of citizens are consumers and producers of mathematics and its applications. OBE pedagogy is therefore driven by the integration of education and training with the workplace, and by its ability to articulate non-formal and informal education processes, especially in the workplace (Vithal, 2005). This concern with non-formal mathematics and broad-based mathematical literacy is to some extent a product of the rhetoric of people's mathematics. But although OBE has a strong pedagogy, it is based on a weak epistemology, with the curriculum spending little to no time defining what mathematics is and how we gain mathematical knowledge. The Curriculum and Assessment Policy Statement (CAPS) which was later introduced for each subject was an attempt to provide a better epistemological grounding to OBE.

Because OBE is philosophically thin on the ground, teachers following the new curriculum come into this pedagogy with half-articulated theoretical underpinnings from either fundamental pedagogy/behaviourism or people's mathematics/constructivism, depending on how they were trained. This leads to teachers either claiming that they have been implementing the progressive pedagogy implied in the new curriculum all along, or being unable to implement the new outcomes. These two responses are reflective of different socio-economic positions in society: the physically and humanly well-resourced schools leave their pedagogic practices unchanged, while the poorer schools with poorly trained teachers flounder under the demands of OBE (Vithal, 2005). Note that a hybrid philosophy of mathematics is not necessarily problematic in itself. Instead, a prescribed philosophy of mathematics becomes problematic when one cannot reconcile the different ontological and epistemological claims that it contains. I think that the historical and political agenda of the mathematics curriculum is important, and that it is trying (but failing) to articulate a way of making sense of mathematics for the citizen while holding on to the belief that specialist and high-level mathematical performance is important. But the furthering of this historical and political agenda should not happen at the expense of students.

We can therefore see that philosophies of mathematics change according to the historical and political realities of a country. Most new philosophies have not been chosen because they have been researched and proven to be better, but are rather implemented as a reaction to unacceptable social or power dynamics. The structural inequalities that still exist in South Africa mean that any incoherence in the philosophies of mathematics that influence pedagogy will systematically disadvantage certain groups in society disproportionately. The current curriculum framework, OBE, is particularly susceptible to this problem since, while it provides a strong account of which pedagogy teachers should embrace, it is unclear as to the mathematical or epistemological content of this pedagogy. This theoretical lacuna means that teachers have to fill in the content with no support from policy makers.

The absence of a coherent philosophy of mathematics in the South African schooling system can therefore be seen to be a historically-created problem. Different philosophies and ideologies have clashed, and only fragments of the different accounts remain. These fragments are the implicit assumptions that teachers and policy makers work from, but it is not enough for a meaningful interaction with mathematics or for successful mathematics teaching.

3.4 Understanding, Intelligibility and Explanatory Potential

It remains to be shown that this lack of a coherent philosophy of mathematics is negatively affecting mathematics students in South Africa, and that the lack of coherent interpretative resources regarding mathematics is related to the poor Trends in International Mathematics and Science Study (TIMSS) score results.

I take it as a given that in order to do well at mathematics, and to get value from mathematics, a student has to have some level of mathematical understanding. This mathematical understanding is necessary to make sense of mathematics. Being able to make sense of mathematics requires one of two levels of mathematical understanding. The first level of mathematical understanding is what Jones refers to as an understanding which makes quite simple contributions to people's lives, and may even take the form of single claims (Jones, 2014: 4:15). When a student understands a simple contribution, for example a single mathematical claim, she attains the potential to *explain* some facet of mathematics or of her mathematical experience. These claims most often take the form of answers to explanation-seeking questions. A lot of these questions are philosophical questions, relating to the ontological and epistemological questions regarding mathematics. It is these that I am concerned with here.

Let us return to the hypothetical cases, discussed throughout this thesis, of Abongile and Buhle in order to exemplify the type of propositions which might have such explanatory potential. Imagine that Abongile has just been introduced to 'unknown' entities in algebra, such as $x = 1/3 + 2$. In order for Abongile to be able to make sense of this type of mathematics, she needs to have an answer to many explanation-seeking questions, like: Can you explain what makes two sides of an equation equal? Can you explain what a fraction is: does it mean divide, does it indicate a proportion or ratio, or does it indicate 'a bit of' something? Can you explain what it means for two entities to be directly or inversely proportional to each other? Can you explain why we graphically represent proportionality? Can you explain why, if we graphed this equation, it would form a straight line?

Buhle, on the other hand, may be working with the Riemann sphere in her course on special relativity. The Riemann sphere is a model of the extended complex plane – the complex plane plus a point at infinity. The extended complex plane is useful because it allows for division by zero in some circumstances, for example $1/0 = \infty$ is well-behaved. In order for Buhle to make sense of the complex plane, she needs to know, amongst other things, what an imaginary number is, what makes non-Euclidean geometry different from Euclidean geometry, when we can and cannot divide by zero and why, what infinity is and what a point at infinity would mean. The explanations that will save the phenomenon for her will be ones that harmonise with her mathematical lived experiences. A teacher's unilateral declaration that one can never divide by zero will not have much explanatory potential for Buhle because her lived experience includes instances where this is not the case and one can divide by zero.

Answers to questions such as what 'infinity', 'equal to' or 'zero divided by zero' are, in the ontological sense, form the basis of an account of what mathematics is. The account which a student holds of what mathematics is will influence what she thinks mathematics can be used for, who can be good at mathematics and how she should go about learning mathematics.

Without a coherent philosophy of mathematics which is able to save the phenomenon for any given student, the answers to explanation-seeking questions do not help this student to make sense of the single claims that form the building blocks of mathematical understanding at this level. If her only mathematical interpretative resource is an incoherent or radically incomplete philosophy of mathematics, she cannot make sense of the mathematical claims that are given to her in the classroom because the explanation she is given does not save the phenomenon or match up with her other mathematical lived experiences. Without a coherent philosophy of mathematics, all 'explanations' lack explanatory power because they do not save the phenomenon for the student. A teacher can 'explain' all she wants, but the student cannot make sense of this 'explanation' because it does not match up with, build on or resonate with her other experiences, understandings or beliefs. An incoherent account cannot explain a phenomenon.

The second level of mathematical understanding is broader than the first and concerned with the coherence of the student's body of mathematical knowledge. As Kvanvig points out:

> Understanding requires the grasping of explanatory and coherence-making relationships in a large and comprehensive body of information. One can know many unrelated pieces of information, but understanding is achieved only when informational items are pieced together by the subject in question" (Jones, 2014: 4:13).

This is a coherentist account of understanding, where a proposition that is 'understood' derives its epistemological status from a suitably unified, integrated, coherent body of information which provides the links between propositions needed for understanding. In this conception of understanding, understanding is closely connected to explanation (See for example Elgin, 2007). In a coherentist account, it is the web of interlocked beliefs that affords a basis for distinguishing between understanding a proposition and just knowing that particular proposition. Research in personal epistemologies indicate that students arrive in the classroom with existing epistemological beliefs and theories that "lead to interpretations of instruction" (Barbara K. Hofer, 2001, p. 372). While coherence might not be a sufficient condition for knowledge, coherence between the epistemological beliefs and the interpretations of what is being taught, is necessary for understanding the subject matter.

Students of mathematics are given many pieces of mathematical information, many of which they are expected to accept *a priori* or as axioms. How well they can make sense of these mathematical claims is the type of understanding that happens at the first level. At the second level of mathematical understanding the aim of interpretative resources is to put these pieces of information together in a way which increases student understanding of mathematics as a whole.

For example, students can memorise and hence know Theorem 9 which states that 'The straight line drawn at right angles to a diameter of a circle from its extremity, is tangent to the circle'. But this does not constitute understanding of the theorem because students lack a suitably unified, integrated, coherent body of information which provides the links between the concepts within the theorem, such as 'tangent' and 'angle', as found by Mwakapenda. In a recent study, Mwakapenda (2005) asked first year students enrolled for tertiary mathematics studies to draw 'concept maps' on mathematical concepts covered in the high school mathematics classroom. Students were asked to draw a map that showed the connections between mathematical concepts by linking related concepts with a drawn line and an explanation. In interviews, the students were asked to explain or elaborate on the relationships they identified between concepts.

What Mwakapenda found is that students who did well at high school mathematics and were enrolled in a mathematics major had more linked concept pairs than students who did not do well at mathematics in high school and were enrolled in the Foundation Mathematics course. In the interviews, the mathematics major students were also able to articulate their links verbally to the researcher, and in some cases draw diagrams to support the verbal analysis of the concept links (Mwakapenda, 2005, p. 258). The Foundation Mathematics students provided “mathematically inadequate” (Mwakapenda, 2005, p. 259) explanations, such as saying that angles formed by a perpendicular to a straight line are perpendicular (259), that angles can be similar (261-2) and that tangents can have angles (263). Regarding the claim that ‘tangents can be found in angles’, one of the students explained (Mwakapenda, 2005, p. 263):

> It is not possible [to know what is meant by ‘tangents can be found in angles’]. We just thought that we can connect even though we did not know how to connect them. But they can connect … We thought of the Grade 11 theorem, Theorem 9. Yah, it was talking about angles and tangent. So we thought angles and tangent can be found in the same place so we just connect here.

Here students are aided by their memory of specific classroom incidents such as ‘words teachers kept mentioning’. The lack of success in recalling detailed information about mathematical principles “makes students unable to make adequate conceptual connections between tangent and angle” (Mwakapenda, 2005, p. 263).

The mathematical concepts and propositions which the students encountered in the mathematics classroom were not understood at the first level of understanding; hence the mathematically inadequate ways of trying to make sense of the mathematical claims, including theorems, concepts and propositions. But additionally, these individual claims are not suitably linked to other parts of mathematics, indicating that for Mwakapenda’s students, mathematics is not a unified, integrated, coherent body of information.
For Mwakapenda’s students a coherent account of what mathematics is, is needed in order to be able to make sense of ‘The straight line drawn at right angles to a diameter of a circle

from its extremity, is tangent to the circle'. Students need some account of how this theorem *proves* something *mathematical* in nature which *has purpose or value;* an account which provides answers to the epistemological and ontological questions regarding mathematics. This account needs to save the phenomenon for them, in other words, it has to mesh up with their mathematical lived experiences. These experiences include what they understand by concepts like 'tangent' and 'angle' from other mathematics classroom experiences, and for any account to have explanatory potential it has to mesh with the mathematical statements that their teachers keep mentioning. Consider again Abongile's attempt to make sense of $x = {}^{1}/_{3} + 2$. Any explanation which will have explanatory potential for her will have to mesh with her belief that a third of a cake is a smaller than a whole cake, her exposure to the concept of a velocity-time graph in physics, her belief that mathematics should be studied for its own sake. Without a coherent account that explains these varied mathematical lived experiences, Abongile is left with only fragments of explanations, and when she understands one of her experiences (e.g. being told that mathematics should be studied for its own sake), she cannot make sense of some of her other beliefs (e.g. that this equation looks similar to the one used in the physics classroom). This would be like Buhle knowing the individual mathematical claims that "Infinity is a number greater than any assignable quantity or countable number" and that "In the extended complex plane, we assign a point at infinity", but having no framework in which these contradictory claims make sense.

As we saw in the discussion of the mathematics curriculum above, we should not be entirely surprised that South African students do not have a coherent account of mathematics as a whole. For historical and political reasons particular to our country, there is at present no coherent philosophy of mathematics for students or teachers to draw upon when trying to make sense of how mathematical claims fit together. There are many competing and incompatible accounts which cannot, without explicitly *doing* philosophy, be employed to facilitate understanding. As a whole, then, students and teachers lack the content which would provide shared interpretative resources which could help them to make sense of or understand the mathematical body of information.

As a result of this lack of understanding there exists an inability to be intelligible. By intelligibility I mean the ability to make your problems known to other people.

Consider this anecdote that will be familiar to most teachers or tutors of mathematics.

> Student (in the mathematics classroom): Teacher, I don't understand.
> Teacher: What don't you understand?
> Student: Everything.

When asked what the problem is, the student cannot make her concerns intelligible to the teacher. She does not have enough understanding of her situation to pinpoint the problem or to find the point where she stopped understanding, and instead feels overwhelmed. Mathematics students are unable to make their mathematical problems intelligible and hence cannot communicate these problems, or the harm which is encapsulated in them, to others.

3.5. Hermeneutical Lacunae

So what do the shared interpretative resources provided by the department of education tell us about hermeneutical injustice in the South African mathematics classroom?

Firstly, it is important to note that the education department does not provide a single skewed and prejudicial philosophy of mathematics which excludes certain mathematics students. Instead, the flaw in the interpretative resources provided by the curriculum is that none of the philosophies of mathematics provided by it are complete and coherent. The only resources students and teachers have, are fragments which clash with one another. Such a fragmented philosophy of mathematics cannot have explanatory power. It does not explain enough of the phenomenon, hence does not save the phenomenon, hence does not enable students to make sense of their mathematical lived experiences. When the department of education does not have a clear and coherent picture of what mathematics is, it seems clear that it will be harder for teachers and students to have a useful conception of mathematics. In fact, if all of the available interpretative resources are fragmented instead of having explanatory potential, then there can be said to be an *absence* of interpretative resources which save the phenomenon for individuals. This is what Miranda Fricker calls a hermeneutical lacuna.

Fricker points out that in the context of unequal relations of power, the "shared hermeneutical resources" are often skewed in favour of the powerful. This means that the powerful tend to have appropriate understandings of their experiences ready to draw on as they make sense of their social experiences, whereas the powerless are more likely to find themselves having some social experiences which they cannot clearly make sense of, with at best ill-fitting meaning to draw on in the effort to render them intelligible (Fricker, 2007, p. 149). The powerless therefore experience a gap in the hermeneutical resources needed to make sense of their social experiences, i.e. a hermeneutical lacuna exists.

The harm here lies in the fact that a subject experiencing "hermeneutical darkness" (Fricker, 2007, p. 150) is wrongfully prevented from understanding a significant area of her social experience, thus depriving her of an important patch of self-understanding. This gap can be a gap of understanding and/or a lack of explanation of the phenomenon. Hermeneutical lacunae concern the cases in which being unable to categorise, incorporate and evaluate experiences can make one's life less good than it would have been without a hermeneutical lacuna (Jones, 2014).

In order to highlight the relevant aspects that constitute a systemic hermeneutical lacuna, I will refer to Fricker's paradigmatic example of the hermeneutical lacuna of 'postpartum depression' (See Fricker, 2007, pp. 149–150). For the purpose of this discussion, we are considering the state of the shared interpretative resources regarding postpartum depression before a variety of social justice activists (through organising feminist speak out campaigns, for example) did work on building up the shared interpretative resources we have *now.* In other words, postpartum depression is an area where the necessary interpretative resources needed to overcome the hermeneutical lacuna exist *now,* but our example dates from a time before these resources existed. This is so that we can draw comparisons between mathematics students *now* and women suffering from postpartum depression then.

In the postpartum depression example, Fricker describes the experiences of women (retold in speak-outs organised by the women's movement) who are blamed by themselves, their husbands and society at large for feeling depressed when they should be happy about having a baby, and who are blamed for being bad mothers to these babies. These women had never told anyone about the experiences before, feeling ashamed and embarrassed and “at a loss to describe the hateful episodes” (Fricker, 2007, p. 151). The only hermeneutical resources these women had for interpreting how they were feeling about themselves and their babies is that they were personally deficient in some way.

The harm in this lack of interpretative resources is twofold. Firstly, the only hermeneutical resources available to the subject are at best ill-fitting: blaming and isolating mothers who experience blaming a woman for the physiological effects of childbirth on her body does not capture what is happening in the subject’s life. This account of postpartum depression does not capture the woman’s feelings of uneasiness, unhappiness and helplessness to change or act any differently. This means that the only interpretation available to the subject is insufficient to help her make sense of her lived experiences. Being left with a patch of experience that she cannot understand leaves the subject “deeply troubled, confused, and isolated” (Fricker, 2007, p. 152) .

Secondly, due to the fact that the subject cannot make sense of her experience, she cannot make her experience intelligible to others. A woman suffering from postpartum depression, but without the interpretative resources to be able to make sense of what she is experiencing, cannot explain to her husband why she feels hopeless, unable to sleep, unable to bond with her baby, unable to concentrate, or why, when she is exercising, taking her vitamins, doing yoga, and generally doing everything right, she can simply not ‘get over this’ or ‘snap out of it’. She knows that something is wrong, but without knowing that she has a perinatal mood or anxiety disorder, she might think that she has ‘gone crazy’. She doesn’t understand why this is happening and feels confused and scared. She is afraid that if she reaches out for help, people will judge her, or that her baby will be taken away (See Postpartum Progress, 2011).

It is important to note that these harms are not the harm of ignorance. If a hermeneutical lacuna simply described a lack of knowledge or of the right vocabulary – a plain cognitive disadvantage – then both the husband and wife would experience the same harms. The husband in this scenario has the same cognitive disadvantage as his wife – he also does not know what postpartum depression is. But he does not experience a harm of being the subject of a hermeneutical lacuna. He has no gap in interpretative resources because the explanation of his wife being to blame for her feelings of sadness, anger, or the absence of bonding with the baby, saves the phenomenon *for* him. The ‘blame’ account of postpartum depression matches his lived experiences, but not hers. What makes this lacuna *unjust* is that it disadvantages one party more than the other. The woman comes starkly into contact with the gap, she has a lived experience with no corresponding explanation and hence no opportunity for making the experience meaningful in her life. She experiences a hole where a piece of self-understanding or self-conception should be.

From the discussion above arises four elements that seem to be constituent of a hermeneutical lacuna which also apply to the mathematics classroom.

Firstly, there are certain individuals who have lived experiences (whether these be of postpartum depression or of alienating experiences in the mathematics classroom) which the subject cannot make sense of because there does not exist adequate shared interpretative resources regarding this phenomenon.

Secondly, this lack of interpretative resources means that the subject cannot communicate the harm that she is experiencing to others. Without the concept of postpartum depression, the new mother suffering from a perinatal mood or anxiety disorder cannot explain to anyone how much harm is being done to her by the 'blame' account of postpartum depression which isolates her further, silences her and gives her an 'explanation' of her situation which does not save the phenomenon for her. Without nuanced ways of talking about the ontological and epistemological aspects of mathematics, students cannot explain to anyone how much harm is done to them by not being able to make sense of individual mathematical topics and being unable to fit these claims into a bigger framework of mathematics, and by receiving mathematical 'explanations' which do not save the phenomenon for them. Furthermore, society isolates and silences the very students experiencing this harm by stereotyping them as being stupid or lazy. The harm in lacking interpretative resources lies not only in a student's inability to realise any of the value of mathematics. The harm includes being seen by society as lacking an essential skill needed to succeed at professional and academic pursuits. Furthermore, students cannot indicate this harm to anyone because they lack the concepts, language and mathematical discourse needed to make the experiences intelligible to others.

Thirdly, a subject's lack of understanding of a phenomenon which it is strongly in her interest to render intelligible, leaves her "deeply troubled, confused and isolated" (Fricker, 2007, p. 152). These emotions are often expressed as outlaw emotions, such as absenteeism, special needs, disruptive behaviour, disengagement and invisibility in the mathematics classroom. This means that when we are looking at outlaw emotions as a warning sign for injustice, we are looking for the emotions that are the *result* of hermeneutical injustice.

Fourthly, hermeneutical injustice is not merely the result of a plain cognitive disadvantage. In the post-partum depression example, it is important to note that both the woman and her husband do not know what postpartum depression is. The husband makes sense of what is happening to his wife by assigning the blame to her and her personal deficiencies. The woman not only gets stuck with an explanatory account of her perinatal anxiety that does not save the phenomenon for her, but also gets assigned the blame. In other words, it is not only that the woman suffering from postpartum depression has the wrong conception of what is happening to her, it is that there did not exist (until the feminist speakouts) an explanation of her experience which saved the phenomenon for her. In the same way, a student who does not have a coherent account of what mathematics is, is not just being lazy by not finding an account that saves the phenomenon for her; such an account does not exist so there is no way in which she could have knowledge of it.

Being the subject of a hermeneutical lacuna is something which disproportionately affects subordinated groups. As Fricker points out, different groups can be hermeneutically disadvantaged for all sorts of reasons, as the changing social world frequently generates new sorts of experience, the understanding of which may dawn only gradually on us (Fricker, 2007, p. 152). But this does not capture what hermeneutical injustice is. Hermeneutical injustice arises when the interpretative resources needed to explain the experiences of subordinated or minority groups *do not exist* since powerful groups create the shared interpretative resources for the lived experiences of their group to the exclusion of less powerful groups. Powerful groups often do not have an active interest in creating interpretative resources for experiences which subordinated individuals have, and often they have an active interest in excluding subordinated individuals through hermeneutical injustice. It is even possible that hegemonic groups cannot create the interpretative resources that subordinated individuals need, perhaps because they do not have the lived experiences of a subordinated individual and therefore cannot know what would save the phenomenon; or perhaps because without these lived experiences it would never occur to them to that there existed a gap in interpretative resources. Therefore, it is often the experiences of female, black or poor individuals which do not get represented.

In the previous chapter I mentioned data from Leslie (2001) and Hannover & Kessels (2004) which points to female and black students being most affected by a lack of diversity in the accounts of mathematics which have explanatory potential for the lived experiences of mathematicians. Jawitz et al. (2000) give us reason to believe that the same thing is happening with South African students.

Female students in their study found mathematics "a total turnoff". Students considered themselves "logical and I've got the talent in that area but when I just think of [mathematics] I just wanna run". They also have the image of mathematicians as not working among people, instead being "isolated by themselves". Furthermore, they choose not to continue with mathematics or mathematics-heavy careers because "everybody was saying that it's too male-dominated. It's not a career for females" (Jawitz et al., 2000).

Furthermore, Jawitz et al. (2000) identified 438 female first year students who achieved the prerequisite high school mathematics marks needed to continue with further study in mathematics or mathematics-heavy fields. Of these students, only 12% were black students.

If it is true in South Africa that black and female mathematicians are still grossly underrepresented, then it will be true that the high-level positions in mathematical fields are dominated by white males. If black people and women never occupy the high-level positions in mathematical journalism, do not sit on policy-making bodies concerned with mathematics education and are not represented in the academy which takes a position on mathematics and mathematics education, then the voices of these subordinated groups are prevented from participating on equal terms in the shaping of collective interpretative resources for mathematics. This means that the voice of subordinated groups is not heard in South Africa when society creates concepts, words, ideas, norms and values that structure how we interpret our lived experiences of mathematics. Even black and female academics, journalists and politicians, if unreflexive, might reiterate the interpretative resources that they were exposed to, no matter how fragmented or inadequate these are. Furthermore, those black and female academics, journalists and politicians who have made it to the higher positions may not see the need to create new resources, since the old interpretative resources worked to get them to the positions they occupy now. South African society is then in the position that there are currently no individuals in a position to help create interpretative resources working together with members of subordinated groups.

3.6 Concluding Remarks

Dunne and Johnston (1994:227) ask us to consider

> The privilege that access to mathematic confers on its 'chosen few', to understand its 'gate-keeping' role in relation to further education and future careers and consider this in the production and reproduction of hierarchical gender (and class and race) relations. What constitutes mathematics, what counts as valued mathematical knowledge, how things came to be this way, and how they are sustained are critical questions.

In this chapter I have shown that in South Africa, black and female students tend not to be part of the 'chosen few' who are privileged to have easy and straightforward access to mathematics. This is because what constitutes mathematics and what counts as valued mathematical knowledge is determined by the norms, values and discourses of the dominant group. These discourses about mathematics track the lived experience of the dominant group and produce accounts that save the phenomenon for members of the dominant group. But because there is no equivalent account to save the phenomenon for members of subordinated groups, they experience a hermeneutical lacuna. A hermeneutical lacuna occurs when:

(a) The subjects only have access to ill-fitting interpretative resources which do not adequately explain the things which they experience.
(b) The subject is unable to make the experience intelligible and hence cannot communicate it or the harm which is encapsulated in it to others.
(c) A hermeneutical injustice is not just a plain cognitive disadvantage. Instead, a hermeneutical lacuna is unjust in that it disadvantages one group of people more than the rest.
(d) Due to a lack of interpretative resources, individuals feel outlaw emotions.

(e) The dominant group has no interest in making the phenomenon intelligible, or the dominant group lacks the ability to generate the right type of interpretative resources.

(f) There are few members of subordinated groups currently in a position to help other members of the subordinated groups to create the interpretative resources that are lacking.

By looking at the history of South African mathematics education I have given an account of how things came to be this way. The 'gate-keeping' role mathematics plays in relation to further education and future careers explains how these one-sided accounts of mathematics are maintained. Because members of subordinated groups do not continue with mathematics as they do not have an account of mathematics that makes the choice to continue viable, they never occupy the high-level mathematical positions. The high-level mathematical positions are usually those positions that determine the mathematical discourses used by a society. Hence the old discourse, and the accompanying hierarchical relations of race, gender and class, are reproduced.

In order to encourage diversity in academic fields like mathematics, therefore, we have to change, or at least diversify, our conception of what it means to be good at mathematics. But this diversification depends on a diversification of what mathematics *is.*

Hence, in order to be self-reflexive of the concepts, assumptions, norms, values, framing perspectives, social ontology and epistemology that are assumed in mathematics, more philosophy of mathematics needs to be practised by mathematicians, and students and teachers of mathematics.

Chapter 4

4.1 Creating Explanatory Accounts

In the previous chapter I examined the lack of a coherent philosophy of mathematics in the South African mathematics curriculum. Because of the fragmented nature of the interpretative resources, students and teachers cannot access an account of mathematics that has explanatory potential. In other words, the mathematical shared interpretative resources that students need in order to make sense of their mathematical lived experiences, are absent.

The solution to the problem of the hermeneutical lacuna in the mathematics classroom is not, I believe, that the Department of Education should choose one philosophy of mathematics and prescribe it to all students.

The first problem with prescribing one particular philosophy of mathematics is that students have such a wide variety of mathematical lived experiences. Every student's mathematical lived experiences are her particular combination of experiences, observations and beliefs about mathematics. As we saw in Chapter 1.3., a student's mathematical lived experiences depend on her social, economic and political context, her day-to-day encounters with mathematical elements in the world, her experiences in the mathematics classrooms, the mathematical ideas/concepts she has been exposed to, as well as the way in which she has heard people talk about mathematics. For an account of mathematics to 'save the phenomenon' of a student's mathematical lived experiences, the account has to be consistent with all of her observations and beliefs about individual instances of mathematics. These different mathematical lived experiences need different accounts of what mathematics is, how we gain mathematical knowledge and how we measure mathematical success, in order for a student to interpret her specific interactions with mathematical ideas and practices, both in her everyday life and in the mathematics classroom. I do not think that there exists one philosophy of mathematics that would save the phenomenon of mathematics for all students.

Instead, we must aim to create a space where students can create their own explanatory accounts of mathematics – accounts which will save the phenomenon for them. An explanatory account of mathematics is a coherent philosophy of mathematics that students can use to explain, to themselves and others, what mathematics is, why it is valuable, how we gain mathematical knowledge and what the characteristics of a good mathematics student are. Without these explanations, students cannot understand mathematical claims or mathematics as a whole, and therefore cannot realise the value of mathematics.

Due to the hermeneutical lacuna in the mathematics classroom, students are lacking the interpretative resources necessary to create explanatory accounts of mathematics. Students do not have access to mathematical concepts and ways of talking about mathematics that will save the phenomenon for them, because these concepts and discourses do not exist in a coherent form in the mathematics classroom. Students therefore have to produce these interpretative resources themselves. There are two ways of doing this: firstly, by working with their own and other students' mathematical lived experiences and, together, build explanatory accounts based on these experiences; and secondly, engage with different philosophies of mathematics as set out by philosophers and academics and apply these to their mathematical lived experiences. Once these mathematical interpretative resources have been created, students can then draw upon them to make sense of and explain their mathematical lived experiences.

4.2 Sharing Mathematical Lived Experience through Dialogue

Miranda Fricker gives us an example of how people can create interpretative resources by sharing their lived experiences: the consciousness-raising 'speak-outs' used by the women's movement. She sees the 'speak-out' as a space where women could come together to share their "scantily understood, barely articulate experiences" – experiences that were "obscure, even unspeakable, for the isolated individual" (Fricker, 2007, p. 149). Women, in the process of sharing the half-formed understandings of their experiences "awakened hitherto dormant resources for social meaning that brought clarity, cognitive confidence, and increased communicative facility … [realizing] resources for meaning that were as yet only implicit in the social interpretive practices of the time" (Fricker, 2007, p. 149). Jones (2010, p. 108) explains how the sharing of lived experiences can create hitherto non-existent interpretative resources through viewing other people's lived experiences as "considerations for inspiration and articulation". In other words, someone else's lived experience may be a consideration which raises for the subject of the lacuna "an alternative way of seeing some feature of the world, or which may bring to her reflective awareness a question or feature of the world upon which she had not previously focused, or which may allow her to make a new connection among phenomena which are important to her". Since the interpretative resource of an individual experiencing a hermeneutical lacuna is fragmented and ill-fitting, she needs new candidate interpretations for filling gaps in her understanding of her self and her world (Jones, 2010, p. 108).

Consider for example a hypothetical dialogue between our familiar characters, Abongile and Buhle. Let us say, for example, that the thing which sparked their conversation was the A4 Example as discussed in Chapter 1.1. The reader will remember that Abongile argued that an A4 piece of paper is not a piece of mathematics because she is a realist about mathematical entities. This is an attractive position to her because it explains mathematics certainty: we can say with certainty that an A5 piece of paper has the ratio 1:√2 (since it is half of an A4 piece of paper) without having to measure any given piece of A5 paper. Buhle, on the other hand, thinks that the 1: √2 ratio was made up by humans to allow us, when we are using the photocopier, to change the size of a picture or diagram, from A4 to A5 for example, without distorting it or having to trim the edges to remove excess paper. In other words, Buhle denies the existence of objective mathematical truth, instead supporting a pragmatist-type account where mathematical concepts are true if they are successful in serving the interests of the humans who created them. The reader will also remember the claim made in Chapter 1.3. that Abongile and Buhle's positions regarding the A4 piece of paper are determined by their mathematical lived experiences: Buhle has been supported in her desire to become an engineer, and she has been exposed to non-Euclidean geometries, while Abongile has never been exposed to some of the big ideas in mathematics and believes that mathematics is just "numbers, +, -, ÷, x" and wants to become a housewife like all of the other women in her family.

When Abongile and Buhle have a conversation about the A4 example, they will not present each other with coherent philosophies of mathematics, or with complete explanatory accounts. With no background in philosophy of mathematics, they will not be able to identify themselves as a realist or an instrumentalist about mathematics. With only the jumbled, confused and fragmented philosophies of mathematics which they get from their parents, teachers and peers, they cannot make their experiences intelligible to themselves or to each other. But what they can do is to share what they have seen or heard or figured out from their own mathematical experiences.

For example, Abongile can point out to Buhle that, if maths is made up to solve human problems, how do you *know* that *this particular A5 paper* has the special ratio? That is like saying that you *know* that the sun will come up tomorrow. While both cases are probably extremely likely, they just use a big set of inductive claims, and might be proved wrong by empirical evidence. Buhle, on the other hand, may introduce Abongile to the notion of non-Euclidean geometries and question the certainty of deductive reasoning if the starting axioms are chosen without absolute justification, in other words, they aren't certain.

Again, it is unlikely that this conversation will happen in this (very philosophical) language. But they are the type of observations students can make.

What is important to note is that for both Abongile and Buhle, the lived experience of the other will introduce "an alternative way of seeing some feature of the world, or which may bring to her reflective awareness a question or feature of the world upon which she had not previously focused, or which may allow her to make a new connection among phenomena which are important to her" (Jones, 2014). By sharing their thoughts, feelings, beliefs and experiences of mathematics with each other in a safe space, students gain access to considerations for inspiration and articulation regarding their own mathematical lived experiences. These new ways of seeing mathematics and new connections between phenomena can then be used to build an explanatory account of mathematics which saves the phenomenon for each student.

In my personal, if anecdotal, experience students gain a lot of understanding through the experience of being exposed to different accounts. A mathematical realist, when challenged by the example of non-Euclidean geometries, is not always convinced because her lived experiences cannot be explained fully by an instrumentalist account of mathematics. But by being shown a different account of mathematics, and being shown where her account falls short or what it cannot explain, a student gains a clearer understanding of *what she believes mathematics (or mathematical knowledge) to be.* In other words, being exposed to someone else's lived experiences and explanatory account allows them to become clearer on what they think the right answers to the ontological and epistemological questions regarding mathematics are.

In order to generate interpretative resources, students need to be engaged in a certain type of discussion; discussions of the type which happen during the 'community of inquiry' method of doing philosophy. According to Haynes & Murris (2012), there are certain, dynamic characteristics built into the design of the community of inquiry that

> "exist to open up the ground of thinking and listening, to encourage student participation, to provide some structure for the processes of reasoning, and to nourish the dialogue. These features include:
>
> - A dedicated time and space, a starting point that *provokes* questioning, and an agenda created by pupils' questions, faithfully presented
> - The teacher holds back her/his views and works to ensure that ideas get a fair hearing and alternative points of view are explored
> - The mainly oral medium of inquiry, removing barriers to thinking and participation for many students
> - Encouragement to risk taking, trying out ideas and playfulness in thinking, freedom to change one's mind, and disagreement normalised
> - Searches for meanings, as well as for truths, and provisional answers based on more rigorous *or* persuasive arguments and examples."

The A4 example discussed in Chapter 1.1. can be used to show the reader what a community of inquiry would look like in real life. The story written by Millar about the A4 piece of paper would be the starting point for the inquiry, with the additional questions asked to give students a chance to share their views and experiences without trying to figure out what the teacher wants them to say. The conversation encourages playing around with the ideas raised by the starting point, and exploring alternative points of view, like the idea of mathematical certainty that Abongile and Buhle discussed above. In a community of inquiry, students would be encouraged to answer the questions raised by the A4 Paper Example by "listening to each others' point of view, thinking out loud, and building on each others' ideas" and "to clarify concepts, to develop lines of inquiry, and to use examples and counter examples to check the validity of their emerging arguments and perspectives" (Haynes & Murris, 2012). Each student's point of view is a reflection of her mathematical lived experiences, and the examples and counter examples which she uses in engaging with the emerging arguments and perspectives, emerge from the mathematical concepts and discourses she has been exposed to. The fragmented and incoherent concepts, ways of talking about mathematics and points of view that form part of each student's mathematical lived experiences can, in the community of inquiry, be clarified, developed, built upon and compared and contrasted with other perspectives. Through this process, a student is exposed to "considerations for inspiration and articulation" (Jones, 2010, p. 108) by being exposed to alternative ways of seeing mathematics, which may bring to her reflective awareness a question or facet of mathematics upon which she had not previously focused. Additionally, a community of inquiry that is based on mathematical ideas, concepts, discourses and lived experiences may allow a student to make a new connection among mathematical phenomena, coming to see connections both between particular mathematical claims and how they fit into the 'bigger picture' of mathematics, and how mathematics fits into the broader scheme of value and meaning-making that she employs in other areas of her life. Buhle, for instance, may come to see how her knowledge of nonEuclidean geometry influences how she views a piece of A4 paper, and thus comes to clarify what she believes mathematics as a whole to be (an instrument used by humans to solve real world problems). But if Buhle's interest in pursuing engineering is motivated by her viewing it as a way to open up opportunities to serve her community, then mathematics, as a means to this end, gains a special significance for Buhle.

The special thing about this sharing of mathematical lived experiences between students is that it has the potential to "[awaken] hitherto dormant resources for social meaning that [can bring] clarity, cognitive confidence, and increased communicative facility … [realizing] resources for meaning that were as yet only implicit in the social interpretive practices of the time" (Fricker, 2007, p. 149). In other words, students in a community of inquiry can generate the type of interpretative resources they need in order to overcome the hermeneutical lacuna they are subject to in the mathematics classroom.

The reader might, at this point, be concerned that If we take seriously the concern that members of marginalised groups have been disadvantaged by not having access to the shared interpretative resources they need in order to make sense of their experiences, then it seems unlikely that putting a whole bunch of misfit interpretations together will necessarily spark the creation of new interpretative resources.

I often find this to be the case when I do philosophy of mathematics with students. Students, never before having examined their mathematical experiences or investigated different mathematical beliefs, and already feeling disaffection towards mathematics, do not have much to say. They cannot articulate *why* (or *why not*) they think that a piece of A4 paper is a piece of mathematics. Or they all repeat the ways in which their teacher has talked about mathematics, without knowing how to examine what *they* think about mathematics. This silence is a result of the hermeneutical injustice and mirrors the type of silence introduced by other kinds of injustice. The students lack interpretative resources to such an extent that critical awareness and response become practically impossible. For example, Paulo Freire found that the dispossessed people that he engaged with had an:

> "ignorance and lethargy [which] were the direct product of the whole situation of economic, social and political domination – and of the paternalism – of which they were victims. Rather than being encouraged and equipped to know and respond to the concrete realities of their world, they were kept 'submerged' in a situation in which such critical awareness and response were practically impossible. … [it became clear] that the whole educational system was one of the major instruments for the maintenance of this culture of silence" (Freire, 2005, p. 30)

Of course, I think that giving people a space in which they *are* encouraged to critically engage and respond to their concrete realities is important. Once participants come to trust that there is not a correct answer which the facilitator is waiting for, that their opinions and ways of thinking are valued, and that they are not going to be judged or made to look stupid, then they are willing to engage in the dialogue needed to create new interpretative resources. In my experience of doing philosophy with students, this can take between six and ten weeks. But even once I have earned their trust, I sometimes encounter groups where students just cannot come up with new explanations or arguments or examples about their mathematical lived experiences. A student can often state her initial position (for example, that a piece of A4 paper *is* a piece of mathematics) but cannot explain this position further, nor comment on the perspectives put forth by other students and compare them with her own perspective. The ‘dialogue’ that these students are meant to be having with each other end up being disjointed statements about mathematics, followed by watching me in silence. The interpretative resources that these students have access to are so fragmented that they just *do not allow the student access to ways of thinking about mathematics.* Their peers are similarly disadvantaged, and cannot provide different ways of thinking and arguing about mathematics. These groups sit and watch me in silence, not because they are not dedicated to the process or invested in the group, but because *they do not know how to break the silence.*

I have argued thus far that if the interpretative resource that an individual experiencing a hermeneutical lacuna has access to, is extremely fragmented and ill-fitting, she needs new candidate interpretations for filling gaps in her understanding of her self and her world (Jones, 2010, p. 108); candidate interpretations that she cannot necessarily come to through discussion with her peers.

At the beginning of this section I mentioned that Abongile and Buhle could have a conversation regarding mathematical certainty without knowing, through philosophical education, what a realist or an instrumentalist view of mathematics is. However, I believe that when a student is exposed to the thoughts that many clever and dedicated philosophers or mathematicians have had about mathematics over the course of the past few thousand years, she is in effect also gaining interpretations of mathematics which can make her own position on mathematics clearer to herself and others.

4.3 Engaging with Philosophy of Mathematics in the Classroom

To get a clearer picture of the value of engaging with other explanatory accounts of a phenomenon, I want to return to Fricker’s example of a particular woman, Wendy Sanford, struggling with postpartum depression. Sanford had the following to say about her experience taking part in a feminist speak-out:

> …[T]hen we broke down into small groups. I had never ‘broken down into a small group’ in my life. In my group people started talking about postpartum depression. In that one forty-five-minute period I realized that what I'd been blaming myself for, and what my husband had blamed me for, wasn't my personal deficiency. It

> was a combination of physiological things and a real societal thing, isolation. (Fricker, 2007, pp. 149–150)

Although Sanford is engaging with the experiences of other women in her 'small group', and in this capacity can hear about the similarities or differences of the postpartum experiences of the other women, she is also being exposed to two *theoretical* accounts of postpartum depression which may not have spontaneously arisen from conversations about the women's lived experiences. Firstly, psychological and physiological discourses are used to make sense of the changes in women after childbirth. Secondly, a sociological or political discourse about isolation is used to examine the effect that isolation can have on how women live their lives. Combining these two theoretical threads gives Sanford a whole new set of meaning-making tools.

In terms of the hermeneutical lacuna in the mathematics classroom, as I argued in the previous section, students can speak to each other about their mathematical lived experiences. But when students are exposed to the theoretical accounts of what mathematics and mathematical knowledge is, this can also be a way for students to gain a whole new set of meaning-making tools.

Philosophy is particularly useful in providing a wide array of potential explanatory accounts for students to engage with. This is because philosophy is a field where disagreement about the true/correct/right account of mathematics is tolerated. Even when these explanatory accounts contradict each other, each is seen to be philosophically valuable. By this I mean that all the plausible philosophical accounts of the nature of mathematics and mathematical knowledge are coherent, endorsed by respected philosophers and well supported by logical argumentation and reasons. In philosophy, we can have different, mutually exclusive live (or current) options where every account is as 'live' as the other. Therefore, all of the plausible philosophical accounts about mathematics are still live options to choose from. Students who are trying to find an account of mathematics that: (1) meshes with all of their (mathematical and other) beliefs, (2) will help them to make sense of their role as a student of mathematics and/or mathematician, and (3) explains why mathematics is taught in the way it is, and what the value of mathematics is and have a body of (well thought-out and logically supported) work ranging over many centuries to explore as options for explaining the phenomenon of mathematics.[1]

Let us refer back to Abongile and Buhle. For Abongile to know that she is a mathematical realist, and that her position gives her access to mathematical certainty, but is vulnerable to stereotypes of women being unable to be good deductive thinkers, gives her a whole new set of interpretative resources. She gets access to new concepts that she can draw upon to make sense of the mathematics that is presented to her in the classroom, a clearer idea of what the value of mathematics really is, new ideas about what is or isn't important mathematical content, new knowledge about the different criteria used to judge mathematical performance, new ways of talking about mathematics, and new ways of talking about herself as a student of mathematics and/or a mathematician. The same is true for Buhle 'discovering' that she is an instrumentalist about mathematics: that is to say, coming to know that an instrumentalist philosophy of mathematics best saves the phenomenon of her mathematical experiences *for her.*

Through doing philosophy of mathematics in a community of inquiry, students can gain access to a wide range of explanatory accounts of what mathematics and mathematical knowledge is. The students can then 'try on' these accounts to see whether the accounts fit in with their lived mathematical experiences. The accounts are tested for their explanatory potential and their ability to 'save the phenomenon' as experienced by each individual.

4.4 Addressing Hermeneutical Injustice in the Mathematics Classroom

My recommendation for empowering students to create the interpretative resources needed for them to overcome the hermeneutical lacuna in the mathematics classroom is to do philosophy of mathematics with them. It is my contention that by giving students the tools they need to understand mathematical claims and mathematics as a whole, they will react to the content taught in the mathematics classroom differently. Further empirical evidence will be needed to substantiate this claim, but I believe that philosophy of mathematics will help students to make the links between mathematical claims and concepts (as described in Mwakapenda, 2005) as well as making mathematics more

---

[1] For the metaphilosophical account of what causes this dissensus in philosophy, and how this affects philosophy outsiders (like mathematics students) which has most influenced my thinking, see (Jones, 2010, 2014)

meaningful for students by showing them different ways in which mathematics can be valuable, therefore decreasing outlaw emotions and increasing motivation to study mathematics.

Although I am not aware of any study measuring the effect of doing philosophy of mathematics with students, there have been a number of studies on the effect of doing Philosophy for Children (P4C) in schools. Most recently, a randomised controlled trial in 48 primary schools in the UK was set up to test the gains in the academic performance of pupils, and also their cognitive ability, after taking part in P4C for one complete academic year. The topics for inquiry were concepts which might be relevant to the lived experience of students and included 'truth', 'fairness' and 'bullying'. The study found that students who had been a part of a community of inquiry were roughly two months ahead on mathematics and reading skills when compared with the control group (Gorard, See, & Siddiqui, 2015, p. 4). In the case of disadvantaged students (measured by whether they qualify for free school meals) the improvement over the control group was three months for mathematics, four months for reading and an additional two months in terms of writing skills. Teachers and pupils generally report improved behaviour and relationships. This is achieved at a cost of around £16 per pupil per year. If there are wider or longer-term benefits to studying philosophy at primary school, then this could make the intervention cost-effective. Presumably, if the topics used as starting points in the community of enquiry would focus specifically on mathematics, rather than on topics like honesty, fairness and bullying, we would expect a bigger increase in the mathematics advances.

Furthermore, I believe that doing philosophy of mathematics will alert the students to the explanatory potential, but also the limitations, of the different philosophies of mathematics. For instance, Buhle might realise that an instrumentalist account of mathematics works really well for explaining parts of mathematics – like non-Euclidean geometry – but is less good at telling us why when we put two pairs of socks into the tumble drier and only three socks come out, we don't think that 2+2=3. Buhle might be okay with this, thinking that, of all the explanatory accounts out there, the instrumentalist account is her best option and explains the parts of mathematics which she values most. Or, Buhle might accept the instrumentalist philosophy of mathematics for now, but keep looking for a better account, come up with a better account herself or search for ways of integrating different philosophies together so that *all* of her beliefs about mathematics cohere.

But coming to know about the strengths and limitations of mathematics can also help us deal with the *systemic* element of hermeneutical injustice – that black, poor and female students are more likely than powerful groups to lack an explanatory account that matches their particular lived experiences. The mathematical claims included in each philosophy of mathematics, which combine with prejudicial beliefs about race, class or gender, is a limitation on that account. By taking these limitations into account, a student can choose the philosophy of mathematics that provides her with the most explanatory potential and the fewest limitations.

4.5 Concluding Remarks

When students' own retellings of their lived mathematical experiences in the safe space of a community of enquiry are combined with the explanatory accounts of mathematics developed by philosophers of mathematics, I believe that the shared interpretative resources needed to overcome the hermeneutical lacuna in the mathematics classroom can be generated. Through such inquiries, a student is presented with many options for making sense of any given mathematical concept, experience or belief and of how it fits into her view of what mathematics is, how it fits in with her other experiences and beliefs, how it fits into the system of what she values, and how mathematics fits into her life. By clarifying mathematical concepts, developing lines of inquiry, and by using examples and counterexamples to check the validity of their emerging arguments and perspectives, students are 'trying on' the potential explanations for mathematics thus produced, to see whether they save the phenomenon for themselves. The newly-found explanations are then shared with others to check for intelligibility. The interpretative resource created at the end of this process can then be used by a student to make sense of her mathematical lived experiences, and to communicate with others about the (often problematic) issues surrounding the experience.

Conclusion

Students stepping into the mathematics classroom have unique mathematical lived experience: unique amalgamations of ideas about and, more broadly, attitudes toward mathematics, in addition to unique backgrounds, inclinations, and abilities. A student's unique set of circumstances means that not all ways of explaining mathematics will serve to further her understanding of mathematics and help her to explain her mathematical lived experiences. Explanations which do not link up to any of her mathematical lived experiences will fail to have explanatory potential for this student and will not help her to better understand the mathematical idea we were trying to explain to her.

We often try to make links between lived experiences and the concept we are teaching when we are engaged in non-mathematical teaching. For example, if we were trying to explain to a child what the colour white looked like, and the child had never seen snow, and we told the child that white is the colour of snow, we know that our explanation would not help the child to understand what white is.

In mathematics, I call our explanatory accounts of what mathematics and mathematical knowledge are, our *philosophies of mathematics.* Different philosophies of mathematics will match up better with different mathematical lived experiences. Students should be exposed to a multiplicity of philosophies of mathematics so that they can endorse the philosophy of mathematics which has the most explanatory potential for their particular mathematical lived experiences. But even if a student does not endorse a specific philosophy of maths, by doing philosophy of mathematics she may come to approach mathematics in a philosophical way. A philosophical approach to mathematics is an approach which gives students an insight into the complexities regarding the ontological and epistemological aspects of mathematics, as well as how different ways of describing mathematics affects teaching and learning. Viewing mathematics philosophically will also help students integrate their own lived experiences with the mathematics education they are receiving.

Which philosophy of mathematics a student endorses changes how she learns mathematics as it changes the mathematical questions which she thinks are most worth asking, what she considers to be appropriate ways of tackling them and what she thinks counts as criteria for appraising mathematical success. Which philosophy of mathematics a teacher endorses changes how she teaches mathematics as it changes the mathematical content she thinks is most worth teaching, what she considers proper mathematical technique for doing mathematics and how she tests this content. When a student understands her own philosophy of mathematics and the philosophy of mathematics endorsed by her teacher – though not necessarily the same – she can appreciate the differences and filter the mathematical content through the lens of her own philosophy of mathematics without getting confused by the explanations which the teacher is trying to provide.

Some philosophies of mathematics include claims which, when combined with other prejudicial claims, work to unjustly exclude certain groups of students, particularly members of subordinated groups like poor, black or female students, from mathematical participation. If we want not to reinforce and reinscribe prejudicial claims about people in the mathematics classroom, we need to be aware of how a certain philosophy of mathematics can exclude certain students based on the fact that they are stereotyped to be stupid or not good at solving real-world, practical problems. We should also diversify the way we see mathematics so that we decrease this unjust exclusion from mathematics.

Because the injustice in the mathematics classroom relates to how we interpret mathematics, it is a *hermeneutical injustice.*

In order to alert students and teachers to different ways of describing mathematics, as well as the pros and cons of each description, and in an attempt to overcome the hermeneutical injustice in the mathematics classroom, members of subordinated groups should be encouraged to share their mathematical experiences in a safe space. These accounts can then be combined with existing philosophies of mathematics as described by philosophers of mathematics to create new ways of making sense of mathematics which do not unjustly exclude members of subordinated groups.

Appendix

The aim of this Appendix is threefold.

In Chapter 4.3. I argued that it is helpful to students to be exposed to a multiplicity of explicitly stated, complete and coherent explanatory accounts of mathematics as set forth by philosophers of mathematics. In this chapter I give a short description of six such philosophies of mathematics.

The first four are roughly based on what Jill Adler identifies as the four dimensions of mathematics “evident in the elaboration of knowledge, skills and values in the [South African] Mathematics Statement for Grades 1-9” (Jill Adler, 2007, p. 193). The other two philosophies of mathematics are accounts which I often hear expressed by my students while we are doing philosophy of mathematics together. I describe, in a nutshell, the answers to the ontological and epistemological questions regarding mathematics that are commonly associated with each philosophy of mathematics. The aim of writing about these accounts is not to argue for which account is most correct, nor to give the reader an exhaustive account of the relevant philosophies of mathematics. Instead, the aim is to provide a starting point to any reader interested in having conversations about the philosophy of mathematics with students – a list of philosophies of mathematics that can be presented to South African students as candidate interpretations for their mathematical lived experiences. Ideally, if the reader is a mathematics educator, this appendix might provide a useful resource for her to examine her own endorsed philosophy of mathematics. Additionally, teachers might come to realise that there are other acceptable approaches to viewing mathematics, and therefore other valid approaches to teaching and learning mathematics. This would hopefully increase a teacher’s consciousness of who she excludes when she explains mathematics in a certain way, and when she tests mathematical competence in a certain way.

In addition, these descriptions will help the reader of this thesis understand the claims I make through the course of this thesis about different philosophies of mathematics.

The second aim of this appendix is to illustrate how endorsing these philosophies of mathematics will change how mathematics is taught and learnt in the classroom. For example, a teacher who endorses the virtue epistemology view of mathematics will test mathematics differently from a teacher who endorses the standard view of mathematics: the first teacher will test to see whether any intellectual virtues have been cultivated, while the second teacher will test to see whether students got the right answer.

The third aim of this appendix is to point out where any given claim in any particular philosophy of mathematics intersects with (commonly held) discriminatory ways of seeing the world. For example, in Chapter 2.4. I mentioned that on the standard view of mathematics, mathematics is seen as a discipline that only people who have the required innate talent for the subject, can be good at. This is not necessarily a problematic way of seeing mathematics, and it reflects many of our intuitions that certain people are more talented in the arts than in the sciences. The belief that mathematics requires innate talent *does* become problematic, however, when it is combined with the prejudicial belief that black or female students will, *qua* black or female student, lack this innate talent. If a student is female with an internalised – stereotypical and unjustified, but still present – belief that she is not innately intelligent and is not good at abstract thinking and deductive reasoning, then the standard view of mathematics will not enable her to match her selfimage with the image which the standard view of mathematics presents of the prototypical mathematics student. If, however, she is exposed to a social constructivist philosophy of mathematics which emphasizes the developing of critical thinking skills in mathematics, she might be able to make sense of herself as learning the necessary skills in order to be a good student of mathematics. Since both of these explanatory accounts of mathematics are seen by philosophers as 'live options', the student should be exposed to both and be able to endorse the one which best saves the phenomenon for her.

There is a two-way flow – beliefs about what mathematics is and who can be good at mathematics affect which groups of students will tend to flourish, and at the same time groups embrace certain beliefs about mathematics precisely because these beliefs serve their interests. As a quick example, consider the belief that mathematical knowledge is gained through deductive reasoning. Coupled with the belief— implicitly held by many — that female and black students are inherently more stupid and less adept at deductive reasoning, this mathematical philosophical belief can change how teachers teach and which groups of students (white males) flourish in the classroom. Female students also internalise these beliefs which means that the stereotype that women cannot be good at mathematics becomes a self-fulfilling prophesy.

At the same time, teachers may resist the philosophical claim that mathematical knowledge is merely a fallible social construction, because that philosophical claim would imply that there is no one correct answer, and further that mathematical answers cannot be bestowed upon students by a sovereign teacher-knower. This means that if teachers were to embrace the philosophical claim that mathematical knowledge is a social construct, they would have to change how they teach since they could no longer be the sole holder of knowledge and authority in the classroom.

Given the broad range of prejudicial beliefs – both overt and conscious, and covert and unconscious – regrettably at work in the South African context, there is an array of mathematical claims which intersect with prejudicial beliefs in such a way that certain mathematics students are unjustly affected by these hermeneutical resources. This appendix points out some of the prejudicial beliefs which each philosophy is vulnerable to.

To sum up, this appendix is concerned with describing six different philosophies and comparing them first in terms of their pedagogical impact, and secondly in terms of their vulnerability to societal prejudice.

a) The Standard View: Universal and Objective

The standard view of mathematics is a realist view. The objects of mathematics (numbers, geometric shapes, etc.) really exist apart from human beings. There are a variety of ways to describe what it means to say that generalised, reified abstractions are real or have existence. This can range from a Platonist description which locates mathematical objects in the world of the forms, to a Popperian picture of 'world 3' which contains the objectivised projections of human thought and language (Brown, 2007; Glas, 2007). In all cases, mathematical objects have their own autonomous properties and relationships which are independent of our awareness of them.

The focus of a standard view of mathematics would be mathematical facts and axioms (like 2+2=4) and the skills needed to do deductive mathematics (like algorithms and computation). The mathematical body of knowledge and techniques is well-defined and complete.

Mathematics is therefore the objective observation of these entities. We get mathematics right when we reflect the reality of these entities to the best of our ability. The best way of doing this is through deduction. Using deductive knowledge, we can work from axioms to conclusions. If one is sufficiently careful and really attends to each step in a proof, and carefully analyses proofs so that each step immediately follows from earlier steps, then one can rig it so that we won't ever make mistakes in mathematics; we can be absolutely certain. This is the Cartesian recipe for certainty, and it seems to work very well on mathematics in the standard view (Glas, 2007).

A side effect of seeing mathematics as a system of purity and rigour can be to appreciate the internal aesthetic of mathematics (Kline, 1964). On this account, if mathematics is done properly it will have a beauty of its own. This produces the tendency within mathematics to value simple and elegant proofs over more unwieldy counterparts which prove the same thing.

If the Standard View of Mathematics is used to describe what mathematics is, mathematics is a completely value-neutral activity since it is independent of human concerns and involves only objective content. The motivation for studying mathematics, therefore, is to get closer to the objective truth. Mathematics should be studied for its intrinsic value.

The aims of mathematical education on this account would be to teach students the axioms and deductive skills in order for them to be able to do mathematics.

Deductive skills are often thought to be innate to some and not to others. This means that some people have built-in raw brilliance and mathematical ability, while others do not. Some people can gain access to the objective world of mathematical entities through pure deductive logic, while others cannot. Mathematical ability might be inherited, but it requires teaching to realise its potential. Thus "although children have particular talents for various topics, a good teacher can make all the difference between a pupil losing interest (which is exceedingly difficult to recover in mathematics), or working conscientiously, enjoying the subject" (Haglund, 2009, p. 164).

Raviv et al. found that teachers of mathematics, more so than teachers of other subjects, perceived themselves to be an epistemic authority to their students (Raviv, Bar-

Tal, Raviv, Biran, & Sela, 2003, p. 37). This indicates that mathematics teachers perceive their disciple differently than other teachers view their own disciplines. Raviv et al. speculate that because mathematics is seen to have "well-defined, unequivocal answers and solutions, as well as … unambiguous rules", students will "tend to perceive teachers of these subjects as epistemic authorities [and experts] in their discipline more than they perceive teachers of social sciences and humanities as such" (Raviv et al., 2003, p. 19). This means that teachers are the only contributors to mathematical knowledge in the classroom. The assessment of mathematics learning will be a process of checking whether the students can reiterate the mathematical content that teachers bestowed upon them, by checking whether students can remember the axioms and use deductive logic to come to a fixed and objectively true answer, usually in written form. Only students who can successfully do this are considered to be good at mathematics.

As mentioned in Chapter 2, female and black students are often stereotyped to be stupid and bad at deductive logic and abstraction. Implementing a pedagogy based on the standard view runs the risk of excluding these students because, as a result of stereotype threat etc., they tend to perform badly at the type of assessment this view encourages. They might also not feel motivated to participate in this type of mathematics because they might feel that 'female ≠ maths' or 'black ≠ maths' (see Banaji & Greenwald, 2013) and therefore they feel that mathematics is not for them.

If mathematics teachers and practitioners are worried about reinforcing and reinscribing unjust black and female exclusion in mathematics, they "might want to downplay talk of innate intellectual giftedness" (Leslie, 2001, p. 265).

b) The Instrumentalist View: Real-World Problem Solving

Over the course of history, scientific and technological problems have often been important driving forces of mathematical development (Glas, 2007, p. 34). In light of these problems, mathematics came with an intended empirical domain of application. Arithmetic and geometry, in particular, come with obviously intended domains of application. When the ontology of mathematics is viewed in this way, as fundamentally related to the physical world, certainty (which is important in the standard view) and mathematical rigour (which is important to those who view mathematics as mathematically correct practices) are not as important as successful application and impressive problem-solving power (Azzouni, 2007, p. 15; Glas, 2007, p. 34).

On this view, 'pure' mathematical skills, procedures, facts and knowledge supply the 'dry bones' of mathematics, which are simply tools to be mastered (Ernest, 1991, p. 162). Applied mathematics, on the other hand, is seen as the vital, living part of mathematics. This philosophy of mathematics sees mathematics as an outward-looking subject where the uses and applications of mathematics (rather than intrinsic values like creativity and pattern) are what make mathematics valuable.

Since the value of 'real-world' mathematics is located in its ability to solve empirical problems, there is no universal best method of application. Rather, experienced professionals and skilled practitioners, the experts on real world mathematics, choose between different approaches and uses of knowledge. There may be disagreements about the best approach, but since the end goal is real-world applicability, many equally valid methods and points of view are acknowledged, and choice between them is made on pragmatic grounds of utility, expediency and self- or group interest" (Ernest, 1991, p. 153).

If the Instrumentalist View of Mathematics is used to describe what mathematics is, mathematics is to be seen as a tool which is useful to solve human problems. The motivation for studying mathematics, therefore, is to solve problems related to science, finance or engineering. If, however, students do not want to pursue these fields, they could come to view mathematics as boring, useless in real life, and increasingly incomprehensible (Haglund, 2009, p. 4).

The aims of mathematical education on this account would be to teach students the 'dry bones' of mathematics so that they could apply this to real-world problems. Mathematics would therefore be assessed with 'word sums' or other real-world scenarios where students need to choose between different approaches and uses of knowledge in order to solve the problem. Only students who can successfully do this are considered to be good at mathematics.

Mathematical ability on this view therefore lies in being practical and efficient. As pointed out by Jawitz et. al (2000), women are stereotyped to lack good spatial and visual projection as well as the skill of being practical, and hence are stereotyped to be bad at applying mathematics to real-world problems. Furthermore, the fields of physics, mathematical finances and engineering are still perceived to be male-dominated, decreasing the motivation of women to enter these fields. If this is the case, then women will also have decreased motivation to study mathematics on the instrumentalist view.

c) Mathematics as Process and Practice

On this view of mathematics, the epistemological question of how we come to know mathematics provides the starting point for what mathematics is. We come to know mathematics through the *processes* of mathematical problem-solving and investigation. Therefore, processes such as generalising, conjecturing, abstracting, symbolising, structuring, carrying out algorithms and justifying are given more prominence than the specific mathematical content. There is often a focus on the individualistic elements of these processes – doing mathematics requires being an autonomous inquirer and knower in mathematics. To understand mathematical ideas, one needs to take the position of the problem solver and enact mathematical processes to get to a solution. (Jill Adler, 2007; Bostock, 2009)

Another way of seeing mathematics-as-practice is to make the claim that mathematics is fundamentally about the *relationships between* mathematical objects. It is these relationships that should be studied. It therefore also does not matter if mathematical objects are not real, because the relations between them can be real regardless of the ontology of mathematical objects.

This answer to how we come to know mathematics, focused as it is on process rather than content, leaves open the ontology of mathematical objects. Someone who believes in the mathematics-as-practice account could equally reasonably see mathematical objects as Platonic or as empirical. The ontology would depend on the type of problems identified. Mathematical practices will then be applied to whatever these objects happen to be.

This view does not entail a clear motivation to do mathematics, unless one goes further to say that the process of doing mathematics is inherently valuable, or if one derives the motivation for doing mathematics from whichever ontology is adopted together with the view.

The aims of mathematical education on this view would be to teach students the processes of mathematics and make sure that they have mastered these practices. Students need to be able to see the relationships between different mathematical entities. These processes are also seen to be individualistic, so a student would need to be able to generalise, conjecture, abstract, symbolise, structure, carry out algorithms and justify her choices. Only students who can successfully do this are considered to be good at mathematics. (Haglund, 2009; Kline, 1964)

Some of the same concerns raised for the Standard View could be raised here, especially if the practices which are emphasised are those practices at which black or female students are stereotyped to be bad.

The individualistic nature of this view might also prove to be problematic for students who have been socialised to solve problems in groups.

Furthermore, the lack of inherent motivation entailed in this view might leave students wondering what the point or value of mathematics is. Perhaps it could be combined with the Virtue Epistemology View below.

d) Social Constructivist View: Mathematics for Citizens

Social constructivist views on mathematics sees all mathematical objects, relations and problems as man-made (Glas, 2007, p. 29). Mathematics, as a human activity, has a history and is culture-bound and value-laden. Since it is a human product, mathematics is fallible and corrigible. This means that, in contrast to the standard view, mathematics is not objective and certain.

All mathematical knowledge is therefore seen as embedded in the culture and the reality of the mathematical practitioner’s situation (Ernest, 1991, p. 222). We come to know these man-made objects, relations and practices by actively engaging with mathematics, posing as well as solving problems, discussing the mathematics embedded in our own lives and environments (ethnomathematics) as well as in broader social contexts (see Gerdes, 1994; Setati & Bangura, 2011). Developing critical thinking in mathematics is therefore part of the broader development of critical thinking necessary for democratic citizenship. Learner (and teacher) conceptions and assumptions need to be articulated, confronted with other perspectives, and challenged, to allow the development of critical thinking. This involves empowering individuals to be confident solvers and posers of mathematical problems embedded in social contexts, and thus understand the social institution of mathematics (Ernest, 1991, p. 222).

The motivation for studying mathematics is the same motivation we have to study any human practice which people use to make sense of the world.

The aims of mathematical education on this view would be to help students understand the progression of mathematics, and show students how mathematicians make mistakes and correct them, so that students can get a better sense of what it means to be a mathematician and can become mathematicians themselves.

Mathematical ability is innate to all students since the mathematics will arise from their environment. Students will therefore be encouraged to think critically about mathematical situations in their own lives and see how this fits into the broader narrative of the mathematical institution. Only students who can successfully do this are considered to be good at mathematics.

This view may tend towards a type of relativism, where students and teachers cannot challenge views that arose from social environments which are different to their own, which could lead to an 'anything goes' view of mathematics where everyone is good at their unique branch of mathematics.

On the other hand, some students may not have the interpretative resources to engage with the mathematics which they encounter in their everyday lives. Black or female students might then feel excluded from the act of doing mathematics.

e) Virtue Epistemology

A Virtue Epistemology View of Mathematics highlights the epistemological and intellectual virtues which mathematics helps students to develop. e.g. "attitudes of selfreliance, independence and curiosity" (Haglund, 2009, p. 8). Students are expected and encouraged to 'appreciate' mathematical ideas as being interesting and able to shape human history and culture. Students are also encouraged to see mathematics as a discipline which includes beauty and creativity. The means that the intellectual virtues of appreciation, acknowledgement of beauty, creativity, curiosity, reflectiveness, open-mindedness, excitement, intellectual humility and intellectual independence are cultivated.

Jason Baehr, in *Educating for Intellectual Virtues: A Fourth Option in Character Education* defines intellectual character as concerning a person's orientation "toward intellectual or epistemic ends like knowledge and truth" (Baehr, 2013, p. 256). "Character-based" virtue epistemology sees a person's intellectual character as a function of her psychological orientation toward epistemic ends like knowledge, truth, and understanding. This fundamental orientation disposes her to think and inquire in ways that reflect the intellectual virtues. Intellectual virtues, like all virtues on the Aristotelian picture, have to be fostered through repetition of virtuous action. Baehr (2013, p. 257) sees this happening through "an active and reflective engagement with the curricular staples of math, science, history, literature, and the like".

Mathematics is therefore the type of activity which is well suited to the cultivation of intellectual virtues. This is done by repetition – active and reflective engagement with mathematics.

This is a view I often get from students. They say that they like or value mathematics because it makes them 'think differently' and helps their 'brains to grow'.

The motivation to do mathematics is the motivation to become a more virtuous person. As Baehr (2013, p. 256) puts it: "[t]o get behind this approach, one need only believe that knowledge and learning are good and worth pursuing and that the personal qualities critical to achieving and making good use of knowledge should be deliberately fostered in educational settings". As mentioned above, students can also easily see the value of improving their thinking. This view of mathematics – as a subject which helps to foster epistemic virtues – allows for reward upon hard mathematical work, or reward upon having the right epistemic attitude towards mathematics. This is similar to Leslie's (2001) recommendation for encouraging increased participation in subjects like mathematics – to emphasise the value of hard work rather than raw brilliance in mathematics.

The aims of mathematical education would be to cultivate, in whichever ways, the intellectual virtues, for which the list above is not exhaustive.

Everyone would be able to do this to some extent or another, and only students who put in a genuine effort to cultivate intellectual virtues are considered to be good at mathematics. Assessment would presumably revolve around participation and student affective attitudes towards mathematics.

One possible concern is that this view does not capture the ontology of what it is that mathematics is. Many different subjects can help to cultivate intellectual virtues. Unless mathematics cultivates a unique virtue, this is not a very helpful account as a philosophy of mathematics which is meant to provide explanatory power to students and teachers.

The second concern would be the subjective way in which mathematical progress is measured.

f) The Wonder Account

The Humanistic Mathematics tradition emphasizes the importance of "[returning] to the educational process the excitement and wonderment of the moments of discovery and creation" (Haglund, 2009, p. 8). In the epilogue to *Heidegger and the Thinking of Place,* Jeff Malpas gives an account of wonder which differentiates wonder from puzzlement or curiosity since wonder entails an encounter that is a revealing, an opening up, of things and of the world (Malpas, 2012, p. 314). Wonder does not entail a need for understanding or explanation – in fact, we can be totally satisfied with our understanding of a phenomenon and still be struck by wonder at it. (For other accounts of wonder, see Bynum, 1997; Hepburn, 1980).

My claim in this section is that mathematics is, by its nature, worthy of wonder. This view claims that mathematics is of such a nature that it is eminently suitable to evoke a sudden presencing of things and the world. Malpas states that wonder is most often evoked by "the beautiful, the tremendous, the elegant or the sublime" and posits that this is

> perhaps because of the way in which these forms of appearance call attention, most immediately and directly, to their own appearing, to the fact of their being encountered. One is brought to a halt by the appearance, and forced to attend to it, not because it shows something else (as it may indicate some use, purpose or cause), nor because of anything that explains how it is (the processes or conditions that give rise to it), but merely by the fact that it is (Malpas, 2012, p. 314)

Mathematics, by its nature, lends itself to being the object of wonder, perhaps because, as the Standard View claims, it is a thing of beauty and perhaps because, as the Mathematics as Process View points out, it is so tied up with generalising, conjecturing, abstracting, symbolising, structuring and justifying.

But while both of these views highlight some aspects of the nature of mathematics that can lead to wonder, neither seem to take the content of mathematics seriously enough. Thinking about mathematical concepts like infinity, fractals, the fourth dimension topology, if taken seriously, may evoke a feeling of wonder – not just curiosity or puzzlement. But even mathematical concepts that are nothing 'special' can bring about wonder. As Malpas (2012, p. 315) points out, wonder may often be evoked by the "self-evidently extraordinary, it may also arise out of the simple, sudden, immediate awareness of the existence of some thing; out of the recognition of the questionability, the strangeness, the wondrousness of things, and of our encounter with them, as it occurs in the most common and ordinary of ways". This explains why even simple mathematical entities, like the ratio of 1:√2 that I mentioned in the first chapter, can bring about wonder. Numbers, equality/identity, etc. can all bring about wonder. Hence, by its very nature, mathematics is prone to being the object of wonder.

For educators endorsing this view, the aim of mathematical education is to get their students to feel wonder at a subject which is so open to being experienced in this way. Wonder is an attitude that students can/should have towards mathematics, but in a deeper sense than the attitudes which concern progressive educators (see Ernest, 1991) when they are promoting positive attitudes and self-esteem with regards to mathematics. Wonder may indeed bring about these positive attitudes and increased self-esteem, but essentially is a paradigm through which to see the world and the things in it, a deep set of dispositional states that shape how everything is seen and experienced, and brings sharply into focus how we are situated in the world. Wonder, in this sense, is a virtue – it is a particular way of experiencing things in the world – of which mathematics would be just one thing – which leads to greater human flourishing. But it is not an epistemic virtue because it is not directed merely at mathematical, or any other, knowledge or coming-to-know. It is therefore distinct in subtle but vital ways from the positions explored above.

The motivation to study mathematics is the motivation to evoke wonder in oneself;

the motivation to teach mathematics is the motivation to evoke wonder in students.

One possible concern is that this view does not capture what we mean when we classify something as being mathematical rather than scientific or linguistic. Many things can be the object of wonder. What is it about mathematical concepts which makes them so successful at evoking wonder?

The second concern would be the subjective way in which mathematical progress in measured.

g) Concluding Remarks

This appendix is meant to illustrate some philosophies of mathematics and how they affect pedagogy. But it can also be used as a tool for philosophical engagements in the classroom. This appendix should be seen as a resource to be used in pursuit of coming to see mathematics in different ways, and for coming to realise the advantages and disadvantages of different accounts. Students should be exposed to the plurality of these accounts so that they are enabled, through seeing which philosophies other people are using, to find the philosophy which best saves the phenomenon for them individually.

Bibliography

Adler, J. (2007). Mathematical Practices in and Accross School Contexts. In B. van Kerkhove (Ed.), *Perspectives on Mathematical Practices: Bringing Together Philosophy of Mathematics, Sociology of Mathematics, and Mathematics Education2* (pp. 191–213). The Netherlands: Springer.

Azzouni, J. (2007). How and Why Mathematics is Unique as a Social Practice. In B. van Kerkhove & J. P. van Bendegein (Eds.), *Perspectives on Mathematical Practices: Bringing Together Philosophy of Mathematics, Sociology of Mathematics, and Mathematics Education* (pp. 2–23). The Netherlands: Springer.

Baehr, J. (2013). Educating for Intellectual Virtues: From Theory to Practice. *Journal of Philosophy of Education*, *47*(2), 248–262.

Banaji, M. R., & Greenwald, A. G. (2013). *Blind Spot: Hidden Biases of Good People*. New York: Random House.

Bostock, D. (2009). *Philosophy of Mathematics: An Introduction*. West Sussex: Wiley-Blackwell.

Brown, J. R. (2007). *Philosophy of Mathematics: A Contemporary Introduction to the World of Proofs and Pictures* (2nd ed.). Taylor & Fracins e-Library: Routledge.

Brownlee, J. (2011). *Personal Epistemology and Teacher Education*. (J. Brownlee, G. Schraw, & D. Berthelsen, Eds.). New York: Routledge.

Bynum, C. W. (1997). Wonder. *The American Historical Association*.

Cao, Y. (2015). A Chinese Young Adult Non -Scientist's Epistemologies and Her Understandings of the Concept of Speed. *International Journal of Mathematical Education in Science and Technology*, *46*(6), 895–915.

Chittka, L., & Geiger, K. (1995). Can honey bees count landmarks? *Animal Behaviour*, *49*(1).

Day, A. (2012). *The Numberverse: How Numbers Are Bursting Out Of Everything And Just Want To Have Fun*. (P. Worley, Ed.) (Kindle Ebo). Crown House Publishing Limited.

DoE, D. of B. E. (2011). *Curriculum and Assessment Policy Statement (CAPS), Mathematics: Senior Phase*. Pretoria.

DoE, D. of E. (2003). *National Curriculum Statement Grades 10-12 (General): Mathematics*. Pretoria.

Elby, A. (2009). Defining Personal Epistemology: A Response to Hofer & Pintrich and Sandoval. *Journal of the Learning Sciences*, *18*(1), 138–149.

Elgin, C. Z. (2007). UNDERSTANDING AND THE FACTS? *Philosophical Studies*, *132*, 33–42.

Ernest, P. (1991). *The Philosophy of Mathematics Education*. Taylor & Francis.

Fleisch, B. (2008). *Primary Education in Crisis: Why South African Schoolchildren Underachieve in Reading and Mathematics*. Johannesburg: Juta.

Freire, P. (2005). *Pedagogy of the Oppressed* (3rd ed.). New York: Continuum International Publishing Group.

Fricker, M. (2007). *Epistemic Injustice: Power and the Ethics of Knowing*. Oxford Scholarship Online. http://doi.org/10.1093/acprof:oso/9780198237907.001.0001

Gerdes, P. (1994). Reflections on Ethnomathematics. *For the Learning of Mathematics, Special Issue on Ethnomathematics in Mathematics Education*, *14*(2), 19–22.

Giroux, H. (1997). *Pedagogy and the Politics of Hope: Theory, Culture, and Schooling: A Critical Reader*. Colorado: Westview Press.

Glas, E. (2007). Mathematics as Objective Knowledge and as Human Practice. In B. van Kerkhove & J. P. van Bendegem (Eds.), *Perspectives on Mathematical Practices: Bringing Together Philosophy of Mathematics, Sociology of Mathematics, and Mathematics Education*. The Netherlands: Springer.

Gorard, S., See, B. H., & Siddiqui, N. (2015). *Philosophy for Children: Evaluation Report and Executive Summary*. Durham University.

Graven, M., & Heyd-Metzuyanim, E. (2014). Primary Learner Descriptions of a Successful Maths Learner. In P. Webb, M. G. Villanueva, & L. Webb (Eds.), *New Avenues to Transform Mathematics, Science and Technology Education in Africa*. Port Elizabeth: SAARMSTE.

Gustafsson, J.-E. (2006). Understanding Causal Influences on Educational Achievement Through Analysis of Within-county Differences Over Time. In *2nd IEA Research Conference*. Washington.

Hagen, H. A. (1889). Do Animals Count? *Nature*, *40*(1031), 319.

Haglund, R. (2009). Using Humanistic Content and Teaching Methods to Motivate Students and Counteract Negative Perceptions of Mathematics.

Hammer, D., & Elby, A. (2002). On the Form of a Personal Epistemology. In B. K. Hofer & P. R. Pintrich (Eds.), *Personal Epistemology: The Psychology of Beliefs about Knowledge and Knowing* (pp. 169–190). Mahwah, NJ: Erlbaum.

Hanna, G. (2002). *Toward Gender Equity in Mathematics Education, An ICMI Study*. (G. Hanna, Ed.). New York: Kluwer.

Hannam, Echeverria, P. E. (2009). *Philosophy with Teenagers: Murturing a Molar Imagination for the 21st Century*. New York: Continuum International Publishing Group.

Hannover, B., & Kessels, U. (2004). Self-to-prototype matching as a strategy for making academic choices: Why high school students do not like math and science. *Learning and Instruction*, *14*, 51–67. http://doi.org/10.1016/j.learninstruc.2003.10.002

Haynes, J., & Murris, K. (2012). *Picturebooks, Pedagogy and Philosophy*. New York: Routledge Research in Education.

Hepburn, R. W. (1980). The Inaugral Address: Wonder. *Proceedings of the Aristotelean Society*, *45*, 1–23.

Hofer, B. K. (2001). Personal Epist emology Research: Implications for Learning and Teaching. *Journal of Educational Psychology Review*, *12*(4), 353–383.

Howie, S., & Hughes, C. A. (1998). *Mathematics and Science Literacy of Fina-Year School Students in South Africa: A Report on the Performance of South African Students in the Third International Mathematics and Science Study (TIMSS)*. Pretoria.

Howie, S., & Plomp, T. (2005). TIMSS-Mathematics Findings From National and International Perspectives: In search of explanations. *Educational Research and Evaluation: An International Journal on Theory and Practice*, *11*(2).

Irvine, A. D. (2009). *Philosophy of Mathematics*. Amsterdam: Elsevier.

Jablonka, E. (2003). Mathematical Literacy. In *Second International Handbook of Mathematics Education* (pp. 75–102). Netherlands: Springer.

Jaggar, A. M. (1989). Love and Knowledge: Emotion in Feminist Epistemology. *Inquiry*, *32*(2), 151–176.

Jawitz, J., & Case, J. (1998). Exploring the Reasons South African Students Give for Studying Engineering. *International Journal of Engaged Education1*, *14*(4), 235–240.

Jawitz, J., Case, J., & Tshabalala, M. (2000). Why NOT Engineering? The Process of Career Choice amongst South African Femaile Students. *International Journal of Engaged Education*, *16*(6), 470–475.

Jones, W. E. (2010). Philosophers and the Poor. *Theoria*.

Jones, W. E. (2014). *Dissensus and the Value of Philosophy*.

Kline, M. (1964). *Mathematics in Western Culture*. New York: Oxford University Press.

Leslie, S.-J. (2001). Expectations ofbrilliance underlie gender distributions across academic disciplines. *Science*, *347*(6219), 23–34. http://doi.org/10.1081/E-EWS

Malpas, J. (2012). *Heidegger and the Thinking of Place: Explorations in the Topology of Being*. Cambridge: The MIT Press.

Mathume, B. (2012). The South African New Mathematics Curriculum: People's Mathematics for People's Power?

Mendus, S. (2000). *Feminism and Emotion: Readings in Moral and Political Philosophy*. St. Martin's Press.

Meyers, D. T. (1997). Emotion and Heterodox Moral Perceptions, An Essay in Moral Social Psychology. In *Feminist Theory and Politics*. Boulder, Colorado: Harper and Collins.

Mills, C. W. (2005). " Ideal Theory" as Ideology. *Hypatia*, *20*(3), 165–184.

Muller, R. (2014, June). Truth about SA's mathematics, science education quality. *My Broadband*.

Murris, K. (2000). CAN CHILDREN DO PHILOSOPHY? *Journal of Philosophy of Education*, *34*(2), 261–279.

Mwakapenda, W. (2005). Examining understanding in mathematics: a perspective from concept mapping. In J. Adler (Ed.), *Researching Mathematics Education in South Africa: Perspective, Practices and Possibilities* (1st ed., pp. 249–280). Cape Town: HSRC Press.

Nardi, E., & Steward, S. (2003). Is Mathematics T . I . R . E . D ? A Profile of Quiet Disaffection in the Secondary Mathematics Classroom. *British Educational Research Journal*, *29*(3), 345–367. http://doi.org/10.1080/0141192031000155999

Postpartum Progress. (2011). The Symptoms of Postpartum Depression & Anxiety (in Plain Mama English).

Raviv, A., Bar -Tal, D., Raviv, A., Biran, B., & Sela, Z. (2003). Teachers' epistemic authority: Perceptions of students and teachers. *Social Psychology of Education*, *6*, 17–42. http://doi.org/10.1023/A:1021724727505

Reddy, V. (2006). *Mathematics and Science Achievement at South African Schools in TIMSS 2003*. Cape Town: HSRC Press.

Schoenfeld, A. H. (1988). When Good Teaching Leads to Bad Results: The Disasters of "Well - Taught" Mathematics Courses. *Educational Psychologist*, *23*(2), 145–166.

Setati, M., & Bangura, A. K. (2011). *African Mathematics: From Bones to Computers*. Maryland: University Press of America.

Stanley, S. (2012). *Why Think?: Philosophical Play from 3-11*. London: Continuum International Publishing Group.

Unknown. (2007). Maths, science teachers a priority.

Unknown. (2015a). Maths, science outcomes improving.

Unknown. (2015b). Take up maths and science urges Minister.

van der Berg, S. (2008). How Effective Are Poor Schools? Poverty and Educational Outcomes in South Africa. *Studies in Educational Evaluation*, *43*(3), 145–154.

van Kerkhove, B. (2007). Introduction. In B. van Kerkhove (Ed.), *Perspectives on Mathematical Practices: Bringing Together Philosophy of Mathematics, Sociology of Mathematics, and Mathematics Education2* (pp. vii–xv). The Netherlands: Springer.

Vithal, R. (2005). Mathematics curriculum research: roots, reforms, reconciliation and relevance. In J. Adler (Ed.), *Researching Mathematics Education in South Africa: Perspective, Practices and Possibilities* (1st ed.). Cape Town: HSRC Press.