An analysis of selected grade 11 learners' interactions with geometry tasks using visualization processes: A case study in Namibia

A full thesis submitted for the degree of

MASTERS IN EDUCATION

(MATHEMATICS EDUCATION)

OF

RHODES UNIVERSITY

by

BRIAN S. KABUKU

DECEMBER 2016

ABSTRACT

This case study was conducted at a secondary school where I teach, situated in the semi-rural setting of Bukalo village in Namibia, and sought to gain insights into the nature and role of visualisation processes employed when selected grade 11 learners interacted with selected geometry problems.

According to Mariotti and Pensci (1994), visualisation takes place when “thinking is spontaneously accompanied and supported by images”, and helps students to understand the problem at hand. Visualisation is regarded as “making the unseen visible and imagery as the power to imagine the possible and the impossible” (Mason 1992).

The study is located within an interpretive research paradigm in order to obtain in-depth understanding of the participants’ visualisation processes. Within this paradigm, both quantitative and qualitative approaches were adopted. The eight Grade 11 participants engaged with 12 items of the Geometry Visualisation Tasks (GVT) worksheets. Data was collected using video-recorded learners’ interactions with the GVT, observations, stimulated recall interviews and post-GVT interviews with the learners.

During the data analysis stage, I used inductive analysis to determine patterns evident in learners ‘thinking processes’. My analytical framework consisted of indicators that were used to identify and classify visualisation processes for each task of the GVT for each participant. I adapted this framework from Ho (2010) and Ho, Ramful and Lowrie’s (2015) clarification of the representations.

The findings from this study revealed that the use of visualisations facilitated meaningful learning when learners made use of these to develop and scaffold their conceptual understanding. The findings revealed that most learners used visualisation processes fairly to very accurately when solving geometry problems. They used visualisation processes by using sketches and diagrams that transformed a mathematical problem pictorially, connected their thinking to previous knowledge and experience, clarified the algebraic task and assisted them to understand the spatial relationships within each task.

ACKNOWLEDGEMENTS

My prime gratitude and appreciation goes to the almighty Jehovah God for the loving care and continued gift of life extended to me throughout the course of my studies.

My profound appreciation goes to my first-rate supervisor, Prof Marc Schäfer. Your support and guidance since 2014 during the second year of my B.Ed. Honours programme through to the inception and completion of this study will never be forgotten. Kindly accept my endless appreciation for your tireless efforts in reading through my work and giving prompt and constructive feedback. You continuously guided my thinking and provided many opportunities for me to be a better researcher, academic writer and critical thinker. I am indeed grateful for your patience, inspiration and encouragement throughout this programme.

My further appreciation goes to the entire Rhodes University team of lecturers assigned to the Namibian programme for moulding me during my B.Ed. Honours level. I also wish to applaud Mr. Robert Kraft, better known to his friends and colleagues as Rob, for hosting me in a loving and caring manner during the research design course and the writing-up retreat. Dr Kenneth Ngcoza, your motivating and persuasive words that I would manage to complete a Masters is highly appreciated.

I would also like to thank my Director of Education in the Zambezi region of Namibia (Mr Austin M. Samupwa) for granting me permission to carry out this study at the school where I work.

I am very grateful to Dr Bennett K. Kangumu for his encouraging words in the sphere of academia. I also want to express appreciation for my mentor and former lecturer Mr. Bosman M. Simasiku for introducing me to Rhodes University and continually encouraging me to work hard in my studies.

To the Inspector of Education for my circuit, the school principal and your two esteemed heads of departments, thank you so much for allowing me to conduct the study at your school. I deeply appreciate your enabling support in all ways noticeable.

I would also like to thank the eight Grade 11 learners who were the participants in this research. Thank you once more for your willingness to share your precious time in the process of completing the GVT worksheets and interviews.

To my wonderful sister Sharon N. Kabuku, your support during my studies was immense and that in itself entails love. I could not have done my studies with ease if you were not there, sis. My one and only little daughter Keiler Kazungwe Kabuku, thank you for giving me time to study when you needed my attention the most.

To my fellow Masters' students, thank you so much for burning the midnight oil together and for the fun we shared while working on our VISNAMZA project.

Finally, I thank my father, Robson Kabuku Nyambe and mom Esther Masiye Nchindo for their support and understanding during this time of my studies.

DEDICATION

The sleepless nights and chair-bound moments that led to the completion of this thesis leaves me to dedicate this thesis to my daughter **Keilar Kazungwe Kabuku.** Above all, I dedicate it to my God Jehovah for sustaining us and giving us the gift of life.

DECLARATION

I, **BRIAN SIMASIKU KABUKU** declare that the work contained in this thesis is my original work. It has not been submitted for assessment to any institution of higher learning before. The ideas and directly quoted thoughts of other people have been acknowledged and put as references as indicated on the list in this thesis.

Signature: ________________________ Date: ________________

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER ONE

1. RESEARCH CONTEXT

There is value in emphasising visual representations in all aspects of the mathematics classrooms (Bishop, 1989).

1.1 INTRODUCTION

The purpose of this study was to explore the nature of visualisation processes employed when selected Grade 11 learners interact with geometry problems. This chapter introduces the study by presenting the background of the study, research goals and questions, research methodology employed, significance of the study and limitations to the study. Lastly, I present the structure of the thesis by giving a brief overview of each chapter.

1.2 BACKGROUND TO THE STUDY

This study examines how learners use visualisation processes in the context of engaging withof geometry tasks. I selected geometry as the relevant mathematical domain in the study because it has a prominent place in the school mathematics curriculum.
A useful contemporary definition of geometry is that attributed to the highly-respected British mathematician, Sir Christopher Zeeman who writes that: "geometry comprises those branches of mathematics that exploit visual intuition (the most dominant of our senses) to remember theorems, understand proof, inspire conjecture, perceive reality, and give global insight" (Royal Society/JMC 2001). This definition aligns well with what this study attempts to achieve and the Namibian curriculum for grade 11.

However, it is well known that the education reform project in Namibia began in 1990 (National Curriculum for Basic Education, 2010). The curriculum has subsequently undergone many reforms and changes. The primary goals for these reforms were identified as ensuring access, equity, quality, and democracy in education (Ministry of Education [MoE], 1993). One of the vehicles for achieving the educational goals was the adoption of learner-centred education (LCE) (Namibia: MoE, 2003). The National Curriculum for Basic Education (NCBE) (Namibia: MoE, 2010a) suggests that learners learn best when they are actively involved in the learning process through a high degree of participation, contribution and production. The document further outlines a variety of classroom techniques that a teacher should use. These techniques include direct questioning, eliciting responses, explaining, demonstrating, challenging the learners' ideas, checking for understanding, helping and supporting, providing for active practice, and encouraging problem solving (p. 26).

The LCE approach to teaching and learning according to the curriculum also encompasses "a text-rich and visually- and tactile-rich learning environment" (Namibia: MoE, 2010a). The national curriculum further outlines that knowledge production is "shared through displays of learners' work, charts, posters, and easily accessible information sources" (p. 27). This approach emphasizes the value and use of visualisation in teaching and learning of mathematics. In addition to the call for the use of visuals in teaching by the NCBE, the Namibian Senior Secondary Mathematics syllabus (Namibia: MoE, 2010b) states that one of its aims for all learners "is to help them recognize when and how a situation may be represented mathematically, identify and interpret relevant factors and, where necessary, select an appropriate mathematical method to solve the problem" (p. 2).

According to Arcavi (2003, p. 217) visualisation is defined as:

> the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understanding.

In comparison to Arcavi's definition of visualisation, Makina (2010) regards visualisation "as a very important cornerstone in 'teaching for understanding' in mathematics because it helps the teacher with facilitation of lessons and with the ability to engage learners in realistic situations". According to them (Arcavi and Makina) , visualisation can enhance conceptual understanding hence provide a rationale for the current study which focuses on visualisation in geometry learning

.

In addition, visualisation is considered to play different functions or roles as students use it to solve different mathematical tasks. From Ho's (2010) perspective, the functions of visualisation are:

> understand the problem, simplify the problem, help learners see connections to a related problem, allow individual learning styles, be a substitute for computation, be a tool to check for solutions and lastly play a function in transforming the problem into a mathematical form" (p. 3).

Additionally, Olkun, Sinoplu and Deryakulu (2005) emphasise that improving students' geometric thinking is one of the major aims of mathematics education since geometric thinking is inherent in so many scientific, technical and occupational areas as well as in mathematics itself (p. 1). Visualisation is a key component in geometric thinking as it enables learners to represent abstract concepts in a visual manner. It is further argued that as learners draw or mentally form images (visualisation), they construct their own ideas and concepts (constructivism).

Constructivism and visualisation are interlinked and intersect each other neatly at the level of developing conceptual understanding for learners. As learners construct diagrams, pictures and shapes whether on paper or in their mind, this study argues that conceptual understanding is enhanced. This is argued because both visualisation and constructing knowledge are intrinsically related to each other as the one cannot happen without the other. In order to understand a mathematical concept conceptually, some form of visualisation takes place, and vice versa. In line with this Arcavi's definition of visualisation suits this study very neatly because it embraces the fundamental notions of constructivism and visualisation. I was thus interested to explore in greater depth how Grade 11 learners made use of visualisation processes when they solved geometry tasks.

1.3 IDENTIFICATION OF THE PROBLEM

In my observation, the teaching operates at different levels – teaching occurs at the level of parental guidance, at school by teachers and colleagues, in the media and in many other contexts. Effective teaching requires an awareness on the part of the teacher of how individual learners learn best. For this reason, mathematics teachers should analyse, evaluate and reflect on what factors promote and/or affect effective teaching and learning of mathematics in general, and geometry in particular. Being one of the important branches of mathematics education, geometry aims at developing learners' ability to think critically and creatively when solving mathematics problems in a spatial context – this also enhances their understanding of other mathematical domains. In order for learners to form ideas of basic shapes, they rely on the way they perceive and/or visualise these shapes. But do teachers harness the visualisation processes that learners employ when they solve geometric problems?

According to van Hiele's five levels of geometric thought, visualisation is considered to be a key component in learning geometry (Van Hiele 1999). It is further claimed that as learners visualise by drawing or mentally forming images, new knowledge and concepts are constructed. This is in contrast to the Namibian situation where examiners regularly detect and report that grade 12 learners perform poorly in the mathematics national examinations. (MoE, Examiners'report, 2011, 2012, 2013, 2014). I often have wondered whether the absence of any formal teaching in how to use visualisation processes has perhaps contributed to the current situation of under-performance.

In this study therefore, my focus was to understand how individual learners interacted with geometry problems using diagrams and other visualisations when solving these problems. To achieve this, the participants of this study interacted with a set of 12 geometry tasks that were assembled in the geometry visualisation tasks (GVT) activity. Chapter two provide more detailed information on the GVT.

1.4 RESEARCH GOALS AND QUESTIONS

The aims and objectives of this study were twofold: (1) to analyse the nature of visualisation processes employed when selected Grade 11 learners interacted with selected geometry problems, and (2) to determine how these Grade 11 learners used visualisation processes in their interactions with the geometry problems.

The fundamental research questions that guided this research study are:

- What is the nature of the visualisation processes employed when selected Grade 11 learners interact with geometry problems?
- How do these Grade 11 learners use visualisation processes in their interactions with geometry problems?

1.5 RESEARCH METHODOLOGY

Creswell (2006, p. 4) defines a research methodology as the underpinning philosophical framework and fundamental assumptions that frame any research studies. This case study took the form of a mixed method (Quantitative & Qualitative) interpretive case study. The mixed method approach helped me to analyse my data from different perspectives. This means data analysed quantitatively complemented the data analysed qualitatively. Using this approach added to the validity and reliability of my study. A quantitative analysis using descriptive statistics provided me with a broad overview of how participants employed visualisation processes in their engagement with the Geometry Visualisation Tasks (GVT) worksheets.

A qualitative method was used to analyse how learners interact and engage with the (GVT) worksheets. This approach provided me with rich, detailed understanding of the character of the visualisation approaches learners used, and why they employed or did not employ certain visualisation processes in their engagement with the GVT.

It is interpretive because it recognizes that "individual' assumptions and experiences contribute to the on-going construction of reality" (Wahyuni, 2012, p. 71). The case in this study was a group of eight Grade 11 learners interacting with a set of geometry tasks. The unit of analysis associated with this case is specifically their engagement with the GVT tasks with reference to the visualisation processes they employed

Data was collected using video-recorded learners' interactions with the GVT worksheets and with me, from observations, stimulated recall interviews and the post-GVT interviews with the learners. The data sets were analysed in relation to the responses to the GVT and to the research questions posed by this study.

To validate the entire process, I used a variety of data gathering techniques. These included an independent on-looker, who video-recorded the learners while they engaged with the GVT and with me, and video-recorded post-GVT interviews. I also translated the responses that learners gave in Subia, their mother tongue, into English.

1.6 SIGNIFICANCE OF THE STUDY

The importance of visualisation in mathematics teaching and learning has been emphasised by many writers (Gorgorio & Jones, 1996; Rodd, 2010; Rosken & Rolka, 2006). Furthermore, the Namibian mathematics curriculum for the senior secondary phase expects "learners to use mathematics language and representation as a means of solving problems relevant to everyday life and to their further education and future careers upon completion of the phase (Namibia: MoE, 2010a, p. 23). The examiners of the Grade 12 national examinations consistently identify geometry as an under-performed mathematical domain in these examinations (NSSCO, 2014; NSSCO, 2013; NSSCO, 2012). From my own experience, this led me to speculate that perhaps the lack of visualisation practices and skills of the learners may be a contributing factor to this poor performance, hence the emphasis in this study on visualisation in geometry. Another reason for researching visualisation processes in the Namibian context is that relatively little is known about these processes in the classroom.

In light of the reform taking place in the Namibian education system, teachers, curriculum designers, researchers and policy makers would gain insightful information about the importance of recognising visualisation processes that learners employ in the learning and teaching of mathematics. Furthermore, this study hopefully would help and inform Namibia's teacher training institutions and curriculum designers to incorporate visual processes into their curriculum. Moreover, this study is part of the VISNAMZA suite of studies looking at visualisation in mathematics in Namibia and Zambia, and will make an individual contribution to the collective aims of the project.

The findings of this study consistently showed that the use of visual representations helped learners to understand that various representations can be used to clarify the task at hand. There are thus strong grounds for exploring the role of visualisation in teaching and learning of geometry. It is hoped that the findings of this study will provide insights into the role of visualisation, which could potentially enhance conceptual understanding in Namibian schools.

1.7 LIMITATIONS

The learners who participated in this study at the school where this research was conducted do not represent the whole population of Grades 11 at the school, region or in Namibia as a whole. For this reason, the findings cannot be directly generalized to other learners and learning situations. The study was also conducted with a specific grade and on a particular topic. This made it difficult to generalize.

The power issue was another possible limitation of this study in that my position as a teacher could have influenced learners' responses given that the study was conducted at the school where I work.

Another possible limitation was time, given that I was expected to carry out my research activities only after normal school hours. Maybe my findings would have been different if I had had more time or carried out the research during school holidays.

1.8 STRUCTURE OF THESIS

This thesis consists of five chapters:

Chapter one outlines the background/context of the study, the research goals and questions, the research methodology, the rationale for and the significance of the study, and possible limitations, as well as detailing the chapters of the thesis.

Chapter two contains a review of literature relevant to the study. It was in this chapter that I reviewed literature on what visualisation is and its role in mathematics teaching and learning in general and geometry in particular. This chapter also delineated the types of visualisation that can be manifested by learners when representing an algebraic solution visually.

Chapter three delineated the methodology I employed to frame, plan and carry out this case study. It also described the interpretive research paradigm and the quantitative and qualitative research approaches that were adopted. I also discussed the research site and sampling issues in detail in this chapter. Explanations of data gathering techniques as well as how data was analysed and validated were also given in this chapter. Lastly, I considered the ethical issues pertinent to this study.

Chapter four presents the analyses and discusses the findings of the data generated from my observations and the participants' responses to the GVT worksheets.

Chapter five provides a summary of findings. Also discussed in this chapter is the significance of the study, recommendations to the study, limitations of the study, suggestions for further research and my personal reflections on the study.

1.9 CONCLUDING REMARKS

In this chapter, I introduced the study, presented the background, research goals, research questions, the methodology, significance of the study, possible limitations and the structure of the whole thesis. The next chapter reviews the literature relating to visualisation in teaching and learning in general and in geometry in particular.

CHAPTER TWO

2. LITERATURE REVIEW

2.1 INTRODUCTION

Like many other post-independent African countries, Namibia went through many political, social, economic and educational reforms to determine its own future after independence in 1990. It is well known that the education reform project in Namibia began in 1990 (National Curriculum for basic education, 2010). The primary goals for this reform were identified as ensuring access, equity, quality and democracy in education (Ministry of Education [MEC], 1993). In addition MOE (1993, p. 74) notes, "as we make the transition from educating the elite to education for all, we also make a shift from teacher-centred to learner-centred which leads to constructivism".

In light of the above, the Ministry of Education sought to launch several strategic initiatives in order to achieve the desired goals. Amongst others, the Education Ministry introduced a new Senior Secondary School Programme, which led to the International General Certificate of Secondary Education (IGCSE) and its variant, the Higher International General Certificate of Secondary Education (HIGCSE). These levels were introduced in 1994, replacing the then South African Cape Education system. According to the Ministry of Education (MEC, 1993), the new programmes were launched to prepare students for entry to the University of Namibia and other tertiary institutions. In the Cape Education system, the approach to teaching was informed by the view that learners were unknowledgeable and thus needed to be imparted with knowledge by their teachers. This system emphasised that teachers were the main source of knowledge while learners were regarded as passive recipients of knowledge.

The Cape Education System was replaced by the current system of two different syllabuses of mathematics at Senior Secondary phase, namely the Namibia Senior Secondary Certificate Ordinary (NSSCO) level Mathematics and the Namibia Senior Secondary Certificate Higher (NSSCH) level Mathematics. The NSSCO Mathematics syllabus is divided into core and extended mathematics. These syllabuses were designed to meet the requirements of the Curriculum Guide for Formal Senior Secondary Education for Namibia and have been approved by the National Examination, Assessment and Certification Board (NEACB [NSSC, Mathematics syllabus], 2009).

Education reform in independent Namibia was necessary for many reasons. According to MEC (1993) and the Ministry of Basic Education, Sport and Culture (MBESC, 1996), the Cape Education system had a number of flaws. Firstly, the Cape Education system was seen to be inefficient in terms of low progression and achievement rates. It was argued that the examinations under the Cape Education system were discriminatory in that, rather than being criterion-referenced, they were norm-referenced. Secondly, it was further argued in MEC (1993) that the Cape Education system was found to be irrelevant to the needs of the indigenous Namibian people. The system was seen to be segregative on the basis of racial and ethnic background. Thirdly, it was characterised by unequal access to education and training at all levels. Fourthly, the Cape Education system was characterised by poor classroom practice (MEC, 1993). The Journal for Education Reform in Namibia (2009, p.12) states that education reform in independent Namibia was necessary due to poor classroom practice that could not be relied upon to promote quality education as it was based on rote learning and memorisation rather than understanding. In view of this, one of the vehicles for achieving the educational goals was the adoption of learner-centred education (LCE) (Namibia: MoE, 2003).

The National Curriculum for Basic Education (NCBE) (Namibia: MOE, 2010a) under the current education reforms, suggests that learners learn best when they are actively involved in the learning process through a high degree of participation, contribution and production. The document further outlines a variety of classroom techniques that a teacher should use. These techniques include direct questioning, eliciting responses, explaining, demonstrating, challenging the learners' ideas, checking for understanding, helping and supporting, providing for active practice, and encouraging problem solving (p. 26). Schrenko (1994) notes that in an LCE approach to teaching and learning, the learner should be at the centre of the teaching and learning process, where learners' interests and needs are taken into account when a teacher is planning or presenting a lesson. The LCE methods to be used in the Namibian classrooms should therefore encourage teachers to teach for understanding, in order to help learners to make connections between what they learn in school and what they do outside of school. This approach to teaching and learning, according to the curriculum, also encompasses "a rich-text and visually and tactile-rich learning environment" (Namibia: MOE, 2010a). This means that learners are expected to use mathematics as a means of communication with an emphasis on the use of clear visual expressions. A visually and tactile-rich learning environment refers to learners' ability to produce and appreciate imaginative and creative work arising from mathematical ideas.

The National Curriculum further outlines that knowledge production be "shared through display of learners' work, charts, posters, and easily accessible information" (p.27). This approach also emphasises the value and use of visualisation in teaching and learning of mathematics in general and geometry in particular. In addition to the call for the use of visuals by the NCBE, the Namibian Senior Secondary Mathematics syllabus (Namibia: MoE, 2010b) states that one of its aims for all learners is to help them "recognise when and how a situation may be represented mathematically, identify and interpret relevant factors and, where necessary, select an appropriate mathematical method to solve the problem" (p .2). In an effort to develop learners' abilities to reason logically, to clarify, to recognise, to generalise and to prove their solutions, there is a need to emphasise and recognise visualisation in the teaching and learning of mathematics.

2.2 VISUALISATION

Visualisation is increasingly being accepted as an important aspect of mathematical reasoning. Wheatley and Brown (1994) states that activities encouraging the construction of images can greatly enhance mathematics learning. Mariotti and Pensci (1994) acknowledges that visualisation that takes place when 'thinking is spontaneously accompanied and supported by images', helps students to understand the problem at hand. Visualisation is regarded as "making the unseen visible' and imagery as 'the power to imagine the possible and the impossible" (Mason 1992). In Lowe's (2000) view, the use of pictures to represent technical subject matter is not a new idea: "ancient pictures from many different countries show that visual information has long been an important means of communicating ideas about the world and how it works. Borst and Kosslyn (2008, p. 849) advances the argument that mental images arise from perceptual representations that are created from stored information instead of information which is currently being registered by the senses.

As an example, Mudaly (2010) states that from birth, children identify their objects based on the meanings they attach to what they see. They recognize whom their father and mother is, suggesting that any other male or female would not be accepted despite similar features.

Furthermore, visualisation in the form of road signs helps adults to make necessary changes in their driving as they commute from point A to point B. An immediate appearance of a "works ahead" road sign, represented by a man with a shovel, necessitates a particular driving strategy (Mudaly, 2010, p. 65). These visual images create in the mind of the viewer a particular reaction, not necessary similar to the reaction created by the verbal utterance of the words.

In the literature, visualisation has been described as the creation of mental images of a given concept. As such, and from a teaching point of view, visualisation seems to be a powerful method to utilise for enhancing students' understanding of a variety of concepts in many disciplines such as computer science, chemistry, physics, biology, engineering, applied statistics and mathematics (Rahim & Siddo, 2009, p. 496).

2.2.1 What is visualisation?

Monovich (2010, p. 13) states that the meanings of the word "visualise" include "make visible" and "make a mental image." This implies that until we "visualise" something, this "something" does not have a visual form. It becomes an image through a process of visualisation.
According to Arcavi (2003, p. 217) visualisation is defined as:

> the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understanding.

Similarly, Zimmermann and Cunningham (1991, p. 1) state, "visualisation describes the process of producing or using geometrical or graphical representations of mathematical concepts, principles or problems, whether hand-drawn or computer-generated". Visualisation can be referred to as geometry done with the mind's eye. Zimmermann and Cunningham further states that visualisation involves being able to create mental images of shapes and then turn them around mentally, thinking about how they look from different view-points, and predicting the results of various transformations (Van de Walle, Bay-Williams & Karp 2014, p.452). From a mathematics perspective, Zimmermann and Cunningham (1991, p. 3) alleges that mathematical visualisation is the process of forming images (mentally, or with pencil and paper, or with the aid of technology) and using such images effectively for mathematical discovery and understanding. In addition, Gorgorio and Jones (1996, p. 2) define visualisation in the domain of geometry as an approach involving both the ability to draw an appropriate diagram and the mental manipulation of geometrical images. Moreover, Makina and Wessels (2009, p. 58) define visualisation in mathematics as a science of patterns. In my thesis, however, visualisation is used as a broad term that incorporates a spectrum of cognitive processes, one of which is visual reasoning.

For the purpose of my study, I opt to define visualisation from the standpoint of Arcavi's (2003) definition as above. In addition to Arcavi and many other researchers' adopted definitions of visualisation, Abrahamson, Lee, Negrete and Gutierrez (2014) emphasise that visualisation is multimodal. In their view, visualisation is considered as perceptuomotor coupling with the affordances of the environment. Their view resonates with genetic epistemology and in particular, the construct of the schema that emerged from micro-ethnographic studies of cognitive development. Furthermore, Abrahamson et al. (2014) conceptualizes visualisation as treating either actual or mentally stimulated images, which may be deployed in a variety of media and semiotic systems. They further state that when attention is oriented toward perceptual information with a goal to drawing inferences, visualising is the natural epistemic orientation.

2.2.2 Types of visualisation

According to Guzman (2002, p.1), visualisation is an interpretation of what is presented for our contemplation, that we can only do when we have learnt to appropriately read the type of communication it offers us. Therefore, it is natural to take into account the different types of visualisation. However, Guzman (2002) acknowledges that mathematical visualisation is not a univocal term, but will depend on the degree of correspondence. This means what goes on in the mind of one person may not necessarily be the same in the mind of another or the picture that two people draw may be different based on their experiences and skill. In light of this, there can be many types of visualisation. In what follows, I have adapted the types of visualisation from Guzman (2002, p. 2).

2.2.3 *Isomorphic visualisation*

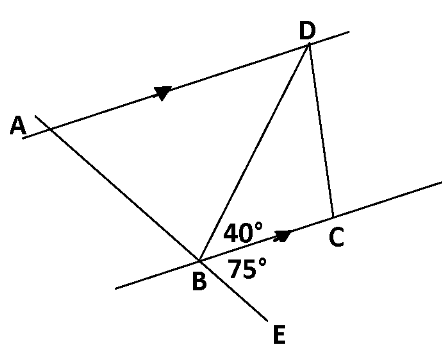
Isomorphic visualisation depicts objects and preserves relationships between them (Miroslav, 2007, p.318).

In this type of visualisation, the objects may have an "exact" correspondence with the representations that make them. This means that, in principle, it would be possible to establish a set of rules to translate the elements of our visual representation and the mathematical relations of the object. The usefulness of this type of visualisation is manifest in everyday life. This is argued because the manipulation of the objects that we perceive with our senses or with our imagination is usually easier and more direct than the handling of abstract objects. As an example, a teacher teaching total surface area of cuboids to a grade 10 class should bring a rectangular box, a pair of scissors and a ruler so that the learners can see the object in its physical state. He or she should guide learners on how to cut out the six faces and measure each of their faces' dimensions. Following that, learners should calculate the area of each of the rectangular faces and then add together the areas of the 6 faces. This presentation should come before introducing the formula method of calculating the surface area of a cuboid to learners as: T.S. A = 2 (l x b) +2 (l x h) + 2 (b x h). Interestingly, Guzman (2002) claims that our visualisations in mathematical analyses are, to a large extent, of this isomorphic kind.

2.2.4 *Homomorphic visualisation*

In this type of visualisation some of the elements of mathematical objects have certain mutual relations that imitate sufficiently well the relationships between the abstract objects and so they can provide us with support, sometimes very important, to guide our imagination in the mathematical process of conjecturing, searching and proving. Below is an example of homomorphic visualisation in a grade 11 class.

In the diagram, AD is parallel to BC. Angle DBC = 40° and angle CBE = 75°



Not drawn to scale

(a) What is the geometrical name given to the quadrilateral ABCD?

(b) Find the value of (i) angle CDB (ii) angle BAD

The success in answering this type of task depends on the learners' understanding and use of properties of angles formed within parallel lines. Learners should understand types of angles created by the transversal line such as line BD in the above example and the angles that form with other lines, for instance angles ADB and DBC, if they are to answer part (b) (i) and (ii). This type of visualisation in many cases can become quite a personal and subjective process, perhaps often not easily communicable, but in any case, the effort to hand it over to our students is worth doing (Guzman, 2002, p.3).

2.2.5 *Analogical visualisation*

Analogical visualisation uses the relation of analogy between objects, for example the use of a ball for a sphere (Miroslav, 2007, p.318).

Under this type of visualisation, one mentally substitutes the objects that he/she is working with by others that relate between themselves in an analogous way. Analogies require learners to use analytical skills. The success of the use of analogies in helping learners to understand mathematical content depends on whether learners are familiar with the vocabulary and if they understand relationships such as comparing, contrasting and sequencing. As an example, the ratio 5:25 is the same as 25:625. This is argued because the first term of each ratio is the square root of the second term and the second term of each ratio is the square of the first term. Using this example, a teacher can ask learners to find as many equivalent ratios as possible. Interestingly, Eratosthenes used an analogy to determine prime numbers between 0 and 100. According to him, finding prime numbers required that you circle 2 because it is a prime number and cross out all its multiples because they are not prime numbers. The next stage is to circle 3 because it is a prime number and cross out all multiples of 3 because they are not prime numbers. One repeats the pattern with other prime numbers and crosses out each prime number's multiples because they are not prime numbers. Ultimately, the numbers that remain circled are the prime numbers. The use of the analogical method has often been put to work in mathematics and the uses of such analogies are capable of the most rigorous development if one strives for it.

2.2.6 *Diagrammatic visualisation*

Diagram visualisation intersects with all the above-mentioned types of visualisation. (Miroslav 2007, p. 318).

Diagrammatic visualisation deals with one's mental object and its mutual relationship concerning the aspects that are of interest to one and is represented by a diagram that merely constitutes a useful help in one's thinking processes. An example of diagrammatic visualisation is the tree diagram we use in probability theory, but there are many others that mathematicians develop for their own use. In many cases, diagrammatic visualisation is communicated with little effort to those that find it extremely useful.

However, some people think that such images and diagrams constitute real obstacles for the development of individuals in mathematics, since what matters, they say, is only the formal justification of our arguments (Guzman, 2002, p. 6). For this reason, Guzman contends that the success that is experienced by the great teachers in mathematics is often due to the efforts they make to transmit to others and to share with them what they know.

2.2.7 How the types of visualisation relate to my study.

The four types of visualisations discussed above related directly to my study. For example, under isomorphic visualisation I looked for diagrams that had an exact correspondence with the representations that the participants made. This aligned very well with the visualisation indicator three (as discussed in section 3.6.2) - transforming the task into a mathematical form. Under the homomorphic type of visualisation, I looked for specific spatial relations such as parallel lines that guided the learners' thinking in solving each task presented. This type of visualisation related to the first indicator of visualisation (see sections 3.6.2) which is understanding spatial relations inherent in the geometry task encountered. Analogical visualisation was the third type of visualisation which aligned very well into this study by enabling me to analyse how learners used their analytical skills when solving the geometry visualisation tasks. It assisted me to see where learners made connections to previously solved tasks to solve the task at hand. The fourth type of visualisation called diagrammatic visualisation intersected with all the other types of visualisation. This type of visualisation is about learners making use of the opportunity to represent their ideas of each task diagrammatically to justify their solutions. This also related very well with the last indicator (see section 3.6.2) of visualisation - illustrating the problem scenario.

2.3 VISUALISATION IN MATHEMATICS EDUCATION

Presmeg (2006, p. 206) reveals that the period of the 1980s was an important watershed in that constructivism was on the rise, countering the influence of behaviourism; qualitative research methodologies were beginning to be accepted as valuable for addressing complex questions in mathematics education. It was in this period that renewed interest in the role of visual thinking in the teaching and learning of mathematics became evident.

According to Presmeg (2006), mathematics is a subject that has diagrams, tables, spatial arrangements of signifiers such as symbols, and other inscriptions as essential components, and as such, visualisation can play an important role in achieving effective teaching and learning. In the learning and teaching of mathematics, which is different from the usual concrete world, the learning object according to Chiappini and Bottino (2007) cannot be shown in an ostensive way, but can only be conjured up by means of external representations. There is no possibility of accessing the thing (object) that can directly hold the meaning of the representation.

In advancing their claim Chiappinni and Bottino assert that mathematical concepts such as numbers, functions and vectors are not directly accessible through everyday experience nor with intuitive perception as real or physical objects are, but have to be represented by signs or symbols. Representations and symbols of mathematics establish a semiotic system, which is of fundamental importance for any mathematical activity (Chiappini & Bottino, 2007). A computer-based system for mathematics learning is also another way of visualising in mathematics education. This computer-based information visualisation can allow the student to access mathematical knowledge integrating the symbolic re-constructive approach with a motor perspective.

In Pape and Tchoshanov (2001)'s view, representations (visualisations) within the domain of mathematics may be thought of as internal abstractions of mathematical ideas or cognitive schemata that are developed by a learner through experience. In addition, mathematical concepts are learned through the gradual building of mental images for primary concepts such as the number of objects in a set, or complex natural phenomena such as the relationship between the flow rate of water and the amount of pollutants.

According to Makina (2010), visualisation is regarded as a very important cornerstone in teaching for understanding in mathematics, because it helps the teacher with facilitation of lessons and with the ability to engage learners in realistic situations. It is recognised as an important aspect of mathematical reasoning (Elliot, 1998, p. 45). In addition, Chappini and Bottino (2007) propose that visualisation refers to the complex phenomenon of visual imagery that plays a central role in all meaning and understanding. Furthermore, Rivera, Steinbring and Arcavi (2014, p. 1) defines visualisation as an epistemological learning tool in mathematics. Miroslav (2007) suggests that visualisation is the transformation of mathematical phenomena into picture form.

Zimmermann and Cunningham (1991) insist that mathematical visualisation is not merely 'maths appreciation through pictures' - a superficial substitute for understanding; instead, they maintain that visualisation supplies depth and meaning to understanding, serving as a reliable guide to problem solving, and creating inspiring creative discoveries. They further state that understanding can be achieved by the use of symbolical, numerical and visual representations of ideas connected together.

2.3.1 Visualisation and learning

Within a mathematical perspective, Makina and Wessels (2009, p. 56) define visualisation as a science of patterns.
As an example, teachers should foster within learners a diligence in finding hidden patterns in for instance, a table of squares of the integers between 1 and 100. As to why visualisation plays an important role in mathematical education, there is a range of reasons:

Wheatley and Brown (1994) states that 'activities encouraging the construction of images can greatly enhance mathematics learning'. Visualisation does not only play a supportive role in mathematical learning but can have an epistemological value, often as a means of discovery, understanding and even as a proof itself (Giaquinto, 2007). Furinghett, Morselli and Antonini (2011) claim that visualisation allows one to simultaneously control a larger number of hypotheses before drawing conclusions (p. 226). Scholars such as Alcock and Simpson (2004), Harrel and Sowder (1998) and Presmeg (1986) have described the role of visual displays as crucial to the work of both experts and students; such displays can condense information, suggest new results or propose potential approaches to proof.

It is further claimed that visualisation extends beyond the graphical, economic and dramatic illustration of an idea to include many other roles associated with higher order visual thinking; thus it may serve as an alternative and powerful resource for mathematics teaching and learning (Giaquinto 2007; Hitt 2002; Manicosu et al., 2005; Presmeg 2006; Rivera 2013; Van Garderen, Scheuermann & Poch, 2014). Accumulating evidence links visual reasoning with deeper understanding of concepts in various mathematical areas such as word problems (Abdullah, Zakaria & Halim, 2012).

In mathematics and its teaching and learning, visualisation in the form of diagrams has a function of describing whole processes and structures at levels of great complexity (Jones 2013). This claim finds support from Samkoff, Lai and Weber (2012) who say that 'diagrams are viewed by mathematicians and mathematics educators alike as an integral component of doing and understanding mathematics' (p. 49); what is more 'drawing diagrams is commonly cited as a heuristic technique for mathematical problem solving that students should engage in' (p. 50). In addition, the National Council of Teachers of Mathematics (NCTM, [2000, p. 67]) states that the ways in which mathematical ideas are represented are fundamental to how people can understand and use those ideas. Therefore, as learners develop a clear and sophisticated visualisation of mathematical concepts, they acquire a deeper understanding of those concepts, and in turn develop what Tall and Vinner (1981) refer to as a concept image. Visualisation in mathematics can help learners to manipulate for example the symbol 2/3 by constructing a square divided equally into three parts, two of which can be shaded. This visual model enables learners to develop and maintain the part-whole meaning of the fractions. Van Garderen's (2006) findings indicate that visualisation skills correlate significantly with students' ability to understand mathematics. It is further argued that high-achieving students often display the highest line of spatial visualisation, likewise low-achieving students benefit from working with given visual static models (Moyer-Packenham, Ulmer and Anderson, 2012). Anderson-Pence, Moyer-Packenham, Westenskow, Shunway and Jordan, (2014, p.3), further claim that when students interpret and create visual static models, they develop new knowledge that can be applied to other problem-solving situations. They add that proficient problem solvers typically develop complex representations for instance, pictures, diagrams or tables to organise and keep track of their solution strategies.

Anderson-Pence et al. (2014) argue that when learners develop clear and sophisticated visualisation of mathematical concepts they will have a deeper understanding of those concepts. They add that visualisation supports meaningful connections with different types of representations and abstract mathematical concepts (p. 3). According to Jones (1998, p. 123), visualisation and imagery has the potential to enhance a global and intuitive view and understanding of various areas of mathematics. Using and applying visualisation in mathematics classroom learning, limits the problem situation that could arise when the basic skills needed to solve the problems have not been mastered by the students (Jones, 2013). He adds that visuals are powerful tools in learning mathematics and can be alternative mass resources. In the literature, cognitive science suggests that we learn to see, and we create what we see; visual reasoning or 'seeing' to think is learned, it can also be taught and it is important to teach it (Whitely, 2004, p. 3). As such, one can argue that visualisation does not only help learners but also teachers to teach well. This claim finds support from Rahim and Siddo (2009), p. 496) who posit that teachers who have learned to become visually skilful would be able to reinforce mathematical concepts and improve the learning process in the classroom.

Furthermore, van Garderen et al. (2014) reiterates that diagrams as a representation strategy demonstrate great versatility as they can be used for solving various types of problems in many topic areas such as geometry, numbers and operations, and probability, and all at great levels. Moreover, they add that diagrams are powerful ways to facilitate communication about critical ideas in mathematics as well as providing a platform for sharing problem-solving strategies with others.

When diagrams are used as mathematical solving strategies, they can serve as a means to understand the problem situation, and as a way to record information both of the problem situation itself and of the ideas as the problem is being solved. A diagram can also be used as a tool to facilitate exploration of critical concepts of the problem being solved and as a way to monitor and evaluate progress (p. 136).

2.3.2 Visualisation and teaching

The effective teaching and learning of mathematics according to the National Council for Teachers of Mathematics [NCTM], (2000) focuses on ensuring that students master problem-solving skills, particularly mathematical word problems. In an effort to enable learners to master mathematical word problem-solving, learners need the support of thinking strategies that will grow the interpretation and manipulation of information through language skills and visual capabilities (Abdullah et. 2012, p. 30). They argue this because mathematical word problems include worded items and their structure sometimes makes them difficult to solve. Furthermore, the basis for selection and decision-making is to analyse and interpret the mathematical problems and in an effort to achieve this goal, students need to be guided and exposed to strategic thinking and representation skills so that mathematical problem-solving skills can be achieved effectively (Abdullah et al. 2012, p. 30). According to Kickbusch (2000), teaching for understanding requires teachers to educate students to exhibit what they know and what they can do with what they know in real time dimension. In whereas teaching for performance is to believe in the capacity of students to create and construct knowledge, and to assign meaning to what they have learnt and experienced as they interact with mathematical tasks.

Given that mathematics is the field of study that seeks patterns in numbers, space, computers and imagination, it is natural to find the most effective way to visualise the patterns and learn how to use visualisation creatively as a tool for understanding (Makina and Wessels (2009, p. 58). This is the process by which learners are helped to learn mathematics by doing it.

According to Usiskin (2012), understanding is viewed as something that goes on in the brain as a response to tasks presented to it. As mathematics is an activity involving objects and the reaction among them, it is important to note that these objects may be abstract or an abstraction from real objects. Since mathematics is founded on concepts from which questions are formulated, Usiskin posits that full understanding of mathematics requires an understanding of the concepts, and the abilities of reasoning and communication are required to solve mathematical problems. Another way to help learners understand mathematics is by the use of representations (visuals).

In further support, Denis (1991) and Kossylyn and Koenig, (1992) reiterate that the use of visualisation has often been cited as a powerful representation process for solving problems. It is further argued that the teaching of pupils to acquire the habit of visualising mathematical realities allows the teacher to gain a powerful tool to achieve goals in educational process, such as successful problem solving, imagination development, fighting formalism in learning and others.

2.4 VISUALISATION PROCESSES

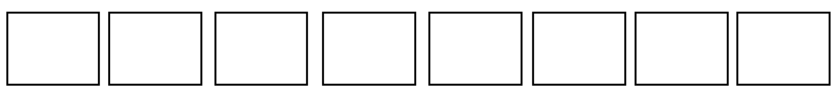
Cohen, Manion and Morrison (2011) claim that in abiding by the principles of fitness for purpose, the researcher must be clear on what he/she wants the data analyses to do, as this will determine the kind of analyses that is undertaken (p. 538). For the purpose of this study, visualisation processes were identified and adapted from Ho (2010), who argues that students go through these visualisation processes when solving mathematical problems. According to Ho (2010), "visualization is at the heart of mathematical problem solving". Visualizing a situation or an object involves "mentally manipulating various alternatives for solving a problem related to a situation or object without benefit of concrete manipulatives". In her view, humans depend on their sight for many things in life, for example they use maps to find their way, pictures to aid recognition, or use diagrams to better describe what their words fail to communicate. She adds that in the mathematics classroom, solutions to problems are sometimes right before learners' eyes. It is in the light of the above that I expanded on Ho's visualisation processes that students go through to fit the purpose of this study in relation to the research questions above. I articulated Ho's processes in such a way that I could transform them into observable indicators. The adopted version of these visualisation processes enabled me to confirm the extent of whether visualisation processes were applied whilst learners interacted with the GVT tasks. For this study I made use of pencil and paper techniques as my participants were not converse with computer technology such as dynamic geometry software.

2.4.1 Processes of visualisation

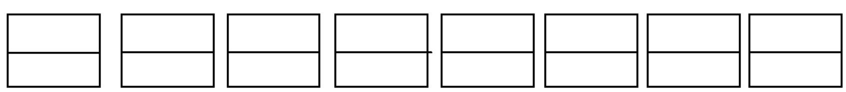
Understanding spatial relations of the elements in the problem: These aspects will be demonstrated by learners if they are able to correctly present the given tasks visually. As an example, if learners are asked to calculate the sum of the interior angles of a pentagon, are learners able to draw the correct shape and identify the appropriate angles inside the pentagon, in other words, is the sketch representing and illustrating the problem/task?

Connecting to previously solved problems: As will be noted at the end of this discussion, visualisation allows students to relate the current problem to previous ones and identify a simpler version of the problem and methods that work for the set problem. For example, in establishing the interior angles of a polygon, one traditionally uses the result that the sum of the interior angles of a regular polygon is (n-2) 180^0 where n is the number of sides of the polygon. This formula allows learners simply to substitute the number of the sides given. Under this process, I will look for evidence that prior knowledge was used to draw a supporting sketch.

Transforming the task into mathematical form (constructing a visual representation): Under this visualisation process, I will specifically be looking for any visuals that the participants construct when solving a given geometrical visualisation task. This can include diagrams, scribble or even images in the mind expressed in words. For example, if learners are asked to solve $8 \div \frac{1}{2}$, some learners may give 16 without showing their working while others may draw eight rectangles and divide each rectangle into 2 parts. Thereafter he or she counts the number of shapes found.



$8 \div \frac{1}{2}$ means 8 rectangles divided into halves. The result is illustrated in the shapes below:



The answer that the learner who divides the eight rectangles gets is 16 halves. Here, I will look at the sketch to see if it has transformed the task into mathematical symbols and constructions.

Clarifying the task at hand (using visualisation to solve the problem): Under this process, the focus is to look specifically on how the visual representation crafted by the participants is used in solving the given task. This process will help me to compare whether the solution to the given task can be obtained directly from the visual representation itself, without the need for computation, or whether the visual is used as a scaffolding mechanism to solve the task at hand. As an example, a volleyball court is 40 metres long and 20 metres wide. Sketch this volleyball court and then draw as many lines (curved or straight) that will divide the court into two halves. In each case what is the area of each half? Under this process, I will see if the sketch simplifies or clarifies the task.

Illustrating the problem scenario (encoding the answer to the problem): This visualisation process focuses on the link between the solution and the visual representation crafted by the participants. Encoding the answer to the problem as a visualisation process helps in measuring the reasonableness of the answer in relation to the visual presented. As an example, the diameter of a cylindrical tin of fish is 5cm and its height is 8cm. Calculate the volume of the tin of fish. The focus is on whether the tin will be accurately labelled with given dimensions or if the dimensions are interchanged, which might give an incorrect answer if followed. This visualisation process will help me to understand if the sketch accurately mirrors the task at hand.

2.4.2 Roles of visualisation

Visualisation has always been an essential aid in the communication of mathematics. Zimmermann and Cunningham (1991) states that visualisation is an important way to concretise concepts, to develop abstraction skills, and to motivate learning, for example in topology and geometry, and in the application of numerical methods to simulations of the real world. It is common knowledge that mathematics teaching is generally communicated in a verbal way, where the teacher orally engages his or her learners with new or old concepts. These words are often abstract or consist of academic cues, which the learners have to make sense of, and together with the language problems that learners naturally have, these cues can confuse learners (Mudaly, 2010, p. 65). In view of this, Ball and Ball (2007), Naidoo (2007), Singh (2007), and Waisel, Wallace and Willemain (1997) indicate that learning is aided by visual or pictorial images.

It is thus important for a teacher to understand how learners use visual and pictorial images to shape their understanding. This study particularly focusses on those diagrams that the learners themselves draw to make sense of the geometric problem at hand. Teaching approaches that encourage learners to recognise connections between different ways of representing geometric ideas and between geometry and other areas of mathematics play an important role in learning geometry.

A key question raised by the intensified study of visualisation according to Hanna and Sidoli (2007, p. 73) is whether or to what extent, visual representations can be used, not only as evidence or inspiration for a mathematical statement, but also in its justification. They add that diagrams and other representations have for a long time been considered as heuristic accompaniments to proof, where they not only facilitate the understanding of a given theorem and its proof, but also can often encourage and inspire the rule to be proved and can highlight approaches to the construction of the proof itself. Visualisation tools are essential to the mathematics curriculum (p. 73).

Visualisation in a form of dynamic software is seen to be very successful in enhancing the ability of students to notice details, to conjuncture, to reflect on and interpret relationships and to offer tentative explanations and proofs (Hanna & Sidoli, 2007, p. 77). Furthermore, Owen (1999) states that some of the visualisation in computing aids in gaining insight by using visual machinery. It transforms the symbolic into the geometric and offers the methods for seeing the unseen. In Owen's view, visualisation enriches the process of scientific discovery and fosters profound and unexpected insights. Further, visualisation essentially has a role to play in promoting a deeper level of understanding of the data under investigation, and fostering new insights into the underlying processes, relying on man's powerful ability to visualise. Moreover, Hearst (2009) highlights another role of visualisation as translating abstract information into the visual form, which provides new insight into that information. This claim finds support in Berkeley (2010), which states that information visualisation unveils the underlying structure of large or abstract data sets using visual representations that utilise the powerful processing capabilities of the human visual perceptual system.

Rosken and Rolka (2006, p. 457) confirm that the role of visualisation in mathematics learning has been the subject of much research (Arcavi, 2003; Bishop, 1989; Dreyfus Eisenberg and , 1986; English, 1997; Kadunz, & Straesser, 2004; Presmeg, 1992; Stylianou & Silver, 2004). According to Guzman (2002), visualisation in the form of images is also a very powerful tool to grasp in a unitary and holistic way the different contexts that constantly arise in different tasks connected with the theory. It is argued further that images are effective vehicles for the communication of ideas.

Clark, Nguyen and Sweller (2006) assert that visual representations alleviate the cognitive load during problem solving and allow learners to work on one part of the model without having to keep track of the entire model in their minds. If learners are able to transform the task into mathematical symbols and constructions, they in turn can interpret the solutions to the given task. In addition, when learners interpret and create visual static models, they develop new knowledge that can be applied to other problem-solving situations.

According to Van Garderen (2006), the use of visualisation has often been cited as a powerful problem representation process for solving mathematical problems (p. 496). Visual imagery has a role in establishing the meaning of the problems channelling problem-solving approaches and influencing connective constructions (Owens and Clements, 1998). Additional support for the nature and role of visualisation and imagery, according to Jones (1998), is that it has a potential to enhance a global and intuitive view and understanding of various areas of mathematics. Fischbein (1987, p. 104), for example, comments that ‘’a visual image not only organises the data at hand meaning structures, but is also an important factor in guiding the analytical developments of a solution’’. It is further argued by Jones (1998) that visualisation has a role in the development of geometrical reasoning and as such, there is value in emphasising visual representations in all aspects of the mathematics classroom.

In addition, visualisation plays a role not only in realising effective teaching and learning of geometry but also in other domains of mathematics, such as algebra, transformation and number sentences. By way of an example in algebra, imagining and drawing the curve of an algebraic equation $y= 10 + 2x - x^2$ is fundamental to understanding the curve in transformation, imagining and drawing a transformation object line. Drawing a right-angled triangle to a given scale helps learners to understand translation beyond the abstract algorithmic procedures, while in a number sense, the imagining and drawing of number lines when adding or subtracting integers helps learners to build visual structures to interpret and understand addition and subtraction.

Moreover, visualisation helps learners to demonstrate, with diagrams, their mental processes or reasoning when given a mathematical task. Serpil, Cihan, Sabri and Ahmet (2002) reiterates that using a visualisation approach in teaching and learning many mathematical concepts in general and geometric concepts in particular, can make the lesson concrete and clear for students to understand (p. 2). Visualisation supports meaningful connections with different types of representation and abstract mathematical concepts (Anderson-Pence et al., 2014. p. 3).

Advancing the role of visualisation in teaching and learning is the matrix from which concepts and methods arise (Guzman, 2002, p. 3). It is a stimulating influence for the rise of interesting problems in different ways. Visuals or manipulatives, as Shaw (2002) calls them, play an important role in reducing conflict of mathematical ideas in the learner's mind. Shaw adds that when there is less conflict in the learner's mind, deeper understanding can begin to take hold, develop and grow, thereby laying the groundwork for future mathematics learning. Shaw further contends that visualisation helps learners not to rely on rules to understand what symbols represent.

In further support of the role of visualisation in teaching and learning, Wheatley (1991) indicates that mathematics tasks should be presented in a familiar setting where students have a greater opportunity to use their prior experience in giving meaning to the tasks. For example, rather than just pose the problem 36 – 29 in an abstract form, the teacher could present the task in a potentially meaningful setting: 36 chairs are needed for a school party, 29 chairs are already in the room, how many more chairs are needed? In so doing, Whitely argues that learners can construct images for the numbers and use this imagery in developing a procedure for subtracting (p. 35). Kashefi, Alias, Kahar, Buhari, Zakaria and Mirzaei (2015) are of the view that visualisation among students must be enhanced in order to boost their insight into mathematics. As cognitive science suggests, we learn to see, we create what we see, visual reasoning or 'seeing to think' is learned; it can be taught and it is actually important to teach it (Whitely, 2004 p.3).

Although in recent decades visualisation in teaching and learning of mathematics has been emphasised to have much value for conceptual understanding, David and Tomaz (2012, p. 415) believes that merely using more visual representations in the classroom does not guarantee the facilitation of visualisation. They state, for example, that when students are just following a series of steps to construct a predetermined geometrical object, without understanding what they are doing, learning does not take place. Steenpass and Steinbring (2014) claim that mathematical visuals do not convey a mathematical concept directly into the students' heads, but that they have to be actively interpreted and embedded in a social-cultural milieu (p. 4).

Bishop (1989) concludes his review by saying that ''there is value in emphasising visual representations in all aspects of the mathematics classrooms". Yet it is also recognised that there are difficulties concerned with visualisation and images (Love, 1995). If mathematics visualisation is taken to be "the process of forming images (mentally or with pencil and paper, or with the aid of technology) and using such imagery effectively for mathematical discovery and understanding" (Zimmermann and Cunningharm 1991, p. 3), then such difficulties could relate to the process of forming images as well as using them for solving problems. Similarly, if mental imagery is taken as involving ''constructing an image, pictures, words or thoughts, representing that image as needed, and transforming that image'' (Wheatley 1991), then difficulties can arise from the process of constructing, representing and transforming. In view of the reviewed literature above, visualisation enhances mathematics understanding only if it is employed appropriately and effectively in a context of connecting with different mathematical domains.

Despite claims by some researchers that visualisation is memorable, Presmeg (2013, p. 153) asserts that visualisation is not self-explanatory, but is a cultural occurrence that unfolds its meaning in interaction processes between teacher and learner (Steenpass and Steinbring 2014). Presmeg further suggests that it is inadequate for a teacher to present a visual display in teaching some mathematical topic, and expect learners to make the connections with the mathematical principle that is in the teacher's mind. Along with the agreement about the contribution of incorporating visual reasoning into mathematics instruction, a suite of studies has also shown that analytical approaches seem to dominate the way students learn mathematics (Eisenberg & Dreyfus, 1994; Tall, 1991). This is a phenomenon that may be attributed to pervasive instructional models emphasizing analytic over visual reasoning, or to beliefs about symbolic form as being the most legitimate (Natsheh & Karsenty, 2014).

Worth noting is that an incorrect use of visualisation can lead to errors in different ways. This could happen if the figure relied upon suggests a situation that in fact does not take place (Guzman, 2002). In some other cases, the visual situation can mislead students from accepting certain relationships that appear so highly obvious but have never come to their minds.

These claims find support in Zimmermann and Cunningham (1991, p. 3) who contend that visualisation is not an end in itself but a means towards an end: to reach understanding.

2.4.3 Namibian Curriculum

According to Mateya (2013), Namibia recognises mathematics as one of the crucial subjects necessary to realise the country's full potential (p. 49). In further support, The NCBE (Namibia: MoE, 2010, p.12) highlights mathematics as one of the key learning areas. A key learning area is a field where essential knowledge and skills can be found and developed. Furthermore, Namibia's National Curriculum stresses that acquiring mathematical skills, knowledge, concepts and processes enables the learner to investigate, model and interpret numerical and spatial relationships and patterns that exist in the world (p. 12). This aligns to the broader goal of Basic Education, which aims to empower learners for the development of Namibia's future as a knowledge-based society.

In the mathematics curriculum, "geometry'' as a domain of mathematics is considered as a key element which should be taught proficiently (Mateya, 2013, p. 50). However, Namibia Senior Secondary Certificate Ordinary (NSSCO) examiners' report (2014), states that questions on geometry appear to be only moderately answered because some learners struggle with circle geometry (p. 314). In addition, the NSSCO examinations report for the past academic years, (NSSCO, 2011, 2012 and 2013) repeatedly encourage learners to show complete methods of working especially if they are asked to prove mathematical results.

In an addition, the NCBE (Namibia: MoE, 2010) emphasises numeracy as one of the core skills under the LCE system. It is stated in this document that this skill involves creating logical models for understanding, and being able to think in terms of relationships of quantity, size, shape and space, and computation. Essentially, the NCBE aims that upon completion of the Senior Secondary phase, learners should be able to use mathematical language and representation as a means of solving problems relevant to everyday life and their further education and future careers. The question at stake is how Namibian schools can best prepare learners to learn, given that these classes have a wide range of mixed ability learners.

The NCBE (Namibia: MoE, 2010) states that one method by which Namibian schools can prepare their learners to learn and understand mathematics fully, is for teachers to support these learners by employing methods of teaching that enable them to follow the lessons and participate in the learning process, and through the provision of the necessary learning aids and support materials (p. 28). Despite the NCBE not making specific mention of the use of visualisation as another method, one can argue that it is necessary to employ visualisation in the teaching and learning especially to the learners with hearing impairments, or learners with speech and language impairments.

2.5 GEOMETRY

Bassarear (2012, p. 463) defines geometry as the study of shapes, their relationships, and their properties. The history of geometry arises from the practical measurement of land in ancient Egypt and the study of the properties of the shapes in Greek geometry (Luneta, 2014). Geometry is an explanatory field of mathematics that has links with human culture, history, art and design. In this study, 'basic geometry' refers to triangles and their properties, measures (area, perimeter and volume of shapes), circle theory and the Pythagoras theorem.
Luneta (2014) further states that geometry deals with the spatial relationships between real things. Knowledge of geometry and geometric reasoning is not acquired through passive consideration, but rather through active interaction with and exploration of shapes. According to Frobisher, Frobisher, Orton and Orton (2007, p. 19), "in their learning of shapes and space, children experience and understand the connections between knowledge, concepts and skills in different facets of geometry". This means that teachers should be mindful of a child's environment and adapt their instructional approach to teaching geometry by employing a more practical teaching strategy that draws on the child's sense of space, which is defined as the understanding of shapes, by describing their characteristics and their relationships to one another (Battista, 2007).

Spatial sense consists of two important components of geometric knowledge, namely spatial visualisation, which is the ability to visually compare shapes that have changed position on the plane-transformation geometry (Bassarear, 2012), and spatial orientation which operates when a fixed object is viewed from different points or when the position of an object is acknowledged (Battista, 2007). Spatial visualisation and spatial orientation play an important role in learners' ability to understand shapes and their properties through geometric reasoning and the visualising of images, their properties and physical representations.

According to Luneta (2014), learners use spatial reasoning to analyse and compare shapes, estimate quantities, and make mental computations. Furthermore, van Hiele (1999, 1986) proposed five levels of geometric thought. In this view, learners make progress and develop their knowledge of geometry in accordance with the developmental trajectory suggested by these levels. For example at level 0 (visualisation or recognition level), learners can identify a shape but are unable to list its properties and judge the shape by its appearance. The next level (level 1) is the analysis level at which learners are able to identify the properties of particular shapes, but not in a logical order. Level 2 is the abstraction or relationship level where learners can combine shapes and their properties to provide a precise definition, as well as relate different shapes to one another. At this level, learners can also present a logical ordering of the properties and these are deduced from one another. At level 3 (deductive level), learners apply formal deductive arguments such as proofs and theorems within an axiomatic system. The fifth and final level according to van Hiele, is called rigor, which is characterised by formal reasoning about mathematical systems by manipulating geometric statements such as theorems.

Understanding these levels enables teachers to identify the general direction of their students' learning and the level at which they are operating geometrically. According to Kilpatrick, Swafford and Findell (2001), the first three levels show the development of procedural fluency in geometry and the last two levels demonstrate the development of conceptual understanding. To date, van Hiele's theory of geometric thought is regarded as a powerful framework known for deconstructing and rewarding teaching and learning geometry (NCTM, 2000). See figure 2.1 below.

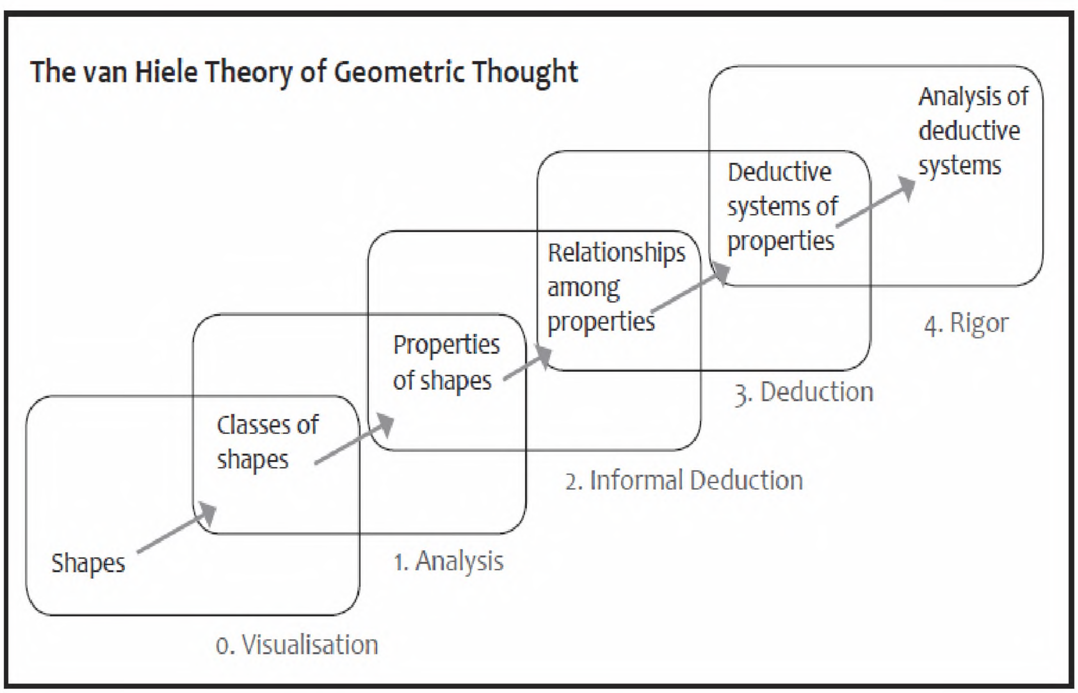


Figure 2.1: The van Hiel theory of geometric thought. Source: van de Walle (2004, p. 347).

At each level of geometric thought, the ideas created become the focus or object of thought at the next level.

In addition, Ding and Jones (2006) posit that effective instruction in geometry requires teachers to develop sound instructional strategies and knowledge of useful resources and activities. Effective mathematics teachers reflect on their connected mathematical knowledge bases, which include content knowledge, pedagogical content knowledge, conceptual knowledge and procedural knowledge (Luneta, 2013). Content knowledge is the main knowledge base that a teacher should possess in order to be effective. Schneider and Stern (2010, p. 178) define content knowledge as a way of providing an abstract understanding of the principles and relations between pieces of knowledge in certain domains. Zakaria and Zaini (2009) describe procedural fluency as the learner's ability to provide mathematical answers without giving reasons for certain steps, methods, operations or formulae that were used. Furthermore, McCormick (1997) explain that procedural fluency is the know-how-to-do-it knowledge of mathematics, which involves the ability to quickly recall and accurately execute procedures.

In view of the above literature, geometry is seen to be one of the important branches of mathematics education, because its teaching aims at developing students' ability of critical thinking and problem solving, and a better understanding of the other themes in mathematics (Aydogdu & Kesan, 2014). A good geometry course is related to peoples' daily lives.

Idris (2009, p. 94) states that geometry is a unifying theme throughout the entire mathematics curriculum and as such is a rich source of visualisation for arithmetical, algebraic, and statistical concepts. Jones (2013) states that one component of mathematics education that makes great use of diagrams is the teaching and learning of geometry (p. 37). A cornerstone of geometry is the visual study of shapes, sizes, patterns and position. In addition, Van de Walle, Karp and Bay-Williams (2014) state that teaching and learning of geometry is based on two related frameworks: (1) Spatial sense and geometric reasoning and (2) Specific geometric content found in distinct objectives (p. 427). They further assert that ''the first framework has to do with the way students think and reason about shape and space while the second framework is content in the more traditional sense - knowing about symmetry triangles, parallel lines and so forth'' (p.426). Solving geometrical tasks in the context of this study refers to the meaningful solution strategies that learners show in obtaining their geometry answers.

Olkun, Sinoplu and Deryakulu (2005) emphasise that improving students' geometric thinking is one of the major aims of mathematical education since geometric thinking is inherent in so many scientific, technical and occupational areas as well as in mathematics itself (p.1). Visualisation is a key component in geometric thinking as it enables students to represent abstract concepts in a visual manner. For this reason, this study focuses on visualisation in geometry learning.

In an effort to understand why learners struggle with solving geometry tasks, Giaquinto (2007) postulates that reliable justifications of beliefs can come from direct visual appraisal; he explains this as follows:

> Our initial geometrical conceptualisation of basic shapes depends on the way we perceive those shapes. In having geometrical concepts for the shapes we have certain belief-forming dispositions. These dispositions can be triggered by the experience of seeing or visual imagining, and when that happens, we acquire geometrical beliefs. The beliefs acquired in this way constitute knowledge, in fact a synthetic a priori knowledge provided the belief-forming dispositions are reliable (p. 12).

Zodik and Zaslavky (2007) argue that geometric problems are often accompanied by figures/diagrams that represent specific mathematical ideas, but that a diagram can be accurate, sketchy or even misleading (p. 265). So too does Arcavi (2003) argue, that 'in different contexts, the same visual object may have different meanings to the experts'' (p. 232). Employing visual support materials such as diagrams and sketches in the teaching and learning of geometry can assist learners to solve geometrical problems, present the solution clearly, and check and interpret their results. Therefore, it is very important that teachers use visual support material strategically and correctly to teach geometry effectively.

In an effort to achieve this goal, it is important to encourage learners to use visualisation tools such as diagrams to illustrate their thinking and understanding (Presmeg and Nendurandu, 2005, p.106). As a mathematical domain, geometry is largely concerned with specific mental entities, i.e. geometrical figures. It deals with spatial relationships between real things (Luneta, 2014). Knowledge of geometry and geometric reasoning is not acquired through passive consideration, but rather through active interaction with and exploration of shapes.

According to Deliyianni, Elia, Gagatsis, Monoyiou, and Panaoura (2009), the understanding of geometry requires that there be no confusion between mathematical objects and their respective representations. Furthermore, educators are encouraged to investigate how pupils use and react to each teaching tool or procedure, and what beliefs or conceptions they develop. In geometry, the use of representations is a necessary component of learning and there are not many studies on students' conceptions about its usefulness nor on their self- beliefs about using them. For this reason, it is considered that the present study will contribute to the extensions of theoretical approaches of cognitive and non-cognitive processes that underlie understanding in the learning of geometry as a mathematical domain.

2.5.1 Geometry and visualisation

According to Battista (1990, p. 47), several National Assessment of Educational Progress found that males significantly outperformed females in the areas of geometry and measurement. On the basis of these assessments, he adds that it was hypothesized that one possible factor that may underlie these gender differences is spatial visualisation. As such, the current curriculum guidelines in school mathematics in many countries highlight the importance of developing students' conceptual understanding of mathematics. According to Huang and Witz (2011), seeking an effective curriculum and instruction that would facilitate children's conceptual understanding is a crucial issue for mathematics education.

As a way of example, children's ability to handle area measurement can be enhanced by employing a teaching curriculum that connects 2-dimensional (2-D) geometry motions (visualisation) and area measurement numerical calculations. Geometry and visualisation are complementary to each other in developing children's conceptual understanding of geometry (Huang & Witz, 2011).

With reference to geometry teaching and learning, students encounter difficulties in making sense what they have learned if they are not given enough time to understand the geometry concepts (Idris, 2009. P.95). This point is argued given that many learners are not introduced to visualisation in their learning of geometry and area measurement (Idris, 2009). She further asserts that the lack of understanding geometry is often caused by the absence of using visualisation among the students, which invariably will lead to poor performance in the subject area. In Cangelosi, (1996) and Idris (2006)'s views, geometry language, visualisation abilities, and ineffective instruction are some of the factors inhabiting conceptual understanding of geometry.

In an effort to attain comprehension of geometrical concepts, operations and relations (conceptual understanding), teachers are encouraged to assist learners to visualise mathematical problems. According to Idris (2009), effective learning (conceptual understanding) of geometry occurs as students actively experience the object of study in appropriate contexts of geometric thinking and as they engage in discussion and reflection using the language of the learning period.

2.5.2 Geometry problem solving strategies

In general, a problem-solving strategy is a technique that may not guarantee solution, but serves as a guide in the problem-solving process (Gick, 1986). She gives an example in geometry of a strategy for proving that two triangles are congruent, as showing that their corresponding angles and sides are equal. Problem solving is considered the most significant cognitive activity in everyday and professional environments (Elia, van der Heuvel-Panhuizen & Kolovou, 2009). An attribute, which is considered integral to problem solving, is strategic behaviour. Elia et al. (2009) continue to assert that what is true for solving problems in general also applies to mathematical problem solving.

Pape and Wang (2003) and Verschaffel, Corte, Lasure, Van Vaerenbergh, Gogaerts and Ratinckx (1999) hold strategy use as central to processing mathematical problems. Some literatures say that success in solving a mathematical problem is positively related to the student's use of problem solving strategies (Elia et al., 2009). Although students continuously face new situations and unfamiliar problems that require them not only to know and apply various strategies but also to be flexible, it is argued that what children learn in one situation and what applies to one problem, will not necessarily fit another situation or be appropriate for another problem. It is therefore to this effect that this study focuses on learners' geometry problem solving strategies related to visualisation. For the purposes of this study, the use of the term problem is interpreted on the basis of Schoenfeld's (1992) definition that a problem is an unfamiliar situation in which an individual does not know how to carry out its solution. In other words, he or she is unable to solve the situation comfortably using routine or familiar procedures. Furthermore, the term *strategies* according to Verschaffel et al. (1999) refers to approaches in solving a problem that include drawing a picture, making a list or a table, guessing and checking.

A fundamental aspect of mathematical thinking is constituted by problem solving strategies. In the view of Elia et al. (2009), students' use of heuristic strategies is positively related to performance in problem solving tests. The problem solving strategies include: guess-check-revise, draw a picture, act out the problem, use of objects, choosing an operation, making tables, looking for a pattern, making an organised list, writing equations, using logical reasoning and working backwards.

Besides heuristic (cognitive) strategies, Schoenfeld (1992) and Verschaffel et al. (1999) note that solving a problem also requires metacognitive strategies. Metacognitive strategies involve self-regulatory actions such as decomposing the problem, monitoring the solution process, and evaluating and verifying results. Schoenfeld (1992) and Verschaffel et al. (1999) stress that these strategies play a crucial role in achieving problem solving success. Alternative solution strategies in problem solving may occur when the students experience difficulties with a problem or at any stage of the problem solving process (Kaizer & Shore, 1995).

Similarly, when solving geometry word problems, the use of diagrams can be an extremely "powerful" visual representation strategy (van Garderen, Scheuermann & Poch, 2014, p. 136). The use of imagery considered schematic (images that contain relational information of the problem), has been positively related to success in mathematical problem solving.

Problem solving in geometry classes provides an important contribution to mathematics education by helping students develop their reasoning and problem solving skills - skills to be used later in life - which is one of the aims of mathematics teaching. (Yilmaz 2007). Duval (1998) highlights that geometrical reasoning involves three kinds of cognitive processes, which fulfil specific epistemological functions, and these cognitive processes include:

Visualisation process: This is the visual representation of a geometrical statement, or the heuristic exploration of a geometrical situation.
Construction process: This is a geometry problem solving strategy where the solver of the given problem uses tools to construct geometrical representations in their solutions to the problem.

Reasoning process: This strategy, particularly the discursive process, is concerned with the extension of knowledge, the explanation of ideas, and proving a geometrical task.
Duval further points out that these different processes/strategies can be performed separately depending on the nature of the task at hand. He contends that even if a construction leads to a visualisation, construction between relevant mathematical properties and the constraints of the tool being used as geometry problem solving strategy, students are enabled to negotiate the meaning of the form they have produced as well as the meaning of the standard representational forms (Pape & Tchoshanov, 2001).

In pictorial representations, construction may be used to help students explain and justify an argument. As a geometry problem solving strategy, visualisation processes in the form of diagrams are generally taken to be an integral component of doing and understanding mathematics. In the teaching of geometry, the use of diagrams is not only due to the nature of the geometrical objects, but also because a diagram is often a particularly effective problem representation that enables complex geometric processes and structures to be represented holistically (Jones, 2013, p. 37). Being a geometry problem solving strategy, the discursive process (talk) is "actually the true foundation of learning" (Zhang, 2009. Zhang further asserts that it is through talk that children actively engage and teachers constructively intervene for the extension of knowledge. It is contended that the quality of student learning is closely associated with the quality of classroom discourse. When a reasoning process is encouraged in learners as a problem solving strategy, this can help learners to generate their questions and to explore alternative answers.

Essentially, there are many factors that affect problem solving but one of the most important of these factors is to choose and use the appropriate strategy. As many different strategies can be used for a particular type of problem, a strategy can also be used for many problems (Rahin, 2007). Some of the strategies that are often used both for the solution of geometry problems and their definition, are adapted from Aydogdu and Kesan (2014):

- *Making a drawing:* what is meant by the word *drawing* here is all the drawings that aid in the representation and correlation of the data given in the problem. These can be simple lines, geometrical shapes, dots etc. (Altun 2002).
- *Intelligent guessing and testing strategy:* With this strategy, the answer of the problem is guessed and the guess is tested for correctness. If it is correct, the problem is

solved; if it is incorrect, new guesses are made. This process goes on until the correct answer is found (Altun, 2002).

- *Simplifying the problem:* When widely complex problems are encountered, using this strategy involves dividing the problem into sub-problems. Each sub- problem solved, simplifies the solution of the original problem. The successive simplifying process goes on until all the sub-problems are solved. These separated parts are then re-combined for the solution of the original problem (Dhillon, 1998).
- *Using known information:* When solving a problem, we sometimes use the formula, correlation or relationship of which we have prior knowledge.
- *Brainstorming:* Brainstorming is a good strategy for raising the quality and number of solutions. Once the problem is defined, all the possible solutions are put forward uncritically. Thereafter, by critical analysis, the most applicable and practical solution is estimated and the best one is chosen (Dhillon, 1998).

2.6 CONCLUSION

This chapter focused on what literature has to say regarding visualisation in the teaching and learning of geometry. Most literature shows that visualisation promotes imagination and creativity in mathematics and other fields of knowledge. Furthermore, visualisation has been considered by some researchers as central to the teaching and learning of mathematics. This argument is advanced by literature citing the role computers play on the development of thinking when learners solve mathematical problems. Moreover, this chapter provided me with the insight to understand that if learners are given the opportunity to "talk through" their thought processes and represent their solutions on paper or in mind, learning of complex mathematical ideas can become meaningful to them.

In this study, it was also revealed that visualisation helps teachers to represent mathematical tasks in multiple ways that can enable learners to understand the task at hand. Essentially, this chapter demonstrated that 'seeing' the diagrams, graphs, tables and pictures drawn, enhance the learning through creation of meaningful opportunities for the learners to grapple with unfamiliar concepts, and content presented. In addition, the literature under this study highlighted that the exploring of geometry can develop problem-solving skills.

Given that problem solving is so important for studying mathematics, it is against this background that geometry is considered to play an essential role in the study of other areas of mathematics. For these reasons, I conclude that the use of visualisation in the teaching and learning of geometry can assist learners to acquire comprehension of mathematical concepts, operations and relations.

CHAPTER THREE

3. METHODOLOGY

3.1 INTRODUCTION

A research methodology refers to the underpinning philosophical framework and fundamental assumptions that frame any research studies (Creswell, 2006, p. 4). Given that the philosophical framework one uses influences the procedures of research, the methodology relates to the entire process of research.

For the purposes of this thesis, I chose a mixed method approach in analysing how selected Grade 11 learners interact with geometry tasks using visualisation processes. I wish to analyse the nature of visualisation processes employed when these learners interact with geometry problems, and to determine how these selected Grade 11 learners use these processes in their interactions with geometry problems. My fundamental research questions that guided this research study are stated below:

1. What is the nature of the visualisation process employed when selected Grade 11 learners interact with geometry problems?
2. How do these Grade 11 learners use visualisation processes in their interactions with geometry problems?

In this chapter, research design, methods of data collection and analysis procedures, research techniques, selection of participants, validity and ethical issues are addressed.

3.2 RESEARCH PARADIGM

Guba and Linkoln (1994, p. 105) defines a research paradigm as a basic belief system or worldview that guides any research investigation. In this study, I adopted an interpretive research paradigm in order to obtain an in-depth understanding of my participants' visualisation processes. The choice to select this paradigm stems from the premise that the interpretive research paradigm is concerned primarily with generating context-based understanding of people's thoughts, beliefs, values, and associated social actions (Taylor, Taylor & Luitel, 2012,

.p. 5). They further argue that this paradigm is appropriate for an open-ended research design process that enables new and still developing research questions.

According to Photongsunan (2010, p. 1), the interpretive paradigm takes different ontological and epistemological positions to the positivist paradigm. For this reason, interpretive researchers do not regard the social world as “out there” but believe that it is constructed by human beings. Given that people give meaning to their social world, interpretive researchers seek to investigate how humans perceive and make sense of this world. In particular, I closely looked at different ways in which my participants visualised mathematical tasks presented to them in order to understand the nature of these visualisations. Additionally, I interrogated and analysed diagrams, pictures and texts that the participants used to construct meaning and understanding. Through interviews and observation, rich descriptions of participants’ responses were observed and analysed.

3.2.1 Research approach

According to Creswell (2006), a research approach refers to the underlying philosophical and empirical assumptions that guide the inquirer. In this thesis, the research approach undertaken was a mixed method approach. As Creswell (2006) suggests, this approach focuses on collecting and analysing both quantitative and qualitative data. Furthermore, it is argued that the central importance of a mixed method is that the use of quantitative and qualitative approaches in combination enhances the understanding of the research problem, more so than either approach alone. For the purposes of this thesis, the data analysed quantitatively complements the data analysed qualitatively. This claim finds support from Reams and Twale (2008, p. 133) who argue that mixed methods are ‘necessary to uncover information and perspective, increase corroboration of the data, and render less biased and more accurate conclusions’.

Denscombe (2008, p. 272) argues that mixed method research can (a) enhance the accuracy of data; (b) provide a more complete picture of the phenomenon under study than would be yielded by a single approach, thereby overcoming the weaknesses and biases of single approaches; (c) allow the researcher to develop the analysis and build on the original data; and (d) assist in sampling. Advancing reasons for the choice of employing a mixed method approach in collecting and analysing data, Taylor, Taylor and Luitel (2012, p. 1) posit that “there is a crack in everything, that is how the light gets in”. This explains why using quantitative over qualitative or vice-versa was seen to be less effective if appropriate data was to be collected and analysed.

In this study, I used an analytical tool to collect and analyse my quantitative data. Quantitative inquiry looks for both verbal accounts and observations in numbers. Burns and Grove (2003, p. 19) describe qualitative research as “a systematic subjective approach used to describe life experiences and situations to give them meaning”. My qualitative data was generated through interviews.

3.3 RESEARCH METHODS

This study followed a case study research method.

3.3.1 Case study

According to Zaidah (2007, p. 2), “a case study is an empirical inquiry that investigates a contemporary phenomenon within its real-life context; when boundaries between phenomenon and context are not clearly evident, and in which multiple sources of evidence are used”. It is further stated that case studies can allow the exploration and understanding of complex issues.

Given that this study is a combination of quantitative and qualitative approaches, the case study helps me to explain the nature and processes of visualisation through observation, reconstruction and analysis of the cases under investigation as noted by Tellis (1997). Zaidah (2007) divides case studies into three categories, namely exploratory, descriptive and explanatory case studies. In Zaidah’s view, exploratory case studies aim to explore any event in the data, which serves as a point of interest to the researcher. For example, in this case study, in an effort to understand what participants imagined in their mind that helped them to draw a representation, I asked them “Does what you see or imagine in your mind help you to draw a picture that represents the geometrical problem at hand?” This question was meant to open up the door for exploring the link between what they imagined and what they drew. Descriptive case studies describe a natural phenomenon. In this instance, this would refer to the different strategies that were used by my participants to use visualisation processes to solve geometric tasks. A third type of case study is the explanatory case study, which examines the data closely both at a surface and deeper level in order to explain the phenomenon in great depth (Zaidah, 2007). In my case study, I particularly asked the participants to give me reasons why they solved each task the way they did.

In my case study, I adopted elements of all three of Zaidah's categories. My case study was exploratory because I knew little about how my participants used visualisation processes in solving geometric problems. It was descriptive because one of the aims of this study was to obtain a picture of learners' understanding of the nature and processes of visualisation. It was also explanatory because the preparatory steps or the planning phase of data collection prior to discussing the actual data gathering procedures needed to be un-packed and explained.

For my study, I selected a group of eight Grade 11 learners who were subjected to a series of geometry visualisation tasks in order to investigate and analyse the nature of visualisation processes that these learners employed and how they used these processes in their interactions with these tasks. My unit of analysis was their interaction with these tasks and the visualisation processes that were evident.

3.4 THE GEOMETRY VISUALISATION TASKS

Geometry visualisation tasks (GVT) is a set of 12 adapted geometry items from different sources which included mathematics text books and previous mathematics national examination papers. The objective of the individual tasks was to illicit visualisation processes ie I wanted to observe how the participants solved the tasks using visualisation processes and skills. The tasks ranged in difficulty from simple to complex. They were all based on the Namibian curriculum and on the articulated Grades 8 – 11 competencies of the curriculum (Namibia: MoE, 2010). The GVT can be found in Appendix 4.

Task 1

In this task, I primarily looked for what visualization processes learners used in identifying directions from a perspective of bearing. This is a requirement of the Namibian mathematics curriculum. Furthermore. Ho (2010) points out that when learners represent the problem visually, they can understand how the elements in the problem relate to each other. This task was aimed at finding out whether the answer to the problem could be obtained directly from the visual representation itself, without the need for computation.

Task 2

In this task, I wished to research whether and how learners applyied the theorem of Pythagoras in visual terms. I wanted to see whether learners drew the respective squares on the sides of the right angle triangle. I also wished to observe whether the learners visualized the problem in relation to previous problem-solving experiences as noted in Ho's processes of visualization.

Task 3

The aim of this task was to observe how learners interact/engage visually with composite shapes having different properties. The Namibian curriculum expects teachers to treat individual shapes first and later combine these shapes to allow learners to unpack these shapes using their properties. This task relates well with Ho's conclusion that by representing the problem visually, learners can understand how the elements in the problem relate to each other.

Task 4

This task looked for learners' awareness of the many types of triangles having a perimeter of 12 units. It was expected of the learners to use sketches of different types of triangles with the same perimeter. This aspect is used in the Namibian curriculum where the competency requires learners to find perimeter of different shapes. The task aligns well with one of Ho' s indicators of visualization by relating the given problem to previous problem-solving experiences.

Task 5

In this task, the aim was to relate visualization to other domains of learning where leaners were expected to explain their thought processes on the question as they figured out the centre of the given circle. Their skill in finding the centre reflected each learner's own preference when it comes to the use of visual representations when solving problems.

Task 6

In this task, I wanted to gain insight on whether learners could understand how the dimensions of the triangle in the problem relate to each other. I specifically wanted to see how the learners made use of a diagram to make sense of this problem.

Task 7

This task was adapted from a grade 9 *maths for life* textbook. It was included in the worksheet to determine whether learners would use visualisations to transform one shape into another. This task relates well with Ho's visualisation process where mathematical forms may be obtained from visual representations.

Task 8

In this task, I wanted to understand how learners compared and contrasted perimeter and area of quadrilaterals using sketches or diagrams. This task is in line with Namibia's curriculum which requires learners to find perimeter and area of different quadrilaterals. Their skill to draw and find a numerical equal perimeter and area finds support from Ho's (2010) paper which suggests that both teachers and learners should use visualisation to help them in their problem-solving process.

Task 9

In this task, I wanted to know how learners can apply what they know to the unknown. This task is in line with the Namibian curriculum which emphasises the need for application as a domain mathematics. Specifically I wanted to observe how the learners used diagrams to come up with as many triangles as possible.

Task 10

In this task I wished to see whether learners made use of diagrams to represent a real life situation and illustrate the mathematical problem realistically. I was hoping that the diagrams they drew would enable the learners to 'see' the problem. Again the curriculum encourages teachers to allow learners to explore and construct their knowledge as the task requires.

Task 11

In this task, the aim was to investigate the understanding of leaners' cross-curricular issues and how they can relate such issues to mathematics learning. As learners reflected on and drew the volleyball court, the visual representations were used to check for the reasonableness of the answer obtained. In so doing, visualization helped learners to understand what the question required of them. This task is rooted in the domain of application in the curriculum where learners are encouraged to apply what they learn into real life situations.

Task 12

In this task, I was interested in observing how learners used visualisations in constructing as many shapes out of the piece of string as possible. I was hoping that the learners would draw a plethora of different shapes with the conclusion that the perimeter of each shape remains constant.

I wish to emphasise that my analysis of the individual GVT tasks was not to look for the correct answer, but to make sense how the learners used visualisations to solve each task. The GVT was first piloted for the purposes of identifying potential practical problems in following the research procedure. The purpose of piloting the GVT was also to find out whether the worksheet was appropriate to answer the research questions. I only made superficial changes such as correcting typographical errors.

3.5 RESEARCH TECHNIQUES

In this case study, participant observations and in-depth interviews were employed as techniques of data collection.

3.5.1 Observations

I observed each participant solve each of the 12 geometry tasks of the GVT. I sat next to each participant as he/she solved each problem. My observation data consisted of a video recording of each participant interacting with each task of the GVT and with me. I was specifically interested in data that showed the nature of the visualisation processes employed as they interacted with each task of the GVT. Observations were video-recorded by an independent on-looker while I used the observation checklist to collect data that supplemented the filmed observations.

3.5.2 Interviews

I employed two types of interviews. These were the GVT stimulated recall-interviews and the post-GVT interviews. They were semi-structured in nature because the focus of my interview was to collect information that was not apparent during observations. My role, as suggested by Henning, van Rensburg and Smit (2011) was to guide the interview guarding against asking leading questions that could result in contaminating data generated.

GVT Stimulated recall-interviews

Using the GVT worksheet as a stimulus, I observed while an independent on-looker video-recorded how each participant engaged with each task of the GVT. I asked each of the participants to 'talk through' their thought processes as they interacted with the GVT and with me. This also afforded them the opportunity to justify how they solved each GVT task. In particular, I encouraged each participant to explain the link between the question asked and the diagram or sketch they drew in their effort to answer the GVT. The open-ended nature of this technique allowed participants to contribute as much detailed information as they desired. It also allowed me to follow-up on their responses.

Post-GVT interviews

Specific time slots were arranged to conduct an interview with each participant after their interaction with the GVT. The aim of this interview was for the participant to reflect on the GVT process as a whole, and for me to ask follow-up questions and seek clarity on issues and ambiguities that may have arisen in the GVT interactions above. The environment where the interview took place was conducive in that I introduced myself to each participant and assured him or her that we would not use cell phones during the interview process for purposes other than capturing data. Furthermore, each participant was addressed with respect. The audio recording device was tested before commencing with the interview, which took place in the computer laboratory. As a backup, batteries were inserted in the audio recorder for fear of a power failure. The audio recordings for the interview were well labelled with dates and pseudonyms, such as p7/06/16F4. This represented the date of the interview, the gender and the number allocated to each participant interviewed.

As noted by Holloway and Wheeler (2002, p. 237), note-taking is integral in the process of an interview, although it might disturb the participants. In the context of this study, I informed each participant that I would take notes to save as backup for the information obtained from the audio recorder. The other reason for taking notes was to ensure that I kept track of every response from each participant to be able to decide if there was a need to probe further on an issue.

The use of open-ended questions such as “Does what you see or imagine in your mind help you to draw a picture that represents the geometrical problem at hand?” enabled participants to respond to such questions from their own understanding and perspective. I posed some pre-prepared questions and some arose naturally during the interview. I used phrases such as “could you elaborate more on that point?” while maintaining eye contact to encourage participants to continue speaking. The maximum duration for each interview was about 30 minutes per interviewee. Furthermore, a “silence” sign was pasted on the door of the interview room to inform staff members that an interview was in progress. In closing the interview, I asked each participant if he or she had other comments on the subject. All the interviews were transcribed.

In relation to my analytical tool and Acarvi’s (2003) definition, the interview questions were crafted to elicit responses from participants about the processes of using visualisations to solve a particular problem. I also used the interview questions to get the participants to reflect on their diagrams and how they used them to advance their understanding. For example in interview question four I asked: How do you find the pictures that you draw or scribble on paper when answering geometrical problems helpful in:

(i) Understanding the spatial relations pertinent to the problem that you are trying to solve?
(ii) Constructing the visual representation of the problem?
(iii) Using visualisation to solve the problem?
(iv) Connecting the present task to previously solved problems?
(v) Linking the solution and the visual representation?

The responses to these questions allowed me to draw conclusions on the visualisation processes learners used in relation to the indicators of visualisation of the analytic framework and Acarvi’ definition of visualisation.

3.6 THE RESEARCH DESIGN

This research study was designed in four phases:

3.6.1 Phase 1- Consent and pilot

In this phase, I requested permission from the Regional Director of Education and from the school principal to conduct this research study at the identified site. I also requested permission from parents and learners to engage in the process of selecting the participants as detailed below. The final designing and piloting of the GVT also took place in this phase.

3.6.1.1 Selection of Participants

The school where this case study was conducted was conveniently selected because I taught there. This school is loctaed in the Zambezi Region of Namibia. I purposefully selected eight learners from three Grade 11 classes. As I wished to select a range of achievers (ie low achievers to high achievers) I consulted with their old Grade 10 teachers to make an appropriate selection. This selection was therefore purposive. Cohen, Manion and Morrison (2011) state that "in purposive sampling, often a feature of qualitative research, researchers hand-pick the cases to be included in the sample on the basis of their judgement of their typicality or possession of the particular characteristics being sought" (p. 156). The reason for using purposive sampling was to focus on specific, unique issues, cases, or sample characteristics and to generate rich data from these. Another reason for using purposive sampling was to allow me to have ample time working with these learners who I knew well and who would be willing to give me the required time instead of working with learners from a different school who could not have trusted me.

3.6.1.2 Selection criteria

I wished to work with participants from a broad cross-section of performance. I thus selected learners who performed at different levels, that is, below average, at average and above average, in their mathematics lessons. An even number of male and female participants was considered because I wished to ensure a gender-balanced sample. This was purely an ethical consideration and did not have anything to do with my research rationale and questions.

3.6.1.3 Process of selection

The selection process was started by introducing all learners from the three Grade 11 classes to the intended visualisation project. I gathered these learners in one place and ran a workshop to explain the selection criteria, their rights and role as potential participants, ethical issues to be followed, and the way data would be collected. At the end of this introductory workshop, 25 learners indicated their willingness to participate in the project and with the assistance of the mathematics teacher responsible for these classes, 15 learners who met the desired criteria were selected. Three of these learners participated in piloting the GVT while the desired eight participants were engaged with the GVT tasks after the refinement of the tasks. The other four participated in piloting the Interview questions leading to their refinement. The desired eight participants who initially participated in the GVT were also subjected to the interview questions.

3.6.1.4 Piloting the GVT and interview questions

The pilot study of both the GVT tasks and the interview questions were aimed at ensuring that instructions are comprehensible. I also piloted the instruments to check the wording of the questions for purposes of reliability and validity. In all, the pilot study enabled me to determine if the items of the GVT and those of the interview questions could yield the kind of information that was needed. Fortunately, the outcome of the pilot studies revealed that careful attention was given to both issues and that only few spelling errors and punctuation marks were observed and had to be changed. In order to avoid the bias of collecting data from participants who were already introduced to the GVT and interview questions in the pilot stage, I opted to work with eight learners who were not introduced to the items before. This exclusion was meant to elicit reliable and valid information among the first-time takers of the items.

3.6.2 Phase 2 - GVT interactions

During this phase, the selected participants interacted with the 12 items of the GVT. The GVT took the form of worksheets and interviews. Each task on the worksheet had sufficient space for written and drawn solutions. The participants were equipped with pencils, erasers and sheets of paper to solve the problems and draw their sketches and diagrams. Each task was analysed as per my analysis template/instrument discussed below.

Each of the participants interacted with the GVT at different times after regular academic teaching hours. I interacted with each participant as he / she worked through the GVT on the worksheet and interviewed him / her during each question. They were required to “talk through” their thought processes and how they solved each task. Participants were not set any time limit to solve the tasks. The emphasis was on providing evidence of the visualisation processes they used to solve the tasks. In this way, I gained rich and deep insights into the visualisation processes that the learners employed. In order to capture data obtained from the above process, the independent on-looker video-recorded the participants interacting with the GVT and with me for each task. The analysis of this answered research question Number 1.

3.6.3 Phase 3 - Post-GVT interview

In this phase, I conducted a one-on-one interview with each participant after he / she completed the GVT. This took place after I completed an initial analysis of the video recording of each participant from phase 2. Participants were asked to reflect and elaborate on how each GVT task was solved using their visualisation processes. I also used this interview to clarify uncertainties that arose in phase 2.

Figure 3.1 below shows a flowchart of the entire data collection process.

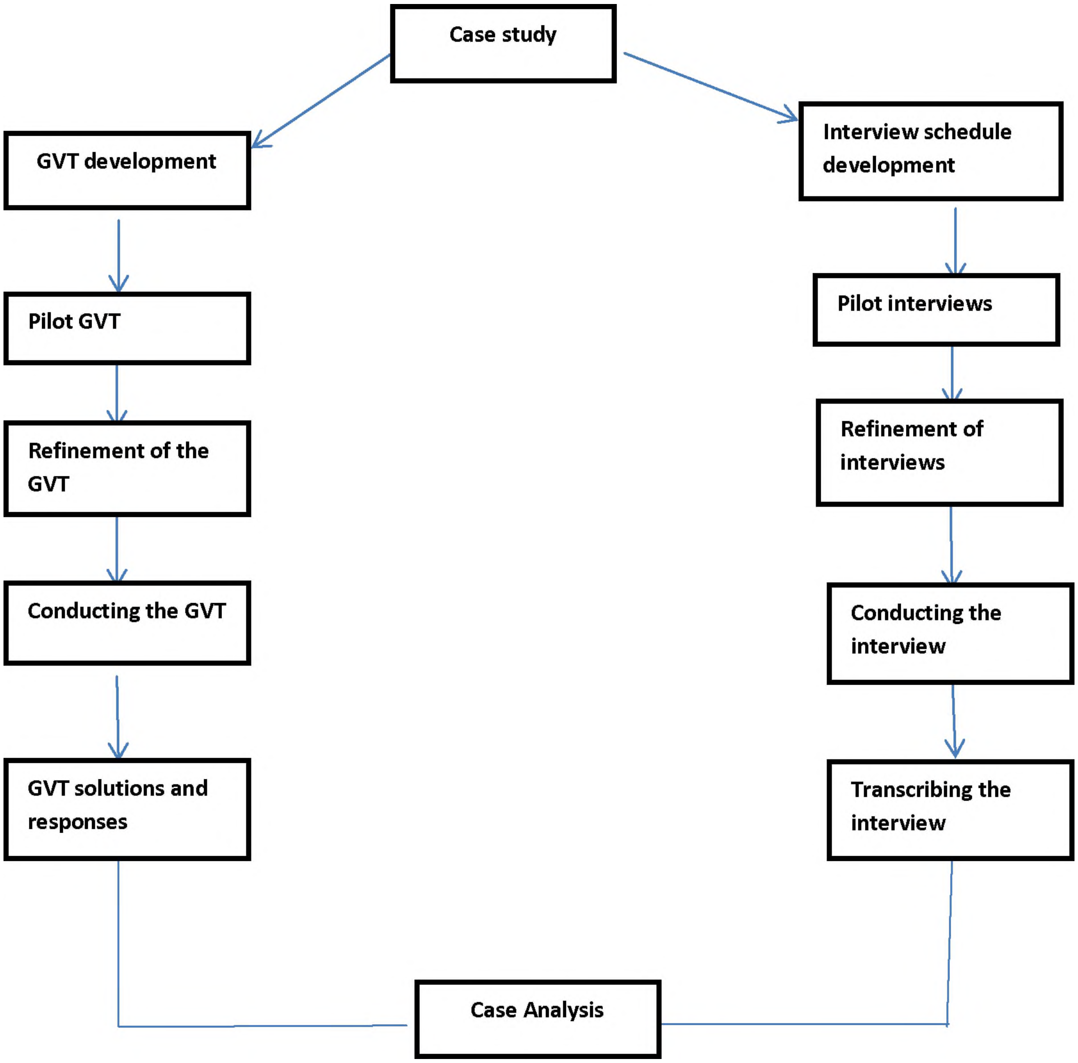


Figure 3.1: The flowchart of the data collection process

3.6.4 Phase 4 - Analysis

During this phase I analysed the data – see section 3.7 below for details.

3.7 DATA ANALYSIS

3.7.1 Data Management

Management of data is as important as its collection in any given research study. In this study, the data collected was arranged in labelled files for the purpose of accessing them easily. The privacy of each participant's results on the GVT needed to be protected and kept safe. I kept the scripts in a file at my home, I did not discuss the results of these tests or interview responses with any of the teaching staff, and a private place was used during the data collection stage while I took notes for each participant who spoke. Data from the completed GVT, post-GVT interviews and the observation, were considered for analysis.

3.7.2 Data Analysis

Cohen et al. (2011) claims that "in abiding by the principle of fitness for purpose, the researcher must be clear what he / she wants the data analysis to do as this will determine the kind of analysis that is undertaken (p. 538).

A template (see Table 3.1 below) of observable indicators adapted from Ho (2010) assisted me to identify and classify visualisation processes evident in the participants' interaction with the GVT and with me. The responses of the GVT were then summarised according to the 12 tasks that each of the eight participants undertook. For example, responses of all the eight learners for task one were grouped together to identify levels of mathematical accuracy and visual representations respectively. The template also enabled me to code the data for each of the 12 tasks.

Table 3.1 Analytical instrument - Table of indicators to identify and classify visualisation processes for each task of the GVT for each participant.

No	Indicator of visualisation	Visual schematic mathematical representation: focuses on evidence of mathematical accuracy.			Visual and pictorial representation: focus is on evidence in the use of pictures to represent mathematical problems			
	The visualisation process/sketch assists in, and there is evidence of:	**1 In-accurate**	**2 Fairly accurate**	**3 Very accurate**	**1 No visual representation**	**2 Poor visual representation**	**3 Fair visual representation**	**4 Very rich visual representation**
1	Understanding the special relations/properties inherent in the geometry tasks encountered, i.e the sketch represents and illustrates the problem/task.							
2	Making connections to previous tasks encountered, i.e there is evidence that prior knowledge was used to draw the sketch.							
3	Transforming the task into a mathematical form, i.e the sketch transformed the task into mathematical symbols and constructions.							
4	Clarifying the task at hand; i.e the sketch simplifies or clarifies the task.							
5	Illustrating the problem scenario, i.e the sketch mirrors the task accurately.							

Source: Indicators adapted from Ho (2010) and Ho, Ramful and Lowrie's (2015) clarification of the representations to analyse the data collected in the study.

My own experience as a classroom mathematics teacher helped me to set the levels of accuracy and richness that the participants used in each task that was solved. Levels of accuracy related to how precise the visualisation processes were in relation to the mathematical concepts involved in each task. The levels of visual representations related to the visual accuracy and precision of the visualisations used in each task. Evidence of mathematical accuracy was displayed if; the sketch represented and illustrated the problem task, there was evidence that prior knowledge was used to draw the sketch, the sketch transformed the task into mathematical symbols and constructions, the sketch simplified the task and if the sketch mirrored the task accurately.

The aspect of very rich visual representation in relation to fair visual representation and poor visual representation was indicative of details visible on the produced sketch. These included marks contrasting relationships of dimensions and directions respectively. The more detailed the sketch, the richer it is in terms information.

Below are the indicators as adapted from Ho 2010.

The indicators are:

1. **Understanding the problem.** Under this criterion, I sought evidence from the learners of whether they could correctly represent the given task visually. For example, if learners were asked to calculate the sum of the interior angles in a pentagon, were learners able to draw the correct shape and identify the appropriate angles inside the pentagon? In addition, I looked at the drawn sketches to understand if they represented and illustrated the problem/task.

2. **Connecting to previously solved problems.** Visualisation allows students to relate the current problem to previous ones, and then identify a simpler version as well as a method of solving that works for the current problem. For example, in establishing the interior angles of a polygon one traditionally uses the result that the sum of the interior angles of a regular polygon is $(n - 2)\ 180^0$ where *n* is the number of sides of the polygon. This formula allows learners to simply substitute the number of sides given. Under this criterion, I specifically wished to see how the participants used a visualisation process to connect to a previously solved problem. In the example of a regular polygon, the observable indicator would thus be to see learners dividing the given polygon into triangles that they know have sums of interior angles of 180^0. Moreover, I looked for evidence of learners using prior knowledge to draw their sketch.

3. **Transforming the task into a mathematical form (constructing a visual representation).** Here I specifically looked for any kind of visual that the participants constructed when solving the tasks of the GVT. These could include diagrams, scribbles or even images in the mind expressed in words. I further looked for the sketches that transformed the task into mathematical symbols and constructions.

4. **Clarifying the task at hand (using the visual representation to solve the problem).** Here I looked specifically at how the visual representation crafted by the participant was used in solving the given task. Can the answer to the given task be obtained directly from the visual representation itself, without the need for computation, or was the visual used as a scaffolding mechanism to solve the task at hand? Under this indicator, I also looked at the sketch to determine whether it clarified the task.

5. **Illustrating the problem scenario (encoding the answer to the problem).** Here I observed the link between the solution and the visual representation crafted by the participants. This indicator helped me measure the reasonableness of the answers in relation to the visuals presented. This indicator helped me to see if the sketch mirrored the task accurately.

I used the template illustrated in Table 3.1 above to code the indicators as I observed each of my participants in action and in the videos. I then used descriptive statistics such as frequency tables and bar charts to illustrate the trend of the data represented in the table. To fully understand the complexity of visualisation and to overcome discrepancies between what my participants said and what they did, observational data was necessary. I collected descriptive data using observation notes when each participant interacted with the GVT.

3.8 VALIDITY

According to Cohen et al. (2001), validity is “an important key to effective research” (p. 106). Maxwell (1996) states that the readers and users of research reports expect assurances that the data and research findings are valid and reliable and not skewed by the researcher’s own perspectives and ideologies. In an effort to address the issues of validity and reliability, I used a variety of data generation methods to triangulate the data through deliberately seeking evidence from the GVT, interviews and observations and comparing findings from those different sources.

The GVT was validated through piloting it. Piloting the GVT helped me to identify the shortcomings of each task regarding the way questions were crafted / set. Furthermore, the pilot of the interviews increased my experience of interviewing as well as improving my interpersonal skills. It helped me to approach my participants with sensitivity and open-endedness.

Another strategy I used to validate my data was member checking. I did member checking by giving feedback to my participants on the way they answered the GVT, through post-GVT interviews for example, and assessing how they scored their responses from their perspectives. I also validated data by examining deviant cases and critically finding reasons as to why they differed. As an example, when participant 1 drew a diagram to represent what he thought would help him answer the task, participant 2 used the formula to answer the same task. In this case, the two participants have produced deviant findings and as such, I had to search in detail to account for why they differed to this extent.

3.9 ETHICS

This section is a summary of the ethical issues that I took into account in my research study. It aims to highlight procedures for safeguarding my research participants' interests. Upholding professional ethics involves collaborative relationships between researcher and participants.

3.9.1 Respect and dignity

From the beginning, I represented the study as a project that participants felt part of through two-way communication. I explained the objective of the project honestly to gain their trust and their contribution through their participation. Participants' anonymity was ensured and their participation was entirely voluntary. This means participants were free to withdraw at any time without any consequences attached. Furthermore, I coded the identities of the participants to ensure confidentiality of the data. In addition, I worked on this project after school academic hours to avoid engaging learners in the project during teaching time.

3.9.2 Transparency and honesty

To ensure transparency, informed consent in writing from district officials, the principal, participants and parents was sought. The participants were involved in every phase of the project by affording them the opportunity to verify their responses in the stimulated recall-interview. On-going conversations between participants and myself during and after the

Geometry Visualisation Tasks (GVT) interactions also provided a sense of transparency and honesty to the participants of the project.

3.9.3 Accountability and responsibility

The power relationship issue between participants and I, which might have influenced the outcome of the research, was considered by avoiding leading participants into giving responses that were not entirely honest. I allowed each participant to use the language they were comfortable with and also give them enough time to respond to questions without interrupting them. To overcome power-related bias, I established a positive rapport between the participants and myself. To a large extent, this had already been established, as the community I work in is relatively small.

3.9.4 Integrity, academic professionalism and researcher positionality

In order to conform to the standard of academic professionalism, I used multiple data collection techniques, which encompassed observations, follow-up interviews, and the GVT to ensure that data collected was authentic. The findings of this study were documented whether they were in conflict with my initial expectations and assumptions or not. Furthermore, the final product of this project shall be made available to the public (teachers, curriculum designers and policy implementers) to serve its intended purpose of sharing the findings on the significance of using visualisation in mathematical learning. I am aware that my participants were minors and thus engaged with them with integrity, at all times upholding the ethical standards of Rhodes University. I acknowledge that the thesis is my own work and that I adhered to the protocols of the Rhodes University Referencing Guide to reference the work of others appropriately.

3.10 CONCLUSION

In this chapter, I described and discussed the methodology of my research study, which adopted a mixed method research approach. I outlined the research objectives and described the research design. I also provided details about the selection of the participants and the criteria used to select them.

Furthermore, I described the four phases of my research and provided details of ensuring validity and adhering to ethical procedures.

CHAPTER FOUR

4 DATA PRESENTATION, ANALYSIS AND DISCUSSION

4.1 INTRODUCTION

This chapter describes the analysis of data followed by a discussion of the research findings. My findings relate to the research questions that guided this study. I analysed data in order to identify, describe and explore the nature of visualisation processes employed when learners interacted with geometry problems, and to determine how these selected grade 11 learners used these processes in their interactions with the GVT geometry problems.

Data was generated from observing how my participants engaged with the 12 tasks. The data source for each task was the completed task document. In addition, I obtained data from my observation notes and videos made of each participant while engaging with the GVT task. I also obtained data from the notes and video records of the post-GVT Interviews conducted with eight participants.

I firstly engaged in a vertical analysis by analysing each task separately as I wanted to foreground the manifested visualisation processes that became evident for each task. I then analysed the data horizontally by analysing across the individual task by following the critaria or themes of the analytical framework. This then facilitated a global interpretation of the data within and across the tasks.

4.2 ANALYSIS OF QUANTITATIVE DATA

The quantitative data that was analysed consisted of the responses of the participants who completed the 12 GVT tasks. The data sets were analysed using bar graphs. Using the analytical instrument discussed in the previous chapter, the trends in learners' responses to the GVT tasks were established, and an analysis undertaken of the premises of two main sets of categories: firstly, the visual schematic mathematical representations category, and secondly, the visual and pictorial representations category. Each task was analysed separately.

As a reminder, for both sets of categories the following criteria were used:

1. **Understanding the spatial relations of the elements in the problem (task).** This criterion established if learners were able to correctly represent the given task visually. For example, if learners were asked to calculate the sum of the interior angles of a pentagon, were they able to draw the correct shape and identify the appropriate angles inside the pentagon? In addition, I looked at the drawn sketches to understand if they correctly represented and illustrated the problem/task.
2. **Connecting to a previously solved problem.** Visualisation allows students to relate the current problem to previous ones and identify a simpler version of the problem and method that works for the set problem. For example, in establishing the interior angles of a polygon one traditionally uses the result that the sum of the interior angles of a regular polygon is $(n - 2)\ 180^0$ where *n* is the number of sides of the polygon. This formula allows learners to simply substitute the number of sides given. Under this criterion, I specifically wished to see how the participants used a visualisation process to connect to a previously solved problem. In the example of a regular polygon, the observable indicator would thus be to see learners dividing the given polygon into triangles that they know have sums of interior angles of 180^0. Moreover, I looked for evidence of learners using prior knowledge to draw their sketch.
3. **Transforming the task into a mathematical form (constructing a visual representation).** Here I specifically looked for any kind of visual that the participants constructed while solving the tasks of the GVT. These could include diagrams, scribbles or even images in the mind expressed in words. I further looked for the sketches that transformed the task into mathematical symbols and constructions.
4. **Clarifying the task at hand (using the visual representation to solve the problem).** Here I looked specifically for how the visual representation crafted by the participant was used in solving the given task. Could the answer to the given task be obtained directly from the visual representation itself, without the need for computation, or was the visual used as a scaffolding mechanism to solve the task at hand? Under this indicator, I also looked at the sketch to determine whether it clarified the task.
5. **Illustrating the problem scenario (encoding the answer to the problem).** Here I observed the link between the solution and the visual representation crafted by the participants. This indicator helped me measure the reasonableness of the answers in relation to the visuals presented. This indicator helped me to see if the sketch mirrored the task accurately. The criteria above formed the horizontal axis of the bar graphs for each task.

4.2.1 Task 1

4.2.1.1 Mathematical accuracy

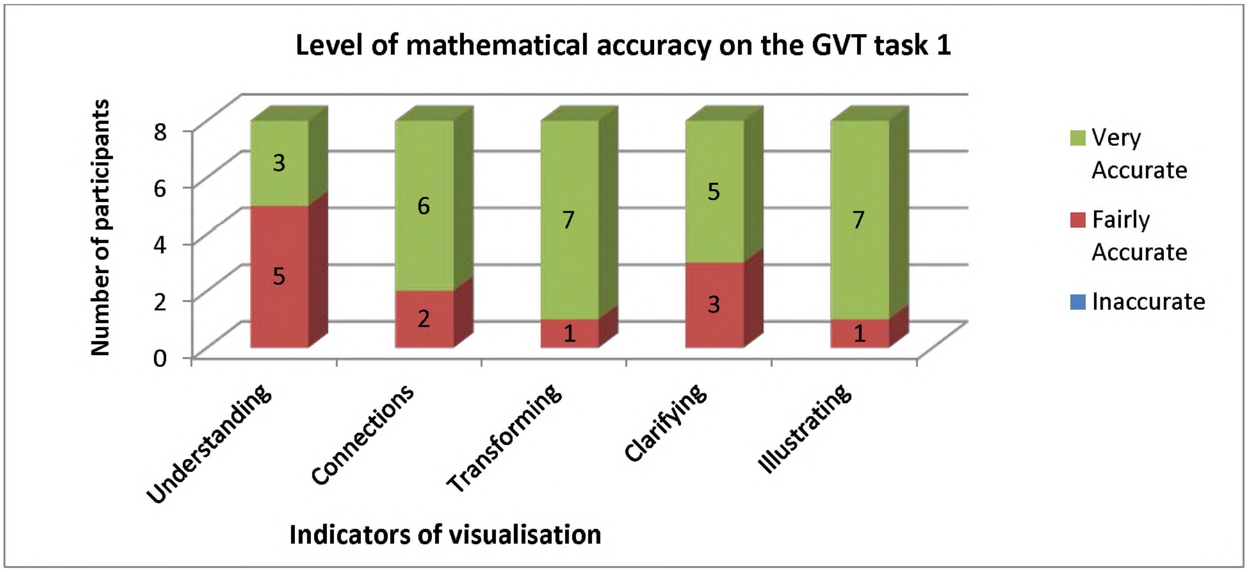
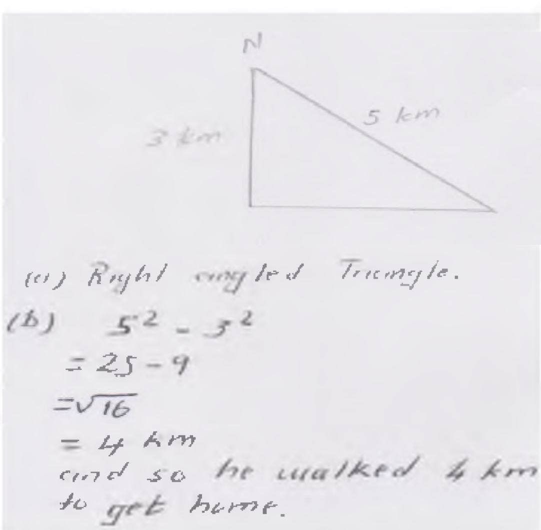


Figure 4.1: Learners' use of visualisation processes for task 1 in terms of mathematical accuracy

Figure 4.1 shows that the majority of the eight learners very accurately used sketches that transformed task 1 into mathematical symbols and constructions. In particular, seven out of the eight participants used very accurate sketches that illustrated the task. See figure 4.2 below as an example of a sketch that participant 2 used to answer task 1.



When solving this task, the learner used prior knowledge to draw and use information from the sketch. This information clarified the task and in turn helped in finding the desired solution. Interestingly, the learner used the theorem of Pythagoras appropriately

Figure 4.2: Sketch used by participant 2 showing mathematical accuracy

4.2.1.2 Visual representation

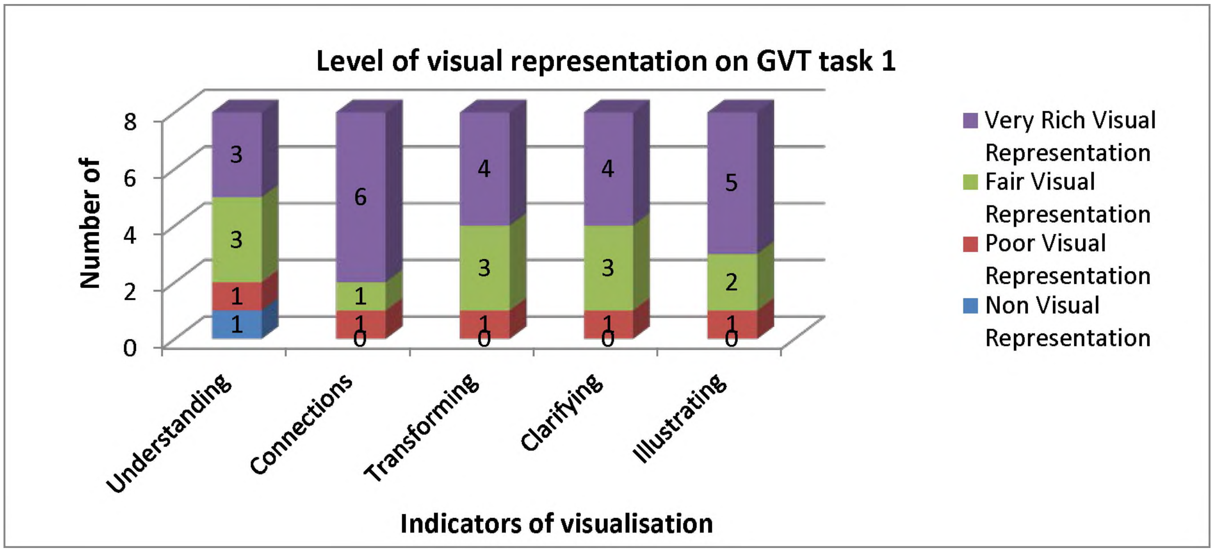
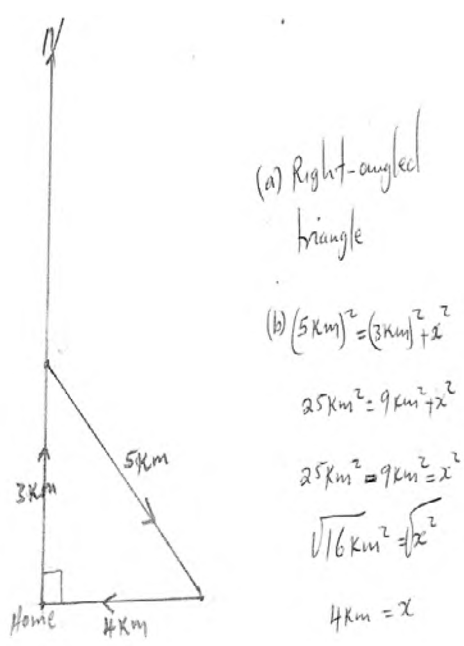


Figure 4.3: Learners' use of visualisation processes for task 1 in terms of visual representations

Figure 4.3 shows that the majority of the participants used fair to very rich visual representation when solving task 1. In particular when making connections, six out of the eight participants used very rich visual representations as illustrated in figure 4.4 below. Interestingly, only one learner did not use a sketch to reflect her understanding of the task.



The sketch presents the direction walked from home and back. The learner illustrated this direction using arrows. The distance covered was also illustrated although not drawn to scale. In addition, the learner stated that the sketch depicts a right-angled triangle and is evident from the sign shown.

Figure 4.4: Shows an example of a very rich visual representation used by participant 4 to solve task 1

4.2.2 Task 2

4.2.2.1 Mathematical accuracy

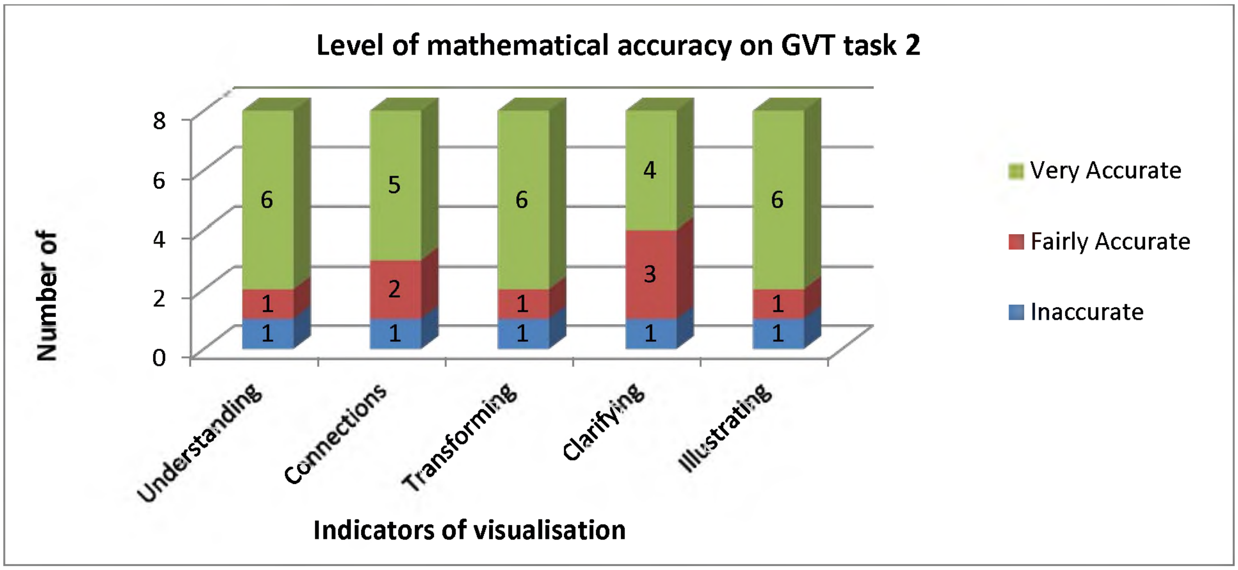


Figure 4.5: Learners' use of visualisation processes for task 2 in terms of mathematical accuracy

Figure 4.5 shows that six out of the eight learners used very accurate sketches to demonstrate their understanding of task 2. The same number of learners also used very accurate sketches in transforming and illustrating the task at hand. This contrasts to one learner whose sketch did not accurately reflect the task. See figures 4.6(a) and 4.6(b) showing contrasting sketches of how task 2 was solved.

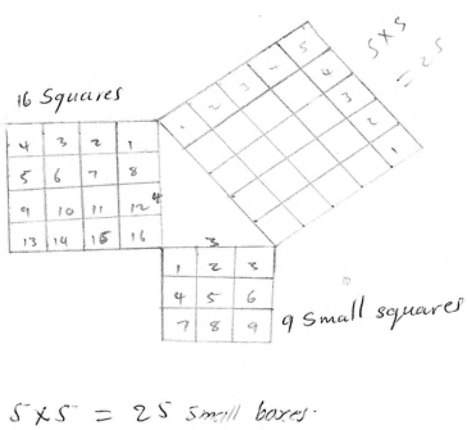


Figure 4.6(a)

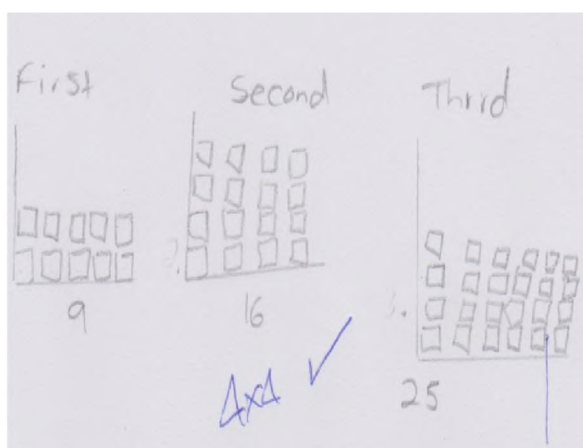


Figure 4.6(b)

Figure 4.6: 4.6(a) and 4.6(b) shows contrasting solutions of task 3, one accurate and the other inaccurate

4.2.2.2 Visual representation

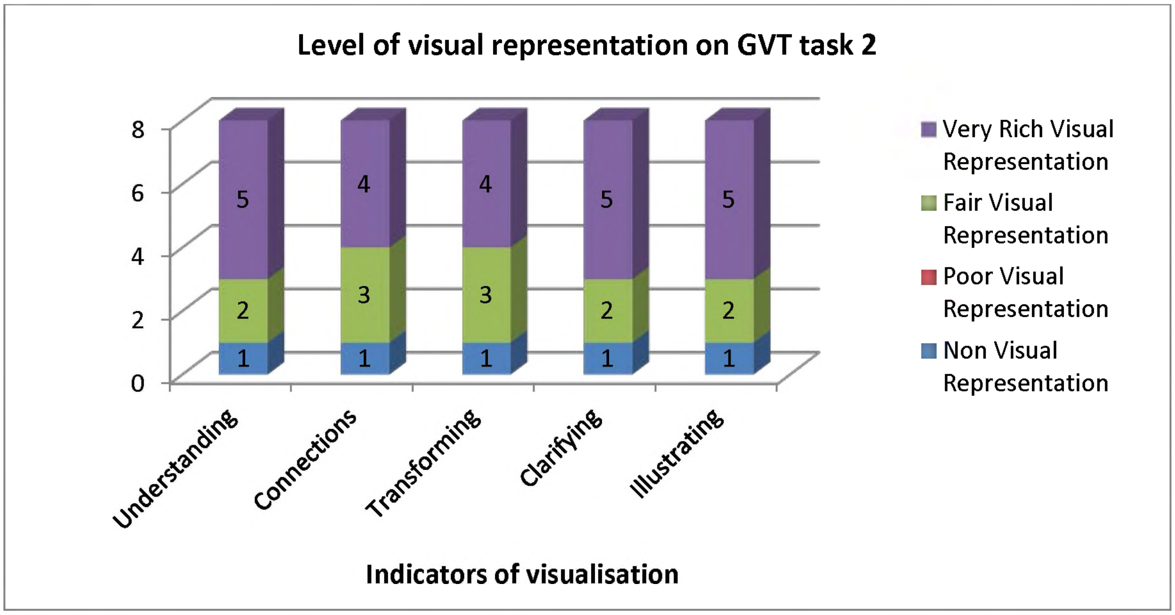
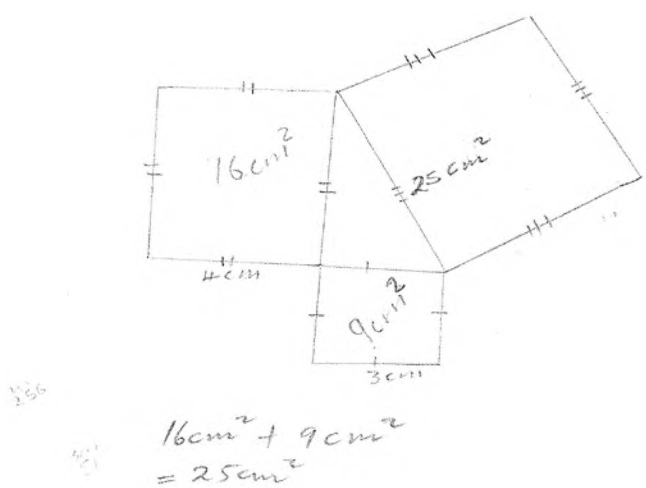


Figure 4.7: Learners' use of visualisations processes for task 2 in terms of visual representation

Figure 4.7 shows that five out of the eight learners used very rich visual representation to illustrate, clarify and four learners transformed task 2 into mathematical constructions. The five learners also used very rich visual representation to show their understanding of the task in relation to four learners who made connections to previous encountered tasks when solving the task as illustrated in figure 4.8 below.



This sketch contains very rich information used by the participant when solving task 2. For example, the same number of marks indicates that the sides are equal as shown on the sketch.

Figure 4.8 shows an example of a very rich visual representation used to answer task 2

4.2.3 Task 3

4.2.3.1 Mathematical accuracy

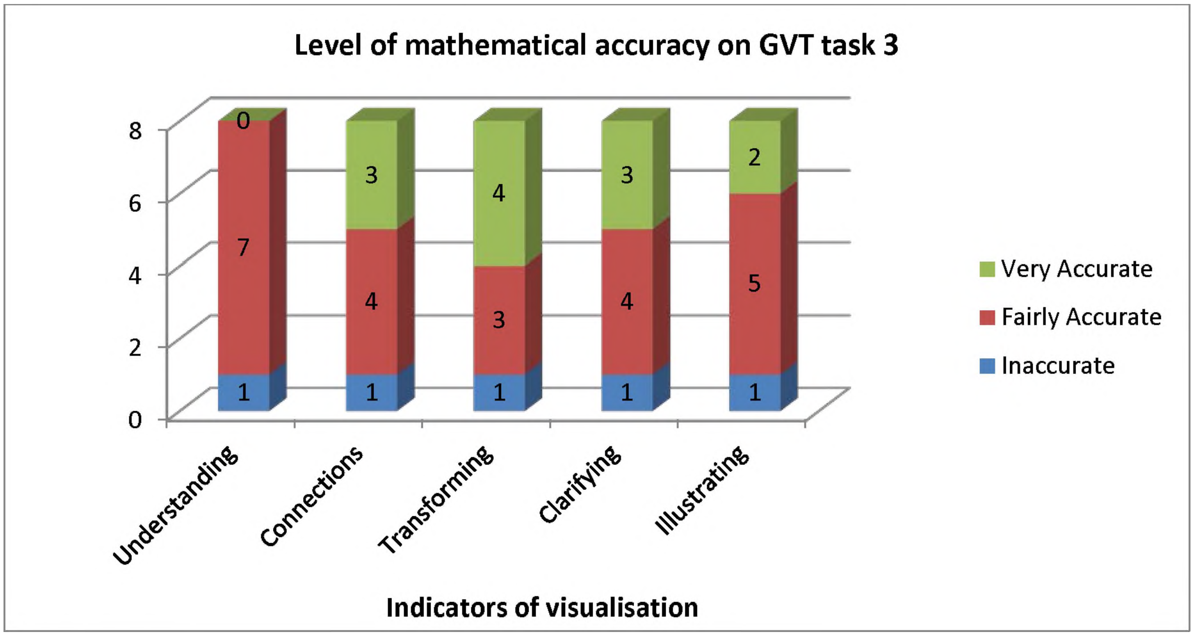
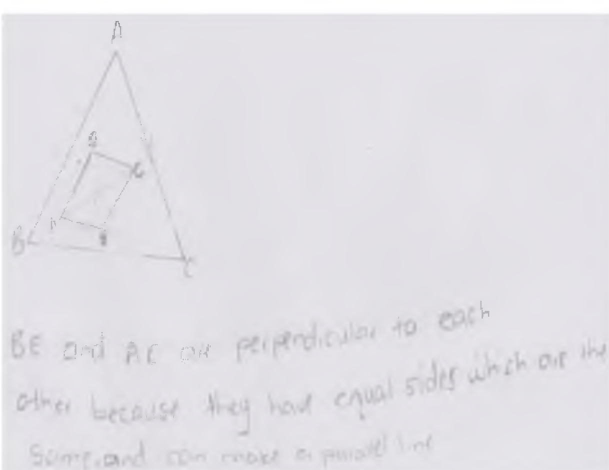


Figure 4.9: Learners' use of visualisation processes for task 3 in terms of mathematical accuracy

Figure 4.9 shows that a greater number of participants used fairly accurate sketches when solving task 3. Specifically, seven out of the eight learners used fairly accurate sketches to demonstrate their understanding of the task. Four learners made connections to previous tasks, five illustrated the problem task using fairly accurate sketches. For all visualisation criterions, only one learner used an inaccurate sketch when solving task 3. See figure 4.10 illustrating how participant 4 solved task 3.



This task is one of the fairly accurate solved tasks. Most learners' responses in terms of mathematical accuracy were fair as can be seen from the sketch used. It is clear from the responses that these learners struggled to make sense of what the question required of them.

Figure 4.10 illustrates a fairly accurate sketch used to answer task 3

4.2.3.2 Visual representation

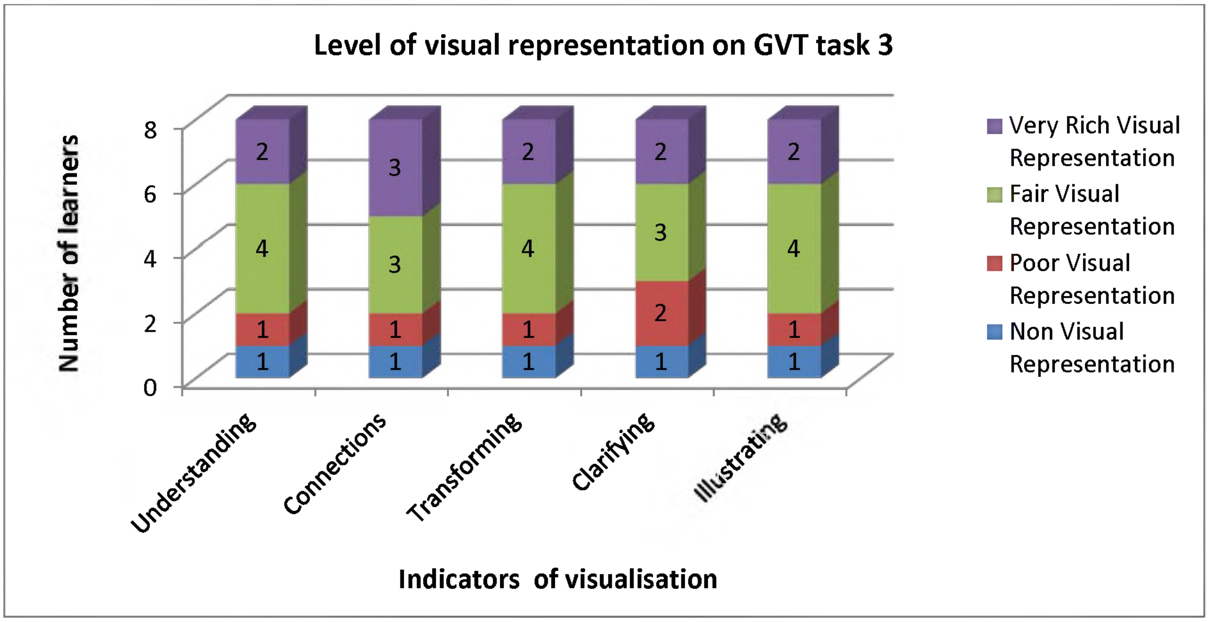
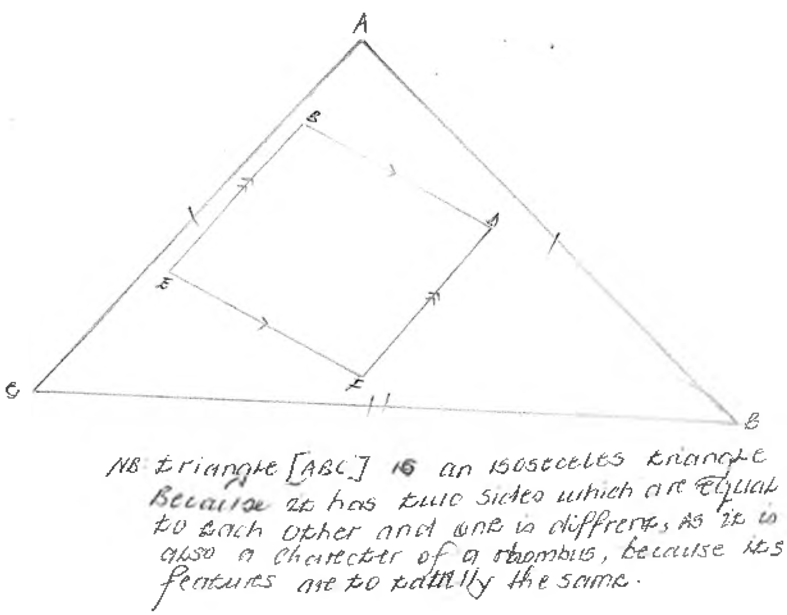


Figure 4.11: Learners' use of visualisation processes for task 3 in terms of visual representation.

Figure 4.11 shows that on average four learners used fair visual representation in transforming, illustrating and representing (understanding) task 3. In this figure, only one of the eight learners did not use any visual representation to solve this task. See figure 4.12 that depicts participant five’s sketch for task 3.



Despite that most learners used fair visual representation to solve task 3, their sketches lacked detailed information. For example, the rhombus’ sides in figure 4.12 should have been marked to be equal.

Figure 4.12 shows a fair visual representation used when solving task 3

4.2.4 Task 4

4.2.4.1. Mathematical accuracy

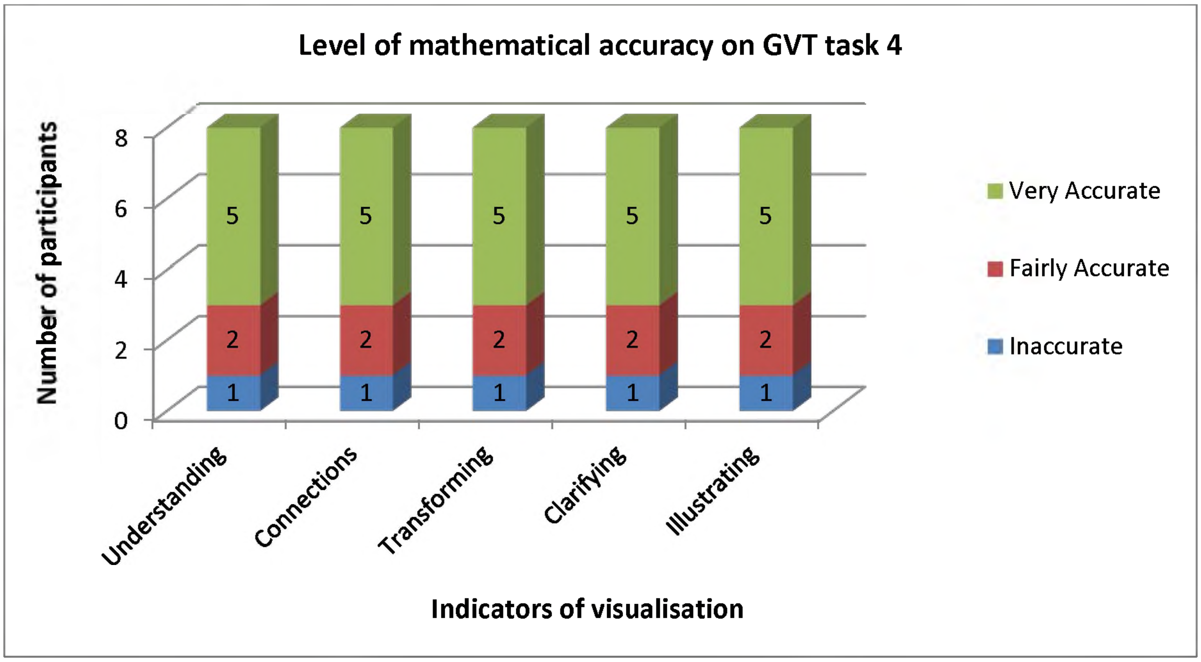
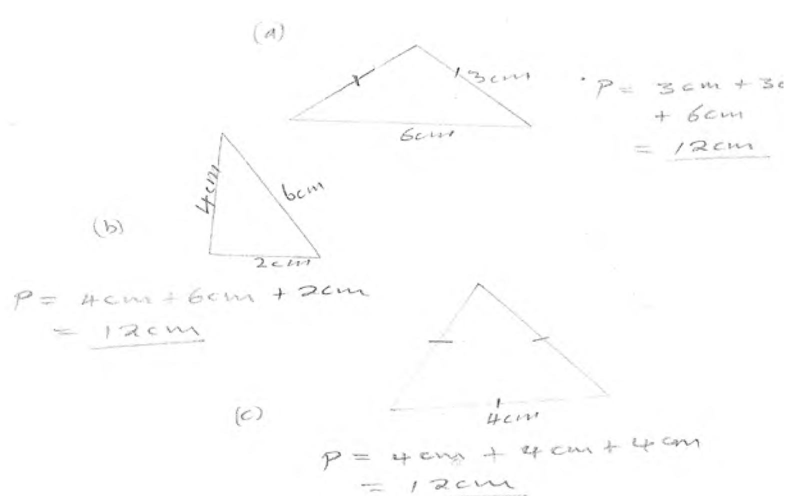


Figure 4.13: Learners' use of visualisation processes for task 4 in terms of mathematical accuracy

Figure 4.13 reveals that five out of the eight participants used very accurate sketches when solving task 4. In so doing, they portrayed their robust understanding and ability to connect this task to previously solved tasks. See figure 4.14 as an example of how task 4 was solved.



It is interesting to note how this learner came up with the dimensions used when solving task 4. The concept of perimeter was linked very well to each sketch in order to accurately solve the task.

Figure 4.14: A sketch showing the mathematical accuracy of task 4

4.2.4.2 Visual representation

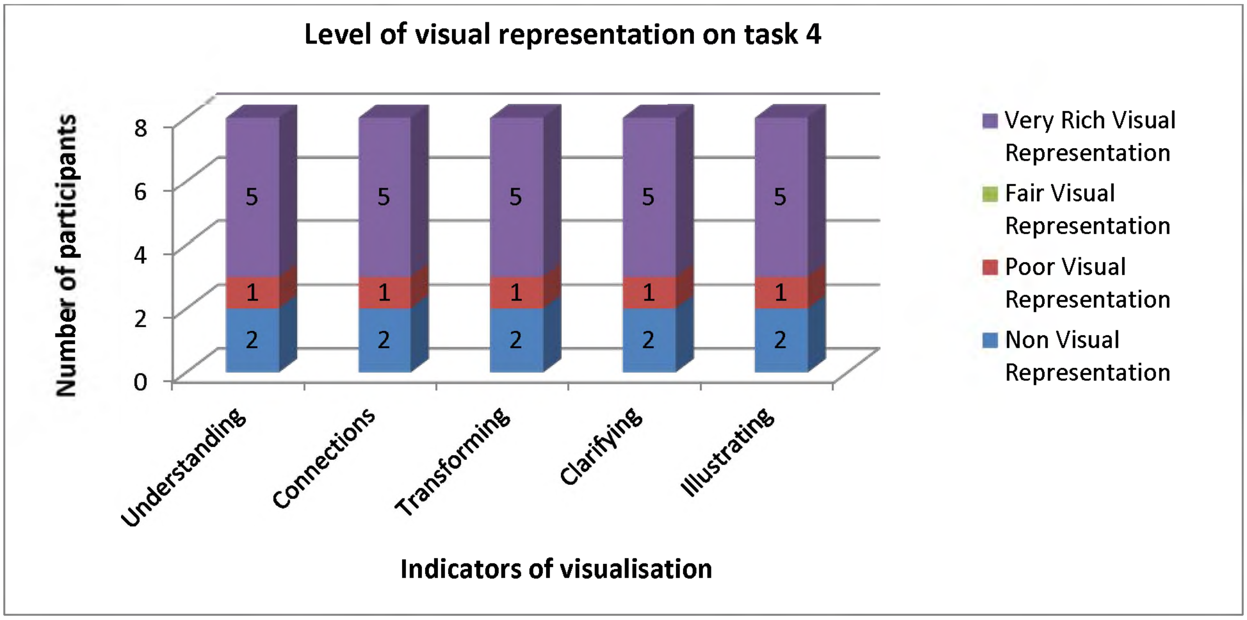
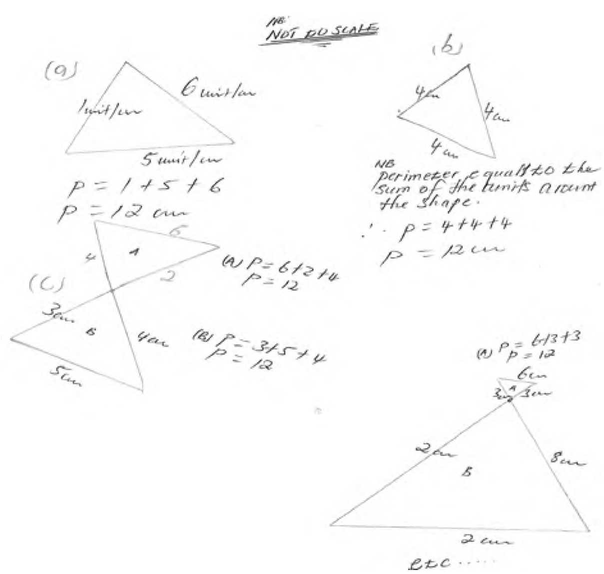


Figure 4.15: Learners' use of visualisation processes for task 4 in terms of visual representations.

Figure 4.15 shows that five participants used very rich visual representation when solving task 4 as illustrated in figure 4.16. In contrast to the five participants who used very rich visual representation, three learners used non-to poor visual representation. To be precise, two learners did not use any visual representation when solving task 4.



The sketches that this learner drew are rich in information. These visual representations were drawn in an attempt to clarify the reasonableness of the answers obtained. They also stand to support the solutions found.

Figure 4.16 shows very rich visual representations used to answer task 4.

4.2.5 Task 5

4.2.5.1 Mathematical accuracy

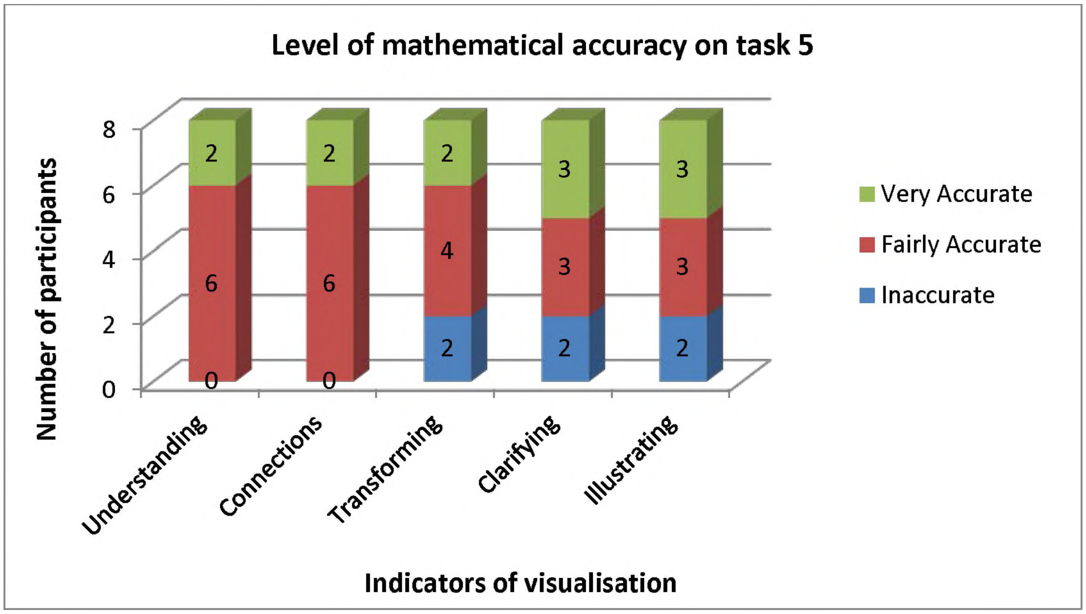
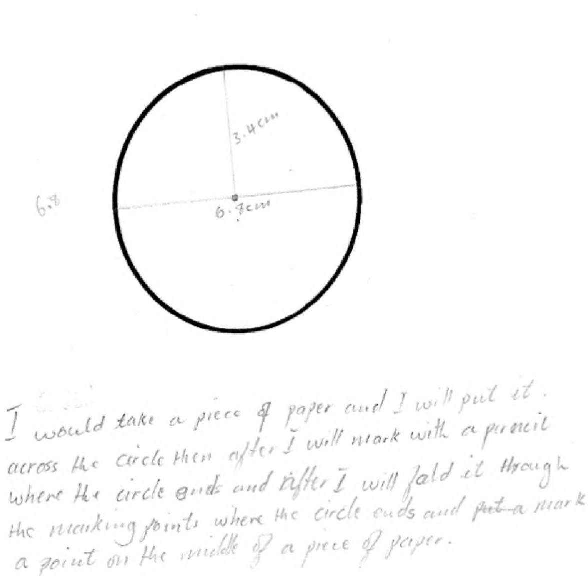


Figure 4.17: Learners' use of visualisation processes for task 5 in terms of mathematical accuracy.

Even though figure 4.18 demonstrates that the majority of the participants (6 learners) used sketches that were fairly accurate, a small fraction of them used very accurate sketches in clarifying and illustrating the task at hand. For example, three out of the eight participants showed evidence of very good mathematical accuracy on task 5 as illustrated by participant 8.



As can be seen from figure 4.18, task 5 required learners to explain and show a sketch in relation to their explanation, how they can find the centre of the circle. Most learners failed to connect this task to the circumference of a circle where they would then find its radius.

Figure 4.18 shows an example of a mathematically accurate sketch of task 5.

4.2.5.2 Visual representation

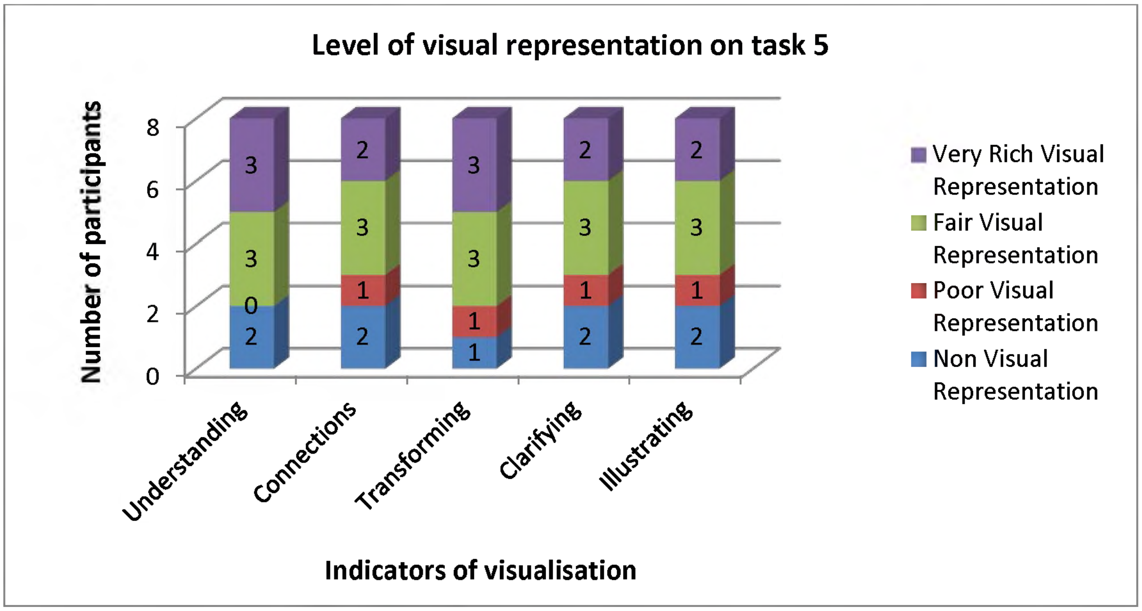
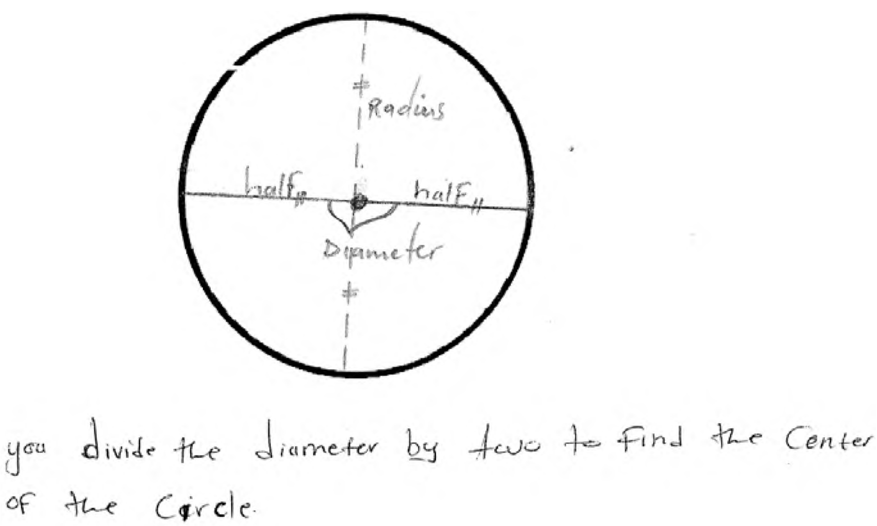


Figure 4.19: Learners' use of visualisation processes for task 5 in terms of visual representation.

Although Figure 4.19 shows a slight difference in the way task 5 was visually represented by the eight participants, three-quarters of them used fair visual representations when solving task 5. Worth noting also is that three learners used sketches that showed evidence of all five visualisation criteria. Figure 4.20 represents sketches that are rich in visual representations, as used by a few learners.



The learner here points to the fact that the centre of a circle lies halfway along its diameter. The marks show that the two sides from the centre are equal in length. These details show understanding.

Figure 4.20 shows an example of fair visual representations used to solve task 5.

4.2.6 Task 6

4.2.6.1 Mathematical accuracy

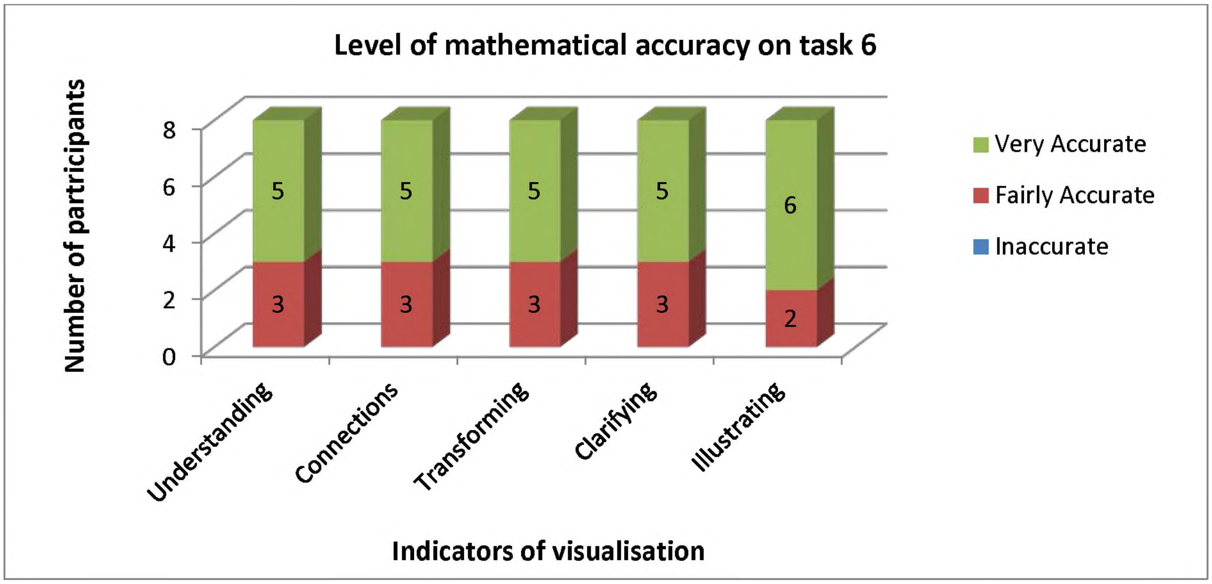
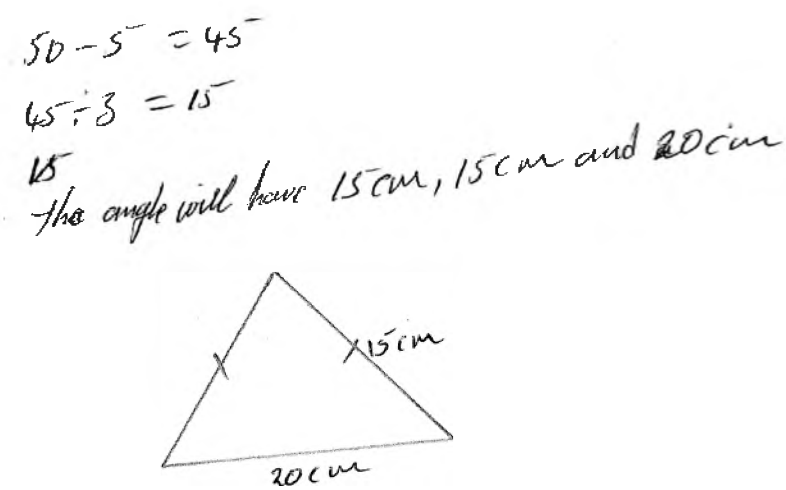


Figure 4.21: Learners' use of visualisation processes for task 6 in terms of mathematical accuracy.

Figure 4.21 shows that most participants used very accurate sketches in illustrating task 6. In particular, six out of the eight learners used very accurate sketches when solving this task. On the other hand, five participants showed evidence of making connections, transforming and clarifying the task through the use of these sketches. See Figure 4.22.



This solution shows that the learner first visualised a triangle, then used that triangle to derive a formula that helped in finding the answer. This can be seen in the 3 dimensions shown (15cm, 15cm, 20cm).

Figure 4.22 shows a sketch of task 6 showing mathematical accuracy.

4.2.6.2 Visual representation

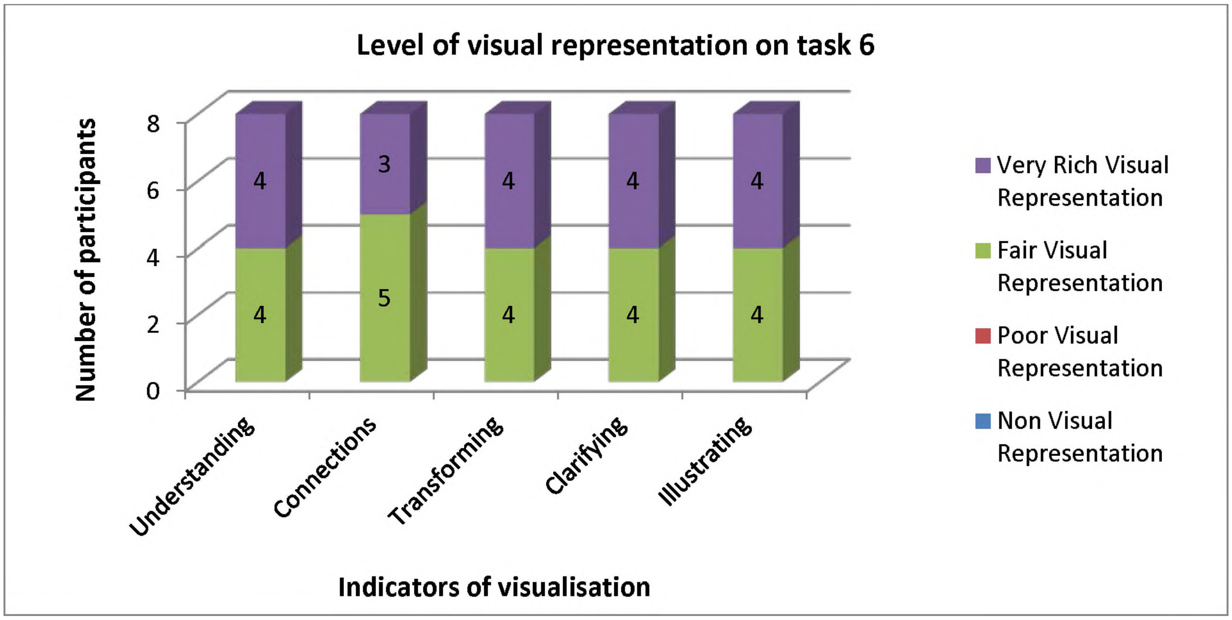


Figure 4.23: Learners' use of visualisation processes for task 6 in terms of visual representation.

Figure 4.23 shows that five out of the eight learners used fair visual representations in making connections to previous tasks solved. In addition, four out of the eight participants used very rich visual representations in transforming and clarifying the task at hand. This is evident in figure 4.24(a), whereas figure 4.24(b) shows a fair sketch used when solving the same task.

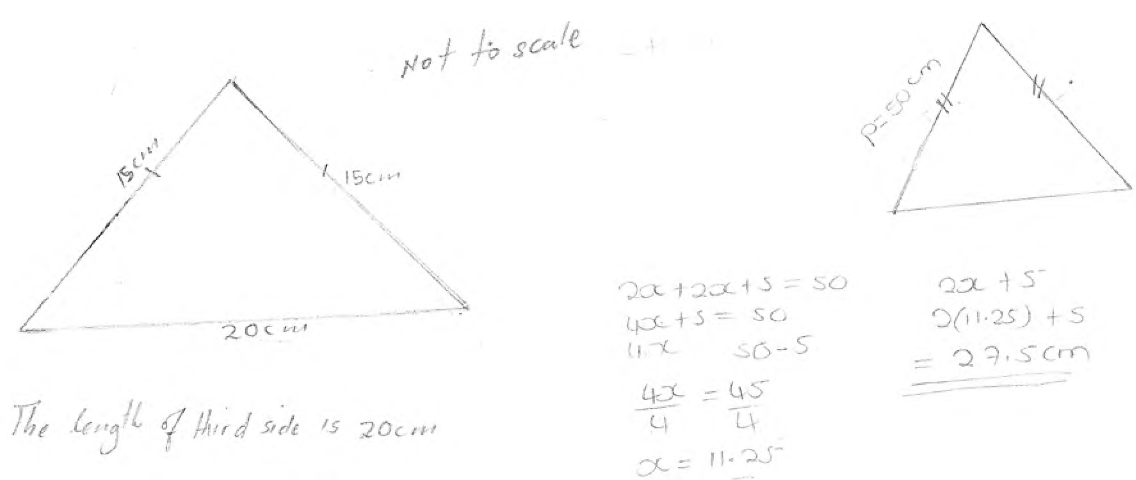


Figure 4.24(a) and 24(b) show contrasting visual representations.

4.2.7 Task 7

4.2.7.1 Mathematical accuracy

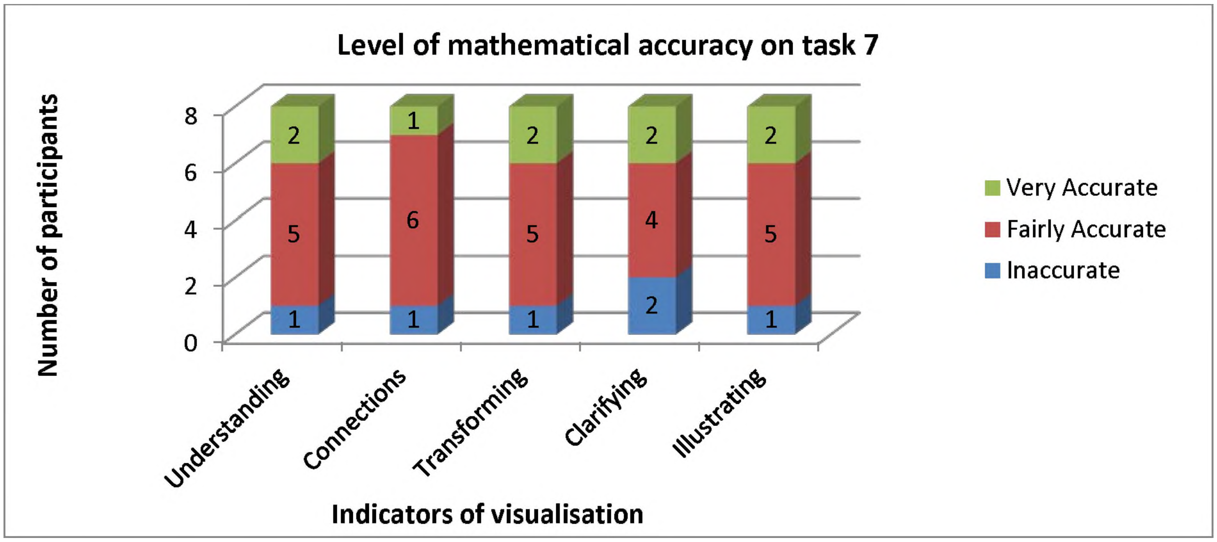
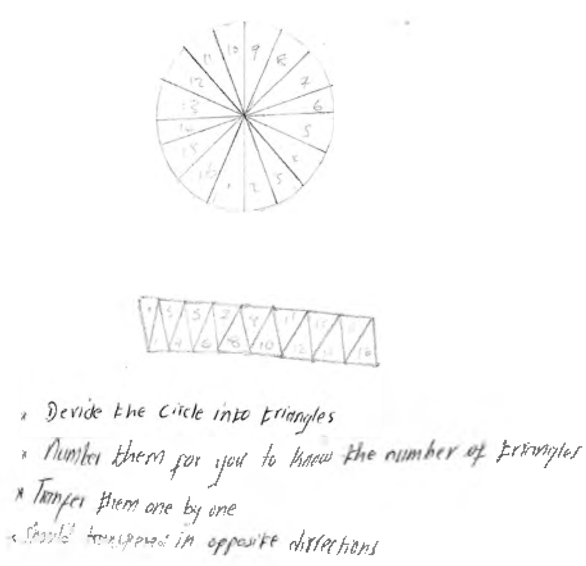


Figure 4.25: Learners' use of visualisation processes for task 7 in terms of mathematical accuracy.

Figure 4.25 shows that a significant proportion of participants used fair to very accurate sketches to depict their understanding of task 7. In particular, an average of two participants used very accurate sketches when solving task 7, as seen in Figure 4.26 below.



This sketch is not only a very accurate but also a very rich visual representation. This is argued because the cut-out–pieces of the circle fittingly illustrate a parallelogram. The steps used are also appropriate for the solution of this task.

Figure 4.26 shows a very accurate sketch demonstrating mathematical accuracy on task 7.

4.2.7.2 Visual representation

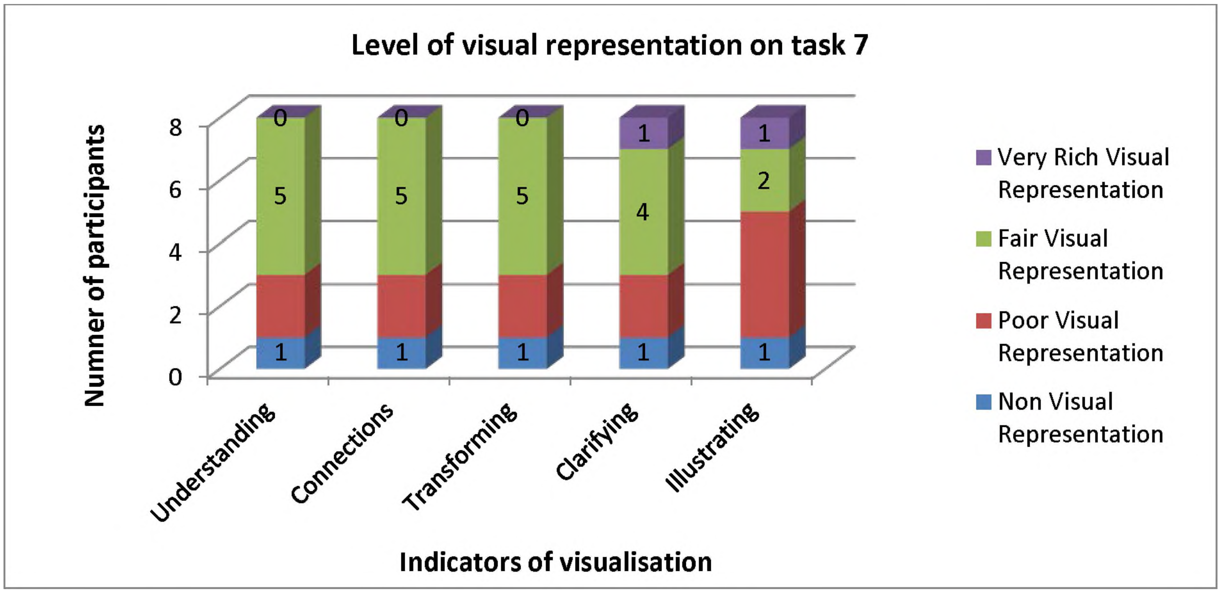


Figure 4.27: Learners' use of visualisation processes for task 7 in terms of visual representation.

Figure 4.27 shows the levels of visual representations that the eight participants used when solving task 7. Only one learner out of the eight used a very rich visual representation in clarifying and illustrating task 7, whereas an average of five participants used fairly visual representations across all five visualisation criterions. See figure 4.28(a) as an example of a very rich visual representation, in contrast to figure 4.28(b) on the right used for this task.

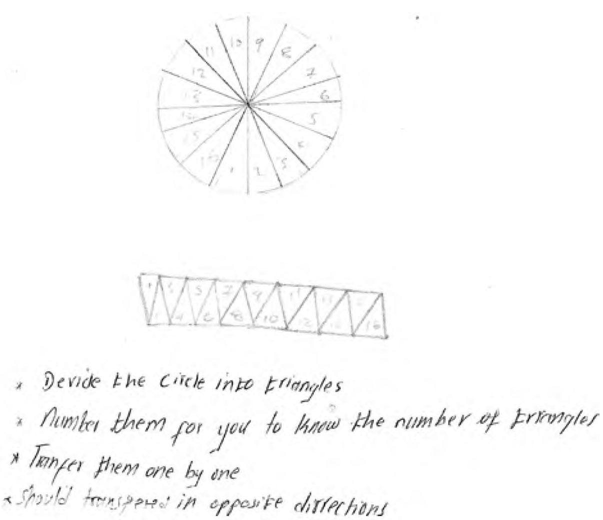


Figure 4.28(a)

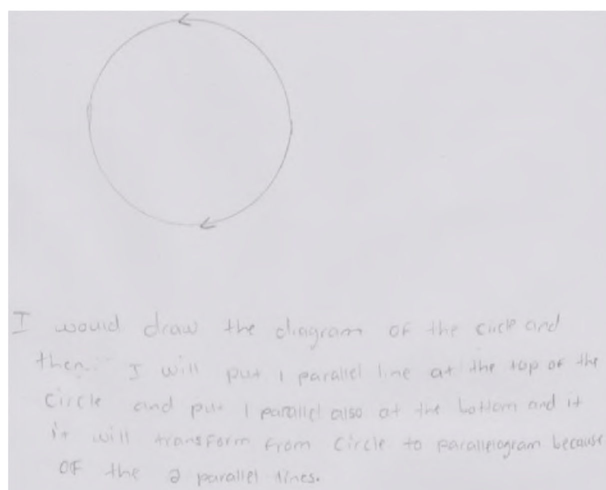


Figure 4.28(b)

Figure 4.28(a) and 28(b) show contrasting sketches illustrating solutions to task 7.

4.2.8 Task 8

4.2.8.1 Mathematical accuracy

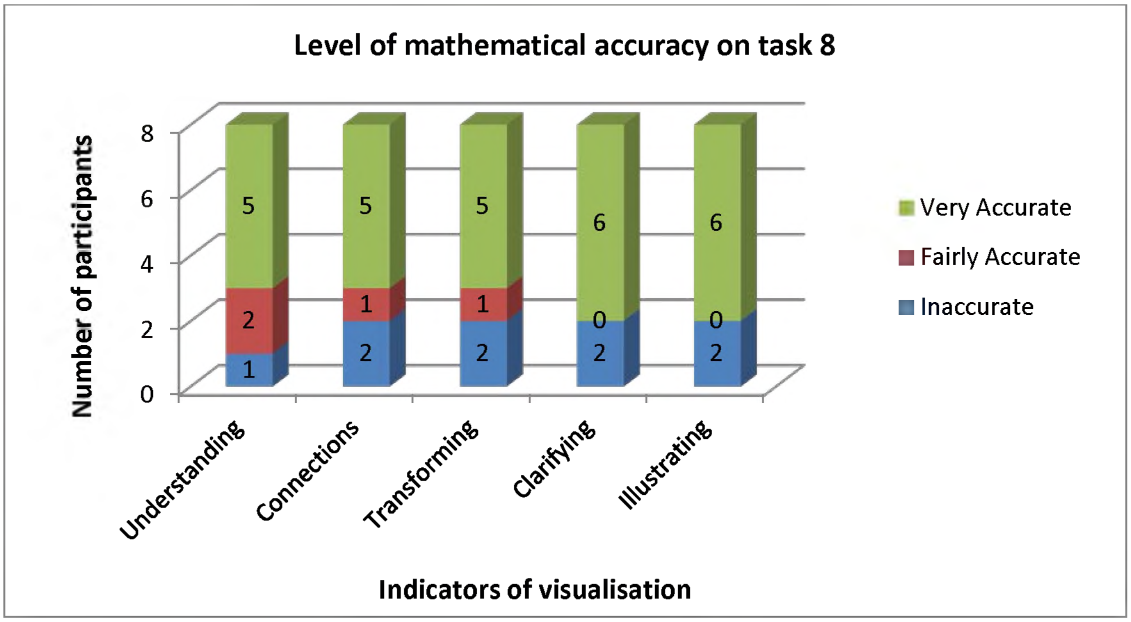
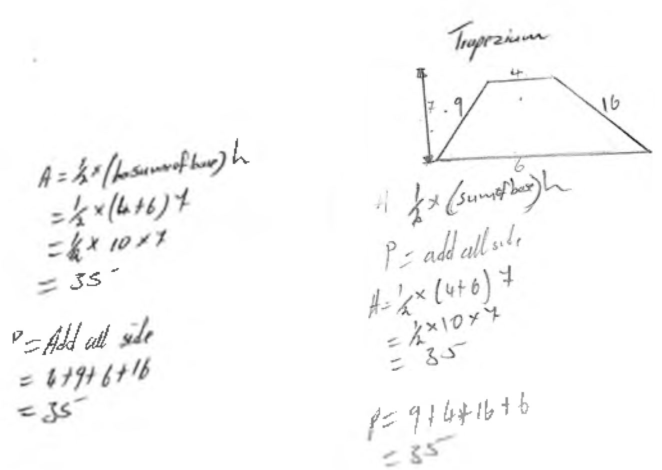


Figure 4.29: Learners' use of visualisation processes for task 8 in terms of mathematical accuracy.

Figure 4.29 shows that most of the participants used very accurate sketches in their response to task 8, in fact six out of the eight learners drew very accurate sketches that classified and illustrated the problem at hand as figure 4.30(a) illustrates. Figure 4.30(b) is an example of inaccurate sketches drawn by the other two learners.



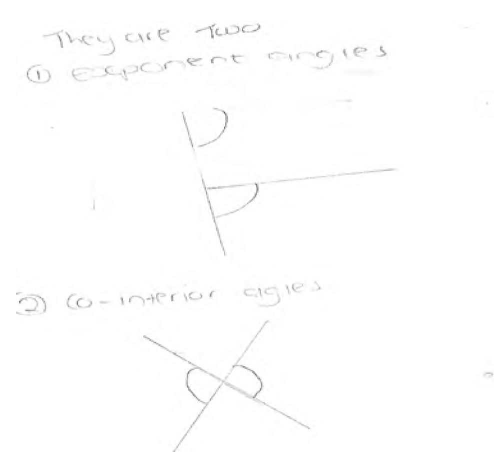


Figure 4.30(a) *Figure 4.30(b)*

Figure 4.30 shows mathematically accurate and inaccurate sketches used for task 8.

4.2.8.2 Visual representation

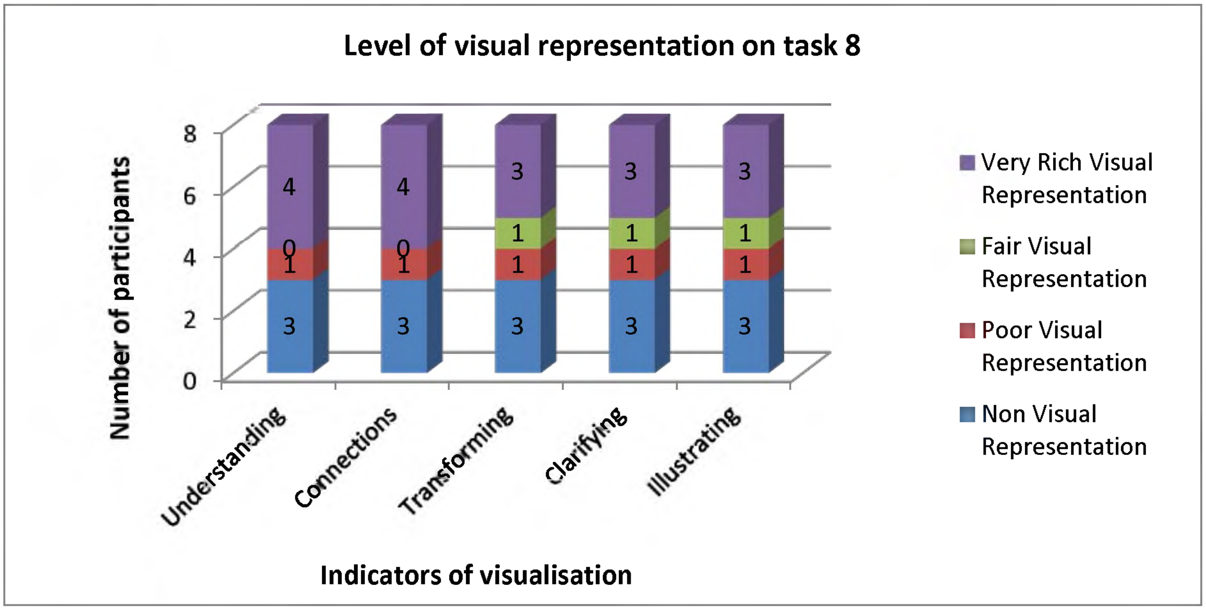
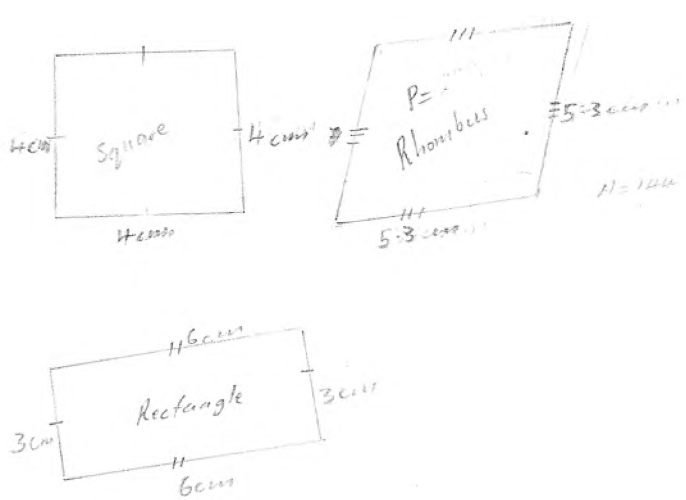


Figure 4.31: Learners' use of visualisation processes for task 8 in terms of visual representation.

Figure 4.31 shows the extent to which visual representation was used when solving task 8. It shows that four participants used very rich visual representations in connecting and showing understanding of this task to previous encountered tasks. It also shows that four out of the eight participants used very rich visual representations that demonstrated their understanding. Figure 4.32 illustrates sketches of very rich visual representation.



These sketches show that the learner linked the concepts of perimeter and area to the visual representations. It also shows that rich visual representations help in mathematical accuracy.

Figure 4.32 shows sketches of very rich visual representation for task 8.

4.2.9 Task 9

4.2.9.1 Mathematical representation

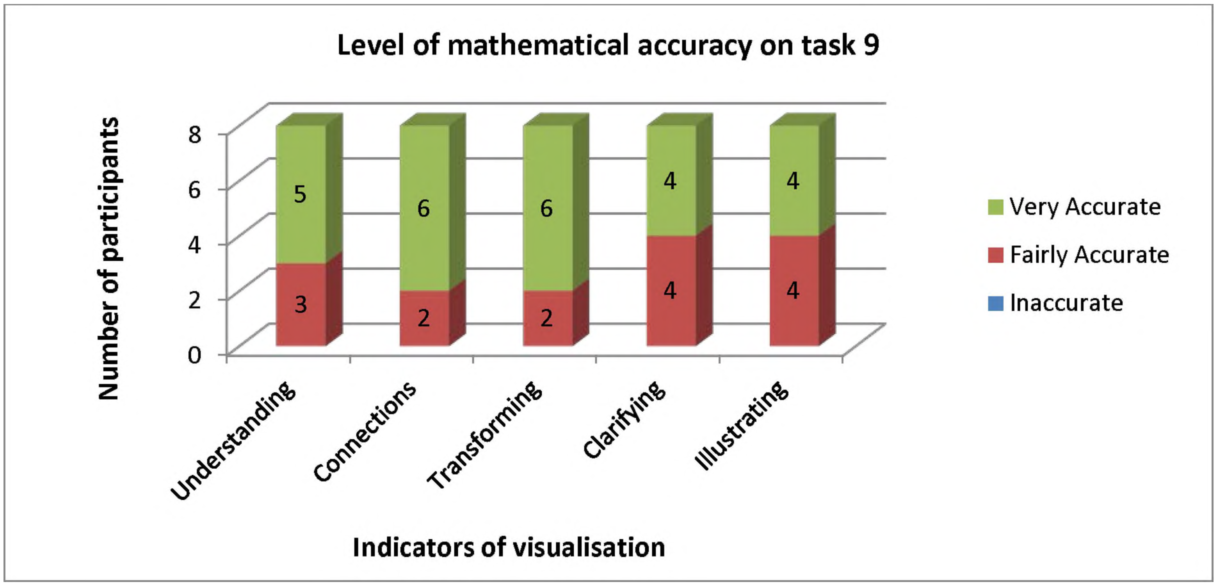
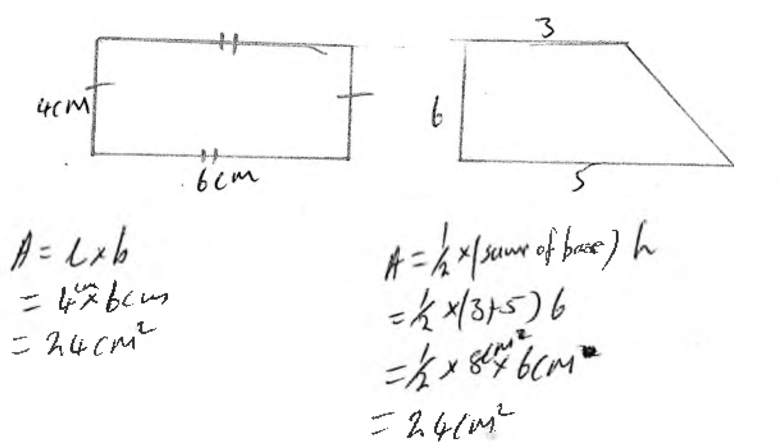


Figure 4.33: Learners' use of visualisation processes for task 9 in terms of mathematical accuracy.

Figure 4.33 shows that the two levels of mathematical accuracy (very accurate and fairly accurate) differ in the number of participants per level. For example, six out of eight participants used very accurate sketches that showed evidence of connecting and transforming the task into mathematical constructions, as shown in figure 4.34, while only 2 learners out of 8 used fairly accurate sketches to transform task 8 into mathematical constructions.



The two sketches confirm the reason why the majority of learners are seen to have transformed task 9 into mathematical symbols.

Figure 4.34 shows an example of a very accurate sketch used to answer task 9.

4.2.9.2 Visual representation

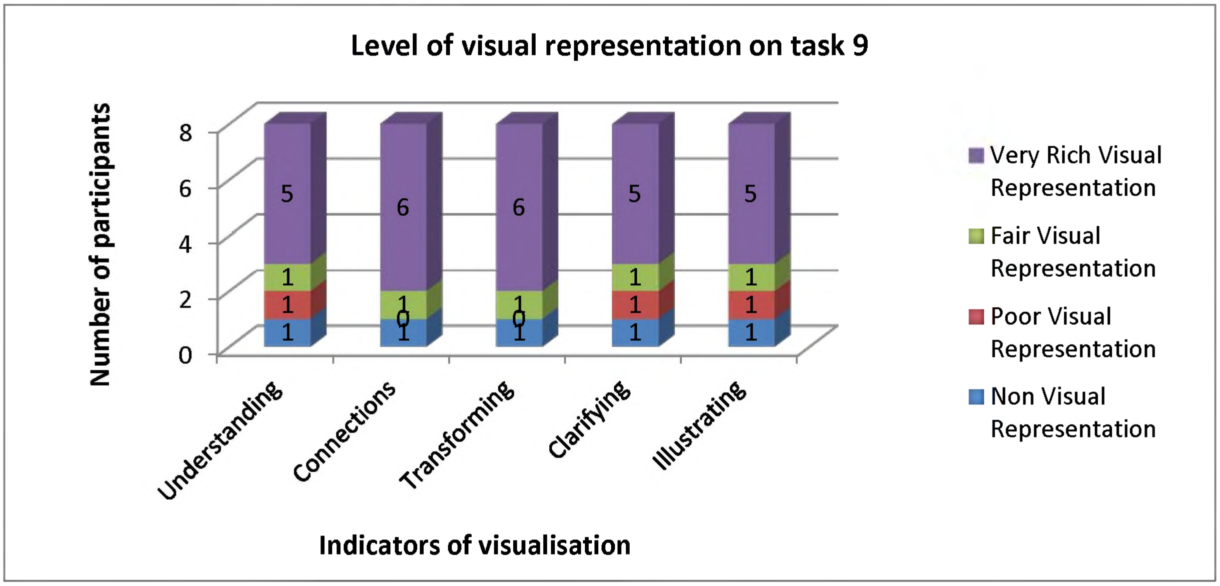
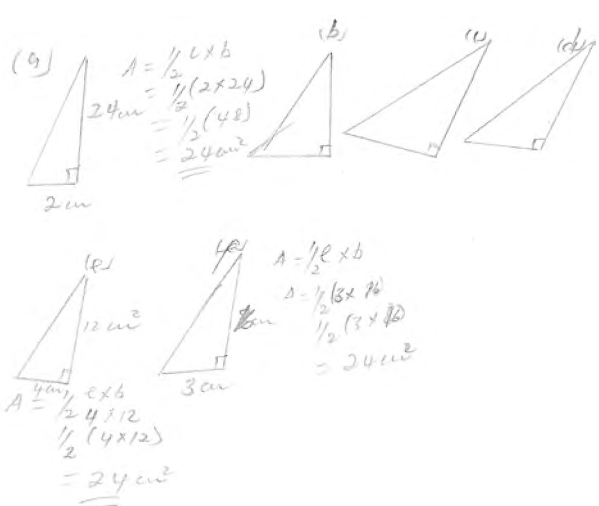


Figure 4.35: Learners' use of visualisation processes for task 9 in terms of visual representation.

Figure 4.35 shows that an average of five participants used very rich visual representations when they solved task 9. They also managed to give responses in relation to all five visualisation criteria, although one of them failed to use any visual representation. Figure 4.36 shows an example of a very rich visual representation for task 9.



These sketches served as scaffolds in the minds of the learners. This is argued because the ones that are solved are mathematically inaccurate in the manner they are drawn.

Figure 4.36 represents a rich visual representation of task 9.

4.2.10 Task 10

4.2.1.1 4.2.10.1 Mathematical accuracy

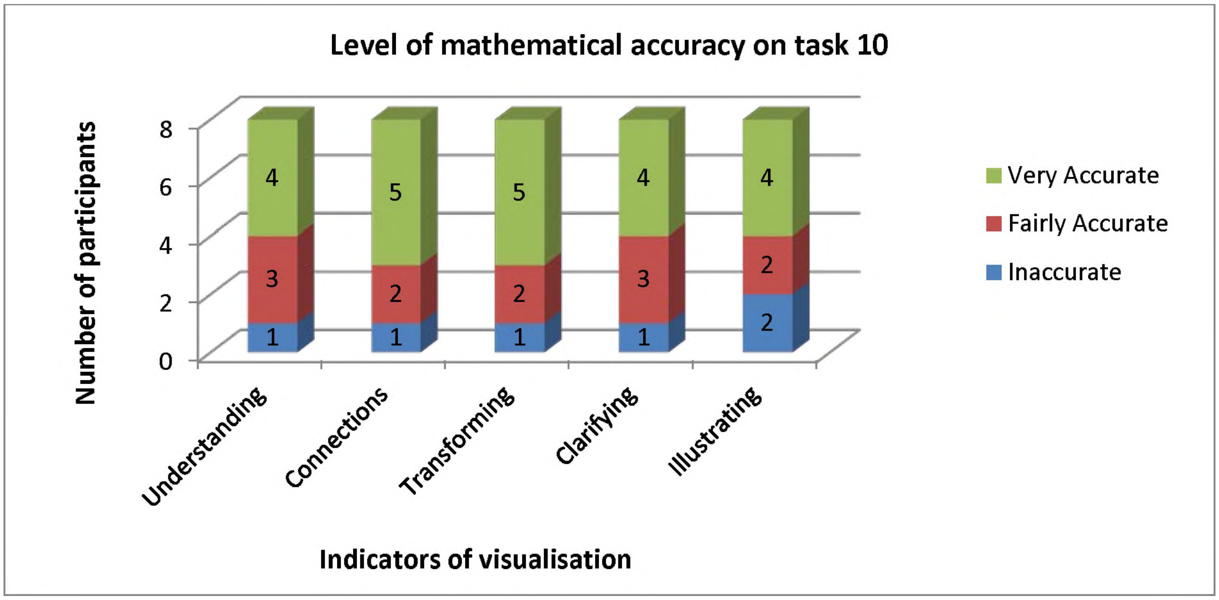
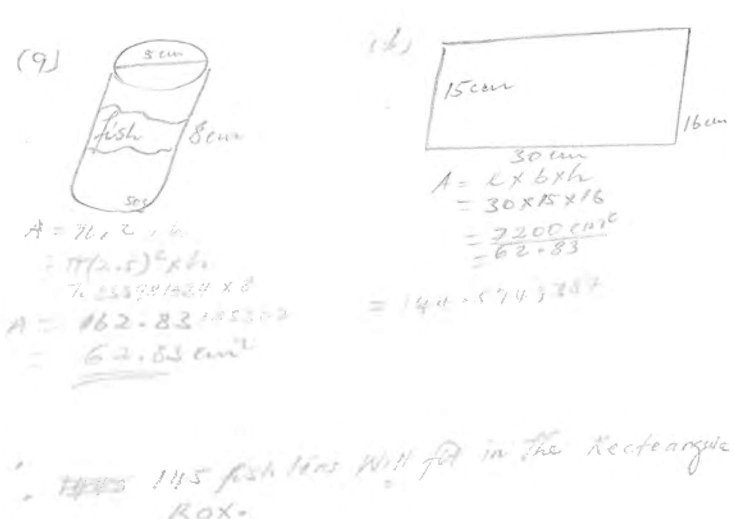


Figure 4.37: Learners' use of visualisation processes for task 10 in terms of mathematical accuracy.

Figure 4.37 shows that the number of participants who used inaccurate to fairly accurate sketches was less than the number of participants who used very accurate sketches when solving task 10. To be precise five learners used very accurate sketches in their solutions especially in making connections and transforming the task into mathematical constructions. Figure 4.38 shows an example of a very accurate sketch that some participants used.



This sketch shows that this learner successfully made connections between this task and the previous ones.

Figure 4.38 exemplify sketches used to answer task 10.

4.2.10.2 Visual representation

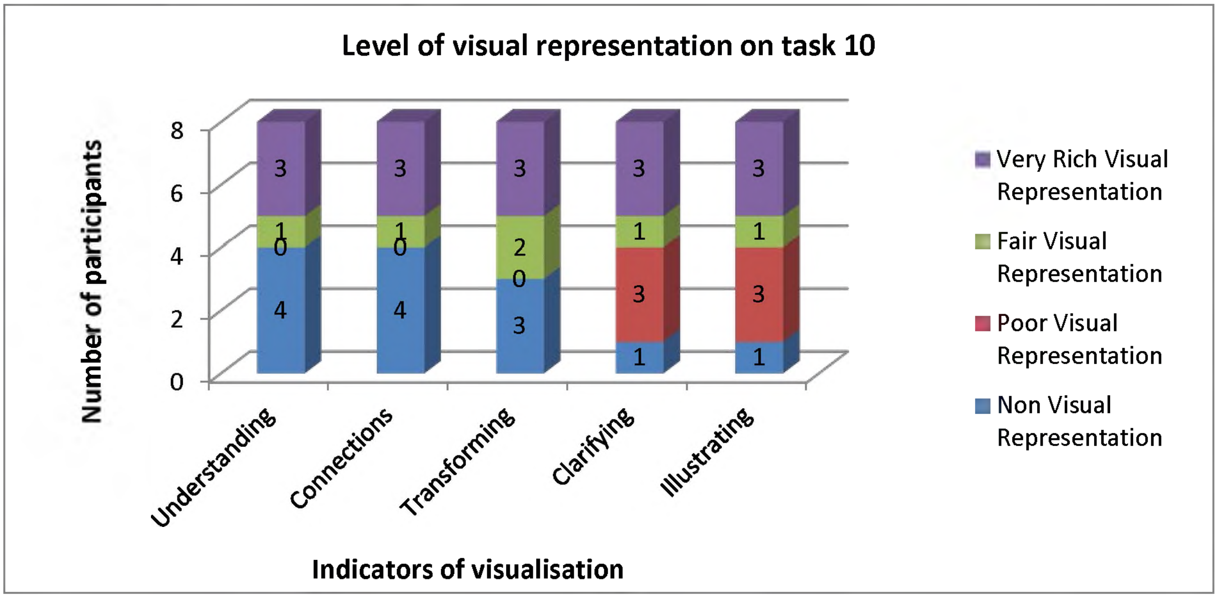


Figure 4.39: Learners' use of visualisation processes for task 10 in terms of visual representation.

Figure 4.39 shows that three out of the eight learners used very rich visual representation across all five visualisation criteria when solving task 10, while three to four leaners failed to use, or used poor visual representation. Figures 4.41(a) and 4.41(b) show a very rich visual representation and a poor visual representation respectively, used to solve task 10.

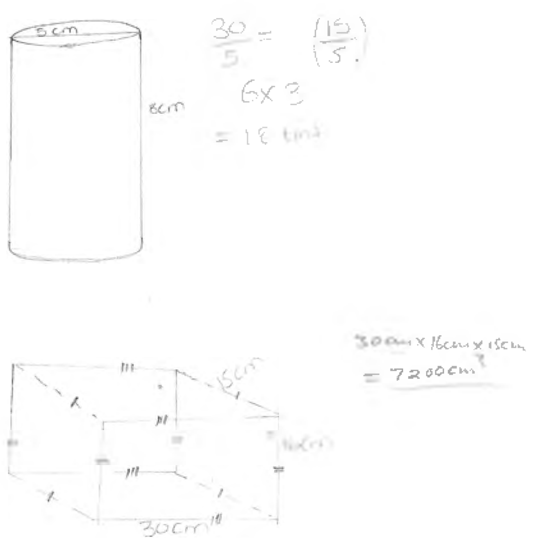


Figure 4.40(a)

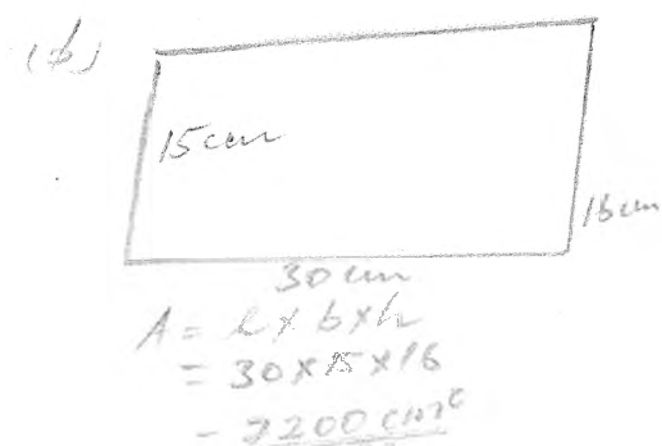


Figure 4.40(b)

Figure 4.40(a) and 4.40(b) show sketches that were used by two learners to solve task 10.

4.2.11. Task 11

4.2.11.1 Mathematical accuracy

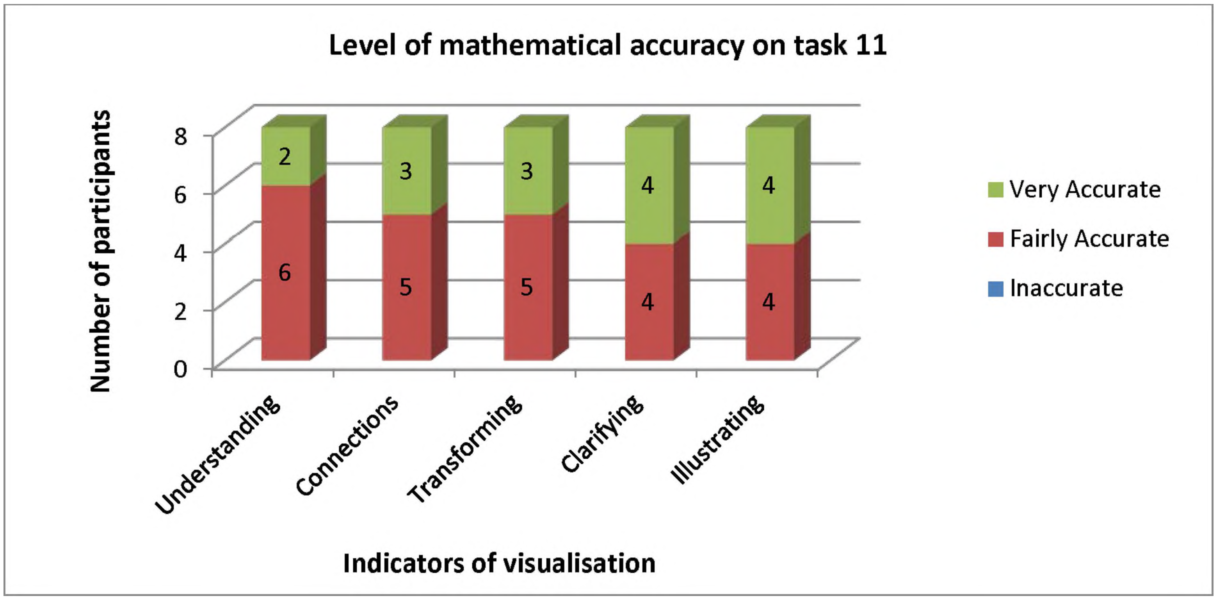
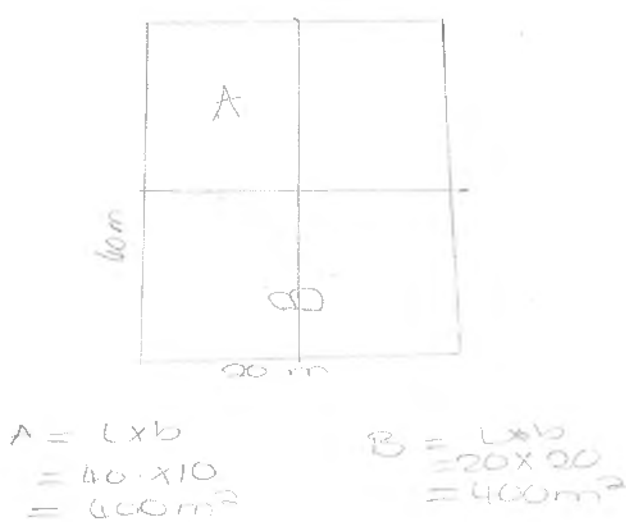


Figure 4.41: Learners' use of visualisation processes for task 11 in terms of mathematical accuracy.

Figure 4.41 shows that most of the participants used fairly accurate sketches in their responses to task 11. For example, six out of the eight participants used fairly accurate sketches whereas two out of the eight used very accurate sketches to reflect their understanding. Regarding evidence in illustrating the task at hand, an equal number of participants (4) used fairly and very accurate sketches. See figure 4.42 showing the level of fairly accurate below.



From the solutions presented by this learner, the formula was derived from each part of the sketch. The learners solved the task by considering each part as a rectangle. Each side length was divided by 2 as shown in the sketch.

Figure 4.42 shows an example of a very accurate sketch used to solve task 11.

4.2.11.2 Visual representation

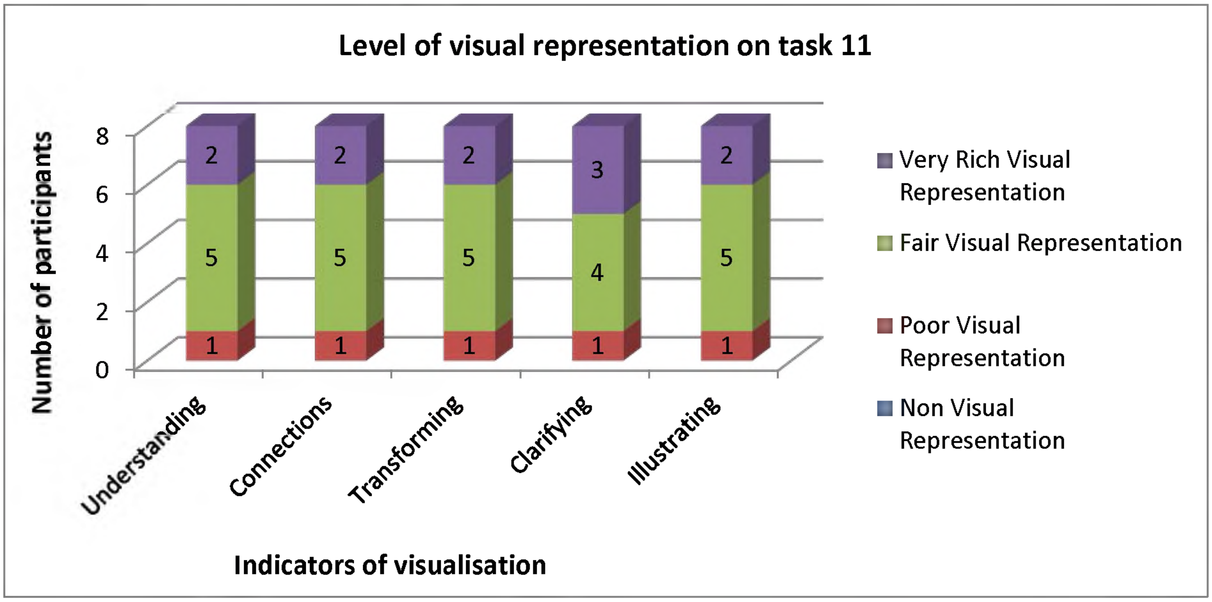
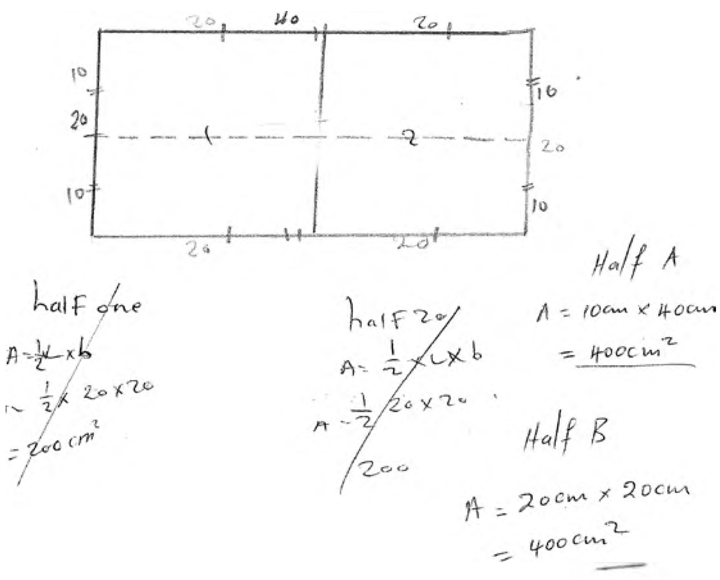


Figure 4.43: Learners' use of visualisation processes for task 11 in terms of visual representation.

Figure 4.43 shows that five of the eight participants managed to use fair visual representation sketches. Nonetheless, it is also clear that on average, two out of the eight participants used very rich visual representation in their solutions to task 11. See figure 4.44 for details in terms of fair visual representations.



This sketch demonstrates that the learner was able to correctly present this task visually. The learner showed that the rectangular sketch has the same area for each of the two halves. Initially, he used the method of area of a triangle but realised it was not appropriate for the rectangles. This shows understanding of the task.

Figure 4.44 shows a fairly rich visual representation used when solving task 11.

4.2.12 Task 12

4.2.12.1 Mathematical accuracy

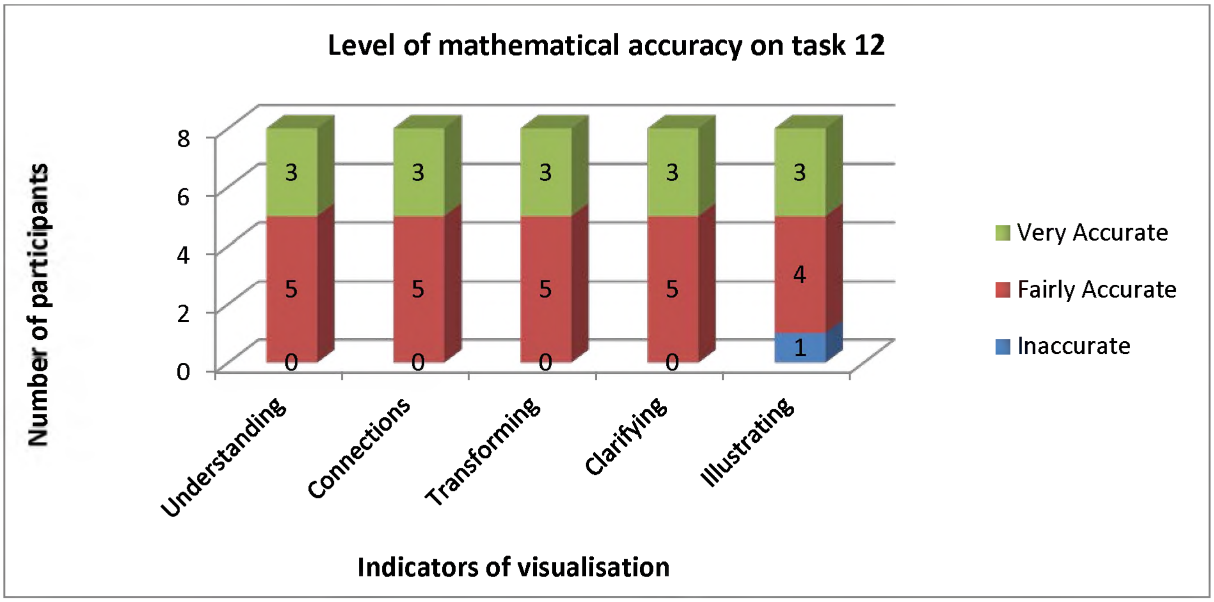
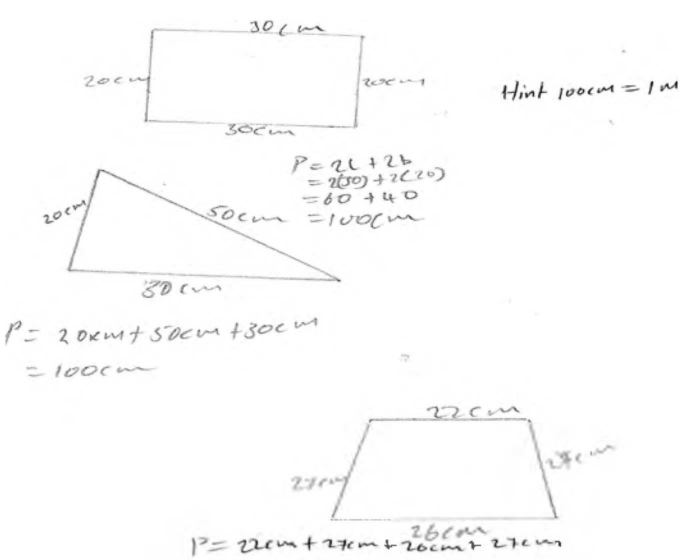


Figure 4.45: Learners' use of visualisation processes for task 12 in terms of mathematical accuracy.

Figure 4.45 shows that one out of the 8 participants used an inaccurate sketch when solving task 12. While five out of the eight participants represents those who used fairly accurate sketches, what tends to be interesting is that a reasonable proportion (three) of participants used very accurate sketches across all five visualisation criteria as illustrated by figure 4.46 below.



The sketches drawn by this learner show that the task was transformed into mathematical constructions. Each side is labelled with dimensions. These dimensions were used to find the answer without computation.

Figure 4.46 shows an example of very accurate sketches used when solving task 12

4.2.12.2 Visual representation

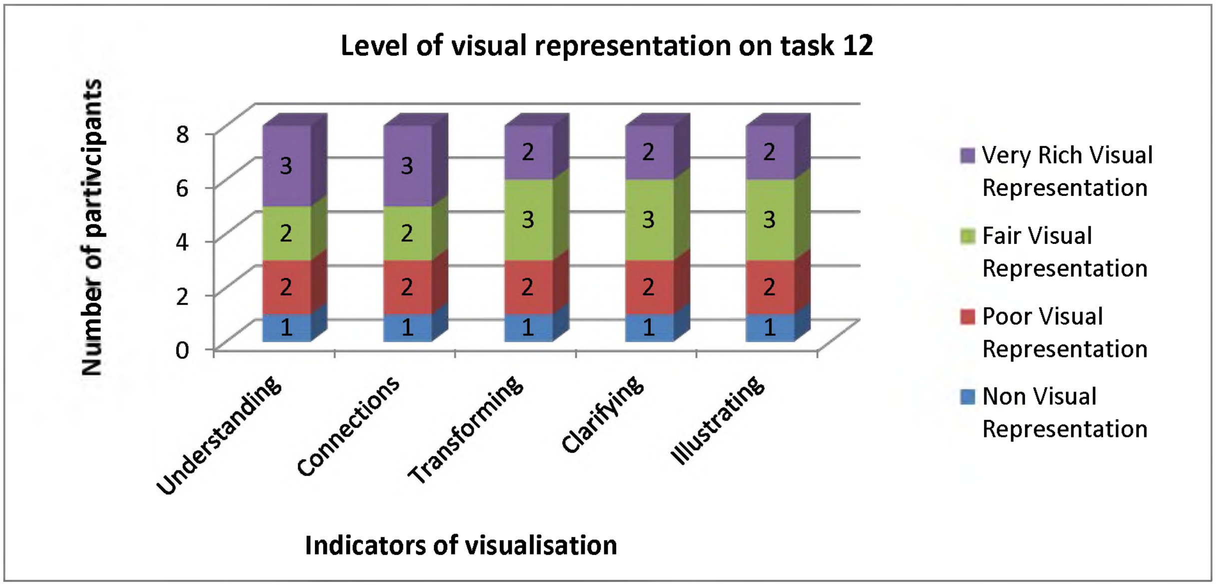
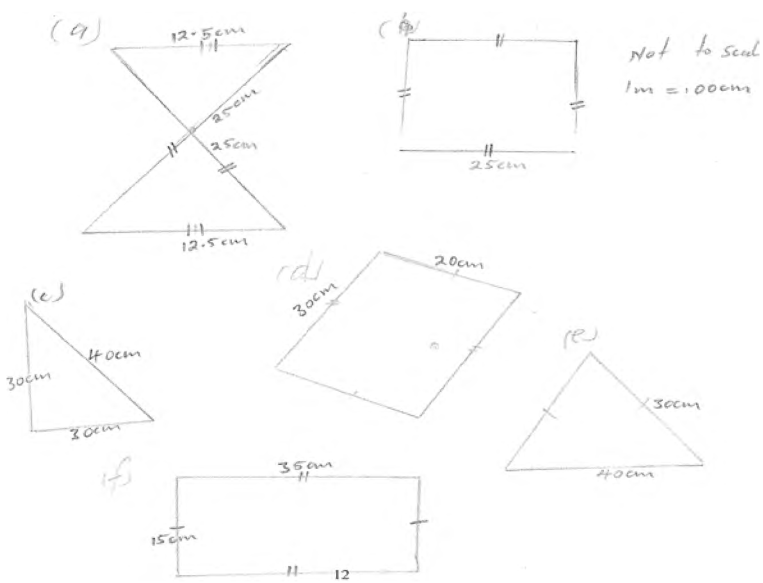


Figure 4.47: Learners' use of visualisation processes for task 12 in terms of visual representation.

Figure 4.47 shows that the majority of learners used fair to very rich visual representation in their solutions to task 12. In particular, five out of the eight participants used moderately rich to very rich visuals when solving task 12. Among the few learners who used very rich visual representations are the three (3) who portrayed appropriate understanding and made very good connections to previous solved tasks. See figure 4.48 as an example of these sketches.



These sketches illustrate the problem situation of task 12. The learner also transformed 1m into 100cm to be consistent with all dimensions given in the sketches.

Figure 4.48 shows very rich visual representations used to solve task 12.

4.3 SUMMARY OF QUANTITATIVE FINDINGS

The results are summarised in the order of how the research questions were asked and how the data was presented in this study. The quantitative results in section 4.2 are discussed as findings to answer research questions 1 and 2 respectively. This summary is based on the five visualisation criteria.

4.3.1 Understanding the spatial relations of the elements in the task

Based on the individual responses to the GVT tasks, about half of the eight participants used visual representations to understand the GVT tasks. Some learners used very accurate sketches. However, three out of the eight participants used only fairly accurate sketches and fair visual representations. The significant proportion of participants who used fair to very accurate sketches confirmed Ho's (2010) claim that understanding a problem by representing it visually enables learners to understand how the elements in the problem relate to each other. It is therefore deduced in this study that learners who used fair to very accurate sketches and fair to very rich visual representations used visualisation processes effectively to solve the tasks.

4.3.2 Connecting to previously solved problems (tasks)

One of the visualisation criteria with the highest number of participants in terms of using very accurate and very rich visual representations was connecting to a previously solved task. This conclusion was reached after checking the level of responses for all five criteria as can be seen in Figures 4.1, 4.3, 4.5, 4.11, 4.13, 4.15, 4.32, 4.34, 4.36, 4.38, 4.40, 4.46 & 4.48. However, a relatively small proportion of participants also made connections when they displayed fair visual representations and fairly accurate mathematical solutions. This indicates that learners made connections to their previous encountered problems. Despite the variability learnt from their solutions, the participants' responses align with what Zimmermann and Cunningham (1991) report on the role of visualisation. According to them, visualisation involves relating a given problem to previous solved tasks and in so doing can understanding be achieved.

4.3.3 Transforming the task into a mathematical construction (Constructing a visual representation)

This visualisation criterion is one of the five where most learners managed to transform the given GVT task into mathematical symbols and constructions. This was evident at both the levels of mathematical accuracy and visual representation. Participants showed that they transformed tasks into mathematical constructions by using very accurate and very rich visual representations when solving the tasks at hand. These responses (sketches) related very well to what scholars such as Alcock and Simpson (2004), Harrel and Sowder (1998), and Presmeg (1986) have reported regarding visualisation. According to them, displays can condense information, suggest new results and propose potential approaches to proof. Since most sketches showed evidence of transforming the task into mathematical symbols and constructions, it is clear that visualisation is part of learners' problem-solving process.

4.3.4 Clarifying (using the visual representations to solve the problem)

Responses in the form of sketches from most learners showed that they were able to simplify and clarify the task by drawing simplified sketches. This is illustrated in Figure 4.6. Using their understanding, they identified the method that worked for them. This trend is similar to those observed by Jones (1998) who posits that the powerful tool in learning mathematics is through visuals, which offer an alternative mass resource almost throughout the media as the presentation of a simplified version of mathematics language especially In the process of solving problems.

4.3.5 Illustrating the problem scenario (encoding the answer to the problem)

In comparing responses of the eight learners, most participants used sketches that illustrated fairly accurately the task at hand. These visual representations were used to confirm the reasonableness of the solutions obtained. This can be seen in Figure 4.4. In this figure, the arrows show movement to and from home. This indicates that these learners were able to illustrate the task by mathematical constructions and symbols. This approach is similar to Wheatley and Brown's (1994) argument that "activities encouraging the construction of images can greatly enhance mathematics learning". They further state that visualisation does not only play a supportive role in mathematical learning but can have an epistemological value, often as a means of discovery, understanding and even as a proof itself (Giaquinto, 2007).

The patterns that emerged from the responses to the GVT tasks revealed that conceptual understanding could effectively be achieved when learners are given the opportunity to use visuals in their solutions to geometry problems. It was also clear from the responses that the five visualisation criteria cannot exist independently from one another. They are interconnected and as Ho (2010, p. 3) reiterates, "visualisation plays different functions or roles as students use it to solve problems".

4.4 ANALYSIS OF QUALITATIVE DATA

Data from the semi-structured interviews of the eight selected participants were analysed qualitatively. The findings of this analysis answer research questions 1 and 2 of this study. The structure of the analysis below follows the five criteria of my analytical tool i.e. understanding, connecting, transforming, clarifying and illustrating.

4.4.1 **Understanding the spatial relations of the elements in the task**

All eight participants that were interviewed used different visual representations when solving geometry problems. These learners explained that working with the GVT encouraged them to use visual representations to understand and solve the GVT tasks. They also pointed out that when a geometry question is asked, they develop initial images of the question in their minds. Two of them stated, "*The images in my mind help me to identify the properties of the diagram in the question. In the process, I can get a formula to find the answer easily. It represents the figure that was initially not drawn. It also guides me to picture the new shape asked"* (P3no10).

"Diagrams help me to prove my answers" (P4no10)

The interviewees further stated that the use of diagrams in solving geometry tasks scaffolds their understanding in geometry. Interestingly, the responses obtained even from the participants who were selected on the basis of performing below average, shows that they enjoyed using visualisation in their solutions. An example is the following statement by one of the learners, *"pictures help me to represent what I am thinking on paper"* (P7no13(i)). This finding aligns with what Arcavi (2003, p. 217) defined visualisation to be:

> the ability, the process and product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understanding.

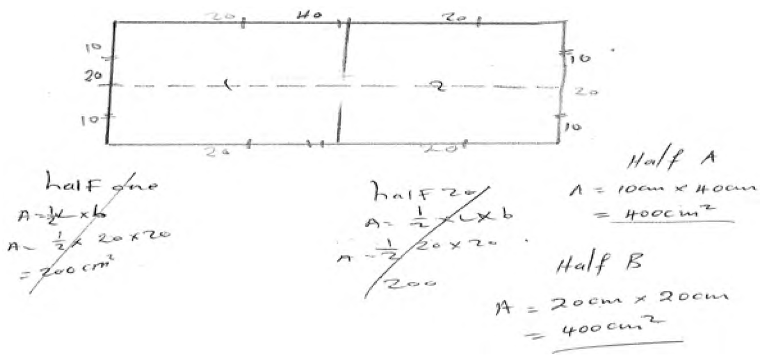
Similarly, Mariotti and Pensci, (1994) acknowledges that visualisation that takes place when "thinking is spontaneously accompanied and supported by images" helps students to understand the problem at hand.

In addition, the majority of the learners indicated that the use of pictures and/or diagrams was beneficial to them. They hinted that the experience of engaging with the GVT was interesting and that they learned a lot from it. They further acknowledged that the GVT tasks were different from their previous experience of geometry. One out of the eight participants pointed out that, *"Diagrams usually assist me to prove if my answer is correct or wrong. They also help me to understand the question. I mean if a question has a diagram, it is clearer than one that does not have a diagram" (P6no10).*

In the same line of argument, another learner commented that, *"Diagrams help me to represent familiar information, for example, if I want to write that these lines are parallel, then I can use this symbol '| |' or if I want to write that triangle ABC has an area of 24cm^2, then I can write △ABC has area 24cm^2"* (P3no 13(i)).

Furthermore, some learners mentioned that the use of visual representations enhanced their understanding of a variety of geometric concepts. These concepts included area, perimeter, visualisation, isosceles triangle, equilateral triangle, rhombus and quadrilaterals. Some participants further highlighted that the use of diagrams when solving geometry problems is important. One of these learners stated, *"They are helpful because sometimes the answer to the question asked is that diagram itself, for example if the question is to identify a parallelogram among other shapes, the properties that you saw before can help you to identify a parallelogram"* (P6no13(iii)).

This finding finds support from Battista (2007) who claims that spatial visualisation and spatial orientation play an important role in learners' ability to understand shapes and their properties through geometric reasoning and visualising of the images, their properties and physical representations. This is illustrated in the learners' extracts above supported by Figure 4.50 below.



As an example, the sketch shows how the learner used visual representation to answer task 11.

Figure 4.49 illustrates the role of visual representation in relation to participant 3's comments above.

Despite the great benefits that the majority of the learners gained from the use of visualisation in learning geometry, two learners were critical. One learner cited that, *"I think visuals are good but they can't help a lot if you don't study a lot for you to be familiar with many shapes"* (P2no20).

The insights in the extracts from the participants above indicate that the use of visualisation in geometric problem solving enhanced their ability to represent geometric problems visually. This conclusion finds support from Abdullah, Zakaria and Halim (2012) who asserts that accumulating evidence links visual reasoning with a deep understanding of concepts in various mathematical areas such as word problems.

4.4.2 Connecting to previously solved tasks

All eight participants demonstrated an application of connecting to previously solved problems (tasks), They expressed finding it helpful to relate present tasks to previous ones. Most of these participants described making connections to previously solved tasks as an aid to simplify the given task. As an example, when participant 2 was asked how he finds pictures that he draws helpful in constructing visual representations, he stated, *"it makes the questions simple, a diagram helps me to give reasons why I solve the task the way I do because some pictures are related to others"* (P2no13(iii)). In the same perspective of argument, another learner mentioned, *"Diagrams usually remind me of the tasks I solved alone or with a friend, or that my teacher used when presenting a lesson"* (P1no14 (iv)).

The learners described the role visual representations play when solving geometry problems as vital in connecting the present to the past. The diagrams that some of these learners drew were self-explanatory and related to the concepts of the respective tasks. Learners showed that they valued the use of visual representations when solving the GVT tasks. Their appreciation of the use of visual representation was observed through their interaction with the GVT and me. This finding connects well with Battista (2007), who advises teachers to be mindful of a child's environment and adapt their instructional approach to teaching geometry by employing a more practical teaching strategy that draws on the child's sense of space, which is defined as understanding of shapes by describing their characteristics and their relations to each other.

Furthermore, one of the participants explained the role of visualisation in geometry by stating, *"I always find diagrams easy to work with because they help me identify and use the appropriate formula depending on the type of question asked"* (P4no14). He adds, *"it helps me to think about … or reminds me of similar questions that we did in other grades"* (P4no10).

The use of visualisation when solving geometric tasks improves relational thinking of learners. Learners become motivated knowing that each geometric question is connected in some way to other previously solved questions. Learners anchor their geometric thought on the knowledge that questions are inter-connected. The findings resonate with that of Frobisher, Frobisher, Orton and Orton's (2007, P.19) study that reported, "in their learning of shapes and space, children experience concepts and skills in different facts of geometry".

In addition, learners tend to like questions that are accompanied by diagrams. In this study, two out of the eight learners pointed out that they enjoy diagrammatic questions because *"Diagrams displays information that can be hidden in the word problems"* (P5no13 (v)).

"Visualisation improves how a learner thinks when he or she is given geometry questions. For example a triangle has three sides and the teacher draws it when teaching, this makes learners not forget this shape because they saw it drawn" (P7no16).

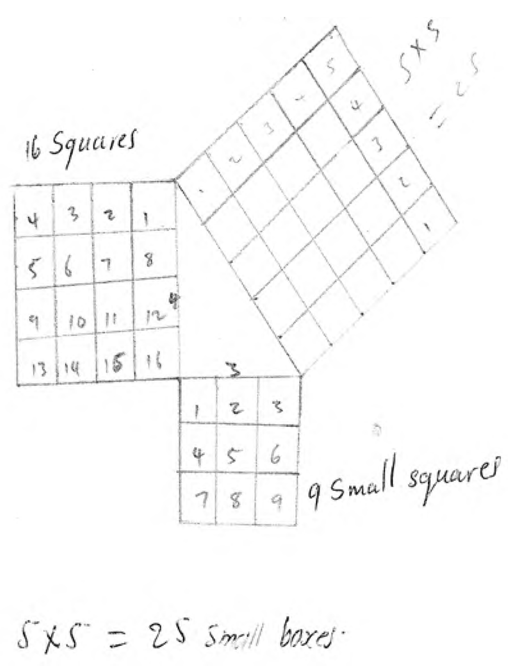
These findings confirm that effective mathematics teaching and learning in general and geometry in particular can be enhanced through the use of visual representations. This is argued because learners have expressed how using visuals to answer geometry problems have made some of them successfully connect some tasks to previously solved ones. Learners also said that sketches could lead to exploration as they look for patterns in the sketch. Additionally, learners explained that they developed an interest in geometry because they realised that the use of visual representations was useful. They also discovered that there was a link between geometry concepts and their everyday life. The following extracts reflect some learners' responses about how they use what they see or imagine in their minds:

"Yes sir, I use the knowledge of things that I learned or experienced, this helps me to see if they have connection" (P8no12).

"Yes sir, for example you can use Pythagoras theorem to find the length of one side of a right-angled triangle if two sides are given or use trigonometric ratios if one side and angle are given" (P2no13(iv)).

"I enjoyed some tasks of the GVT because they helped me to show properties of the diagram which I could not show without a diagram. These properties made me substitute values in my formula to find the known" (P8no16).

These extracts revealed that learners learn well when visual representations are used in the teaching and learning of geometry. As an example, I wish to refer you to figure 4.51 showing a sketch that participant 4 used in the GVT tasks to solve task 2.



Here, the learner exhibited what he knows about task 2 by using very rich visual representation to solve the task accurately.

Figure 4.50 shows how participant 4 used visual representation to solve task 2.

The findings displayed in the sketch and in the verbal responses, indicate that learners can learn geometry with conceptual understanding if visualisation plays a central role in the teaching and learning of geometry.

4.4.3 Transforming the task into a mathematical construction (constructing a visual representation)

Most participants were able to use visual representations that transformed geometric questions into mathematical symbols and constructions. The sketches that they drew in the GVT stage in relation to their responses at the interview clearly indicate that learners were able to associate geometric concepts with their everyday experiences. When asked whether they see or imagine something in their minds when a geometrical question is posed, learners had this to say, *"When I am asked a geometrical question, I see different shapes and figures in my mind depending on the question. For example, if the question is about circles, what comes to my mind is a ball"* (P3no4).

"I see triangles, rectangles, circles, squares, and other shapes. Maybe a trapezium also" (P1no6).

From the above, I concluded that there are many types of visual representations. Theses visual representations therefore can be represented differently by each learner. It is to this effect that most learners drew fair to very rich visual representations for different tasks of the GVT. Their solutions in the form of sketches communicated information about the tasks they solved. It emerged from the findings that most of the interviewees appreciated the use of visual representations in transforming geometric tasks into mathematical symbols and constructions. Two learners stated, *"I liked the idea that I could use signs to represent information on the diagram. I also learned that the diagrams I drew were expected to represent the concept being discussed"* (P4no14).

This finding aligns with Usiskin's (2012) report that full understanding of mathematics requires an understanding of concepts, and an ability to reason through, communicate about and solve mathematics problems. The use of visual representations is another way of teaching mathematics with understanding. This suggests that where possible, encouraging learners to use visual representations enables them to transform word problems into visuals, thus enhancing understanding. This is because visualisations help learners to learn mathematics by doing it.

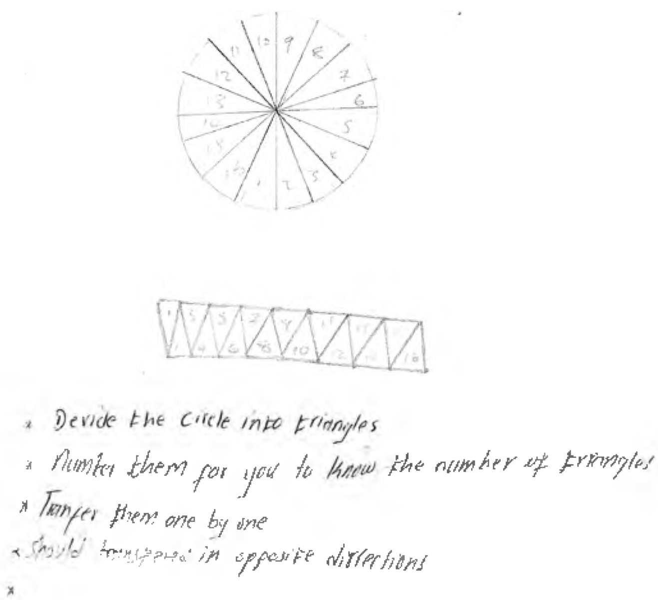
Furthermore, it was observed from the responses of these learners that they appreciated the role visualisation plays in transforming geometric tasks into mathematical constructions, as shown by one learner stating, *"It helps me because information can be obtained from the* *diagram"* (P8no14). *"It helps me to interpret the task given and helps me to use given dimensions to find a suitable method to be used in order to find the answer to the question"* (P8no13 (ii)).

Similarly, Jones (2015) affirms that using visualisation in mathematics classroom learning limits the problem situation that could arise when the basic skills needed to solve the problems have not been mastered by the students. In further support of using visual representations in geometry teaching and learning, Miraslov (2007.325) states that:

> Visualisation helps in transforming a mathematical problem into the form of an image. This image enables the solver to grasp a number of problems whose solution has been accessible using other approaches. Visualisation may fulfil solving, information, imagination education, motivation, control, diagnostic and other functions.

For these reasons, it is important to encourage learners to use visual representations when solving geometric problems. In advancing the argument of the role of visualisation in transforming mathematical tasks into mathematical symbols and constructions, one learner stated, *"I learned that it is easy to represent the task if you understand what the question expects you to do. I also heard that using diagrams makes someone remember what he or she experienced before"* (P4no13 (iii).

The statement above revealed that in order to use visual representation in geometry successfully, learners should be exposed to geometric language and application. This claim was illustrated by participant 4 when he managed to transform task 7 into a mathematical construction as illustrated in figure 4. 52 below.



This figure is an example of how some learners used visualisation to successfully transform a mathematical task into mathematical symbols and constructions, when solving task 7.

Figure 4.51 shows a rich and accurate sketch used to answer task 7.

4.4.4 Clarifying the task at hand (using the visual representations to solve the problem)

Most of the participants demonstrated their ability to clarify geometric tasks when they interacted with the GVT task. They expressed their solutions to tasks using visual representations to solve each of the 12 tasks. Asked what they liked with the approach of using visualisation in geometry, three out the eight participants replied, *"Visuals make my solutions true and clear for me when I do revision. The diagrams display information that can be hidden if using an algebraic method"* (P1no17).

"Visualisation is good to people who have a hearing problem because it gives them a chance to see or observe how geometry tasks can be represented in a visual form. It also helps them to see shapes or diagrams that they should find answers to" (P2no16).

"Smiles…, I think it gives more information. It also simplifies the question especially if details are given on the diagram" (P3no10).

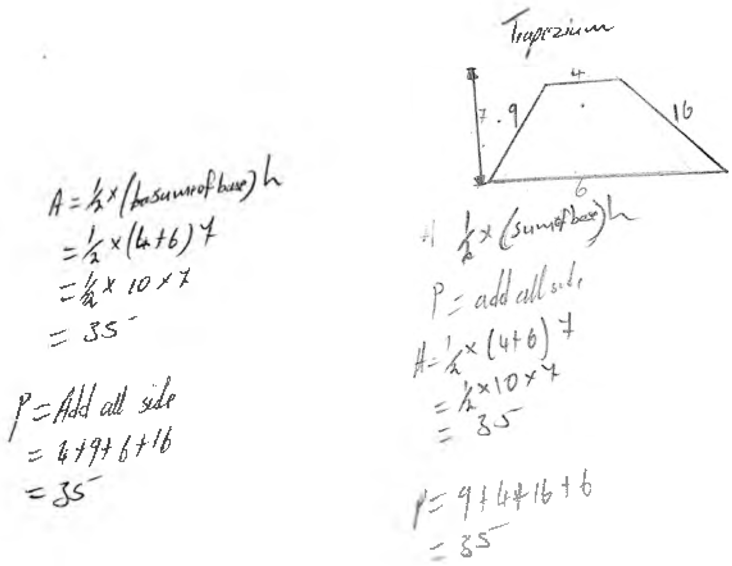
With reference to the above findings, it is clear from their responses that encouraging and teaching learners to use visualisation in their geometry learning can improve their conceptual understanding. Learners' comments indicate awareness of the link between visualisation and learning with understanding. Their responses connect with Kashefi, Alias, Kahar, Buhari, Zakaria Mirzaei (2015) who report, “by using visualisation, students’ understanding of the problems can be seen before they solve the problems”. The comments by learners find support from Shaw’s (2002) report that visualisation helps learners not to rely on the rules to understand what symbols represent. She emphasises that manipulatives and models are valuable resource tools for engaging students in the language and communication of mathematical ideas and concepts. This suggests that the use of visual representations in learning geometry clarifies and simplifies tasks at hand. It can also broaden learners’ thinking pathways.

Participant 6 finds support for his claim that visualisation benefits hearing-impaired learners from Whitely (2004, p. 3), which states that we learn to see, we create what we see, therefore, visual reasoning or ‘seeing’ to think is learned, it can be taught and it is important to teach it.

The findings further reveal statements of other participants in terms of how using visual representations help or do not help them answer geometric questions. They put it this way, *“Sometimes what I imagine confuses me because certain diagrams that I think about makes me not use the right information”* (P3no12). She adds, *“Sometimes diagrams are complicated and can take a lot of time to interpret and also to draw them if you are asked to draw”* (P3no14).

These findings confirm Steenpass and Steinbring (2014, p.4) who report, “Mathematical visuals do not convey mathematical concepts directly into the students’ heads, but they have to be actively interpreted embedded in a cultural milieu”. In light of this, effective learning of geometry can be attained if learners are supported and taught how to use visual representations accurately. This goal will help learners to use visuals in order to clarify and simplify the geometric task at hand. According to participant 6, the use of visual representations should be looked at from both sides. He stated, *“It helps but you should be careful that you don’t substitute values wrongly. For example, when finding the area of a trapezium, you use dimensions for slanting sides in place of the height”* (P6no14).

The argument that the learner puts forward, shows that he is interested in the use of visualisation. However, he points out that both teachers and learners should have a thorough understanding to be able to use visual representations when solving geometry problems. Figure 4.53 illustrates the sketch that this participant used to clarify the GVT task.



The learner's solution to task 8 shows awareness that 9 and 16 units on both slanting lines cannot be substituted for height of the trapezium in finding area.

Figure 4.52 shows a rich and accurate sketch used to clarify task 8.

4.4.5 Illustrating the problem scenario (encoding the answer to the problem)

The findings of the responses from both the GVT solutions and the post GVT-interviews show that nearly every participant of each task illustrated the task with a sketch. These sketches varied from one participant to the next. When asked how they found pictures that they drew or scribbled on paper helpful in understanding the content, some learners stated, *"I think a diagram stands in place of the answer given in a number form because pictures help to clarify the question"* (P4no139(i)).

"I use diagrams as back-ups. It is like a storage of information that I can go back to if it is needed" (P1no14 (iv)).

The lesson I learned from these findings is that pictures or diagrams used as visual representations are valued by learners when solving geometric problems. The fact that diagrams are used to represent mathematical solutions and are a storage of information as remarked by learners, makes it important to illustrate geometric solutions as visual representations. Doing so will help in measuring the reasonableness of the answers in relation to the visual presented. This argument resonates with the statements that two learners gave by stating, *"It helps when the question requires me to give a proof for my answer because the diagram that I will draw will show if my answer is correct or wrong"* (P4no13 (iv)).

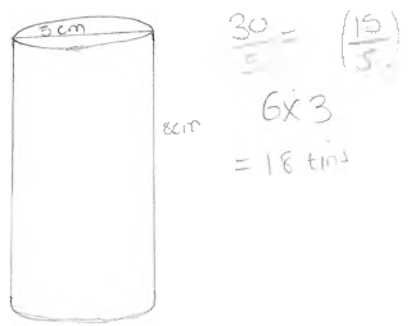
"It helps me to give reasons why I solved the task the way I did since pictures represent the solution to the question" (P6no13 (iii)).

The findings also advance reasons why illustrating a problem scenario is an effective approach of using visualisation in learning geometry. In agreement, Hearst (2009) emphasises that visualisation translates abstract information into the visual form, which provides new insight about that information. The approach of illustrating the answer using visual representation helps learners to demonstrate their reasoning when given a geometry task through the use of diagrams or mental processes. Illustrating the solutions as an approach to learning supports meaningful connections. One learner pointed out, *"Diagrams communicate information that they carry. They also contain rich details about the solutions that the question expects"* (P8no13 (v)). He further states, *"They are good because they have extra information, if they are shown next to the question"* (P8no13 (i)).

The statements above revealed that using a visualisation approach of illustrating the solutions to geometric problems can enable learners to make geometric concepts concrete and clear in their minds. As learners draw sketches that represent their solutions, they internalise the properties of that sketch and deepen their understanding of the concept. This is argued because some learners may not manage to draw an accurate sketch once but might draw-and-erase as they look for suitable patterns. In so doing, they communicate their ideas in the mind. Another leaner noted, *"There is a relationship between what I draw and the answer that I find; I think diagrams or pictures represent the final answer that is needed"* (P2no13 (i)).

This finding unveils the process of visualisation when learners illustrate the problem scenario using visual representations. In the same line of argument, another learner comments that illustrating solutions using visual representations compares the answer given numerically to the one given in visual form. She states it this way, *"It compares the answer given in numbers with the one drawn"* (P3no13 (v)).

Another learner remarked, *"The picture with which I represented my solution indicated the dimensions found in the question. This helps me to understand the question better. I mean if the questions talk circles, I draw the circle or a square, I draw it. This is how I understand your question sir"* (P7no13 (i)). These statements show that learners appreciate illustrating their solutions using visual representations. Furthermore, the statement shows that in illustrating solutions, practice is involved as shown in figure 4.54 below.



Interesting to note is that although not drawn to scale, the sketch in figure 4.54, an example of some solutions obtained from the GVT interaction, illustrates the problem task in a rich manner.

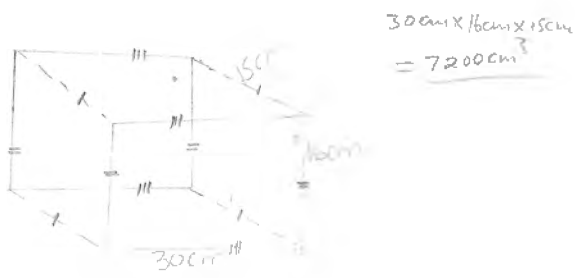


Figure 4.53 Shows a sketch used to illustrate task 1(

The above statements revealed that despite some learners being critical and uncertain about the use of visualisation in illustrating their solutions, learners were motivated and encouraged to use visual representations to illustrate their solutions to geometric problems as illustrated by figure 4.54.

4.5 CONCLUDING REMARKS

This chapter began with presentations of the quantitative and qualitative results collected from the three data gathering tools used.

Most of the eight participants showed varied use of visualisation processes in terms of mathematical accuracy according to the analytical framework of visualisation processes. Task 1 for example revealed that seven participants solved this task very accurately in terms of transforming and illustrating the task at hand. In the same perspective, on average six participants very accurately solved task 2 across the visualisation criteria. Regarding task 3, an average of three participants very accurately solved the task. This compares well with task 4 where five participants very accurately represented the task at hand. It is worth noting that two learners solved task five very accurately across the visualisation indicators.. The findings also show that five participants solved task six very accurately. With respect to task seven, an average of two participants appear to have solved this task very accurately. Tasks 8 and 9 were very accurately solved by six participants. The findings also show that task 10 was solved very accurately by five participants. For tasks 11 and 12 three participants solved them accurately.

Along similar lines, the majority of the learners showed varied use of visualisation processes in terms of visual representations. This was evident in tasks 1 and 2 were four participants used very rich representations. Tasks 3, 5, 11 and 12 revealed that only two participants used very rich visual representations across the visualisation criteria. Interestingly, five participants solved task 4 using very rich visual representations across the five visualisation criteria. This is in contrast to task 6 where four participants solved it using very rich visual representations. Task seven only has one participant who solved it using very rich visual representations particularly in clarifying and illustrating the problem. Furthermore, the findings also revealed that tasks 8 and 10 had three participants who used very rich visual representations across the five visualisation criteria while task 9 had six participants who used very rich visual representations in making connections and transforming the problem at hand. The post-GVT interview responses depicted how learners felt about the use of visualisation in their geometric problem solving. The results of the post-GVT interviews also revealed that learners are interested in the use of visual representations. Most of them stated that visualisations help them to think broadly and help them to anchor their thoughts while solving geometry problems.

However, some indicated that exposure to different geometric problems and guidance on how to use these visuals are pre-requisite interventions if visualisation is to fully yield its value in geometry teaching and learning.

Each of the themes used was discussed in support and comparison to the literature reviewed. In the next section, conclusions are drawn from the study, and some implications and limitations are presented.

CHAPTER FIVE

5 SUMMARY OF FINDINGS, RECOMMENDATIONS AND CONCLUSIONS

5.1 INTRODUCTION

Studies have emphasised the value of using visual representations in teaching and learning mathematics in general and geometry in particular. They have shown that the use of visual representations in mathematics boosts the learners' insights into mathematics. (Kashefi et al. 2005; Zimmermann & Cunningham, 1991; Serpil, Cihan, Sabri, Sabri & Ahmet, 2002; Jones, 1998). Research also shows that the use of visual representations can mediate learners' understanding of mathematical concepts (Usiskin, 2012).

My case study explored the nature of the visualisation processes employed when selected grade 11 learners interacted with geometry problems. I also explored how these learners used visualisation processes in their interactions with given geometry problems. I employed a mixed method research approach in answering my two research questions.

5.2 SUMMARY OF FINDINGS

Research question 1: What is the nature of the visualisation processes employed when selected grade 11 learners interact with geometry tasks?

In response to research question one, the results presented in Chapter 4 (Section 4.2) revealed the nature of visualisation processes employed when selected grade 11 learners interacted with geometry tasks in terms of the seven sub-scales examined. The findings indicated that most of the learners used fairly to very accurate sketches and fairly to very rich visual representations when solving geometric problems. The GVT results showed that the majority of the learners understood the link between the algebraic solutions and the diagrams that they drew to represent those solutions. This may be due to their ability to link geometric concepts with their environment. In addition, the results presented in chapter four should have been influenced by their interest in mathematics.

The approach by learners showed that they valued the use of visual representations in geometry and found them helpful. The nature of visualisation processes employed by learners linked very well with the five visualisation criteria as follows:

Understanding the spatial relations of the elements in the task: Under this criterion, most learners drew new diagrams and often adapted them to previously constructed ones. Subsequently, the leaners obtained critical information from the sketches that they drew to understand the task fully. As an example, most learners used the directions in task 1 to draw the right-angled triangle. Later, they realised that the theorem of Pythagoras was applicable in finding the length of the third side. The approach of constructing a triangle enabled the learners to gain additional information from the diagram. It was the drawing of the diagram that revealed to the learners that they could use Pythagoras theorem. It became evident that as learners kept practicing drawing sketches, they affirmed their understanding of the sketches, and their understanding of the task.

When it came to *understanding the spatial relations in the task* the participants provided evidence that they could recognize, label, and generate examples of concepts; use and interrelate diagrams and varied representations of concepts; identify and apply principles; know and apply facts and definitions; compare, contrast, and integrate related concepts and principles; recognize, interpret, and apply the signs, symbols, and terms used to represent concepts. This was reflected in their ability to reason in settings involving the careful application of concept definitions, relations, or representations of either.

Connecting to the previously solved problems: In the interviews, learners stated that they used information from the sketches to explore each task presented to them. It was revealed that once learners realised certain patterns in the diagram that they drew, they used those patterns to make explicit links between geometrical concepts and prior knowledge they had already gained. Furthermore, the majority of learners used varied visual representations to convey mathematical information. The findings also showed that learners were able to connect between different representations such as diagrams and algebraic expressions.

Transforming the task into mathematical form: The results of the GVT tasks showed that there was overwhelming evidence that the participants engaged in a process of transforming sketches over and over again. In the learners' views, they also said that diagrams simplified the task by transforming word problems into mathematical symbols relevant to the solution of the problem.

Clarifying the task at hand: It was evident that visualisation processes were used to clarify the learners' mathematical thinking. These visual representations assisted learners to reflect upon and clarify ideas, relationships and geometrical arguments.

Illustrating the problem scenario: Many of the sketches drawn by learners illustrated their thought processes and understanding of the given tasks. Each learner illustrated each of the 12 tasks differently.

Research question 2: How do these grade 11 learners use visualisation processes in their interactions with geometry?

The findings from this study showed that most learners used visualisation processes fairly to very accurately when solving geometry problems. With respect to understanding spatial relations, the majority of the learners managed to produce visual representations of geometrical statements by making recognizable diagrams or sketches. Their ability to mentally manipulate objects and understand spatial relationships is indicative of how these learners used visualisation processes in a range of tasks.

Another way in which learners used visualisation processes with respect to geometry word problems was by identifying shapes that were connected to each other and were intrinsic to the particular problem. Their thinking, while engaging with the GVT was often connected to previous knowledge and experience. They used these experiences to draw sketches that linked to the task they were solving.

In relation to how learners used visualisation processes in their interactions with geometry tasks, they also used algebraic expressions to assist them to find answers to the problem. The algebra would often assist in clarifying the problem.

The sketches drawn by most learners simplified the task at hand by way of accent marks illustrated on them. It was also revealed in this study that the majority of the learners transformed each task pictorially. This was done by displaying detailed sketches representing the solutions to the problem. As they transformed given tasks into mathematical constructions, they examined relationships that existed between what they knew before and the task at hand. In doing this, they used visualisation processes to illustrate their thought processes when solving geometry problems. In this context, solving geometric tasks using visualisation processes assisted learners in understanding the spatial relationships within each task. This spatial relation was arrived at when learners successfully made connections between different representations such as diagrams and algebraic expressions.

Overall, the findings of this study showed that learners made use of visualisation processes to develop and scaffold their conceptual understanding of geometric problems.

5.3 SIGNIFICANCE OF THE STUDY

The value of visualisation in mathematics teaching and learning has been emphasised by many writers (Gorgorio and Jones, 1996; Rodd, 2010; Rosken & Rolka, 2006).
My study showed that the appropriate use of visual representations in teaching and learning of geometry could lead to conceptual understanding. For this reason, the insights that I gained of how learners visualise mathematical problems are significant to teachers, curriculum designers, teacher training institutions and policy makers.

Moreover, this study corroborates what a large body of studies emphasise regarding the effective use of visualisation in teaching and learning. This study consistently showed that the use of visual representations helped learners to understand that various representations can be used to clarify the task at hand. It was also revealed in this study that learners represent mathematical ideas and relationships using a variety of images, pictures and sketches.
It is thus incumbent on teachers to harness the visualisation processes to enrich and enhance their teaching for conceptual understanding.

5.4 RECOMMENDATIONS

From the perspective of my findings, I recommend the following to teachers, curriculum designers, teacher training institutions and policy makers:

- Teachers of mathematics should create opportunities for leaners to demonstrate their solutions visually.
- Mathematics lessons should be allocated adequate time on timetables to allow both teachers and learners to use visual representations in their teaching and learning. This will deepen the learners' conceptual understanding of mathematics.

- Continuous professional development programmes for teachers should recognise the value of the use of visualisation and its integration in teaching and learning of mathematics.
- Continuous professional development programmes should motivate and encourage teachers to critically review their practice of using visualisations in an attempt to improve their teaching.
- Curriculum designers should make use of visual representations more explicitly in the curriculum and facilitate the creation of teaching and learning materials respectively.
- Training institutions should incorporate the use of visualisation in the training of teachers.
- Policy makers should affirm and persuade curriculum designers to incorporate the use of visualisation strategies in all phases of schooling.
- It is further recommended that parents support their children's learning by providing them with required instruments and resources that will enable them to use visual representations in their everyday activities at home.

5.5 LIMITATIONS OF THE STUDY

The following are taken to be the possible limitations of this study:

- *Small sample size:* The findings of this study are not generalizable to other learners and their learning situations. I argue this given the small sample size of my participants.
- *Specific locality:* The findings of this study may not apply to other localities given that different settings have varied circumstances. It is also apparent that learners in different settings respond differently to different situations.
- *Power issue:* The issue of power between participants and me was another possible limitation. My position as a teacher could have influenced learners' responses since the study was undertaken at the school where I work. Some learners might thus not have shared their innermost and honest feelings.
- *Time:* Time was another possible limitation in that I was expected to collect data only after normal school hours.

5.6 SUGGESTIONS FOR FURTHER RESEARCH

Exploring the use of visual representation was an exciting and very educative exercise for me. As such, this case study could form a useful platform for future research in the following areas:

- Conducting a similar study with a larger sample of participants drawn from two or more schools taught by different teachers.
- Conducting a similar study with teachers as data sources. This could generate data on how best the use of visual representations can be applied in a classroom situation.
- Conducting a study that establishes teachers' attitudes towards the everyday use of visualisation in teaching and learning of mathematics.
- Exploring whether boys' and girls' use of visualisation differ in geometry problem solving situations.

5.7 PERSONAL REFLECTIONS

Since the inception of this study, I learnt many different approaches to conducting a research study. This study has helped me in many ways:

Firstly, it has taught me how to become a researcher and what is involved in the research process. I learnt that doing research requires the researcher to be ethical with everything that revolves around gathering and analysing data. I also learned that each research study that one undertakes should relate to previous research in the same field in order to grow that field. This adds to the validity and legitimacy of the study.

Secondly, I learned that to undertake a research study requires patience, discipline and sacrifice in order to successfully accomplish the goals and the objectives of the study. This means I had to forego a multitude of activities so as to devote appropriate time for the reading of different texts and literature that related to my study.

Thirdly, this study has helped me to improve my academic literacy in terms of writing academically and coherently.

Fourthly, this research study provided me with an opportunity to work insightfully with learners and their thinking processes. Most importantly, this research study helped me to acquaint myself with many unfamiliar ideas and gave me meaningful ideas for my own practice.

Additionally, I found the participation in the research design course at the beginning of my research journey very informative. I also found in this study that any research process should contribute to the expanding and informing of the broader field, and not merely enriching one's own personal perspective.

Furthermore, the experience of interacting with colleagues was educative and enriching. Above all, the first-rate support and guidance from my supervisor was much appreciated.

5.8 CONCLUSION

In this chapter, the summary of findings, significance of the study, recommendations, limitations, suggestions for further research and personal reflections were presented. The study revealed that on the whole the participating learners made meaningful use of visualisation processes in solving selected geometry tasks.

REFERENCES

Abdullah, N., Zakaria, I. & Halim, I. (2012). The effect of a thinking strategy approach through visual representation.

Abrahamson, D., Lee, R. G., Negrete, A. G. & Gutierrez, J. F. (2014). Coordinating visualisations of polysemous action: Values added for grounding proportion. Imagery and Cognition (103-131). New York: Springer Verlag.

Alcock, I. & Simpson, A. (2004). Convergence of Sequences and Series: Interactions between visual reasoning and the learner's beliefs about their own role. Educational studies in Mathematics, 57, 1-32.

Altun, M. (2002). Teaching of Mathematics. Bursa: alfa publishing, 331-392

Anderson-Pence, K. I., Moyer-Packenham, P. S., Westenskow, A., Shumway, J., & Jordan, K. (2014). Relationships between visual static models and students' written solutions to Fraction tasks. *International Journal for Mathematics teaching and learning*, *15*, 1-18.

Arcavi, A. (2003). The role of visual representations in the learning of Mathematics. *Educational studies in mathematics*, *52*(3), 215-241.

Aydoğdu, M. Z., & Keşan, A. (2014). A research on geometry problem solving strategies used by elementary mathematics teacher candidates. *Journal of Educational and instructional studies in the world*, *4*(1).

Ball, B. and Ball, D. (2007). Visualising geometry in *mathematics teaching incorporating micromath* 205, 4-5.

Bassarear, T. (2012). Mathematics for Elementary School Teachers. Fifth edition. Belmont, CA: Brooks/Cole. Battista, M.T. 2007. The development of geometric and spatial thinking. In F. Lester (Ed.), Second Handbook of Research on Mathematics Teaching and Learning. Charlotte, NC: NCTM/Information Age Publishing. 843-908.

Battista, M. T. (1990). Spatial visualisation and gender differences in high school geometry. *Journal for research in mathematics education*, 47-60.

Battista, M. T. (2007). the development of geometric and spatial thinking. *second handbook of research on mathematics teaching and learning*, *2*, 843-908.

Bishop, A. J. (1989). Review of research on visualisation in mathematics education. focus on learning problems in mathematics. 11(1), 7-16.

Borst, G & Kosslyn, S. M. (2008). Visual mental imagery and visual perception: structural equivalence revealed by scanning processes. Memory and cognition, 36(4), 849-862. doi:10.3758/me.36.4.849.

Burns, S., N., & Grove, S., K. (2003). Understanding Nursing Research. 3rd Edition. Philadelphia: Saunders.

Cangelosi, J. S. (1996). Teaching mathematics in secondary and middle school: An interactive approach. Englewood Cliffs, NJ: Prentice Hall.

Chiappini, G., & Bottino, R. M. (2007). Visualisation in teaching-learning mathematics: the role of the computer. *COMPUTER GRAPHICS AND VISUALISATION EDUCATION '99*.

Clark, R., Nguyen, F., & Sweller, J. (2006). Efficiency in learning: Evidence-based guidelines to manage cognitive load: Pfeiffer. S*an Francisco*, 23.

Cohen, L., Manion, L. Morrison, K. (2001). *Research Method in education, Rutledge Flamer.*

Cohen, L., Manion, L., Morrison, K. (2011). Planning educational research. *Research methods in education. New York: Routledge Editors*.

Creswell, J. W. (2006). Understanding mixed method research. Thousand Oaks. Sage.

Cuoco, A., Goldenberg, E. P., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *The journal of mathematical behaviour*, *15*(4), 375-402.

David, M. M., & Tomaz, V. S. (2012). The role of visual representations for structuring classroom mathematical activity. *Educational studies in mathematics*, *80*(3), 413-431.

Deliyianni, E., Elia, I., Gagatsis, A., Monoyiou, A., & Panaoura, A. (2009). A theoretical model of students' geometrical figure understanding. In *proceedings of the sixth congress of the European society for research in mathematics education* (696-706).

Denis, M. (1991). Imagery and thinking. in C. Cornold & M. McDaniel (eds.), imagery and cognition (103–131). New York: Springer Verlag.

Denscombe, M. (2008). Communities of practice a research paradigm for the mixed methods approach. *Journal of mixed methods research*, *2*(3), 270-283.

Dhillon, A.S. (1998). Individual differences within problem-solving strategies used in Physics. Science Education, 82(3), 379.

Ding, I., & Jones, K. (2006). Teaching geometry in lower secondary school in shanghai, china. *Proceedings of the British society for research into learning Mathematics*, *26*(1), 41-46.

Dreyfus, T. (1991), On the status of visual reasoning in mathematics and mathematics education. In: F. Furinghetti (Ed), Proceedings of the 15th annual conference of the International Group for the Psychology of Mathematics Education. Assissi. Vol 1, 33-48.

Dreyfus, T., & Eisenberg, T. (1986). On the aesthetics of mathematical thought. *for the learning of mathematics*, *6*(1), 2-10.

Duval, R. (1998). Geometry from a cognitive point of view. In C mammana and v villani (eds), perspectives on the teaching of geometry for the 21st century: an ICMI study. Dordrecht: Kluwer.

Eisenberg, T., & Dreyfus, T. (1994). On understanding how students learn to visualise function transformations. *Research in collegiate mathematics education*, *1*, 45-68.

Elia, I., Van den Heuvel-Panhuizen, M., & Kolovou, A. (2009). Exploring strategy use and strategy flexibility in non-routine problem solving by primary school high achievers in mathematics. *ZDM, 41*(5), 605-618.

Elliott, S. (1998). Visualisation and using technology in a level mathematics. *proceedings of the British society for future research into learning mathematics, 33*(2), 37-42.

English, I. (Ed.). (1997). Mathematical reasoning - analogies, metaphors and images. Mahwah, NJ, Lawrence Erlbaum associates.

Fischbein, E. (1987). Intuition in science and mathematics. Dordrecht: kluwer.

Frobisher, I., Frobisher, A., Orton, A., & Orton, J. (2007). Learning to teach shape and space. *Cheltenham: Nelson Thornes*.

Furinghetti, F., Morselli, F. & Antonini, S. (2011). to exist or not to exist: example generation in real analysis. zdm – the international journal on mathematics education, 43, 219 – 232.

Giaquinto, M. (2007). V*isual thinking in mathematics*. Oxford University Press.

Gick, M. L. (1986). Problem-solving strategies. *Educational psychologist*, *21*(1-2), 99-120.

Gorgorió, N. and Jones, K. (1996), Elements of the Visualisation Process within a Dynamic Geometry Environment. Invited paper presented to the topic group on The Future of Geometry at the 8th International Congress on Mathematical Education (ICME8), Seville, Spain, July 1996. 6pp.

Guba, E., G., & Linkoln, Y. S. (1994). Competing paradigms in qualitative research.

Guzman, M. (2002). The role of visualisation in the teaching and learning of mathematical analysis. in proceedings of the international conference on the teaching of mathematics (at the undergraduate level). Greece: hersomssos (ERIC document reproduction service no. ed 472 047).

Hanna, G., & Sidoli, N. (2007). Visualisation and proof: A brief survey of philosophical perspectives. ZDM, *39*(1-2), 73-78.

Harrel, G., & Sowder, I. (1998). Students' proof schemes: Results from exploratory studies. In A.N. Schoenfeld, J. Kaput & E. Dubinsky (Eds.), Research in collegiate mathematics education (234 – 283). Providence: American mathematical society.

Hearst, M. (2009). *Search user interfaces*. Cambridge University Press.

Henning, E, Van Rensburg, W and Smit, B. (2011). Finding your way in qualitative research. Van Shaika Publishers.

Hitt, F. (2002). Representations and mathematics visualisation. *Author Speiser, Robert, ed.; Maher, Carolyn A., ed.; Walter, Charles N., ed*, 53.

Ho, S. Y., Ramful, A., & Lowrie, T. (2015). *Representations of the word problems on Ipad and pencil-and-paper test modes.* paper presented at the 7th ICMI-east Asia regional conference on mathematics education, (pp. 569-578), University of Canberra, Cebu.

Ho, S.Y. (2010). Seeing the value of visualisation. Mathematics and mathematics education academic group, national institute of education, Singapore: Nanyang technology university.

Holloway, I., & Wheeler, S. (2002). Research in nursing.

Huang, H. M. E., & Witz, K. G. (2011). Developing children's conceptual understanding of area measurement: A curriculum and teaching experiment. *Learning and instruction*, *21*(1), 1-13.

Idris, N. (2006). Teaching and learning of mathematics: Making sense and developing cognitive abilities. Perak: Utusan Publication Sdn. Bhd.

Idris, N. (2009). The impact of using Geometers' Sketchpad on Malaysian students' achievement and van Hiele geometric thinking. *Journal of Mathematics Education*, *2*(2), 94-107.

Jones, K. (1998). Theoretical Frameworks for the Learning of Geometrical Reasoning. proceeding of the British Society for Research into Learning mathematics. King's college, University of London.

Jones, K. (2013). Diagrams in the teaching and learning of geometry: some results and ideas for future research. In C. Smith (Ed.). Proceedings of the British Society for Research into Learning mathematics 33(2).

Kadunz, G., & Sträesser, R. (2004). Image-metaphor-diagram: Visualisation in learning mathematics. Paper presented at the 28th conference of the international group for the Psychology of Mathematics Education.

Kaizer, C., & Shore, B. M. (1995). Strategy flexibility in more and less competent students on mathematical word problems. *Creativity research journal*, *8*(1), 77-82.

Kashefi, H., Alias, N., A., Kahar, M., F., Buhari, O., Zakaria, S & Mirzaei, F. (2015). *Visualisation in Mathematics problem solving Meta-Analysis research.*

Kickbusch, K. (2000). Teaching for Understanding: Educating Students for Performance.

http:www.weac.org/resource/under.htrn (access date: 12 March 2015).

Kilpatrick, S., & Swafford, J. Findell (2001). *Adding it up: Helping children learn mathematics*.

Kosslyn, S., M., & Koenig, O. (1992). Wet mind: The new cognitive neuroscience. New york: free press.

Love, E. (1995). The Functions of Visualisation in Learning Geometry. In R. Sutherland and J. Mason (Eds). Exploiting Mental Imagery with Computers in Mathematics Education. Berlin: Springer.

Lowe, R. (2000). *Visual literacy and learning in science.* Columbus, OH: ERIC Clearinghouse for Scie.

Luneta, K. (2013). *Teaching elementary mathematics: learning to teach elementary mathematics through mentorship and professional development*.

Luneta, K. (2014). Foundation phase teachers' (limited) knowledge of geometry. *South African Journal of childhood education*, *4*(3), 71-86.

Makina, A. & Wessels, D. (2009). The role of visualisation in Data Handling in Grade 9 within a problem-centred context.

Makina, A. (2010). The role or visualisation in developing critical thinking in mathematics. Perspectives in Education, 28(1), 24 – 33.

Manicosu, P., Jorgensen, K., & Pedersen, S. (2005). Visualisation, exploration and reasoning styles in mathematics. New York: Springer.

Manovich, L. (2010). What is visualization?. *paj: The Journal of the Initiative for Digital Humanities, Media, and Culture*, *2*(1).

Mariotti, M. A., & Pensci, A. (1994). Visualisation in teaching-learning situations. Procceedings of PME 18, 1, 22.

Mason, J. (1992). Towards a Research Programme for Mental Imagery. Proceedings of the November Conference of BSRLM, pp. 24-29.

Mateya, M. (2013). Using the van Hiele theory to analysis geometrical conceptualization in grade 12 students: A Namibian Perspective. In M. Schafer, D. Samson & D. Brown (Eds.). *Namibia counts, stories of mathematics education research in Namibia* (49-64). Cape town: Digital print solutions.

Maxwell, J. A. (1996). Qualitative research design: An interactive approach. Thousand Oaks, CA: Sage

McCormick, R. (1997). Conceptual and procedural knowledge. *international journal of technology and design education*, *7*(1-2), 141-159.

Miroslav, M. (2007). Forms of visualisation as a solving approach in teaching mathematics.

Moyer-Packenham, P. S., Ulmer, I. A., & Anderson, R. I. (2012). Examining pictorial models and virtual manipulatives for third grade fraction instruction. Journal of interactive online learning, 11(3), 103 – 120.

Mudaly, V. (2010). Thinking with diagrams whilst writing with words. Pythagoras, 71, 65-75.

Naidoo, J. (2007). The effect of social class on visualisation in geometry in two KwaZulu-Natal schools, South Africa. Master of philosophy thesis. University of Nottingham: Nottingham.

Namibia. Ministry of Education. (2011). *Namibia Senior Secondary Ordinary Mathematics Examiners Report.* Windhoek: DNEA

Namibia. Ministry of Education. (2012). *Namibia Senior Secondary Ordinary Mathematics Examiners Report.* Windhoek: DNEA.

Namibia. Ministry of Education. (2013). *Namibia Senior Secondary Ordinary Mathematics Examiners Report.* Windhoek: DNEA.

Namibia. Ministry of Education. (2014). *Namibia Senior Secondary Ordinary Mathematics Examiners Report.* Windhoek: DNEA.

Namibia: Ministry of Education, sport and Culture. (1993). Towards Education for All. Windhoek: Gamsberg Macmillan Publishers.

Namibia: Ministry of Basic Education, Sport and Culture. (1996). Curiculum Guide for Formal Basic Education. Windhoek.

Namibia: Ministry of Education. (2003). Learner-Centred Education in the Namibian Context: A Conceptual Framework. Discussion document, January 2003. Okahandja: NIED.

Namibia: Ministry of Education. (2009). Journal for Educational Reform in Namibia. Okahandja: NIED.

Namibia: Ministry of Education. (2010). Namibia Senior Secondary Certificate Grade 12 Ordinary level Mathematics syllabus.

Namibia: Ministry of Education. (2010). The National Curriculum for Basic Education. Okahandja: NIED.

National Council for Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics.* Reston, VA: NCTM.

Natsheh, I., & Karsenty, R. (2014). Exploring the potential role of visual reasoning tasks among inexperienced solvers. ZDM—The International Journal on Mathematics Education, 46(1) (this issue). doi:10.1007/s11858-013-0551-1.

Olkun, S., Sinoplu, N., B. & Deryakulu, D. (2005). Geometric explorations with dynamic geometry applications based on van Hiele levels. *International journal of for mathematics teaching and learning,* 1-12.

Owen, G. S. (1999). HyperVis Teaching Scientific Visualisation Using Hypermedia. ACM SIGGRAPH Education Committee. http://www.siggraph.org/education/materials/HyperVis/hypervis.htm.

Owens, K. D., & Clements, M. K. (1998). Representations in spatial problem solving in the classroom. *The journal of mathematical behavior*, *17*(2), 197-218.

Pape, S. S. & Tchoshanov, M. (2001). The role of representations in developing mathematical understanding. Theory into practice, 40, 118–127. doi: 10.12071515430421tip4002-6.

Pape, S. J., & Wang, C. (2003). Middle school children's strategic behavior: classification and relation to academic achievement and mathematical problem solving. *instructional science*, *31*(6), 419-449.

Phothongsunan, S. (2010). "Communication strategies by Thai University students in English language learning." proceedings of the 2nd international conference on language and communication (iclc 2010) "dynamism of language and communication in society. national institute of development administration (nida), bangkok, thailand. 217-227.

Presmeg, N. C. (1986). Visualisation and mathematical giftedness. *educational studies in mathematics*, *17*(3), 297-311.

Presmeg, N. C. (1992). Prototypes, metaphors, metonymies and imaginative rationality in high school mathematics. Educational Studies in Mathematics, 23, 595–610.

Presmeg, N., C. & Nenduradu, R. (2005). An investigation of preservice teachers' use of representations in solving algebraic problems involving exponential relations. In H. L. Chick & J. I. Vincent (Eds.). *Proceedings of the 29th conference of the international group for the Psychology of Mathematics Education* (pp. 105-112). Melbourne: *PME*

Presmeg, N., C. (2006). Research on visualisation in learning and teaching mathematics. in A. Gutierrenz & P. Boero (Eds.). Handbook of research on the Psychology of Mathematics Education: Past, present and future (pp.205-235).

Presmeg, N. C. (2013). An investigation of relationships between students' mathematical problem-posing abilities and their mathematical content knowledge. *Educational Studies in Mathematics*, *83*(1), 117-132.

Rahim, M. H. & Siddo, R. (2009). The use of visualisation for learning and teaching mathematics.

Rahin, A. A. (2007). Determining the problem solving strategies for the students aged 13-14 years. Master thesis. Balikesir University, institute of natural sciences, Balikesir.

Reams, P. & Twale, T. (2008). The promise of mixed methods: discovering conflicting realities in data.

Rivera, F. (2013). teaching and learning patterns in school mathematics: psychological and pedagogical considerations. Dordrecht: Springer.

Rivera, F. D., Steinbring, H., & Arcavi, A. (2014). Visualisation as an epistemological learning tool: an introduction. *ZDM—The International Journal on Mathematics Education*, *46*(1), 1-2.

Rodd, M. (2010). Geometrical visualisation–epistemic and emotional. *For the learning of mathematics*, *30*(3), 29-35.

Rösken, B., & Rolka, K. (2006). A picture is worth a 1000 words – the role of visualisation in mathematics learning. In *proceedings 30th conference of the international group for the Psychology of Mathematics Education* (vol. 4, pp. 457-464).

Royal Society/ Joint Mathematical Council (2001), *Teaching and Learning Geometry 11-19*. London: Royal Society/Joint Mathematical Council.

Samkoff, A., Lai, Y. & Webber, K. (2012). On the different ways that mathematics use diagrams.

Schneider, M., & Stern, E. (2010). The developmental relations between conceptual and procedural knowledge: A multimethod approach. *developmental psychology*, *46*(1), 178.

Schoenfeld, A. H. (1992). Learning to Think Mathematically: Problem Solving, Metacognition, and Sense-making Mathematics. Grouws, D. (Ed). *Research on Mathematics Teaching and Learning:* 334–370. Macmillan, New York. USA.

Schrenko, L. (1994). *Structuring a Learner-Centered School*. IRI/Skylight Publishing, Inc., 200 East Wood Street, Suite 274, Palatine.

Serpil, K., Cihan, K., A., Sabri, I., A., & Ahmet, I. (2002). The Role of Visualisation Approach on Student's Conceptual Learning. Ataturk University, 25240 – Erzurum.

Shaw, J., M. (2002). Manipulatives enhance the learning of mathematics, 1-4. From www.eduplace.com/ state/pdf/author/shaw.pdf. (retrieved on 6 may 2015).

Singh, R. I. (2007). An investigation into learner understanding of the properties of selected quadrilaterals using manipulatives in a grade eight mathematics class. Master of education thesis. University of kwazulu-natal: Durban.

Steenpass, A., & Steinbring, H. (2014). Young students' subjective interpretations of mathematical diagrams—elements of the theoretical construct "frame-based interpreting competence". ZDM—The International Journal on Mathematics Education, 46(1) (this issue). doi:10.1007/s11858-013-0544-0.

Stylianou, D. A., & Silver, E. A. (2004). The role of visual representations in advanced mathematical problem solving: An examination of expert-novice similarities and differences. *Mathematical thinking and learning*, *6*(4), 353-387.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. Educational studies in mathematics, 12, 151-169.

Tall, D. (1991). *Advanced mathematical thinking* (Vol. 11). Springer Science & Business Media

Taylor, P. C., Taylor, E. and Luitel, B. C. (2012). Multi-Paradigmatic transformative research as for teacher education. an integral perspective. In K. G. Tobin, B. J. Fraser and Mac Robbie (Eds.). Second international hand book of science education (pp.373-387). Dordrecht, the Netherlands: Springer.

Tellis, W. (1997). Introduction to case study. the qualitative report, 3, (2), http://www.nova.edu/ssss/qr3-2/tellis.html.

Usiskin, Z. (2012). The shapes of geometry and their implications for the school geometry curriculum. In *12th international congress on mathematical education*.

Van der Walle, J., Karp, K. S., & Bay-Williams, J. S. (2014). *Elementary and middle school mathematics teaching developmentally* (8th ed.). Newyork: Pearson new international edition.

Van Garderen, D. (2006). Social visualisation, visual imagery, and mathematical problem solving of students with varying abilities. Journal of learning disabilities, 3(6), 496 – 506. doi: 10, 1177/00222194060390060201.

Van Garderen, D., Scheuermann, A. & Poch, A. (2014). Challenges students identified with a learning disability and as a high arching experience when using diagrams as a visualisation tool to solve mathematics word problems ZDM. The international journal on mathematics education, (1).

Van Hiele, P. M. (1986), *Structure and insight: A theory of mathematics education.*

.

Van Hiele, P. M. (1999). Developing geometric thinking through activities that begin with play. *teaching children mathematics*, *5*(6), 310.

Verschaffel, I., de Corte, E., Lasure, S., van Vaerenbergh, G., Gogaerts, H., & Ratinckx, E. (1999). Learning to solve mathematical application problems: a design experiment with fifth graders. *mathematical thinking and learning*, *1*(3), 195-229.

Wahyuni, D. (2012). The research design maze: Understanding paradigms, cases, methods and methodologies. *Journal of applied Management Accounting Research, 10*, 69-80.

Waisel, L. B., Wallace, W. A., & Willemain, T. R. (1997). *Using diagrammatic representations in mathematical modeling:* The sketches of expert modelers. In Technical Report of the

AAAI 1997 Fall Symposium on Reasoning with Diagrammatic Representations. Troy: Massachusetts Institute of Technology.

Wheatley, G. H. (1991). Enhancing mathematics learning through imagery. *The Arithmetic Teacher*, *39*(1), 34.

Wheatley, G., H., & Brown, D. (1994). The construction and re-presentation of images in mathematical activity. *proceedings of PME*, *18*(1), 81.

Whitely, W. (2004). Visualisation in mathematics. www.math.yorku.

Yilmaz, K. (2007). Uurvey about effect of epistemological beliefs and beliefs about mathematical problem solving on problem solving process. master thesis, Marmara University. Institute of Educational sciences, Istanbul.

Zaidah, Z. (2007). Case study as a research method. *Journal Kemanusiaan*, (9), 1-6.

Zakaria, E., & Zaini, N. (2009). Conceptual and procedural knowledge of rational numbers in trainee teachers. *European journal of social sciences*, *9*(2), 202-217.

Zhang, Y. (2009). Classroom discourse and student learning. *Asian social science*, *4*(9), 80.

Zimmermann, W., & Cunningham, S. (1991). Editor's introduction: What is mathematical visualisation. *Visualisation in teaching and learning mathematics*, 1-7.

Zodik, I., & Zaslavsky, O. (2007). Is a visual example in geometry always helpful? In *proceedings of the 31st conference of the international group for the psychology of mathematics education* (vol. 4, 265-272).

APPENDICES

Appendix 1: Consent from the Director

Ministry of Education, Arts and Culture	**REPUBLIC OF NAMIBIA** **ZAMBEZI REGIONAL DIRECTORATE OF EDUCATION, ARTS AND CULTURE** **OFFICE OF THE REGIONAL DIRECTOR**	Zambezi Regional Council

Tel No.: (066) 261962/261931
Fax No.: (066) 253187

Private Bag 5006
Katima Mulilo

Enquiries: [redacted]
Reference No: 11/1/1

07 May 2015

PO Box 673
Ngweze
Namibia

Attention: Mr Brian S Kabuku

RE: PERMISSION TO CARRY OUT AN EDUCATION RESEARCH STUDY WITH LEARNERS AT [redacted] SENIOR SECONDARY SCHOOL: YOURSELF

Your letter to the office of the Regional Director: Zambezi Region date 5 May 2015 with the caption Permission to carry out an Education Research Study with Learners at [redacted] Senior Secondary School was received.

The Ministry of Education, Zambezi Region hereby would like to thank you for willingness to conduct a research at Sanjo Senior Secondary School. Kindly be informed that approval is granted to you to conduct your research as requested, but let me draw your attention to the following aspects: **NOTE!**

a) The granted approval should not disrupt the normal teaching and learning at those schools you intend visiting.

b) Ministry of Education, Zambezi Region hereby would like to request you to share your findings with the Directorate.

By copy of this letter the Inspector of Education is notified accordingly of your presence at the school.

I trust and hope you will find this in order.

DIRECTORATE OF EDUCATION
Zambezi Regional Council
2015-05-08
Director

[redacted]
REGIONAL DIRECTOR: EDUCATION, ARTS AND CULTURE

Appendix 2: Consent from the School principal

REPUBLIC OF NAMIBIA

MINISTRY OF EDUCATION, ARTS AND CULTURE
[redacted]
BUKALO CIRCUIT
ZAMBEZI REGION

[redacted] **SENIOR SECONDARY SCHOOL**
P O BOX 6059
BUKALO
TEL: 0926466252865
TELE-FAX 0926466[redacted]2892

Enquiries: [redacted]

[redacted]

To : **Mr B.S. Kabuku**
P.O. Box 673
Ngweze

Dear Sir

RE: Permission to carry out an Education Research on the school premises and use of some selected grade 11 learners

Permission is hereby granted to your request to carry out some research work at [redacted] Senior Secondary School for your Education Master's degree programme in Mathematics. The approval is based on the following conditions.

- Your research activities should not interfere with the school activities, this means that your research should not be carried during the normal teaching times from 06:30 - 14h45 PM.
- Participants may not leave the school activities due to the commitments by the research activities.
- Consent letters from the participants should be submitted to the Principal's office before the commencement of your research work.

Failure to observe the above given conditions will result in the cancellation of this permission.

Thank you

Yours Faithfully

[redacted]
Principal

SENIOR SECONDARY SCHOOL
P.O.BOX 6059
BUKALO
[redacted]
The Principal
Tell: 066- 252892
REPUBLIC OF NAMIBIA

Appendix 3: Consent from the learners

Learners' consent

By signing this form, I provide consent to participate in this research project from the 30th of March to the 30th of June 2016. The aims and objectives of the research project have been explained to me. I am fully aware of what is expected of me and I know that my participation is voluntary. For this reason, I volunteer to participate in the project. It was further explained to me that I may withdraw from the study at any time without negative or undesirable consequences attached.

	Learner name	*Grade*	*Signature*	*Date*
1	[redacted]	11 A	[illegible]	31 march 2016
2	[redacted]	11 A	[illegible]	31 March 2016
3	[redacted]	11A	[illegible]	31 march 2016
4	[redacted]	11:B	[illegible]	31 March 2016
5	[redacted]	11A	[illegible]	31 March 2016
6	[redacted]	11C	[illegible]	31 march 2016
7	[redacted]	11D	[illegible]	31 march 2016
8	[redacted]	11D	[illegible]	31 March 2016

Appendix 4: GVT tasks.

RHODES UNIVERSITY
Where leaders learn

GEOMETRY VISUALISATION TASKS

Read these instructions first:

- Answer all questions in the spaces provided
- Show all working clearly
- Where possible make use of sketches and diagrams to show your working using a pencil
- You are expected to talk through your thought processes with me as you interact with each task.

Name: ………………………………..

Grade: ……………………………….

Date : ………………………………

Task 1

A man was looking for his cattle. He walked 3km due north. Then he returned and walked roughly south-east for 5km. There, he found his cattle and realised that he can get home by walking directly west.

(a) What kind of triangle did the man walk?

(b) How far did the man walk to reach home after he found his cattle?

Task 2

You can place exactly 9 and 16 small squares of the same size in the two squares formed by the two shorter sides of a right-angled triangle respectively. How many small squares of the same size can you place in the square formed by the third side of this triangle.

Task 3

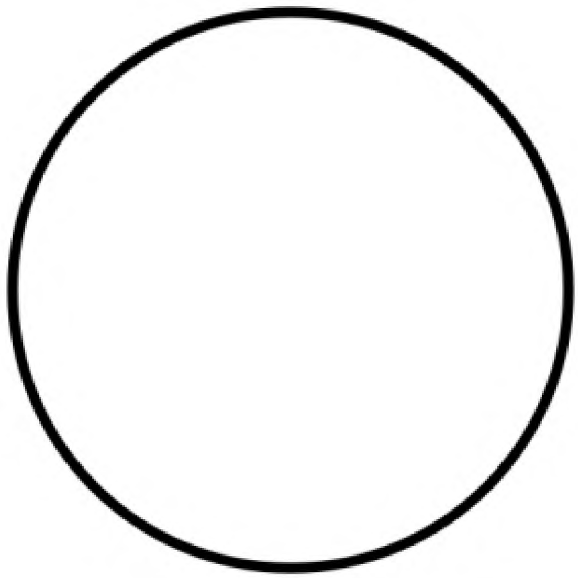
A rhombus BDEF is inscribed in a triangle ABC. Its diagonal BE is perpendicular to the side AC of the triangle. Prove that the triangle ABC is an isosceles triangle.

Task 4

Find and identify as many types of triangles as you can that have a perimeter 12 units.

Task 5

Explain and show how you would find the centre of this circle.



Task 6

A triangle has a perimeter of 50cm. If two of its sides are equal and the third side is 5cm more than the equal sides, what is the length of the third side?

Task 7

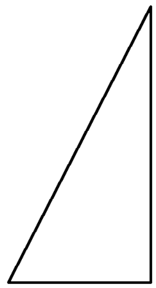
Explain how you would transform a circle into an approximate parallelogram.

Task 8

How many quadrilaterals can you find whose perimeter and area are numerically equal.

Task 9

The right-angled triangle below has an area of $24cm^2$.



Find and draw as many shapes as possible that also have an area of $24cm^2$.

Task 10

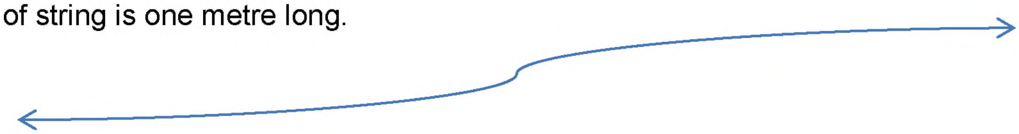
The diameter of a cylindrical tin of fish is 5cm and its height is 8cm. How many tins of fish will fit in a rectangular box whose length is 30cm, width 15cm and height 16cm.

Task 11

A volleyball court is 40 metres long and 20 metres wide. Sketch this volley court and then draw as many lines (curved or straight) that will divide the court into two halves. In each case what is the area of each half?

Task 12

A piece of string is one metre long.



You can use it to make different geometric shapes. With this piece of string construct as many geometric shapes as possible.

(a) What is the perimeter of each shape?

Appendix 5: Post-GVT interview questions

1. When you are asked a geometrical question, do you see something in your mind? Give me an example of such a situation and describe to me what you see or imagine.

   OK – now I will give you a geometry situation and I want you to describe to me what you see in your mind. A three sided figure with equal length has perimeter of 9cm.

   OK – now draw this for me.

2. How does what you see or imagine in your mind help you answer geometrical questions?

3. Does what you see or imagine in your mind help you to draw a picture that represents the geometrical problem at hand? Explain your answer.

4. How do you find the pictures that you draw or scribble on paper when answering geometrical problems helpful in:

   (i) Understanding the spatial relations pertinent to the problem that you are trying to solve?

   (ii) Constructing a visual representation of the problem?

   (iii) Using visualisation to solve the problems?

   (iv) Connecting the present task to previously solved problems?

   (v) Linking the solution and the visual representation?

5. Tell me how using a picture helps or does not help you solve geometric questions.

6. What did you like with the approach of using visualisation in learning geometry? Explain your answer.

7. Tell me any other method/way that you know can help you solve geometrical problems?
8. What did you not like about the use of visualisation in learning? Explain your answer.