TWISTORS IN CURVED SPACE

by

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Chapter 1. Introduction

During the past decade, the theory of twistors has been introduced and developed, primarily by Professor Roger Penrose, as part of a long-term program aimed at resolving certain difficulties in present-day physical theory. These difficulties include, firstly, the problem of combining quantum mechanics and general relativity, and, secondly, the question of whether the concept of a continuum is at all relevant to physics.

Most models of space-time used in general relativity employ the idea of a manifold consisting of a continuum of points. This feature of the models has often been criticised, on the grounds that physical observations are essentially discrete in nature; for reasons that are mathematical, rather than physical, the gaps between these observations are filled in a continuous fashion (see, for example, Schrödinger (1), pp.26-31). Although analysis (in its generally accepted form) demands that quantities should take on a continuous range of values, physics, as such, does not make such a demand.

The situation in quantum mechanics is not all that much better since, although some quantities such as angular momentum can only take on certain discrete values, one still has to deal with the complex continuum of probability amplitudes. From this point of view it would be desirable to have all physical laws expressed in terms of combinatorial mathematics, rather than in terms of (standard) analysis.

As far as the first difficulty mentioned is concerned, it would appear as if quantum mechanics and general relativity cannot satisfactorily be combined by merely applying standard quantization techniques to gravitational phenomena (Anderson (1)). One aspect of this is that it is not really sufficient for the uncertainty principle to "smear out" the light cones of points without "smearing out" the points themselves. Thus again the concept of a continuum of points as primary objects in space-time becomes undesirable.

One way of avoiding these difficulties would be to choose a physical quantity which takes on discrete values as the primary object out of which to construct space-time; points in space-time would then become secondary or derived objects. It would be desirable for such a quantity to have the concept of space direction built into it, if a meaningful type of space-time was to be constructed. The most obvious choice satisfying these criteria is total angular momentum.

It will be seen that a twistor can be visualised (more or less) as a zero-rest-mass particle in free motion . The concept of a twistor is conformally invariant and is therefore ideal for describing physical phenomena which posses conformal invariance (for example, zero-rest-mass systems). On the other hand, there is the drawback that twistors, as such, do not provide a description of phenomena which break conformal invariance, such as nonzero-rest-mass and curved space-time. Consequently, one has to adopt the viewpoint that zero-rest-mass phenomena and conformal invariance are of primary importance in physics and that the conformal-symmetry-breaking part of physics can be derived from this primary structure. There is some evidence to support this idea; for example, nonzero-rest-mass fields can be regarded as arising from the interaction between zero-rest-mass fields (Penrose (5)).

It was pointed out above that the concept of the complex continuum should, if possible, be avoided. In flat space-time, however, it turns out that twistor space possesses a complex structure and that holomorphic functions play an important role in twistor theory. Thus complex numbers seem tied up not only with quantum mechanics, but also with the structure of space-time itself. Penrose is of the opinion that it may eventually be possible to replace holomophic functions with concepts which are purely combinational (Penrose and MacCallum (1)), but this possibility is very much an open question.

The problem of extending twistor theory to conformally curved space-times presents difficulties which are essentially due to the lack of points as primary objects. In the most logical extended theory, namely that of global twistors, the complex structure breaks down - in fact, it appears to "shift" when viewed from distant regions of space-time. All that is left is a weaker, symplectic structure. This symplectic structure, however, ties in very neatly with the Hamiltonian and Hilbert space structure used in quantum mechanics. These matters will not be discussed in this essay; for a detailed description of them the reader may refer to Penrose (5), Crampin and Pirani (1), and Penrose (7). The other known curved-space twistor formalisms ,namely that of asymptotic twistors and their generalisation, hypersurface twistors, will be dealt with in detail.

The discussion of twistor theory presented here derives primarily from Penrose (4) and Penrose and MacCullum (1); the notation of the former paper has been changed so as to conform with that of the latter.

Formulae will be numbered in the form (c.s.n), where c denotes the chapter and s the section. A defining relationship will be indicated by the symbol := (or =:), the colon being written on the same side of the equation as the quantity being defined.

CHAPTER 2. PRELIMINARIES.

§ 2.1. THE ABSTRACT INDEX NOTATION.

For a more comprehensive discussion of the material in this section, the reader may refer to Penrose(1), pp. 135 - 141.

The ranges of the indices that will be used are as follows:

a,b,... range over 0,1,2,3;

A,B,... range over 0,1;

A',B',... range over 0',1';

$\alpha,\beta,\ldots$ range over 0,1,2,3.

Symmetrization and skew-symmetrization will be denoted in the usual way by enclosing the relevant indices in, respectively, round brackets and square brackets. The Einstein summation convention will be used throughout, and will apply to actual indices and to abstract indices in the sense of contraction.

A symbol such as ξ^a is usually taken to denote a component of a vector in some particular basis frame; this interpretation forces one to avoid the use of indices when discussing frame-independent concepts. The great disadvantage of this restriction is that some algebraic operations (such as contraction) can be expressed very usefully in terms of indices, even though the operations are in fact completely frame-independent.

The convention that will be used in this essay is the <u>abstract index notation</u>, whereby tensors and spinors are denoted by symbols such as $\xi^{ab..d}{}_{ef..g}$ and $\psi^{A..BA'..B'}{}_{C..DC'..D'}$ respectively; the indices are to be regarded as abstract, in that they do not have to be interpreted numerically in a particular basis frame. Where an index has to be interpreted numerically, it will be written with a tilde underneath it. Thus ξ^a denotes a cotravariant vector (a frame-independent concept) and $\xi^{\underset{\sim}{a}}$ denotes the $\underset{\sim}{a}$-th component, with respect to a particular basis frame, of the vector ξ^a.

Two symbols with the same kernel letter and the same form of indexing (for example ξ^a and ξ^b) are regarded as denoting the <u>same</u> vector; the two symbols must, however, be regarded as <u>different</u> entities. This distinction is important when considering algebraic relations and operations, which will now be considered in brief.

The notion of equality is only defined if the same indices appear on both sides of the equation (for example, $\xi^a = \eta^a$ is permissible, whereas $\xi^a = \eta^b$ is not). Indices that are summed over are disregarded; in other words, it is permissible to write

$$\xi^a = \eta^{ab} \mu_b .$$

A similar criterion applies to addition: an expression such as $\xi^a + \eta^a$ is meaningful, but the expression $\xi^a + \eta^b$ is not. In a tensor or spinor product, however, the indices are <u>not</u> allowed to be the same, so that $\xi^a \eta^b$, say, is allowed, whereas $\xi^a \eta^a$ is not. Products are also required to be commutative; for example, the identity

$$\xi^a \eta^b = \eta^b \xi^a$$

must hold for all ξ^a and all η^b. Since a tensor product is non-commutative, the reason why an expression like $\xi^a \eta^a$ is not allowed becomes clear: the indices must be kept distinct so as to label the various factors in the product uniquely.

§ 2.2. SPACE-TIME.

The following definitions and formulae cover the details that are required for this essay. For further details the reader is referred to Hawking and Ellis(1), pp. 10 - 44 ; where this reference is inadequate other references will be given.

By a space-time is meant a pair (M,g_{ab}), where M is a connected, Hausdorff, paracompact, 4-dimensional manifold of class C^{∞} and g_{ab} is a C^{∞} metric of signature -2 on M. A space-time with boundary is defined in exactly the same way, except that the word "manifold" is replaced by "manifold with boundary".

Minkowski space-time is the manifold R^4 with the natural pseudo-Cartesian coordinates (x^0,x^1,x^2,x^3) and the metric

$$\eta_{ab} := \mathrm{diag}(1,-1,-1,-1), \tag{2.2.1}$$

i.e.

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2. \tag{2.2.2}$$

Given a space-time (M,g_{ab}), the metric g_{ab} determines a unique torsion-free connection on M; this connection in turn determines a covariant derivative operation. The covariant derivative operator will be denoted by ∇_a. In Minkowski space-time, the operator ∇_a is identically equal to the operator $\partial/\partial x^a$.

A conformal rescaling on M is a replacement of g_{ab} by

$$\hat{g}_{ab} = \Omega^2 g_{ab}, \tag{2.2.3}$$

where Ω is a C^{∞} real-valued function on M.

A metric is conformally flat if it is a conformal rescaling of a flat metric (a flat metric being one which is derived from the Minkowski metric η_{ab} by means of a coordinate transformation).

Let (M,g_{ab}) and $(\hat{M},\hat{g}_{ab})$ be two space-times. A C^{∞} diffeomorphism ϕ of M onto $\hat{M}$ is called a conformal mapping if the metric on $\hat{M}$ induced by ϕ from g_{ab} is a conformal rescaling of $\hat{g}_{ab}$.

A physical theory is said to be conformally invariant if conformal weights can be attached to the quantities and operators appearing in the field equations, in such a way that the equations are form-invariant under a conformal rescaling of the metric.

Let (M, η_{ab}) be Minkowski space-time. A procedure for obtaining a compactification of M will now be sketched; more details may be found in Penrose(1), Penrose(2) and Penrose(3).

There exists a space-time (M', g_{ab}) with boundary and a conformal mapping which maps M onto the interior of M' (this mapping is described in Penrose(1), p. 175). The boundary of M' (denoted by ∂M) can be interpreted by studying the straight lines in M. All timelike straight lines in M become curves having as past endpoint a point I^- on ∂M and as future endpoint a point I^+ on ∂M. All spacelike straight lines in M become closed curves through a point I^0 in M. Let $\mathscr{I}^-$ be the past null cone of I^0 and $\mathscr{I}^+$ the future null cone of I^0. Then it can be shown that $\mathscr{I}^-$ is also the future null cone of I^- and $\mathscr{I}^+$ the past null cone of I^+. In addition,

$$\partial M = \mathscr{I}^- \cup \mathscr{I}^+ \cup I^- \cup I^0 \cup I^+.$$

The null straight lines in M become null geodesics originating on $\mathscr{I}^-$ and terminating on $\mathscr{I}^+$.

Strictly speaking, the space M' as described above is not a manifold with boundary, because the conformal curvature becomes infinite at the points I^0, I^-, and I^+ (Hawking and Ellis(1), p.222). These points should therefore be deleted from M' and from ∂M when a strict definition is given; the boundary of M' will then consist of the two hypersurfaces $\mathscr{I}^-$ and $\mathscr{I}^+$.

Two null geodesics in M' terminate at the same point of $\mathscr{I}^+$ if and only if the two corresponding null lines in M lie in the same null hyperplane (Penrose(1), p.179). Also, this condition is necessary and sufficient for two null geodesics in M' to originate at the same point of $\mathscr{I}^-$. Consequently, to each null hyperplane in M there corresponds a unique point on $\mathscr{I}^+$ and a unique point on $\mathscr{I}^-$; this permits the identification of $\mathscr{I}^-$ with $\mathscr{I}^+$, giving a single hypersurface $\mathscr{I}$ at infinity. So each null hyperplane in M acquires exactly one point on $\mathscr{I}$ in M'. The points I^+, I^-, and I^0 can also be identified: the resulting single point on ∂ M is denoted by I.[1]

The hypersurface $\mathscr{I}$ can be regarded as the null cone of the point I, and will occasionally be called the <u>null cone at infinity</u>. It possesses the important property that each of its generators consists of a set of points (at infinity) corresponding to a system of <u>parallel</u> null hyperplanes in M (Penrose(4)). Conversely, to each such system of parallel null hyperplanes corresponds a unique generator of the null cone at infinity.

Thus M is compactified by the addition of a closed null cone at infinity and may be visualised as in figure 1. The two null hypersurfaces $\mathscr{I}^-$ and $\mathscr{I}^+$ are identified along opposite generators and the space bounded by the two hypersurfaces represents M. Note that this picture is inaccurate in the "equatorial" region, since I^0 is in fact a single point.

In space-times which are <u>not</u> Minkowski, it may still be possible to define a conformal infinity. A space-time (M, g_{ab}) is said to be <u>asymptotically simple</u> if

(a) there exists a space-time $(\hat{M}, \hat{g}_{ab})$ with boundary (denoted by ∂M) and a C^∞ diffeomorphism ϕ mapping M onto the interior of $\hat{M}$;

(b) There exists a C^∞ function Ω on $\hat{M}$, positive on the interior of $\hat{M}$ and zero on ∂M, such that

$$\hat{g}_{ab} = \Omega^2 g'_{ab} \qquad (2.2.4)$$

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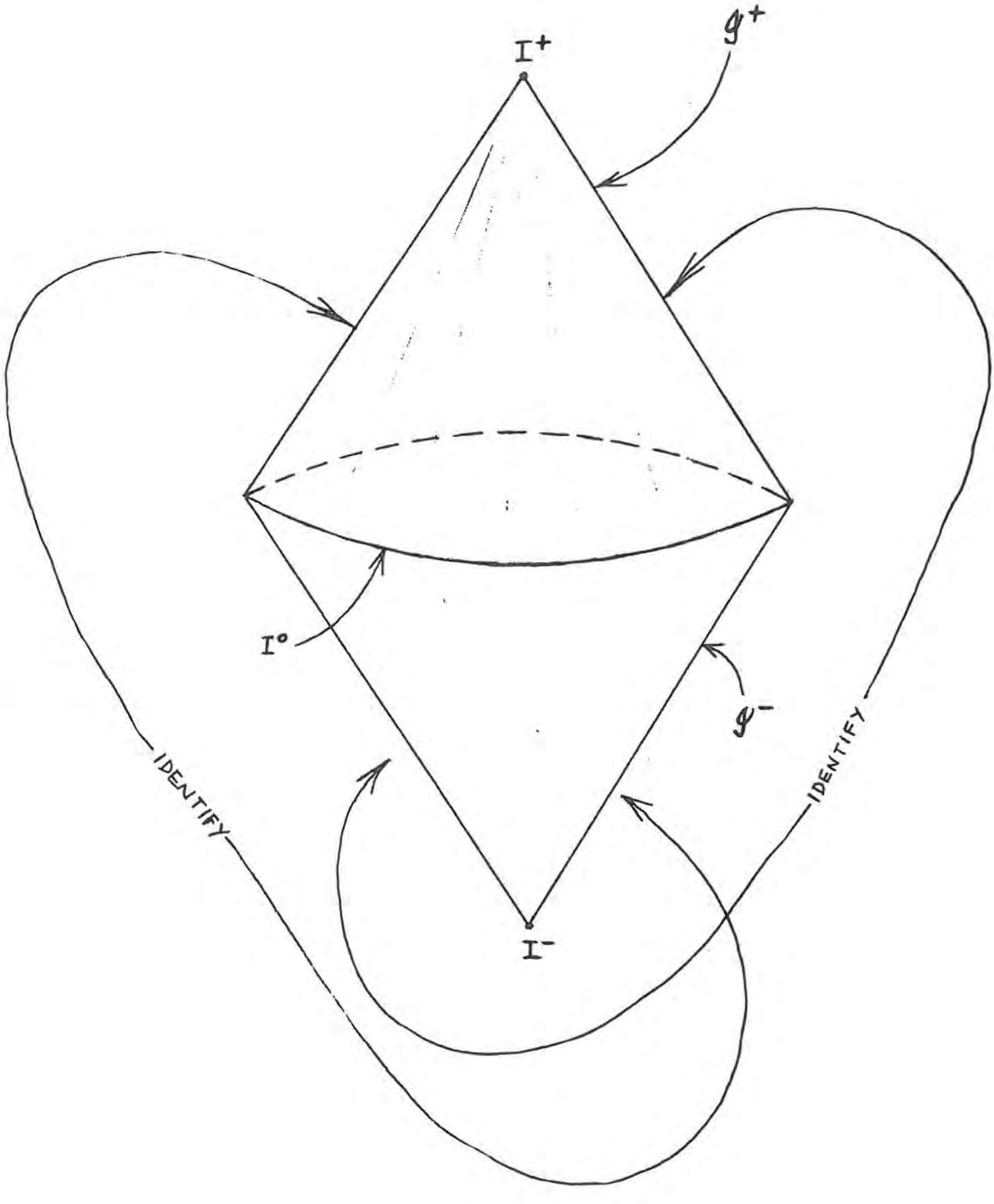
[1] Note that this procedure, although possible in Minkowski space-time, is not necessarily possible in more general space-times.

Fig. 1. Compactified Minkowski Space-time.



on $\phi(M)$, where g'_{ab} is the metric on $\phi(M)$ induced by ϕ from g_{ab};

(c) on $\mathcal{M}$, $\tilde{\nabla}_a \ne 0$; and

(d) every null geodesic in M has two endpoints on $\mathcal{M}$.

An asymptotically simple space-time is said to be asymptotically flat if, in some neighbourhood of $\mathcal{M}$,

(a) Einstein's vacuum field equations (without cosmological term) hold; and

(b) the energy-momentum tensor T_{ab} is zero.[1]

As was the case with Minkowski space-time, the boundary ∂M of an asymptotically flat space-time (M, g_{ab}) can be written as

$$\partial M = \mathscr{I}^- \cup \mathscr{I}^+$$

(Hawking and Ellis(1), p.222), where $\mathscr{I}^-$ and $\mathscr{I}^+$ are disconnected null hypersurfaces at infinity. Every null geodesic in M has a past endpoint on $\mathscr{I}^-$ and a future endpoint on $\mathscr{I}^+$, but $\mathscr{I}^-$ and $\mathscr{I}^+$ cannot, in general, be identified as in the Minkowski case.

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[1] For a less restrictive definition of asymptotic flatness, see Penrose(1), p.184.

§ 2.3. SPINORS.

The most convenient formalism for discussing zero-rest-mass phenomena and conformal invariance is that of two-component spinors, some details of which are outlined in this section. For more details the reader is referred to Penrose(1), pp.141-160 and Pirani(1), pp.305-330.

The correspondence between tensors and spinors is provided by the Infeld-van der Waerden symbols $\sigma_a^{AA'}$. Thus if x^a is a vector and $x^{AA'}$ the corresponding spinor, then

$$x^{AA'} = \sigma_a^{AA'} x^a. \tag{2.3.1}$$

In Minkowski space-time one may use the Pauli spin matrices for the σ's. The correspondence (2.3.1) then becomes

$$\begin{pmatrix} x^{00'} & x^{01'} \\ x^{10'} & x^{11'} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + x^1 & x^2 + ix^3 \\ x^2 - ix^3 & x^0 - x^1 \end{pmatrix}, \tag{2.3.2}$$

where $i = \sqrt{-1}$.[1]

The use of the abstract index notation permits the identification of each tensor index with a pair of spinor indices, one unprimed and one primed. Thus given, say, a vector x^a, the spinor corresponding to it will be denoted by a symbol with the (abstract) contravariant indices AA'. It is therefore permissible to write down an identity such as

$$x^a = x^{AA'}, \tag{2.3.3}$$

where the equality sign is to be interpreted in terms of the correspondence (2.3.2) or (2.3.1).

The complex conjugate of a spinor π_A will be denoted by $\bar{\pi}_{A'}$. The

[1] The correspondence (2.3.2) will be used throughout unless otherwise stated.

spinors ε_{AB}, $\varepsilon_{A'B'}$, ε^{AB} and $\varepsilon^{A'B'}$ are used to raise and lower indices; each of the ε's has coordinate representation

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and they are linked to the Minkowski metric by

$$\eta_{ab} = \varepsilon_{AB}\,\varepsilon_{A'B'}\,. \tag{2.3.4}$$

Note that $\varepsilon_{A'B'}$ should be written as $\bar{\varepsilon}_{A'B'}$, but the bar will be omitted.

A vector x^a is real if and only if its spinor equivalent is Hermitian, i.e. if and only if

$$x^{AA'} = \overline{x^{A'A}} = \bar{x}^{AA'}. \tag{2.3.5}$$

Identities that will be found very useful are

$$\xi_{AB} - \xi_{BA} = \varepsilon_{AB}\,\xi_C{}^C \tag{2.3.6a}$$

and

$$\xi^{AB} - \xi^{BA} = \varepsilon^{AB}\,\xi_C{}^C, \tag{2.3.6b}$$

for any 2-spinors ξ_{AB} and ξ^{AB}.

A spinor π^A has a geometrical interpretation, up to sign, as a null flag, with its flagpole pointing in the (future-pointing) null direction

$$p^a := \pi^A\,\bar{\pi}^{A'} \tag{2.3.7}$$

and its flag representing a 2-plane element given by the bivector

$$F_{ab} := \varepsilon_{AB}\,\bar{\pi}_{A'}\,\bar{\pi}_{B'} + \varepsilon_{A'B'}\,\pi_A\,\pi_B. \tag{2.3.8}$$

The spinor operator corresponding to the covariant derivative operator ∇_a is written as $\nabla_{AA'}$. In Minkowski space-time, as was noted above, the vector covariant derivative operator is $\partial/\partial x^a$; its spinor equivalent, using the correspondence (2.3.2), is clearly $\partial/\partial x^{AA'}$ (since the relevant σ's, being constant, commute with derivative operators).

Consider now the effects on spinors and spinor operators of the conformal rescaling (2.2.3).

The ε's will be taken to transform as follows:

$$\begin{aligned} \hat{\varepsilon}_{AB} &= \Omega\, \varepsilon_{AB} \; ; \\ \hat{\varepsilon}_{A'B'} &= \Omega\, \varepsilon_{A'B'} \; ; \\ \hat{\varepsilon}^{AB} &= \Omega^{-1}\, \varepsilon^{AB} \; ; \\ \hat{\varepsilon}^{A'B'} &= \Omega^{-1}\, \varepsilon^{A'B'} . \end{aligned} \tag{2.3.9}$$

The transformation law for the operator $\nabla_{AA'}$ must be such that the covariant derivative axioms are satisfied (Penrose(1), p.141); this criterion is met by the following definitions.

If ϕ is a scalar function, take

$$\hat{\nabla}_{AA'}\, \phi = \nabla_{AA'}\, \phi . \tag{2.3.10}$$

For $\nabla_{AA'}$ acting on spinors of valence one, take

$$\begin{aligned} \hat{\nabla}_{AA'}\, \pi_B &= \nabla_{AA'}\, \pi_B - T_{BA'}\, \pi_A \; ; \\ \hat{\nabla}_{AA'}\, \pi^B &= \nabla_{AA'}\, \pi^B + \varepsilon_A{}^B\, T_{CA'}\, \pi^C \; ; \\ \hat{\nabla}_{AA'}\, \eta_{B'} &= \nabla_{AA'}\, \eta_{B'} - T_{AB'}\, \eta_{A'} \; ; \\ \hat{\nabla}_{AA'}\, \eta^{B'} &= \nabla_{AA'}\, \eta^{B'} + \varepsilon_{A'}{}^{B'}\, T_{AC'}\, \eta^{C'} \; , \end{aligned} \tag{2.3.11}$$

where

$$T_{AA'} := \Omega^{-1}\, \nabla_{AA'}\, \Omega . \tag{2.3.12}$$

For $\nabla_{AA'}$ acting on spinors of higher valence, treat each index in turn according to the scheme of (2.3.11) using the Leibnitz rule.

The remainder of this section is devoted to stating certain results concerning the curvature tensor and its representation in terms of spinors. More comprehensive treatments may be found in the references mentioned above, and also in Penrose and MacCallum(1).

Suppose that the curvature tensor R_{abcd} satisfies the sign convention

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\, V_d =: R_{abcd}\, V^c \tag{2.3.13}$$

and that R_{ab} and R are defined by

$$\begin{aligned} R_{ab} &:= R^c{}_{acb} \, , \\ R &:= R^a{}_a . \end{aligned} \tag{2.3.14}$$

A spinor representation of R_{abcd} is given by

$$R_{abcd} = \Psi_{ABCD}\, \varepsilon_{A'B'}\, \varepsilon_{C'D'} + \varepsilon_{AB}\, \varepsilon_{CD}\, \bar{\Psi}_{A'B'C'D'} +$$
$$+ \varepsilon_{A'B'}\, \Phi_{ABC'D'}\, \varepsilon_{CD} + \varepsilon_{AB}\, \Phi_{CDA'B'}\, \varepsilon_{C'D'} + \qquad (2.3.15)$$
$$+ 2\,\Lambda(\varepsilon_{AC}\, \varepsilon_{BD}\, \varepsilon_{A'B'}\, \varepsilon_{C'D'} + \varepsilon_{AB}\, \varepsilon_{CD}\, \varepsilon_{A'D'}\, \varepsilon_{B'C'}).$$

The spinor Ψ_{ABCD} is completely symmetric and is related to the Weyl conformal curvature tensor C_{abcd} by

$$\Psi_{ABCD}\, \varepsilon_{A'B'}\, \varepsilon_{C'D'} + \varepsilon_{AB}\, \varepsilon_{CD}\, \bar{\Psi}_{A'B'C'D'} = C_{abcd}. \qquad (2.3.16)$$

Both Ψ_{ABCD} and C_{abcd} vanish if and only if the space-time is conformally flat.

The spinor $\Phi_{ABC'D'}$ and the scalar Λ are both real, in the sense that

$$\bar{\Phi}_{ABC'D'} = \Phi_{ABC'D'}\,,$$
$$\Lambda = \bar{\Lambda}. \qquad (2.3.17)$$

Furthermore, both $\Phi_{ABC'D'}$ and Λ vanish in regions of space-time containing no mass-energy.

Under a conformal rescaling, the spinor Ψ_{ABCD} is invariant and the spinor $P_{ABC'D'}$, defined by

$$P_{ABC'D'} := \Phi_{ABC'D'} - \Lambda\varepsilon_{AB}\, \varepsilon_{C'D'}\,, \qquad (2.3.18)$$

transforms as

$$\hat{P}_{ABC'D'} = P_{ABC'D'} - \nabla_{AC'}\, T_{BD'} + T_{AD'}\, T_{BC'}\,. \qquad (2.3.19)$$

Notice that

$$\bar{P}_{ABC'D'} = P_{ABC'D'} \qquad (2.3.20)$$

by (2.3.17).

CHAPTER 3. TWISTORS IN FLAT SPACE

Throughout this chapter, let (M, η_{ab}) denote Minkowski space-time and (M', g_{ab}) its compactification. The phrase "null straight line" (or "null geodesic") in M or in M' will usually be abbreviated to "null line".

§ 3.1 THE BASIC CONCEPT

The object of this section is to describe the representation of null lines in M (i.e. world-lines of zero-rest-mass particles in M) in terms of twistors. The concepts of momentum, angular momentum and spin will be utilised; details of these may be found in Synge (1), pp.216-227.

Consider a finite system with 4-momentum $P^a \neq 0$, and total angular momentum M^{ab} with respect to the origin 0 ; the tensor M^{ab} is skew-symmetric.

If a displacement x^a is applied to the origin, taking it to $\tilde{0}$, the angular momentum with respect to $\tilde{0}$ is

$$\tilde{M}^{ab} = M^{ab} - 2\, x^{[a} P^{b]} \,. \tag{3.1.1}$$

The momentum vector P^a is invariant under such a change of origin.

Assume firstly that $P^a P_a > 0$. The centre of mass of the system is defined as the locus of origins such that

$$\tilde{M}^{ab} P_a = 0 \,. \tag{3.1.2}$$

Equation (3.1.2) has as its solution a straight line (Synge (1), p.219) and, as can be verified by substitution, this line is given by

$$x^a = M^{ab} P_b \,/\, (P^c P_c) + \lambda P^a \,, \tag{3.1.3}$$

where λ is a real parameter.

The spin tensor H^{ab} is defined by

$$H^{ab} := M^{ab} - 2\, x^{[a} P^{b]} \,, \tag{3.1.4}$$

where x^a is given by (3.1.3). In other words, the spin tensor gives the intrinsic angular momentum of the system about a point on the centre of mass line. The spin s is usually defined in terms of H^{ab} by

$$s := + (\tfrac{1}{2} H^{ab} H_{ab})^{\frac{1}{2}} \quad . \tag{3.1.5}$$

Now consider the case $P^a P_a = 0$, i.e. zero rest mass. Equation (3.1.2) becomes

$$M^{ab} P_a = (x^a P_a) P^b \quad . \tag{3.1.6}$$

The null vector P^a can be represented by a spinor $\pi^{A'}$ (cf.(2.3.7)), according to

$$P^a = \bar{\pi}^A \pi^{A'} \tag{3.1.7}$$

and this representation is unique up to the phase transformation $\pi^{A'} \to e^{i\theta} \pi^{A'}$, with θ real. The skew-symmetric tensor M^{ab} can be represented by a symmetric 2-spinor $\mu^{AB} = \mu^{(AB)}$, such that

$$M^{ab} = \mu^{AB} \varepsilon^{A'B'} + \bar{\mu}^{A'B'} \varepsilon^{AB} \quad . \tag{3.1.8}$$

In terms of these spinors, equation (3.1.6) can be written

$$\mu^{AB} \varepsilon^{A'B'} \bar{\pi}_A \pi_{A'} + \bar{\mu}^{A'B'} \varepsilon^{AB} \bar{\pi}_A \pi_{A'} = (x^{AA'} \bar{\pi}_A \pi_{A'}) \bar{\pi}^B \pi^{B'} \quad , \tag{3.1.9}$$

where $x^{AA'}$ is the spinor corresponding to x^a (cf. (2.3.2)).

Multiplication of (3.1.9) by $\bar{\pi}_B$ gives

$$\mu^{AB} \bar{\pi}_A \bar{\pi}_B = 0 \quad . \tag{3.1.10}$$

Now, since μ^{AB} is symmetric, it is the symmetrized outer product of two 1-spinors (Pirani (1),p.319), say

$$\mu^{AB} = 2 \xi^{(A} \eta^{B)} \quad ; \tag{3.1.11}$$

substitution of (3.1.11) into (3.1.10) yields

$$(\xi^A \bar{\pi}_A)(\eta^B \bar{\pi}_B) = 0 \quad . \tag{3.1.12}$$

Therefore at least one of ξ^A and η^A is proportional to $\bar{\pi}^A$, and consequently a spinor ω^A can be defined by

$$\mu^{AB} = i \omega^{(A} \bar{\pi}^{B)} \quad . \tag{3.1.13}$$

Using (3.1.13), equation (3.1.8) becomes

$$M^{ab} = i\,\omega^{(A}\,\bar{\pi}^{B)}\,\varepsilon^{A'B'} - i\,\bar{\omega}^{(A'}\,\pi^{B')}\,\varepsilon^{AB} \quad , \tag{3.1.14}$$

while substitution of (3.1.13) into (3.1.9) gives

$$-i(\omega^{A}\,\bar{\pi}_{A} - \bar{\omega}^{A'}\,\pi_{A'}) = 2\,x^{AA'}\,\bar{\pi}_{A}\,\pi_{A'} \quad . \tag{3.1.15}$$

Notice that M^{ab} takes the special from (3.1.14) as a necessary condition for equation (3.1.9) (or (3.1.6)) to possess a solution; the equation for the centre of mass then becomes (3.1.15), which is the equation of a null hyperplane. This hyperplane will be denoted by K ; it can be regarded as the analogue of the centre-of-mass line in the $P^a\,P_a > 0$ case.

Carrying this analogy further, let x^a be a point on K and define the tensor H^{ab} by (cf.(3.1.4))

$$H^{ab} := M^{ab} - 2\,x^{[a}\,P^{b]} \quad . \tag{3.1.16}$$

Equation (3.1.5) suggests that the product $H^{ab}\,H_{ab}$ be evaluated; using (3.1.6) and (3.1.14) it is

$$\begin{aligned}
&(M^{ab} - x^a\,P^b + x^b\,P^a)(M_{ab} - x_a\,P_b + x_b\,P_a) \\
&= M^{ab}\,M_{ab} + 4(x^a\,P_a)^2 - 2(x^a\,P_a)^2 \\
&= M^{ab}\,M_{ab} + 2(x^a\,P_a)^2 \\
&= \left(i\,\omega^{(A}\,\bar{\pi}^{B)}\,\varepsilon^{A'B'} - i\,\bar{\omega}^{(A'}\,\pi^{B')}\,\varepsilon^{AB}\right)\left(i\,\omega_{(A}\,\bar{\pi}_{B)}\,\varepsilon_{A'B'}\right. \\
&\qquad \left. - i\,\bar{\omega}_{(A'}\,\pi_{B')}\,\varepsilon_{AB}\right) + 2(x^{AA'}\,\bar{\pi}_A\,\pi_{A'})^2 \\
&= (\omega^A\,\bar{\pi}_A)^2 + (\bar{\omega}^{A'}\,\pi_{A'})^2 - \tfrac{1}{2}(\omega^A\,\bar{\pi}_A - \bar{\omega}^{A'}\,\pi_{A'})^2 \\
&= \tfrac{1}{2}\left((\omega^A\,\bar{\pi}_A)^2 + (\bar{\omega}^{A'}\,\pi_{A'})^2\right) + (\omega^A\,\bar{\pi}_A)(\bar{\omega}^{A'}\,\pi_{A'})
\end{aligned}$$

Now, letting $\omega^A\,\bar{\pi}_A = a + ib$, where a and b are real, it follows that

$$\begin{aligned}
H^{ab}\,H_{ab} &= \tfrac{1}{2}\left(2(a^2 - b^2)\right) + a^2 + b^2 \\
&= 2a^2 \quad ,
\end{aligned} \tag{3.1.17}$$

which is independent of the choice of x^a on K. Comparison with (3.1.5) shows that the spin is given by

$$s = \mathrm{Re}(\omega^A\,\bar{\pi}_A) \quad . \tag{3.1.18}$$

Spin, as defined by (3.1.5), can only take on non-negative values, whereas this is not true of the quantity s in equation (3.1.18). From now on, however, the word "spin" will be used in the wider sense (namely that of (3.1.18)) when applied to zero-rest-mass systems.

If the angular momentum with respect to some point on K is zero, then $H^{ab} H_{ab} = 0$ everywhere on K and hence the spin is zero. Conversely, suppose that $s = 0$. Then equation (3.1.18) implies that

$$\omega^{C} \bar{\pi}_{C} + \bar{\omega}^{C'} \pi_{C'} = 0 \quad ,$$

i.e. that

$$\omega^{C} \bar{\pi}_{C} \varepsilon^{AB} \varepsilon^{A'B'} + \bar{\omega}^{C'} \pi_{C'} \varepsilon^{A'B'} \varepsilon^{AB} = 0 \quad ,$$

whence, using (2.3.6),

$$i\, \omega^{[A} \bar{\pi}^{B]} \varepsilon^{A'B'} + i\, \bar{\omega}^{[A'} \pi^{B']} \varepsilon^{AB} = 0 \quad . \qquad (3.1.19)$$

Addition of (3.1.19) to (3.1.14) yields

$$M^{ab} = i\, \omega^{A} \bar{\pi}^{B} \varepsilon^{A'B'} - i\, \bar{\omega}^{B'} \pi^{A'} \varepsilon^{AB} \quad . \qquad (3.1.20)$$

The two cases $\omega^{A} \bar{\pi}_{A} \neq 0$ and $\omega^{A} \bar{\pi}_{A} = 0$ will now be considered separately. In the first case, (3.1.20) leads to

$$\begin{aligned}
(\bar{\omega}^{B'} \pi_{B'}) M^{ab} &= (\bar{\omega}^{B'} \pi_{B'})(i\, \omega^{A} \bar{\pi}^{B} \varepsilon^{A'B'} - i\, \bar{\omega}^{B'} \pi^{A'} \varepsilon^{AB}) \\
&= i\, \omega^{A} \bar{\pi}^{B} (\bar{\omega}^{C'} \pi_{C'} \varepsilon^{AB'}) + i\, \bar{\omega}^{B'} \pi^{A'} (\omega^{C} \bar{\pi}_{C} \varepsilon^{AB}) \\
&= -\, i\, \omega^{A} \bar{\pi}^{B} (\bar{\omega}^{A'} \pi^{B'} - \bar{\omega}^{B'} \pi^{A'}) - i\, \bar{\omega}^{B'} \pi^{A'} (\omega^{A} \bar{\pi}^{B} - \omega^{B} \bar{\pi}^{A}) \\
&= -\, i\, \omega^{A} \bar{\pi}^{B} \bar{\omega}^{A'} \pi^{B'} + i\, \bar{\omega}^{B'} \pi^{A'} \omega^{B} \bar{\pi}^{A} \quad . \qquad (3.1.21)
\end{aligned}$$

Dividing both sides of (3.1.21) by $\bar{\omega}^{B'} \pi_{B'}$ and letting

$$T^{AA'} := -\, i (\bar{\omega}^{B'} \pi_{B'})^{-1} \omega^{A} \bar{\omega}^{A'} \quad ; \qquad (3.1.22)$$

yields

$$M^{ab} = 2\, T^{[a} P^{b]} \quad , \qquad (3.1.23)$$

where T^{a} is the vector corresponding to $T^{AA'}$ (notice that T^{a} is real, since $T^{AA'}$ is Hermitian).

In the second case (i.e. $\omega^{A} \bar{\pi}_{A} = 0$), there exists a complex constant α such that

$$\omega^{A} = \alpha\, \bar{\pi}^{A} \quad .$$

Thus (3.1.20) becomes

$$M^{ab} = i\,\alpha\,\bar{\pi}^A\,\bar{\pi}^B\,\varepsilon^{A'B'} - i\,\bar{\alpha}\,\pi^{A'}\,\pi^{B'}\,\varepsilon^{AB} \quad . \tag{3.1.24}$$

Let $\eta^{A'}$ be a spinor such that $\pi_{C'}\,\eta^{C'} = 1$. Then (3.1.24) can be written as

$$\begin{aligned} M^{ab} &= i\,\alpha\,\bar{\pi}^A\,\bar{\pi}^B\,(\pi_{C'}\,\eta^{C'}\,\varepsilon^{A'B'}) - i\,\bar{\alpha}\,\pi^{A'}\,\pi^{B'}(\bar{\pi}_C\,\bar{\eta}^C\,\varepsilon^{AB}) \\ &= i\,\alpha\,\bar{\pi}^A\,\bar{\pi}^B\,(\pi^{A'}\,\eta^{B'} - \pi^{B'}\,\eta^{A'}) - i\,\bar{\alpha}\,\pi^{A'}\,\pi^{B'}(\bar{\pi}^A\,\bar{\eta}^B - \bar{\pi}^B\,\bar{\eta}^A) \\ &= \pi^A\,\pi^{A'}\,(i\,\alpha\,\bar{\pi}^B\,\eta^{B'} - i\,\bar{\alpha}\,\pi^{B'}\,\bar{\eta}^B) - \bar{\pi}^B\,\pi^{B'}\,(i\,\alpha\,\bar{\pi}^A\,\eta^{A'} - \\ &\qquad i\,\bar{\alpha}\,\pi^{A'}\,\bar{\eta}^A) \quad . \end{aligned} \tag{3.1.25}$$

Let $R^{AA'} := - i\,\alpha\,\bar{\pi}^A\,\eta^{A'} + i\,\bar{\alpha}\,\pi^{A'}\,\bar{\eta}^A$; since $R^{AA'}$ is Hermitian, it corresponds to a real vector R^a and so equation (3.1.25) is the same as

$$M^{ab} = 2\,R^{[a}\,P^{b]} \quad . \tag{3.1.26}$$

Summarising, it follows form equations (3.1.23) and (3.1.26) that if the spin is zero, then M^{ab} has the form

$$M^{ab} = 2\,\omega^{[a}\,P^{b]} \quad , \tag{3.1.27}$$

where ω^a is a (real) vector.

The angular momentum about a point on the null line given by

$$x^a = \omega^a + \lambda\,P^a \quad , \tag{3.1.28}$$

where λ is a real parameter and ω^a is defined in (3.1.27), is

$$\begin{aligned} \tilde{M}^{ab} &= 2\,\omega^{[a}\,P^{b]} - 2\,x^{[a}\,P^{b]} \\ &= 2\,\omega^{[a}\,P^{b]} - 2\,\omega^{[a}\,P^{b]} - 2\lambda\,P^{[a}\,P^{b]} \\ &= 0 \quad . \end{aligned}$$

Thus, as in the $P^a\,P_a > 0$ case, the system is characterised by the line (3.1.28), which lies in the hyperplane K , since equation (3.1.2) is satisfied.

If, on the other hand, $s \neq 0$, then the angular momentum is non-zero about every point on the null hyperplane K . In this sense, if the system consists of a particle, the particle is not localised (i.e."tied" to a world line) as it would be if its spin were zero.

It has been shown that the pair (P^a, M^{ab}) can be represented by the pair $(\omega^A, \pi_{A'})$. This representation is not unique, since replacing $(\omega^A, \pi_{A'})$ by $(e^{i\theta}\omega^A, e^{i\theta}\pi_{A'})$, with θ real, alters neither P^a nor M^{ab}. Analogously, a twistor is something which can be represented by the pair $(\omega^A, \pi_{A'})$. A symbol such as Z^α is used to denote a twistor; in component form it is given by

$$Z^\alpha := (\omega^0, \omega^1, \pi_{0'}, \pi_{1'}) . \qquad (3.1.29)$$

The complex conjugate of the twistor Z^α is denoted by $\bar{Z}_\alpha$ and defined, in component form, as

$$\bar{Z}_\alpha := (\bar{\pi}_0, \bar{\pi}_1, \bar{\omega}^{0'}, \bar{\omega}^{1'}) . \qquad (3.1.30)$$

The inner product of (3.1.29) and (3.1.30) is

$$Z^\alpha Z_\alpha = \omega^A \bar{\pi}_A + \bar{\omega}^{A'} \pi_{A'}$$

$$= 2s . \qquad (3.1.31)$$

Since s is independent of the spinor frame, equation (3.1.31) can be written as

$$Z^\alpha \bar{Z}_\alpha = 2s . \qquad (3.1.32)$$

If a twistor Z^α is such that $Z^\alpha\bar{Z}_\alpha = 0$, it is said to be null and, as seen above (the $s = 0$ case), it represents a unique null line in M. (This holds provided $\pi_{A'} \neq 0$; the case $\pi_{A'} = 0$ will be considered later.) Thus, in Minkowski space-time, null twistors can be identified with null lines. It must be borne in mind, however, that if Z^α represents a null line, then λZ^α, where $\lambda \neq 0$ is a complex constant, represents exactly the same null line. (The twistor λZ^α is defined "component-wise", i.e. if $(\omega^A, \pi_{A'})$ represents Z^α, then $(\lambda \omega^A, \lambda \omega_{A'})$ represents λZ^α.)

§ 3.2 Twistor Intersections.

Let $(\omega^A, \pi_{A'})$ with $\pi_{A'} \neq 0$ represent a twistor. If the origin 0 is displaced by the vector $x^{AA'}$ to $\tilde{0}$, P^a remains unchanged and so $\pi_{A'}$ is unaltered up to a phase factor; take this factor to be unity, so that the phase of $\pi_{A'}$ is preserved. Suppose that ω^A is defined to transform as

$$\tilde{\omega}^A = \omega^A - i\, x^{AA'}\, \pi_{A'} \quad ; \tag{3.2.1}$$

then it follows that

$$\begin{aligned}
\tilde{M}^{ab} &= i\,\tilde{\omega}^{(A}\,\bar{\pi}^{B)}\,\varepsilon^{A'B'} - i\,\bar{\tilde{\omega}}^{(A'}\,\pi^{B')}\,\varepsilon^{AB} \\
&= M^{ab} + \tfrac{1}{2}\big(x^{AC'}\,\pi_{C'}\,\bar{\pi}^{B}\,\varepsilon^{A'B'} + x^{BC'}\,\pi_{C'}\,\bar{\pi}^{A}\,\varepsilon^{A'B'} + x^{A'C}\,\bar{\pi}_{C}\,\pi^{B'}\,\varepsilon^{AB} \\
&\quad + x^{B'C}\,\bar{\pi}_{C}\,\pi^{A'}\,\varepsilon^{AB}\big) \\
&= M^{ab} - \tfrac{1}{2}\big(x^{AA'}\,\pi^{B'}\,\bar{\pi}^{B} - x^{AB'}\,\pi^{A'}\,\bar{\pi}^{B} + x^{BA'}\,\pi^{B'}\,\bar{\pi}^{A} - x^{BB'}\,\pi^{A'}\,\bar{\pi}^{A} \\
&\quad + x^{A'A}\,\bar{\pi}^{B}\,\pi^{B'} - x^{A'B}\,\bar{\pi}^{A}\,\pi^{B'} + x^{B'A}\,\bar{\pi}^{B}\,\pi^{A'} - x^{B'B}\,\bar{\pi}^{A}\,\pi^{A'}\big) \\
&= M^{ab} - 2\,x^{[a}\,P^{b]} \quad ,
\end{aligned}$$

in agreement with equation (3.1.1).

If $(\omega^A, \pi_{A'})$ represents a null twistor whose corresponding null line Z passes through the origin, then M^{ab}, and hence ω^A, are zero. By applying a displacement of $-x^{AA'}$ to the origin, it follows that if Z passes through the point $x^{AA'}$, then

$$\omega^A = i\, x^{AA'}\, \pi_{A'} \quad . \tag{3.2.2}$$

Suppose now that Z intersects the null cone of the origin and denote the point of intesection by $x^{AA'}$. Then equation (3.2.2) is satisfied, since Z passes through the point $x^{AA'}$. In addition, $x^{AA'}$, regarded as a vector, lies in the null cone of the origin and is therefore null. Consequently, $x^{AA'}$ has the form

$$x^{AA'} = \pm\eta^A\,\bar{\eta}^{A'} \quad , \tag{3.2.3}$$

where η^A is some spinor.

By substituting (3.2.3) into (3.2.2) it is seen that η^A is proportional to

ω^A, whence

$$x^{AA'} = K\,\omega^A\,\bar{\omega}^{A'} \quad , \tag{3.2.4}$$

where K is a constant.

Substitution of (3.2.4) into (3.2.2) gives

$$x^{AA'} = -i\,(\bar{\omega}^{B'}\,\pi_{B'})^{-1}\,\omega^A\,\bar{\omega}^{A'} \quad , \tag{3.2.5}$$

provided $\bar{\omega}^{B'}\,\pi_{B'} \neq 0$ (cf. equation (3.1.22)). If $\bar{\omega}^{B'}\,\pi_{B'} = 0$, then $\bar{\omega}^{A'}$ is proportional to $\pi^{A'}$ and from (3.2.2) it follows that

$$x^{AA'}\,\bar{\pi}_A\,\pi_{A'} = 0 \quad . \tag{3.2.6}$$

In other words, $x^{AA'}$ is orthogonal to the null vector P^a and therefore lies in a null hyperplane through the origin (in fact the null hyperplane K of section 3.1).

Suppose that the two null twistors Z^α and Y^α are represented, respectively, by $(\omega^A, \pi_{A'})$ and $(\xi^A, \eta_{A'})$. This gives, respectively, two null lines Z and Y in M. What is required is a criterion which determines whether or not the null lines intersect.

If they intersect at the point $x^{AA'}$, then both (3.2.2) and

$$\xi^A = i\,x^{AA'}\,\eta_{A'} \tag{3.2.7}$$

must hold.

Thus

$$\begin{aligned}\bar{\eta}_A\,\omega^A = i\,\bar{\eta}_A\,x^{AA'}\,\pi_{A'} &= i\,\bar{\eta}_A\,\bar{x}^{AA'}\,\pi_{A'} \\ &= -\,\bar{\xi}^{A'}\,\pi_{A'} \quad ,\end{aligned} \tag{3.2.8}$$

since $x^{AA'}$ represents a real vector and is therefore Hermitian.

Equation (3.2.8) is the same as

$$Z^\alpha\,\bar{Y}_\alpha = 0 \quad , \tag{3.2.9}$$

which is consequently a necessary condition for intersection.

To show that the condition (3.29) is also sufficient, assume firstly that $\pi_{A'}$ and $\eta_{A'}$ are not proportional (i.e. that $\pi_{A'}\,\eta^{A'} \neq 0$). Then the two null lines will be non-parallel. Define the vector $x^{AA'}$ as

$$x^{AA'} = \left(- i/(\pi_{B'} \eta^{B'})\right) (\omega^{A} \eta^{A'} - \xi^{A} \pi^{A'}) \quad . \tag{3.2.10}$$

Then (3.2.2) and (3.2.7) are satisfied and it only remains to check that $x^{AA'}$ is a real vector, i.e. that

$$x^{AA'} - \bar{x}^{AA'} = 0 \quad .$$

This is done by taking components with respect to the two non-parallel spinors $\pi_{A'}$ and $\eta_{A'}$:

$$\bar{\pi}_{A} \pi_{A'} (x^{AA'} - \bar{x}^{AA'}) = - i (\omega^{A} \bar{\pi}_{A} + \bar{\omega}^{A'} \pi_{A'})$$

$$= - i Z^{\alpha} \bar{Z}_{\alpha} \quad ;$$

$$\bar{\eta}_{A} \eta_{A'} (x^{AA'} - {}^{-AA'}) = - i (\xi^{A} \bar{\eta}_{A} + \bar{\xi}^{A'} \eta_{A'})$$

$$= - i Y^{\alpha} \bar{Y}_{\alpha} \quad ;$$

$$\bar{\eta}_{A} \pi_{A'} (x^{AA'} - \bar{x}^{AA'}) = - i (\omega^{A} \bar{\eta}_{A} + \bar{\xi}^{A'} \pi_{A'})$$

$$= - i Z^{\alpha} \bar{Y}_{\alpha} \quad .$$

So a necessary and sufficient condition for a pair of null non-parallel lines (represented by Z^{α} and Y^{α}) to intersect is

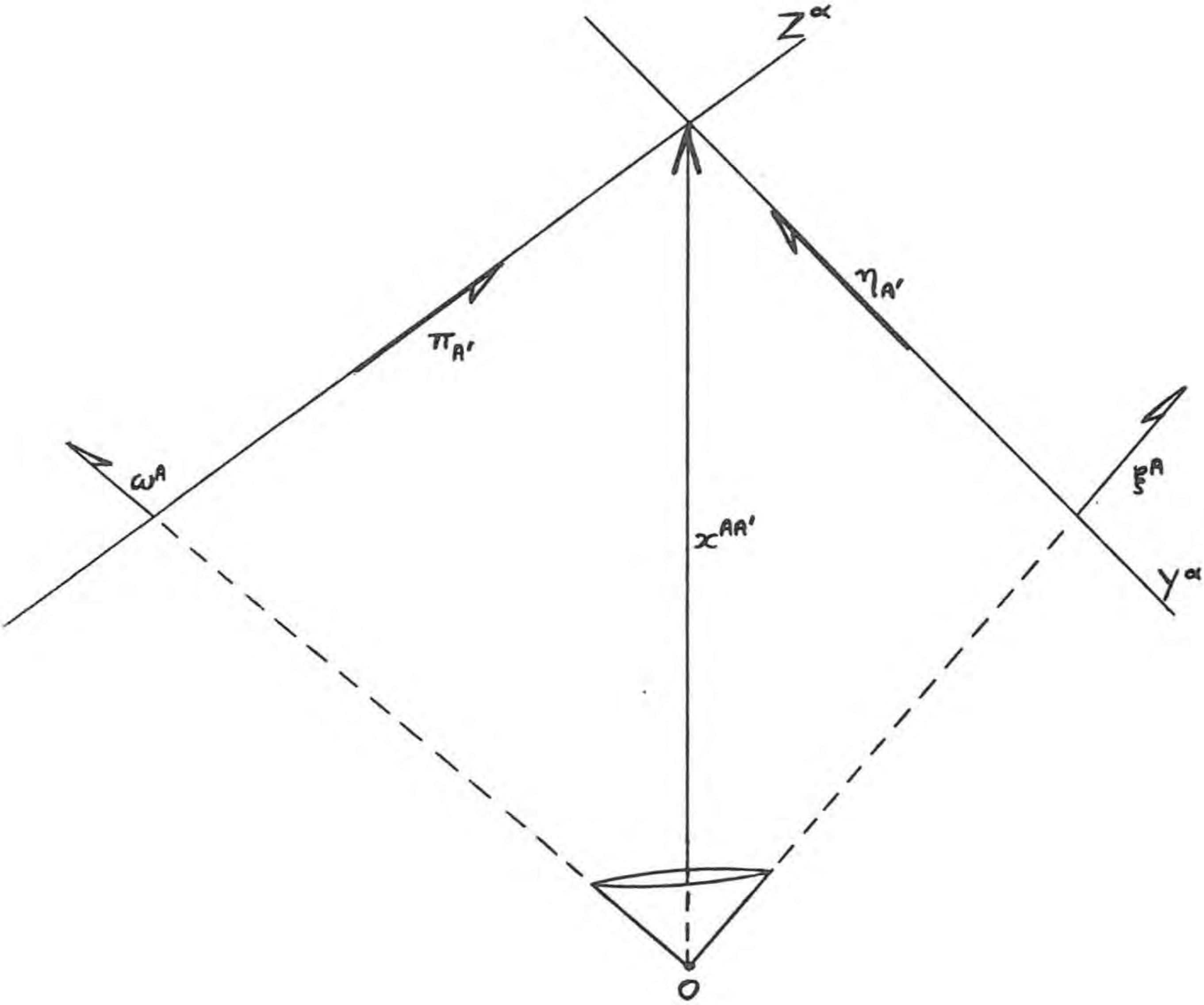
$$Z^{\alpha} \bar{Y}_{\alpha} = 0 \quad .$$

A diagrammatic representation is provided in fig. 2.

If $\pi_{A'}$ and $\eta_{A'}$ are proportional but non-zero, it is possible to reason as follows. Fix Y and move Z , so that it continues to intersect Y and so that its direction approaches that of Y . The point of intersection is the vertex of the null cone containing Z and Y and this point is moved "out to infinity" as the two lines approach parallelism. In the limit, the null cone becomes a null hyperplane and so it is seen that the condition (3.2.9) is a necessary and sufficient for the two parallel null lines to be contained in the same null hyperplane. In other words, condition (3.2.9) is necessary and sufficient for two parallel null lines to intersect at a point at infinity.

It is now possible to interpret the meaning of the twistor Z^{α} whose

Fig. 2. Intersecting Null Twistors.



representation is $(\omega^A, 0)$. The condition (3.2.9) becomes

$$\omega^A \bar{\eta}_A = 0 \quad , \tag{3.2.11}$$

since $\pi_{A'} = 0$. From (3.2.11) it follows that the collection of all null lines which intersect Z is the set of all null lines whose direction coincides with the null direction represented by ω^A. This gives a collection of parallel null hyperplanes; as seen in § 2.2, each hyperplane has a point at infinity and the points at infinity corresponding to the parallel hyperplanes "join up" to form one of the generators of the null cone at infinity. The required interpretation of the twistor Z^α is provided by this null line: indeed, it intersects each of the null lines in each of the set of parallel null hyperplanes corresponding to it.

To summarise: a non-zero null twistor Z^α (i.e. $Z^\alpha \bar{Z}_\alpha = 0$, $Z^\alpha \neq 0$) determines a unique null geodesic in compactified Minkowski space-time M' .

Above the implicit assumption has been made that null geodesics in (M,η_{ab}) are also null geodesics in (M',g_{ab}). This can be substantiated as follows: the condiiton for a null line with tangent vector $K^a = K^A \bar{K}^{A'}$ to be a geodesic is given in Pirani (1), p.343, as

$$K^A K^B \bar{K}^{B'} \nabla_{BB'} K_A = 0 \quad .$$

Since g_{ab} is a conformal resealing of η_{ab}, (2.3.11) can be used to yield

$$K^A K^B \bar{K}^{B'} \hat{\nabla}_{BB'} K_A = K^A K^B \bar{K}^{B'} \nabla_{BB'} K_A - K^A K^B \bar{K}^{B'} T_{AB'} K_B$$
$$= 0 \ ,$$

where $\hat{\nabla}_{BB'}$ is the covariant derivative operator corresponding to g_{ab}. This proves the assertion.

§ 3.3 Robinson Congruences

In sections 3.1 and 3.2 it was found that null twistors could be visualised in terms of null lines; the object of this section is to obtain a geometric interpretation of non-null twistors.

A null line in M' can be represented by the collection of all null lines which meet it. In other words, a null twistor Z^{α} determines, and is determined up to proportionality by, the set of all twistors Y^{α} satisfying $Y^{\alpha}\bar{Y}_{\alpha} = 0$ and $Y^{\alpha}\bar{Z}_{\alpha} = 0$; the twistor $\bar{Z}_{\alpha}$ can be thought of as representing this set of null lines. Similarly, the general twistor R_{α} , whether null or not, can be regarded as a representation of the collection of null lines whose twistors Y^{α} satisfy $Y^{\alpha}\bar{Y}_{\alpha} = 0$ and $Y^{\alpha}R_{\alpha} = 0$. If $R_{\alpha}\bar{R}^{\alpha} \neq 0$, this collection is called a Robinson congruence. If $R_{\alpha}\bar{R}^{\alpha} = 0$, $\bar{R}^{\alpha}$ is taken to denote the null line which is met by all the members of the collection R_{α} ; thus the notation remains consistent with that used above. A Robinson congruence for which $R_{\alpha}\bar{R}^{\alpha} > 0$ is called right-handed, while one for which $R_{\alpha}\bar{R}^{\alpha} < 0$ is called left-handed.

A particular Robinson congruence (suggested by Penrose (4)) will now be investigated: let ε be a real number and take

$$R_{\alpha} := (1,\ 0,\ \varepsilon|\surd 2,\ 0) \quad . \tag{3.3.1}$$

Let Y^{α} be a null twistor represented by the spinor pair $(\omega^{A}, \pi_{A'})$. The condition for the null line Y represented by Y^{α} to belong to the congruence given by R_{α} is

$$\omega^{0} + \pi_{0'}\,\varepsilon|\surd 2 = 0 \quad . \tag{3.3.2}$$

If x^{a} is a point on the line Y , write $x^{0} = t$, $x^{1} = x$, $x^{2} = y$, $x^{3} = z$. From equations (3.2.2) and (2.3.2) it follows that

$$\begin{bmatrix}\omega^{0}\\ \omega^{1}\end{bmatrix} = i|\surd 2 \begin{bmatrix} t+x & y+iz \\ y-iz & t-x\end{bmatrix} \begin{bmatrix}\pi_{0'}\\ \pi_{1'}\end{bmatrix} \quad . \tag{3.3.3}$$

Using (3.3.2), equation (3.3.3) gives

$$. - \pi_{0'}\ \varepsilon|\sqrt{2} = i|\sqrt{2}\big((t+x)\pi_{0'} + (y+i\,z)\pi_{1'}\big) ,$$

that is

$$\pi_{0'} : \pi_{1'} = y + i\,z : i\,\varepsilon - t - x \ . \tag{3.3.4}$$

Now the direction of Y is given by the null vector corresponding to $\pi^{A'}$, and so

$$\begin{bmatrix} \bar{\pi}^0\ \pi^{0'} & \bar{\pi}^0\ \pi^{1'} \\ \bar{\pi}^1\ \pi^{0'} & \bar{\pi}^1\ \pi^{1'} \end{bmatrix} \propto \begin{bmatrix} dt + dx & dy + i\,d\,z \\ dy - i\,d\,z & dt - dx \end{bmatrix}$$

or equivalently

$$\begin{bmatrix} \bar{\pi}_1\ \pi_{1'} & -\bar{\pi}_1\ \pi_{0'} \\ -\bar{\pi}_0\ \pi_{1'} & \bar{\pi}_0\ \pi_{0'} \end{bmatrix} \propto \begin{bmatrix} dt + dx & dy + i\,d\,z \\ dy - i\,d\,z & dt - dx \end{bmatrix} \ . \tag{3.3.5}$$

Combining (3.3.4) and (3.3.5) gives the differential equations for the line Y, namely

$$y + i\,z : t + x - i\varepsilon = dy + i\,d\,z : dt + dx$$

$$= dt - dx : dy - i\,d\,z \ . \tag{3.3.6}$$

The first equation in (3.3.6) gives

$$t + x - i\varepsilon = \alpha(y + i\,z) , \tag{3.3.7}$$

where α is a complex constant; the second equaiton in (3.3.6) then yields

$$(t - x)\alpha = y - i\,z + \beta \ , \tag{3.3.8}$$

where β is another constant.

Multiplying the complex conjugate of (3.3.8) by α and subtracting (3.3.7) from the result, it is seen that

$$(t - x)|\alpha|^2 - t - x + i\,\varepsilon = \bar{\beta}\,\alpha ,$$

the imaginary part of which reads

$$\mathrm{Im}(\bar{\beta}\alpha) = \varepsilon \ . \tag{3.3.9}$$

The constants α and β determine the particular null line Y of the congruence; for consistency they must satisfy (3.3.9). Determination of a line requires three equations, whereas (3.3.7) and (3.3.8) provide four, but there is no problem, since both the imaginary part of (3.3.7) and the imaginary part of (3.3.8) read

$$z\ \mathrm{Re}\,(\alpha) + y\ \mathrm{Im}(\alpha) + \varepsilon = 0 \ ,$$

where use has been made of (3.3.9). Consequently, three (at most) of the four equations are independent.

A useful visualisation of the congruence can be obtained by considering its intersection with the hyperplane $t = \tau$, where τ is a real constant. The tangents to the curves of the congruence are projected onto this hyperplane and integration then yields a collection of curves in the hyperplane. The projection can be achieved in the following way: firstly (noting that $dt^2 = dx^2 + dy^2 + dz^2$ for a null line), replace dt in (3.3.6) by $ds := (dx^2 + dy^2 + dz^2)^{\frac{1}{2}}$; and secondly replace t by τ. This gives

$$(y + i\,z)(dy - i\,dz) = (\tau + x - i\,\varepsilon)(ds - dx) \quad . \tag{3.3.10}$$

Introducing the new real variables w, r and ψ, given by

$$w := \tau + x \quad ,$$

$$y := r \cos\psi \quad ,$$

$$z := r \sin\psi \quad ,$$

it is seen that (3.3.10) is equivalent to the two real equations

$$w(ds - dw) = r\,dr \quad , \tag{3.3.11a}$$

$$\varepsilon(ds - dw) = r^2\,d\psi \quad . \tag{3.3.11b}$$

From (3.3.11) it follows immediately that

$$\varepsilon\,dr = w\,r\,d\psi \quad . \tag{3.3.12}$$

Using (3.3.12) and the definition of ds, equation (3.3.11a) becomes

$$w^2 ds^2 = w^2(dw^2 + dr^2 + \varepsilon^2 dr^2/w^2)$$

$$= r^2 dr^2 + 2\,r\,w\,dr\,dw + w^2 d\,w^2 \quad ,$$

whence

$$2\,r\,w\,d\,w = (w^2 - r^2 + \varepsilon^2)dr \quad . \tag{3.3.13}$$

Multiplying both sides of (3.3.13) by $(w^2 + r^2 + \varepsilon^2)$ gives

$$2\,r\,w^3 dw + 2\,r^3 w\,dw + 2\,r\,w\,\varepsilon^2 dw$$

$$= w^4\,dr + 2\,w^2\varepsilon^2 dr + \varepsilon^4 dr - r^4\,dr \quad ,$$

or, dividing by r^3 and rearranging,

$$r\,dr + 2\,w\,d\,w + 2\,r^{-2}w^3\,d\,w - w^4 r^{-3}dr + 2\,\varepsilon^2 w\,r^{-2}d\,w - 2\,\varepsilon^2 w^2 r^{-3}dr$$
$$- \varepsilon^4\,r^{-3}dr = 0\ ,$$

that is

$$d(r^2) + 2\,d(w^2) + d(w^4 r^{-2}) + 2\,\varepsilon^2 d(w^2 r^{-2}) + \varepsilon^4\,d(r^{-2}) = 0\ .$$

Integration yields

$$r^2 + 2w^2 + w^4 r^{-2} + 2\,\varepsilon^2 w^2\,r^{-2} + \varepsilon^4 r^{-2} = d\ , \qquad (3.3.14)$$

where d is a (real) constant.

Equation (3.3.14) can be rewritten as

$$r^{-2}\left((r^2+w^2)^2 - 2(r^2-w^2)\varepsilon^2 + \varepsilon^4\right) = d - 2\,\varepsilon^2\ ,$$

or

$$(r^2 + w^2 - \varepsilon^2)^2 = (d - 2\varepsilon^2)r^2 - 4\,w^2\,\varepsilon^3$$
$$= 4\,\varepsilon^2(a^2r^2 - w^2)\ , \qquad (3.3.15)$$

where $a := (d - 2\,\varepsilon^2)^{\frac{1}{2}} \mid 2\,\varepsilon$.

Multiplying out equation (3.3.15) leads to

$$w^4 + 2\,w^2(r^2 + \varepsilon^2) + (r^4 - 4\varepsilon^2\,a^2r^2 - 2\,\varepsilon^2 r^2 + \varepsilon^4) = 0\ ,$$

whence

$$w^2 = - r^2 - \varepsilon^2 \pm 2\,|\varepsilon|\,r(a^2+1)^{\frac{1}{2}}\ . \qquad (3.3.16)$$

The minus part of the ± sign can clearly be discarded. Thus w is given by

$$w = \pm\left(- r^2-\varepsilon^2 + 2|\varepsilon|r(a^2+1)^{\frac{1}{2}}\right)^{\frac{1}{2}}\ . \qquad (3.3.17)$$

Substituting (3.3.17) into (3.3.12) and integrating with respect to r

yields

$$\psi + \phi = \pm\ \mathrm{sgn}\ (\varepsilon)\ \arcsin$$
$$\left((2|\varepsilon|r(a^2+1)^{\frac{1}{2}} - 2\,\varepsilon^2)\,/\,2\,r\,|\varepsilon|\ \ |a|\ \right)\ , \qquad (3.3.18)$$

where ϕ is a constant. The uncertainly in sign can be taken care of by an appropriate choice of ϕ , so that (3.3.18) can be written more simply as

$$\psi + \phi = -\arcsin\left((2\,|\varepsilon|\,r(a^2+1)^{\frac{1}{2}} - 2\,\varepsilon^2)\mid 2\,\varepsilon\,a\,r\right)$$
$$= -\arcsin\left((w^2+r^2-\varepsilon^2)\,|\,2\,\varepsilon\,a\,r\right)\ ,$$

by (3.3.16), and hence

$$w^2 + r^2 - \varepsilon^2 + 2\,\varepsilon\, a\, r \sin(\psi+\phi) = 0 \quad . \qquad (3.3.19)$$

From (3.3.15) and (3.3.19) it follows that

$$\begin{aligned} 4\,\varepsilon^2 a^2\, r^2 - 4\varepsilon^2 w^2 &= (r^2+w^2-\varepsilon^2)^2 \\ &= 4\,\varepsilon^2 a^2\, r^2 \sin^2(\psi+\phi) \\ &= 4\,\varepsilon^2 a^2\, r^2 - 4\,\varepsilon^2 a^2\, r^2 \cos^2(\psi+\phi) \quad , \end{aligned}$$

whence

$$w = ar \cos(\psi+\phi) \quad . \qquad (3.3.20)$$

Reverting to the original coordinates, equations (3.3.19) and (3.3.20) become

$$y^2 + z^2 + (\tau+x)^2 + 2\,\varepsilon\, a(y \sin\phi + z\cos\phi) = \varepsilon^2 \quad , \qquad (3.3.21)$$

$$\tau + x = a(y\cos\phi - z\sin\phi) \quad . \qquad (3.3.22)$$

This solution is the same as the one given in Penrose (4). The slightly different appearance arises from a change of notation.

Equation (3.3.21) represents a collection of spheres and equation (3.3.22) a collection of planes, so that the intersection curves are circles. These circles must lie on the surfaces given by (3.3.16), which are nested tori axisymmetric about the w-axis. Their intersection with the (y,w) plane for a few different values of the parameter a , is pictured in figure 3. The relevant equation, which follows immediately from (3.3.16), is

$$w^2 + \left(y - |\varepsilon|\, (a^2+1)^{\frac{1}{2}}\right)^2 = \varepsilon^2 a^2 \quad . \qquad (3.3.23)$$

The particular curve given by $a = 0$ is a circle, centred at $(x,y,z) = (\tau,0,0)$ and with radius $|\varepsilon|$. To reconstruct the original null lines which gave rise to this circle, one must "lift" out the tangent vectors to the circle so that they become null vectors in the full four-dimensional space, and then allow τ to vary. The result of this process may be visualised as in figure 4. It is seen that the null lines twist around the line $t = x$ in the (t,x) plane; the screw sense of the twist will of course depend on the sign of ε , i.e. on whether R_α is right- or left-handed.

Fig. 3. The Nested Tori (with $\varepsilon a \neq 0$).

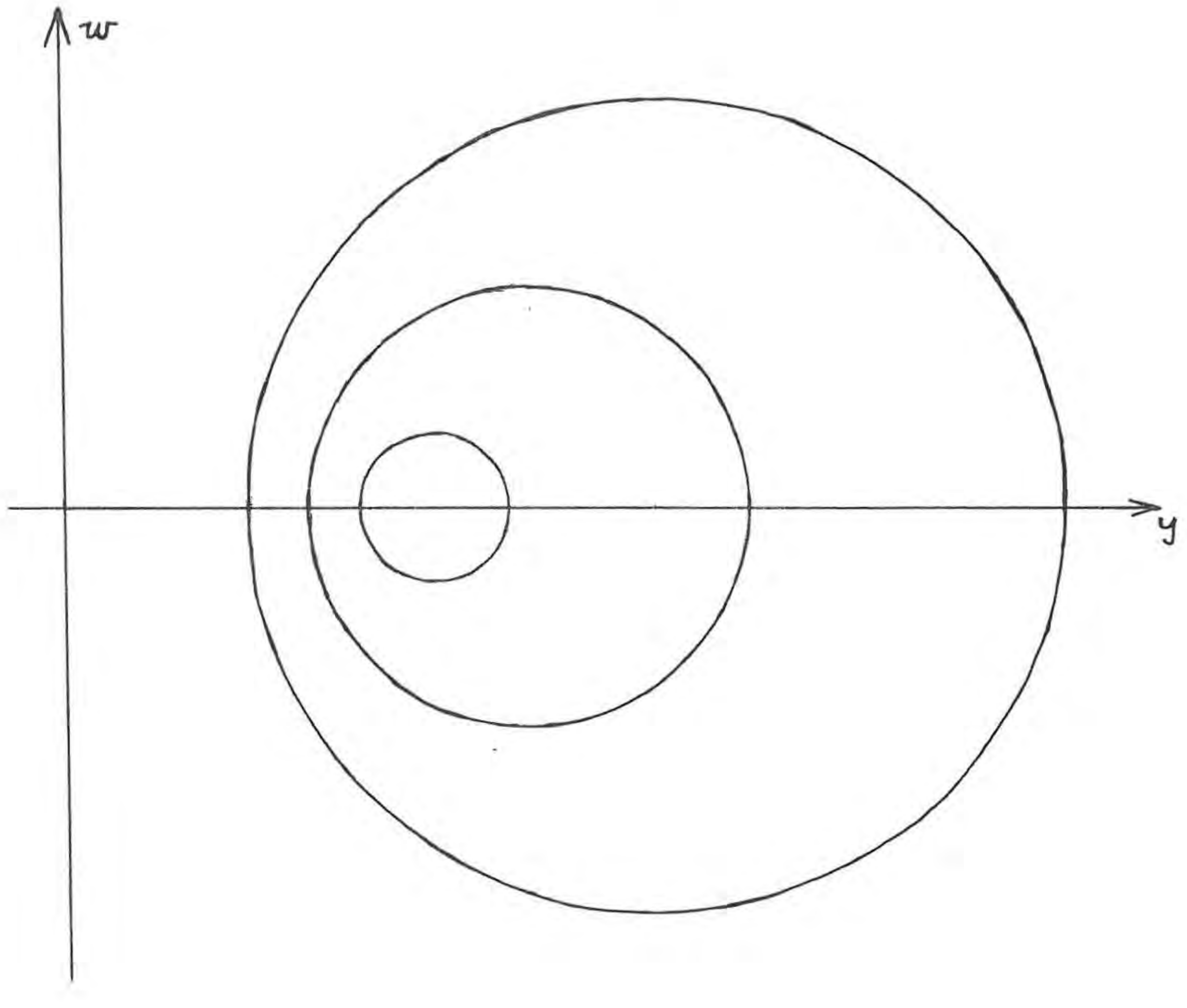
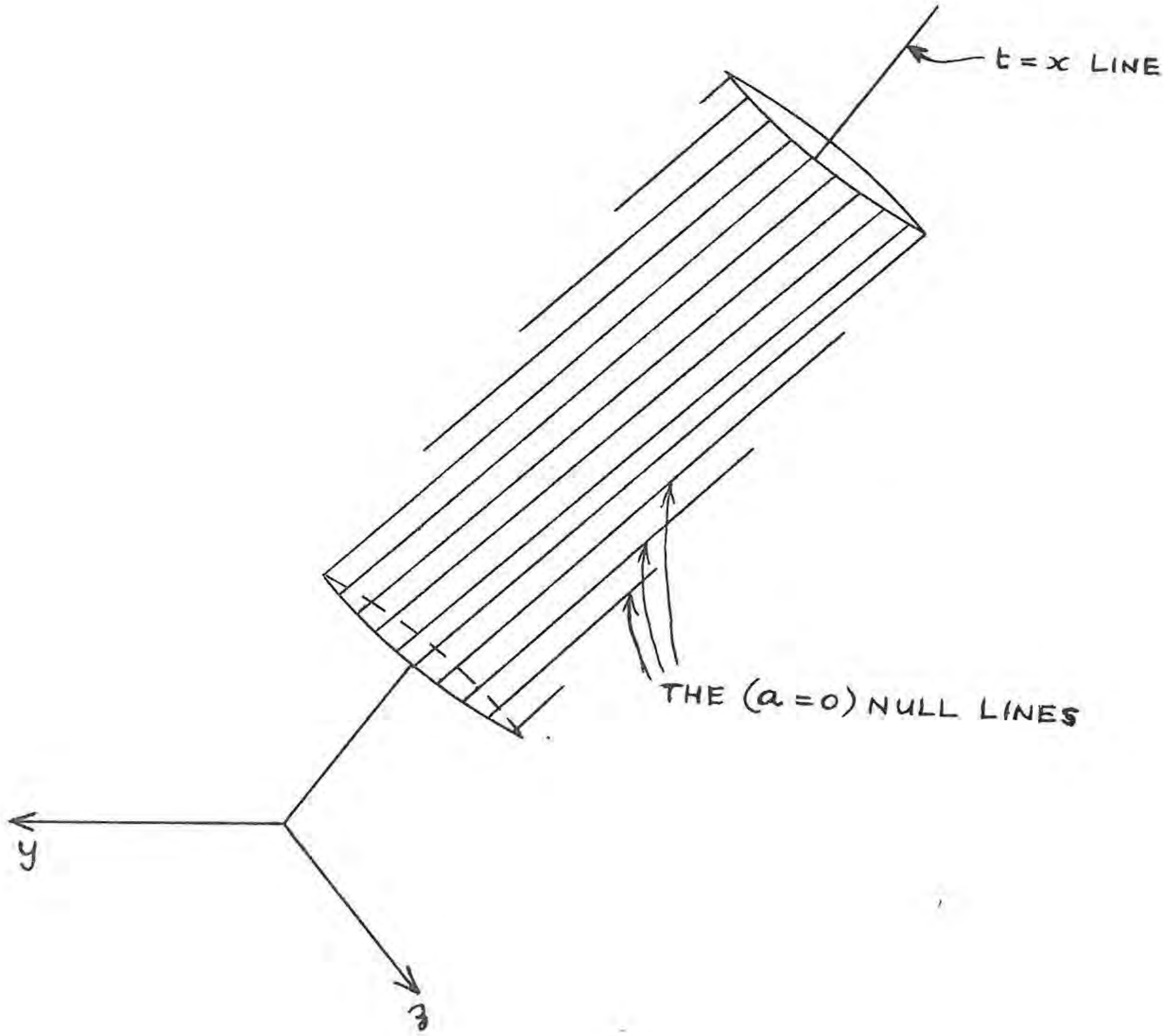


Fig. 4. Some Members of the Robinson Congruence.



Consider, finally, the limiting case when $\varepsilon \to 0$. From equation (3.3.23) it then follows that the entire Robinson congruence collapses to the line $t = x$, $y = 0 = z$. This is exactly the null line corresponding to the twistor given by

$$[\bar{R}^{\alpha}]_{\varepsilon=0} = (0,0,1,0) \quad .$$

§3.4 The Associated Spinor Field

In this section it will be shown how twistors can be interpreted in terms of certain spinor fields. Let the twistor Z^{α} be represented by $(\omega^{A},\pi_{A'})$ and recall equation (3.2.1) which describes the behaviour of ω^{A} under a shift of origin:

$$\tilde{\omega}^{A} = \omega^{A} - i\, x^{AA'}\, \pi_{A'} \quad .$$

This can be regarded as defining a spinor field $\tilde{\omega}^{A}$ on M. Taking the covariant derivative of both sides gives

$$\nabla^{AA'}\, \tilde{\omega}^{B} = -\, i\, \pi_{B}\, \varepsilon^{AB}\, \varepsilon^{A'B'} \quad ,$$

and hence

$$\nabla^{A'(A}\, \tilde{\omega}^{B)} = 0 \; . \qquad (3.4.1)$$

Conversely, given a spinor field $\tilde{\omega}^{B}$ satisfying (3.4.1), unique spinors ω^{A} and $\pi_{A'}$ exist such that (3.2.1) holds; for (3.4.1) implies that $\nabla^{AA'}\, \tilde{\omega}^{B}$ is skew in AB, and therefore $\nabla^{CC'}\, \nabla^{AA'}\, \tilde{\omega}^{B}$ is skew in both AB and CB, since covariant derivative operators commute in Minkowski space-time. But then

$$\begin{aligned}\nabla^{CC'}\, \nabla^{AA'}\, \tilde{\omega}^{B} &= -\, \nabla^{CC'}\, \nabla^{BA'}\, \tilde{\omega}^{A} \\ &= \nabla^{AC'}\, \nabla^{BA'}\, \tilde{\omega}^{C} \\ &= -\, \nabla^{AC'}\, \nabla^{CA'}\, \tilde{\omega}^{B} \quad ,\end{aligned}$$

so that $\nabla^{CC'}\, \nabla^{AA'}\, \tilde{\omega}^{B}$ is skew in CAB and must vanish (by the two-dimensionality of spinor space). Thus $\nabla^{AA'}\, \tilde{\omega}^{B}$ is constant and since it is skew in AB, it can be written as

$$\begin{aligned}\nabla^{AA'}\, \tilde{\omega}^{B} &= \tfrac{1}{2}(\nabla^{AA'}\, \tilde{\omega}^{B} - \nabla^{BA'}\, \tilde{\omega}^{A}) \\ &= \tfrac{1}{2}\, \nabla_{C}{}^{A'}\, \tilde{\omega}^{C}\, \varepsilon^{AB} \\ &= -\, i\, \varepsilon^{AB}\, \pi^{A'} \; ,\end{aligned} \qquad (3.4.2)$$

where $\pi_{A'} := \tfrac{1}{2}\, i\, \nabla_{CA'}\, \tilde{\omega}^{C}$. (3.4.3)

Integration of (3.4.2) leads directly to (3.2.1), where ω^{A} is a constant of integration.

Consequently the field $\tilde{\omega}^B$ determines a unique twistor, in that it determines the two spinors ω^A and $\pi_{A'}$. Furthermore, equation (3.4.1) is invariant under the conformal rescaling (2.2.3) with $\hat{\tilde{\omega}}^B = \tilde{\omega}^B$, for if $\nabla^{AA'} \tilde{\omega}^B$ is skew in AB, then

$$\begin{aligned}\hat{\nabla}^{AA'} \hat{\tilde{\omega}}^B &= \hat{\varepsilon}^{AC} \hat{\varepsilon}^{A'C'} \hat{\nabla}_{CC'} \tilde{\omega}^B \\ &= \Omega^{-2} \ \varepsilon^{AC} \varepsilon^{A'C'} (\nabla_{CC'} \tilde{\omega}^B + \varepsilon_C{}^B T_{DC'} \tilde{\omega}^D) \\ &= \Omega^{-2} (\nabla^{AA'} \tilde{\omega}^B + \varepsilon^{AB} T_D{}^{A'} \tilde{\omega}^D)\end{aligned}$$

is also skew in AB. Thus the spinor field $\tilde{\omega}^B$ provides a conformally invariant description of a twistor.

If this spinor field is to be defined on the compactification M' of Minkowski space-time, a global problem arises, namely the question of how the field is to be defined across the null cone at infinity. It is stated in Penrose (4) that the field must "pick up" the factor i on crossing infinity; for this reason it has to be regarded as four-valued on M'. This will not be proved here, but it will be shown in section 3.6 that twistors of odd total valence are four-valued under the conformal group of transformations.

Strictly speaking, the operator $\nabla_{AA'}$ should be written as $\hat{\nabla}_{AA'}$, since the metric of compactified Minkowski space-time is a conformal rescaling of the usual Minkowski metric η_{ab}. However, the conformal invariance of (3.4.1) means that the "hat" can be dropped without causing ambiguity.

The behaviour of the spinor $\pi_{A'}$ under a conformal rescaling follows immediately from equation (3.4.3): $\pi_{A'}$ becomes

$$\begin{aligned}\hat{\pi}_{A'} &= \tfrac{1}{2} i \ \hat{\nabla}_{CA'} \hat{\tilde{\omega}}^C \\ &= \tfrac{1}{2} i \ (\nabla_{CA'} \tilde{\omega}^C + \varepsilon_C{}^C T_{AA'} \tilde{\omega}^A) \\ &= \pi_{A'} + i \ T_{AA'} \tilde{\omega}^A \ . \qquad (3.4.4)\end{aligned}$$

Notice that $\hat{\pi}_{A'}$ is not a constant spinor like $\pi_{A'}$; in fact $\hat{\pi}_{A'}$ is a spinor field. The spin s , however, depends neither on the position of the origin nor on the conformal factor Ω , since

$$\begin{aligned}
\hat{\tilde{\omega}}^{A}\, \hat{\bar{\pi}} + \bar{\hat{\tilde{\omega}}}^{A'}\, \hat{\pi}_{A'} &= \tilde{\omega}^{A}(\bar{\pi}_{A} - i\, T_{A'A}\, \bar{\tilde{\omega}}^{A'}) + \bar{\tilde{\omega}}^{A'}\, (\pi_{A'} + i\, T_{AA'}\, \tilde{\omega}^{A}) \\
&= \tilde{\omega}^{A}\, \bar{\pi}_{A} + \bar{\tilde{\omega}}^{A'}\, \pi_{A'} \\
&= (\omega^{A} - i\, x^{AA'}\, \pi_{A'})\bar{\pi}_{A} + (\bar{\omega}^{A'} + i\, x^{A'A}\, \bar{\pi}_{A})\pi_{A'} \\
&= \omega^{A}\, \bar{\pi}_{A} + \bar{\omega}^{A'}\, \pi_{A'} \qquad .
\end{aligned}$$

Consider now the collection of null lines $\{Y\}$ whose twistors $\{Y^{\alpha}\}$ satisfy $Y^{\alpha}\, \bar{Y}_{\alpha} = 0$ and $Z^{\alpha}\, \bar{Y}_{\alpha} = 0$ (i.e. the Robinson congruence determined by $\bar{Z}_{\alpha}$, if Z^{α} is not null; or the "congruence" of null lines intersecting the null lines given by Z^{α} , if Z^{α} is null). Let Y^{α} be represented by the spinors $(\xi^{A},\eta_{A'})$ and suppose that the line $Y \in \{Y\}$ passes through the point $x^{AA'}$. Then the equation $Z^{\alpha}\, \bar{Y}_{\alpha} = 0$ implies that

$$\omega^{A}\, \bar{\eta}_{A} + \bar{\xi}^{A'}\, \pi_{A'} = 0 \ ,$$

i.e. $$(\omega^{A} - i\, x^{AA'})\bar{\eta}_{A} = 0 \ ,$$

i.e. $$\tilde{\omega}^{A}\, \bar{\eta}_{A} = 0 \ , \tag{3.4.5}$$

using equations (3.2.7) and (3.2.1). This means that the field of null vectors W^{a} given by

$$W^{a} := \tilde{\omega}^{A}\, \bar{\tilde{\omega}}^{A'}$$

is such that the null vectors are tangent to the null lines in the congruence.

Equation (3.4.5) permits one to write

$$\omega^{A} = \rho\, \bar{\eta}^{A} \quad ,$$

where ρ is some differentable function on M . Also, since $\nabla^{AA'}\, \tilde{\omega}^{B}$ is skew in AB , it follows that

$$\tilde{\omega}_{A}\, \tilde{\omega}_{B}\, \nabla^{AA'}\, \tilde{\omega}^{B} = 0 \quad ,$$

and hence that

$$0 = \rho^2 \bar{\eta}_A \bar{\eta}_B \nabla^{AA'} (\rho \bar{\eta}^B)$$

$$= \rho^3 \bar{\eta}_A \bar{\eta}_B \nabla^{AA'} \bar{\eta}^B \quad . \tag{3.4.6}$$

The condition for a null congruence with tangent vector field $k^a = k^A k^{A'}$ to the geodesic and shearfree is given in Pirani (1) p.343, as

$$k^A k^B \nabla_{BB'} k_A^{\cdot} = 0 \quad .$$

Comparison with (3.4.6) shows that the shear of the congruence vanishes, provided $\rho \neq 0$. The case $\rho = 0$ only occurs when Z^α is null, and ρ is then zero on the null line Z determined by Z^α , i.e. where the lines of the "congruence" meet Z . Therefore every Robinson congruence is shear-free.

§3.5 The C-Picture.

It has been seen that a twistor Z^{α}, represented by $(\omega^{A},\pi_{A'})$, can be interpreted as representing a null line (if Z^{α} is null) or a Robinson congruence (if Z^{α} is not null). Neither of these geometric structures is altered if $(\omega^{A},\pi_{A'})$ is replaced by $(\lambda\omega^{A},\lambda\pi_{A'})$, where λ is a non-zero complex constant. Twistor space C will therefore be chosen to be a three-dimensional complex projective space, each point of which represents a non-zero twistor up to proportionality. The zero twistor, $Z^{\alpha} = 0$, has no geometric interpretation and has no place in the C-picture.

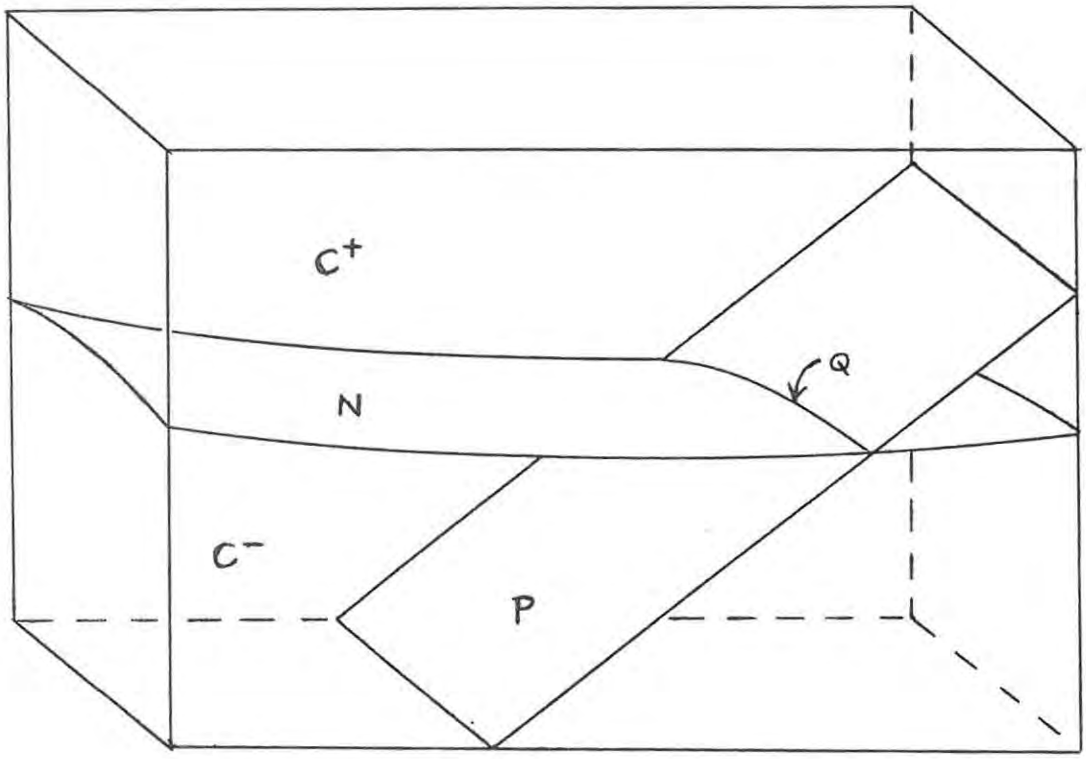
The spin s associated with a twistor Z^{α} is given by (cf. (3.1.32))

$$s = \tfrac{1}{2}Z^{\alpha}\bar{Z}_{\alpha} ,$$

while that assoicated with λZ^{α} is $\tfrac{1}{2}\lambda\bar{\lambda}Z^{\alpha}\bar{Z}_{\alpha}$. Since $\lambda\bar{\lambda} > 0$, each proportionality class of twistors in the C-picture (i.e. each point in C) can consequently be described as having positive, negative or zero spin.

Let N denote the subset of C consisting of those points having zero spin; C^{+} the set of points with positive spin; and C^{-} the set of points with negative spin (see fig. 5). The points of N are in one-to-one correspondence with the null lines in M'. Since C is six-real-dimensional and the equation $Z^{\alpha}\bar{Z}_{\alpha} = 0$ is essentially just one real equation, the subspace N is five-real-dimensional. The system of null lines in M' is also five-dimensional, as can be seen by considering a fixed spacelike hypersurface S in M. Each null geodesic in M intersects S in exactly one point P, so that the null lines in M can be labelled uniquely by specifying a point P on S (∞^3 choices) and a future-pointing null direction at P (∞^2 choices). Thus the null lines in M form an ∞^5 system. In M', the null lines at infinity must be counted as well; but these form an ∞^3 system (being the generators of the null cone at infinity) and do not increase the dimensionality of the

Fig. 5. The C-Picture.



system.

The notion of conjugation of a twistor (i.e. the connection between Z^α and $\bar{Z}_\alpha$) can be interpreted in the C-picture as follows. While a twistor Z^α determines a <u>point</u> in C , a twistor R_α determines a <u>plane</u> R in C , consisting of those points whose twistors Y^α satisfy

$$R_\alpha Y^\alpha = 0 \ . \qquad (3.5.1)$$

The word "plane" here means "complex plane", i.e. a linear four-real-dimensional subset of C; such a plane determines in turn a twistor R_α , unique up to proportionality. So the correspondence $Z^\alpha \leftrightarrow \bar{Z}_\alpha$ becomes, in the C-picture, a point$\leftrightarrow$ plane correspondence.

Let Z^α be a non-null twistor. The conjugate twistor $\bar{Z}_\alpha$ determines a Robinson congruence in M' , and also determines a plane P in C . Let Q denote the intersection of P with N ; then equation (3.5.1) shows that the points of Q represent a collection of null lines in M' which is exactly the Robinson congruence (see fig. 5).

If Z^α is null, it determines a point Z on N and its complex conjugate $\bar{Z}_\alpha$ determines, as above, a plane P in C . In this case, however, the point Z lies in the set $Q = P \cap N$, since the equation

$$\bar{Z}_\alpha Y^\alpha = 0$$

(cf. (3.5.1)) is satisfied for $Y^\alpha = Z^\alpha$. In other words, points with zero spin lie <u>in</u> their corresponding conjugate planes.

Null lines in M' correspond to points in C ; <u>points</u> in M' will now be interpreted in the C-picture. A point P in M' determines, and is determined by, an ∞^2 system of null lines in M' , namely the generators of the null cone of P . Let Z and Y be two of these generators; Z and Y then determine two points in C . Denote the conjugate planes of these points by $\bar{Z}$ and $\bar{Y}$ respectively, and let

$$L := \bar{Z} \cap \bar{Y} \quad .$$

Firstly, notice that L is not the empty set, since Z (being null) lies in $\bar{Z}$, and Z also lies in $\bar{Y}$, because the line Z intersects the line Y . Secondly, the set L , being the intersection of two planes in C , is a complex projective <u>straight line</u> in C . Thirdly, L lies enterely <u>in N</u> , because if Z^α and Y^α are two twistors corresponding to Z and Y respectevely, then a point on L corresponds to a twistor of the form

$$X^\alpha = Z^\alpha + \beta Y^\alpha , \qquad (3.5.2)$$

where β is a complex parameter. That X^α is null now follows from

$$X^\alpha \bar{X}_\alpha = (Z^\alpha + \beta Y^\alpha)(Z_\alpha + \bar{\beta} \bar{Y}_\alpha)$$
$$= 0 ,$$

since both Z and Y are null and since Z intersects Y . The twistor summation operation used in (3.5.2) is defined component-wise, i.e. if $(\omega^A, \pi_{A'})$ and $(\xi^A, \eta_{A'})$ represent Z^α and Y^α respectively, then $(\omega^A + \xi^A, \pi_{A'} + \eta_{A'})$ represents $Z^\alpha + Y^\alpha$.

Returning to the problem of representing the null cone of P , any generator of this null cone intersects both of the lines Z and Y . Therefore the point in C corresponding to this generator lies in both of the planes $\bar{Z}$ and $\bar{Y}$, i.e. it lies in L .

Conversely, let L be a (straight) line in N and suppose that Z and Y are two points on L . If Z^α and Y^α are two twistors corresponding, respectively, to Z and Y , then any other point X on L has a twistor of the form (3.5.2). Since Z , Y and X are all null, it follows that

$$Z^\alpha \bar{Z}_\alpha = 0 ,$$
$$Y^\alpha \bar{Y}_\alpha = 0 ,$$
$$(Z^\alpha + \beta Y^\alpha)(\bar{Z}_\alpha + \bar{\beta} \bar{Y}_\alpha) = 0 ,$$

and hence that

$$Z^{\alpha} \bar{Y}_{\alpha} = 0 \ .$$

Thus Z and Y, considered as null lines in M' , intersect. Since Z and Y were arbitrary points on L , the points of L must therefore correspond to an ∞^2 system of null lines in M' which intersect at a single point.

So, in addition to there being a one-to-one correspondence between null lines in M' and points in N , there is a one-to-one correspondence between points in M' and complex projective lines in N . Furthermore, it follows from the discussion above that a point will lie on a null line in M' if and only if the corresponding line in N passes through the corresponding point in N . This is reminiscent of the duality principle in projective geometry (Semple and Kneebone (1), p.79).

Consider now the collection of all lines in C (not only those lying in N). This is a four-dimensional complex system, which suggests that these lines correspond to points in M^* , the complexification of M' (for details of complexification, see Trautman (1), pp.47-48). This interpretation can be realised by using equation (3.2.10), namely

$$x^{AA'} = \left(- i \mid (\pi_{B'} \eta^{B'}) \right) (\omega^A \eta^{A'} - \xi^A \pi^{A'}) \ ,$$

where Z^{α} , represented by $(\omega^A, \pi_{A'})$, and Y^{α} , represented by $(\xi^A, \eta_{A'})$, are twistors corresponding to two points Z and Y on a given line L . For this to be consistent, it has to be shown that any other pair of points on L determine the same point $x^{AA'}$ in M^* . So replace Y^{α} , say, by $X^{\alpha} = Z^{\alpha} + \beta Y^{\alpha}$, where β is a complex constant. Then the point determined by Z^{α} and X^{α} is

$$\left[- i \mid \left(\pi_{B'} (\pi^{B'} + \beta \eta^{B'})\right) \right] \left(\omega^A (\pi^{A'} + \beta \eta^{A'}) - (\omega^A + \beta \xi^A) \ \pi^{A'} \right)$$

$$= \left[- i \mid (\pi_{B'} \eta^{B'}) \right] (\omega^A \eta^{A'} - \xi^A \pi^{A'})$$

$$= x^{AA'} \ ,$$

as required.

This section is concluded by mentioning a theorem due to Kerr, which bears on the problem of constructing all the shear-free null congruences in M' ; for more details, see Penrose (4). The theorem reads as follows:

A congruence of null lines in M' is shear-free if and only if this congruence can be represented in C as the intersection of N with a complex analytic hypersurface in C (or as a limiting case of such a construction).

A complex analytic hypersurface in C is given by an equation of the form

$$\phi(Z^{\alpha}) \equiv \phi(\omega^{A},\pi_{A'}) = 0 \quad ,$$

where ϕ is an analytic function of the four complex variables ω^0 , ω^1 , $\pi_{0'}$ and $\pi_{1'}$; in addition ϕ must be homogeneous in these four variables, because of the definition of C as a complex projective space.

§3.6 Twistor Transformations

Since twistors can be interpreted in terms of null lines in M', it is natural to consider the continuous transformations of M' onto itself which preserve its null line and null cone structure. Such transformations form a fifteen-parameter group, called the conformal group, which is generated (in the case of Minkowski space-time) by the conformal rescalings together with the Poincaré transformations (i.e. the inhomogeneous Lorentz transformations)(Kuiper (1), Penrose and MacCallum (1)). In the latter reference it is stated that the group SU(2,2) (the pseudo-unitary unimodular group of signature (++--)) is 4-1 homomorphic with the restricted conformal group (i.e. that subgroup of the conformal group connected with the identity). This statement will not be proved, but it is illustrated by the results of this section.

Firstly, notice that the signature of the form $Z^{\alpha}\bar{Z}_{\alpha}$ is (++--); this can be seen by introducing new twistor coordinates u, v, t, w, where

$$2u := Z^0 + Z^2 \quad ,$$

$$2v := Z^0 - Z^2 \quad ,$$

$$2t := Z^1 + Z^3 \quad ,$$

$$2w := Z^1 - Z^3 \quad ,$$

for the form $Z^{\alpha}\bar{Z}_{\alpha}$ then becomes

$$Z^{\alpha}\bar{Z}_{\alpha} = Z^0\bar{Z}^2 + Z^1\bar{Z}^3 + Z^2\bar{Z}^0 + Z^3\bar{Z}^1$$
$$= 2|u|^2 - 2|v|^2 + 2|t|^2 - 2|w|^2 \quad .$$

The (++--) signature suggests that twistors form a 4-1 representation space for the restricted conformal group. In fact, it will now be shown that conformal transformations of M' correspond to elements of the group SU(2,2) acting in twistor space.

Recall that null lines and null cones in M' correspond to points and straight lines in N, in such a way that a null line is a generator of a null cone in M' if and only if the corresponding point in N lies on the corresponding line. Therefore, since the null line structure of M' is conformally invariant, a conformal transformation of M' corresponds to a continuous transformation of N into itself which preserves its linearity structure. This transformation on N can be extended uniquely to a transformation on C, because corresponding to every point of C there is a spinor field $\tilde{\omega}^A$ on M satisfying equation (3.4.1), namely

$$\nabla^{A'(A}\tilde{\omega}^{B)} = 0 \ .$$

But this equation is conformally invariant, so if the conformal transformation on M sends $\tilde{\omega}^A$ into a new spinor field $\tilde{\xi}^A$, then

$$\nabla^{A'(A}\tilde{\xi}^{B)} = 0 \ ,$$

so that ξ^A will determine a unique point of C according to the scheme of section 3.4. In addition, since the linearity sturcture of C corresponds to the conformally invariant null line structure of M', the transformation on C preserves this linearity structure.

It follows that a conformal transformation of M' connected with the identity, becomes a <u>projective point transformation</u> in the C-picture (Semple and Kneebone (1), p.29); this means that twistors Z^α and Y_α transform as

$$Z^\alpha \rightarrow \tilde{Z}^\alpha = t^\alpha_\beta Z^\beta \ , \qquad (3.6.1a)$$

$$Y^\alpha \rightarrow \tilde{Y}_\alpha = T^\beta_\alpha Y_\beta \ , \qquad (3.6.1b)$$

under a (restricted) conformal transformation of M'.

The four-by-four matrices t^α_β and T^β_α are <u>non-singular</u> and are only defined up to proportionality, i.e. up to a constant multiplicative factor. This factor will be chosen so that

$$\det(t^\alpha_\beta) = 1 \ , \qquad \det(T^\beta_\alpha) = 1 \ . \qquad (3.6.2)$$

This choice is consistent with the choice of

$$\delta^{\alpha}_{\beta} := \text{diag}\ (1,1,1,1)$$

as the identity twistor transformation.

Since the orthogonality relation

$$Z^{\alpha}\ Y_{\alpha} = 0$$

between points and planes in C can be interpreted in terms of conformally invariant incidence properties of null line systems in M' , equations (3.6.1) imply that

$$\tilde{Z}^{\alpha}\ \tilde{Y}_{\alpha} = \lambda\ Z^{\alpha}\ Y_{\alpha}\ ,$$

that is

$$t^{\alpha}_{\beta}\ T^{\gamma}_{\alpha}\ Z^{\beta}\ Y_{\gamma} = \lambda\ Z^{\alpha}\ Y_{\alpha}\ ,$$

that is

$$t^{\alpha}_{\beta}\ T^{\gamma}_{\alpha} = \lambda\ \delta^{\gamma}_{\beta}\ ,$$

where λ is some (complex) scalar. Taking the determinant on both sides yields

$$\det\ (t^{\alpha}_{\beta}).\det(T^{\beta}_{\gamma}) = \lambda^{4}\ ,$$

which, using (3.6.2), leads to

$$\lambda^{4} = 1\ .$$

Equations (3.6.2) determine the constant of proportionality only up to the factor i^{n} , where n is an integer. Thus there is just enough freedom left to allow the further condition

$$\lambda = 1$$

to be imposed. It then follows that

$$t^{\alpha}_{\beta}\ T^{\gamma}_{\alpha} = \delta^{\gamma}_{\beta}, \tag{3.6.3a}$$

and, similarly,

$$t^{\alpha}_{\beta}\ T^{\beta}_{\gamma} = \delta^{\alpha}_{\gamma}\ . \tag{3.6.3b}$$

Complex conjugation when more than one twistor index is involved is defined by analogy with single-index twistors (cf.(3.1.30)): upper and lower index positions are interchanged and the pair 0,1 is interchanged with the pair 2,3 . For example, the matrix $[\bar{t}^{\alpha}_{\beta}]$ is given by

$$\bar{t}^{0}_{0} = \bar{t}^{2}_{2} , \quad \bar{t}^{0}_{1} = \bar{t}^{3}_{2} , \quad \bar{t}^{0}_{2} = \bar{t}^{0}_{2} , \quad \bar{t}^{0}_{3} = \bar{t}^{1}_{2} , \; \ldots , \; \bar{t}^{3}_{2} = \bar{t}^{0}_{1} , \quad \bar{t}^{3}_{3} = \bar{t}^{1}_{1} .$$

Using this definition, it is consistent to write the complex conjugate of (3.6.1a) as

$$\tilde{\bar{Z}}_{\alpha} = \bar{t}^{\beta}_{\alpha} \; \bar{Z}_{\beta} \quad ,$$

and comparison with (3.6.1b) yields

$$T^{\beta}_{\alpha} = \bar{t}^{\beta}_{\alpha} \quad ,$$

or

$$t^{\alpha}_{\beta} \; \bar{t}^{\beta}_{\gamma} = \delta^{\alpha}_{\gamma} \quad , \tag{3.6.4}$$

using (3.6.3).

By adopting a procedure similar to that used in "building up" tensors out of vectors (see, for example, Willmore (1), pp.172-179), twistors $A^{\alpha\beta\ldots\gamma}_{\rho\sigma\ldots\tau}$ may be defined; if the indices $\alpha, \beta, \ldots, \gamma$ are r in number and $\rho, \sigma, \ldots, \tau$ are s in number, the twistor is said to be of <u>valence (r,s)</u>. Under a restricted conformal transformation its transformation law will be (cf. (3.6.1))

$$\tilde{A}^{\alpha\ldots\gamma}_{\rho\ldots\tau} = t^{\alpha}_{\kappa} \ldots t^{\gamma}_{\nu} \; T^{\phi}_{\rho} \ldots T^{\psi}_{\tau} \; A^{\kappa\ldots\nu}_{\phi\ldots\psi} \quad . \tag{3.6.5}$$

Equation (3.6.4) means that the form $Z^{\alpha} \bar{Z}_{\alpha}$ is invariant under a conformal transformation (i.e. that the matrix $[t^{\alpha}_{\beta}]$ is pseudo-unitary) and also states that the complex conjugation operation is a <u>twistor operation</u>, in the sense that if $A^{\alpha\ldots\gamma}_{\rho\ldots\tau}$ is a twistor of valence (r,s), then $\bar{A}^{\rho\ldots\tau}_{\alpha\ldots\gamma}$ is a twistor of valence (s,r). The usual tensor operations such as addition and contraction between upper and lower indices are also twistor operations, in that they commute with the transformation (3.6.5).

Consider now the right-handed Robinson congruence R given by the twistor R_{α} with components

$$\underset{\sim}{R_{\alpha}} := 2^{-\frac{1}{2}} \, (1,0,1,0)^{1} .$$

For each real number θ , define a matrix $t(\theta)^{\alpha}_{\beta}$ by

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[1] This is a particular case of a Robinson congruence discussed in Penrose(4).

$$t(\theta)^{\alpha}_{\beta} := e^{i\theta}\,\delta^{\alpha}_{\beta} + (e^{-3i\theta} - e^{i\theta})\bar{R}^{\alpha}\,R_{\beta} \quad . \tag{3.6.6}$$

Firstly (noting that $\bar{R}^{\alpha}\,R_{\alpha} = 1$) it follows that

$$\begin{aligned} t(\theta)^{\alpha}_{\beta}\,\bar{t}(\theta)^{\beta}_{\gamma} &= \left(e^{i\theta}\,\delta^{\alpha}_{\beta} + (e^{-3i\theta} - e^{i\theta})\bar{R}^{\alpha}\,R_{\beta}\right)\left(e^{-i\theta}\,\delta^{\beta}_{\gamma} + (e^{3i\theta} - e^{-i\theta})\bar{R}^{\beta}\,R_{\gamma}\right) \\ &= \delta^{\alpha}_{\beta} + (e^{4i\theta}-1)\bar{R}^{\alpha}\,R_{\gamma} + (e^{-4i\theta}-1)\bar{R}^{\alpha}\,R_{\gamma} + (2 - e^{-4i\theta} - e^{4i\theta})\bar{R}^{\alpha}\,R_{\alpha} \\ &= \delta^{\alpha}_{\beta} \quad , \end{aligned}$$

so that (3.6.4) is satisfied.

Secondly, putting

$$p := p(\theta) := (e^{-4i\theta} - 1)|2 \quad ,$$

the determinant of $t(\theta)^{\alpha}_{\beta}$ becomes

$$\begin{aligned} & e^{4i\theta} \begin{bmatrix} 1+p & 0 & p & 0 \\ 0 & 1 & 0 & 0 \\ p & 0 & 1+p & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= e^{4i\theta} \begin{bmatrix} 1+p & 0 & p \\ 0 & 1 & 0 \\ p & 0 & 1+p \end{bmatrix} \\ &= e^{4i\theta}(1 + 2p + p^2 - p^2) \\ &= e^{4i\theta}(1 + e^{-4i\theta} - 1) \\ &= 1 \quad , \end{aligned}$$

so that $t(\theta)^{\alpha}_{\beta}$ is unimodular and is consequently an allowable twistor transformation.

Thirdly, since

$$\begin{aligned} t(\theta)^{\alpha}_{\beta}\,t(\phi)^{\beta}_{\gamma} &= \left(e^{i\theta}\,\delta^{\alpha}_{\beta} + (e^{-3i\theta}-e^{i\theta})\bar{R}^{\alpha}\,R_{\beta}\right)\left(e^{i\phi}\,\delta^{\beta}_{\gamma} + (e^{-3i\theta}-e^{i\phi})\,\bar{R}^{\beta}\,R_{\gamma}\right) \\ &= e^{i(\theta+\phi)}\,\delta^{\alpha}_{\gamma} + \bar{R}^{\alpha}\,R_{\gamma}\,\big(e^{i(\theta-3\phi)} - e^{i(\theta+\phi)} + e^{i(-3\theta+\phi)} - e^{i(\theta+\phi)} + \\ &\quad e^{-3i(\theta+\phi)} + e^{i(\theta+\phi)} - e^{i(-3\theta+\phi)} - e^{i(\theta-3\phi)}\big) \end{aligned}$$

$$= e^{i(\theta+\phi)}\,\delta^{\alpha}_{\gamma} + \bar{R}^{\alpha} R_{\alpha}\,(e^{-3i(\theta+\phi)} - e^{i(\theta+\phi)})$$

$$= t(\theta+\phi)^{\alpha}_{\gamma} \quad ,$$

these transformations form a one-parameter subgroup of SU(2,2).

Finally, if Z (represented by the twistor Z^{α}) is any member of the Robinson congruence R , then Z^{α} transforms as

$$t(\theta)^{\alpha}_{\beta} Z^{\beta} = e^{i\theta} Z^{\alpha} + (e^{-3i\theta} - e^{i\theta})\bar{R}^{\alpha} R_{\beta} Z^{\beta}$$

$$= e^{i\theta} Z^{\alpha} \quad ,$$

so that every null line in the congruence is invariant under the transformation.

Now, the transformation $t(0)^{\alpha}_{\beta}$ is both the identity twistor transformation and the identity transformation on twistor space C ; the transformation

$$t(\pi/2)^{\alpha}_{\beta} = i\,\delta^{\alpha}_{\beta} \quad , \qquad (3.6.7)$$

however, gives the identity transformation on C (and therefore also the identity transformation on M), while it multiplies every twistor of valence (1,0) by i . Furthermore, the transformation (3.6.7) is continuous with the identity twistor transfromation $t(0)^{\alpha}_{\beta}$, as can be seen by letting θ run from 0 to $\pi/2$ in (3.6.6). This means that twistors of valence (1,0) are <u>four-valued</u> under restricted conformal transformations. The same result clearly holds for any twistors of <u>odd</u> total valence.

Chapter 4. Twistor Description of Zero-Rest-Mass Fields

In this chapter it will be seen how solutions of the zero-rest-mass free-field equations may be written in terms of contour integrals of holomorphic (i.e. complex analytic) functions of twistors. More comprehensive discussions may be found in Penrose (5) and Penrose (6).

§ 4.1 The Contour Integral

In Minkowski space-time, replace the usual coordinates (x^0, x^1, x^2, x^3) by

$$\begin{aligned} u &:= (x^0 + x^1)/\sqrt{2} \ , \\ v &:= (x^0 - x^1)/\sqrt{2} \ , \\ \zeta &:= (x^2 + i\, x^3)/\sqrt{2} \ . \end{aligned} \tag{4.1.1}$$

Thus u and v are real and ζ is complex. The Minkowski metric (2.2.2) becomes

$$ds^2 = 2\, du\, dv - 2\, d\zeta\, d\bar{\zeta} \ . \tag{4.1.2}$$

Let f be a complex-valued function of three complex variables and suppose that f is holomorphic on some region of C^3. Then for $u, v,$ and ζ fixed, and λ a complex variable, the function g of λ given by

$$g(\lambda) := f(\lambda,\ u + \lambda\, \bar{\zeta}\ ,\ \zeta + \lambda\, v)$$

will be holomorphic in some region Γ of C. The region Γ will depend on u, v and ζ ; suppose that Γ is non-empty for $(u,v,\zeta) \in \Omega$, where Ω is some region of $R^2 \times C$.

For each set of values $(u,v,\zeta) \in \Omega$, choose a closed contour γ lying in Γ , in such a way that γ varies continuously with (u,v,ζ). Let s be a real number such that $2s$ is a non-negative integer, and for $r = 0, 1, \ldots, 2s$ define

$$\phi_r = (2\pi i)^{-1} \int_\gamma \lambda^r f(\lambda, u+\lambda\bar{\zeta}, \zeta+\lambda v)\, d\lambda \quad . \tag{4.1.3}$$

Notice that unless Γ is multiply connected and $g(\lambda)$ has non-zero periods (Ahlfors (1), p.146), ϕ_r will be identically zero.

From Ahlfors (1), p.123, it follows that ϕ_r is a differentiable function of u, v and ζ in Ω and that

$$\begin{aligned} \frac{\partial\phi_r}{\partial\bar{\zeta}} &= \frac{\partial\phi_{r+1}}{\partial u} \quad , \\ \frac{\partial\phi_r}{\partial v} &= \frac{\partial\phi_{r+1}}{\partial \zeta} \quad , \end{aligned} \tag{4.1.4}$$

for $r = 0,1,\ldots,2s-1$ and $s > 0$.

Now define a totally symmetric spinor $\phi_{AB\ \ldots.\ K} = \phi_{(AB\ \ldots.\ K)}$ with $2s$ indices by

$$\begin{aligned} \phi_0 &:= \phi_{00\ \ldots\ 0} \quad , \\ \phi_1 &:= \phi_{10\ \ldots\ 0} \quad , \\ &\ \ \vdots \\ \phi_r &:= \phi_{\underbrace{11\ \ldots\ 1}_{r}0\ \ldots\ 0} \quad , \\ &\ \ \vdots \\ \phi_{2s} &:= \phi_{11\ \ldots\ 1} \quad . \end{aligned}$$

Equations (4.1.4) become (using the total symmetry)

$$\begin{aligned} \frac{\partial\phi_{0BC\ldots K}}{\partial\bar{\zeta}} &= \frac{\partial\phi_{1BC\ldots K}}{\partial u} \quad , \\ \frac{\partial\phi_{0BC\ldots K}}{\partial v} &= \frac{\partial\phi_{1BC\ldots K}}{\partial\zeta} \quad . \end{aligned} \tag{4.1.5}$$

If $x^{AA'}$ is the spinor equivalent of x^a , equation (2.3.2) shows that

$$x_{\sim}^{A}{}_{\sim}^{A'} = \begin{bmatrix} u & \zeta \\ \bar{\zeta} & v \end{bmatrix} , \qquad (4.1.6a)$$

whence

$$x_{\underset{\sim}{A}\,\underset{\sim}{A'}} = \begin{bmatrix} v & -\bar{\zeta} \\ -\zeta & u \end{bmatrix} , \qquad (4.1.6b)$$

Using (4.1.6b), the equations (4.1.5) become

$$\frac{\partial \phi_{0\,BC\ldots K}}{\partial x_{0P'}} + \frac{\partial \phi_{1BC\ldots K}}{\partial x_{10'}} = 0 ,$$

or

$$\nabla^{AP'} \phi_{AB\ldots K} = 0 . \qquad (4.1.7)$$

Notice in passing that (4.1.4) also gives

$$\left\{ \frac{\partial^2}{\partial u \partial v} - \frac{\partial^2}{\partial \zeta \partial \bar{\zeta}} \right\} \phi_r = 0 , \qquad (4.1.8)$$

which is the wave equation in the coordinates (4.1.1).

In the case $s = 0$, it follows from (4.1.3) that

$$\frac{\partial^2 \phi_0}{\partial u \partial v} = \frac{\partial^2 \phi_0}{\partial \bar{\zeta} \partial \zeta} ,$$

and hence that

$$\nabla_{AA'} \nabla^{AA'} \phi_0 = 0 . \qquad (4.1.9)$$

Equations (4.1.7) and (4.1.9) are the zero-rest-mass free-field equations for spin $s > 0$ and spin $s=0$ respectively (Dirac (1)). Thus any holomorphic function f determines a zero-rest-mass free field in the region Ω . It is stated in Penrose (5) and Penrose (6) that most (if not all) analytic zero-rest-mass fields can be written in this contour integral form. However, no proof is given and this particular question remains open.

The theory described above is reminiscent of the way in which solutions of ordinary differential equations may be expressed as contour integrals (Burkill (1), pp.59-67). In both cases, the solution depends on the residues of the integrand function at its poles and on the homotopy class to which the contour belongs (Ahlfors (1), p.283).

§ 4.2 Transition to Twistor Notation.

Let Z be a null line passing through the point $x^{AA'}$ (with coordinates as in (4.1.6)); suppose that the direction of Z is given by $du : dv : d\zeta : d\bar{\zeta}$. Since the direction is null, ds = 0 in equation (4.1.2) and so

$$du\, dv = d\zeta\, d\bar{\zeta} . \tag{4.2.1}$$

If complex number λ (possibly infinite) is defined by

$$\lambda := - du/d\bar{\zeta} ,$$

then (4.2.1) shows that the line is given by

$$du + \lambda\, d\bar{\zeta} = 0 = d\zeta + \lambda\, dv ; \tag{4.2.2}$$

thus

$$du : dv : d\zeta : d\bar{\zeta} = \lambda\bar{\lambda} : 1 : -\lambda : -\bar{\lambda} . \tag{4.2.3}$$

Let Z^{α} be a null twistor corresponding to the null line Z and let Z^{α} be represented by $(\omega^{A}, \pi_{A'})$; then (cf.(4.1.6b)) it follows that

$$\underset{\sim}{\bar{\pi}_A}\ \underset{\sim}{\pi_{A'}} \propto \begin{bmatrix} dv & -d\bar{\zeta} \\ -d\zeta & du \end{bmatrix} . \tag{4.2.4}$$

Thence, writing $\underset{\sim}{W_{\alpha}} := \underset{\sim}{\bar{Z}_{\alpha}}$, equation (4.2.2) yields

$$\begin{aligned} \lambda &= \bar{\pi}_1 \mid \bar{\pi}_0 \\ &= W_1 \mid W_0 . \end{aligned} \tag{4.2.5}$$

The complex conjugate of equation (3.2.2) reads

$$\bar{\omega}^{A'} = - i\, x^{AA'}\, \bar{\pi}_A$$

and consequently

$$\begin{aligned} (W_2, W_3) &= (\bar{\omega}^{0'}, \bar{\omega}^{1'}) \\ &= - i\, (\bar{\pi}_0, \bar{\pi}_1) \begin{bmatrix} u & \zeta \\ \bar{\zeta} & v \end{bmatrix} \\ &= - i\, W_o (u + \lambda\bar{\zeta},\ \bar{\zeta} + \lambda\, v) . \end{aligned} \tag{4.2.6}$$

Combining (4.2.5) and (4.2.6) gives

$$(W_0, W_1, W_2, W_3) = W_0(1, \lambda, -i(u+\lambda\bar{\zeta}), -i(\zeta+\lambda v)) , \qquad (4.2.7)$$

and if f is a function of the type considered in section 4.1, then

$$f(\lambda, u+\lambda\bar{\zeta}, \zeta+\lambda v) = f(W_1|W_0, i W_2|W_0, i W_3|W_0) . \qquad (4.2.8)$$

Define the function F of one twistor variable by

$$F(W_\alpha) = (W_0)^{-s-2} f(W_1|W_0, i W_2|W_0, i W_3|W_0) ; \qquad (4.2.9)$$

then F is homogeneous of degree $(-s-2)$ in W_α. Now using (4.2.7), W_α can be written as

$$W_\alpha = W_0(U_\alpha + \lambda V_\alpha) , \qquad (4.2.10)$$

where

$$\underset{\sim}{U_\alpha} = (1, 0, -i u, -i \zeta) ,$$

$$\underset{\sim}{V_\alpha} = (0, 1, -i \bar{\zeta}, -i v) .$$

Then, since both

$$U_\alpha \bar{U}^\alpha = (i u + 0 - i u - 0) = 0$$

and

$$V_\alpha \bar{V}^\alpha = (0 + i v - 0 - i v) = 0 ,$$

U_α and V_α are null twistors; in fact it is easily seen from equation (4.2.5) that $\bar{U}^\alpha$ and $\bar{V}^\alpha$ represent null lines through the point $x^{AA'}$ given by $\lambda = 0$ and $\lambda = \infty$ respectively. Therefore U_α and V_α can be regarded as defining a spin frame at $x^{AA'}$; the values of the function ϕ_r for $r = 0,1,\ldots,2s$ (and hence the components of $\phi_{AB\ldots K}$) depend on the choice of spin frame and so ϕ_r is a function of U_α and V_α. The explicit form of $\phi_r(U_\alpha, V_\alpha)$ derived from equation (4.1.3), using (4.2.8), (4.2.9) and (4.2.10), is

$$\phi_r(U_\alpha, V_\alpha) = \frac{1}{2\pi i} \oint \lambda^r (W_0)^{s+2} F\left(W_0(U_\alpha + \lambda V_\alpha)\right) d\lambda$$

$$= \frac{1}{2\pi i} \oint \lambda^r F(U_\alpha + \lambda V_\alpha) d\lambda ,$$

by the homogeneity of F.

The main object of this discussion has been to illustrate the usefulness of having a twistor space endowed with a complex structure. Thus to deal with, say, scattering problems in curved space-times, it would be desirable to have curved-space twistors possessing such a complex structure.

Chapter 5. Twistors in Curved Space

The preceding chapters dealt with the concept of a twistor in Minkowski space-time, and these considerations carry over directly to twistors in conformally flat space-times, by virtue of the conformal invariance of twistors. The problem now is to extend twistor theory to space-times which are conformally curved (i.e. not conformally flat), so as to deal with the more general physical situations (such as the presence of non-zero mass and gravitation) which break conformal symmetry.

Three different curved-space twistor concepts will be mentioned in this chapter: local twistors, global twistors (briefly) and asymptotic twistors, while hypersurface twistors (a generalisation of asymptotic twistors) will be dealt with in chapter 6.

§ 5.1 Local Twistors

Local twistors, as the name suggests, are twistors defined at each point of a space time (M,G_{ab}) : if $P \in M$, then a local twistor Z^{α} at P may be represented by a pair of spinors $(\omega^{A},\pi_{A'})$ at P . Thus the space of local twistors of valence (1,0) can be regarded as a fibre bundle over M (Hawking and Ellis, p.50), the fibres of which consist of all spinor pairs $(\omega^{A},\pi_{A'})$. Each fibre, therefore, is the direct sum of a spin-space and a conjugate spin-space and is four-complex-dimensional.

If the theory is to be consistent with flat-space twistors, the spinor representation of a local twistor cannot be conformally invariant. Consistency is achieved if the behaviour of ω^{A} and $\pi_{A'}$ under a conformal rescaling is chosen to be (cf. section 3.4)

$$\hat{\omega}^{A} = \omega^{A} ,$$

$$\hat{\pi}_{A'} = \pi_{A'} + i\, T_{AA'}\, \omega^{A} . \qquad (5.1.1)$$

In order to compare local twistors at different points of space-time, the concept of a twistor covariant derivative is needed. This derivative will be required to possess the following properties: it should be linear, satisfy the Leibnitz rule, commute with twistor conjugation[1] and contraction, be conformally invariant and be consistent with flat-space theory. Getting "back and forth" between the twistor and its spinor representation can be effected by defining the operators

$$e_\alpha^{\ A} , \quad e_{\alpha A'} , \quad e_A^{\ \alpha} , \quad e^{\alpha A'} ,$$

such that

$$\begin{aligned} e_\alpha^{\ A} \, Z^\alpha &= \omega^A , \\ e_{\alpha A'} \, Z^\alpha &= \pi_{A'} , \\ e_A^{\ \alpha} \, \bar{Z}_\alpha &= \bar{\pi}_A , \\ e^{\alpha A'} \, \bar{Z}_\alpha &= \bar{\omega}^{A'} , \\ Z^\alpha &= e_A^{\ \alpha} \, \omega^A + e^{\alpha A'} \, \pi_{A'} , \\ \bar{Z}_\alpha &= e_{\alpha A'} \, \bar{\omega}^{A'} + e_\alpha^{\ A} \, \bar{\pi}_A . \end{aligned} \tag{5.1.2}$$

Some consistency relations can be derived from (5.1.2): firstly, $\overline{\omega^A} = \bar{\omega}^{A'}$ implies that

$$\overline{e_\alpha^{\ A} \, Z^\alpha} = \overline{e_\alpha^{\ A}} \, \bar{Z}_\alpha = e^{\alpha A'} \, \bar{Z}_\alpha ,$$

whence

$$\overline{e_\alpha^{\ A}} = e^{\alpha A'} , \tag{5.1.3a}$$

and similarly,

$$\overline{e_A^{\ \alpha}} = e_{\alpha A'} . \tag{5.1.3b}$$

Secondly,

$$\omega^A = e_\alpha^{\ A} \, Z^\alpha = e_\alpha^{\ A} \, e_B^{\ \alpha} \, \omega^B + e_\alpha^{\ A} \, e^{\alpha B'} \, \pi_{B'} ,$$

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[1] Local twistor conjugation is defined by analogy with flat-space twistors (cf. (3.1.30)): if $(\omega^A, \pi_{A'})$ represents Z^α at P , then $(\bar{\pi}_A, \bar{\omega}^{A'})$ represents its conjugate $\bar{Z}_\alpha$ at P .

whence
$$e_\alpha^{\ A}\, e_B^{\ \alpha} = \varepsilon_B^{\ A} \ , \quad e_\alpha^{\ A}\, e^{\alpha B'} = 0 \ , \tag{5.1.3c}$$
and similarly,
$$e_{\alpha A'}\, e^{\alpha B'} = \varepsilon_{A'}^{\ \ B'} \ , \quad e_{\alpha B'}\, e_A^{\ \alpha} = 0 \ . \tag{5.1.3d}$$
Finally,
$$\begin{aligned} Z^\alpha &= e_A^{\ \alpha}\, \omega^A + e^{\alpha A'}\, \pi_{A'} \\ &= e_A^{\ \alpha}\, e_\beta^{\ A}\, Z^\beta + e^{\alpha A'}\, e_{\beta A'}\, Z^\beta \ , \end{aligned}$$
and so
$$e_A^{\ \alpha}\, e_\beta^{\ A} + e^{\alpha A'}\, e_{\beta A'} = \delta_\beta^\alpha \ . \tag{5.1.3e}$$
Notice also, using (5.1.3), that
$$\begin{aligned} Z^\alpha\, \bar{Z}_\alpha &= (e_A^{\ \alpha}\, \omega^A + e^{\alpha A'}\, \pi_{A'})(e_{\alpha B'}\, \bar{\omega}^{B'} + e_\alpha^{\ B}\, \bar{\pi}_B) \\ &= \omega^A\, \bar{\pi}_A + \bar{\omega}^{A'}\, \pi_{A'} \ , \end{aligned}$$
as expected.

Under a conformal rescaling the twistor Z^α is required to be invariant, while ω^A and $\pi_{A'}$ transform according to (5.1.1). It follows that
$$\begin{aligned} \hat{e}_\alpha^{\ A}\, Z^\alpha &= \hat{e}_\alpha^{\ A}\, \hat{Z}^\alpha \\ &= \hat{\omega}^A \\ &= \omega^A \\ &= e_\alpha^{\ A}\, Z^\alpha \ , \end{aligned}$$
whence
$$\hat{e}_\alpha^{\ A} = e_\alpha^{\ A} \ ; \tag{5.1.4a}$$
and that
$$\begin{aligned} \hat{e}_{\alpha A'}\, Z^\alpha &= \hat{e}_{\alpha A'}\, \hat{Z}^\alpha \\ &= \hat{\pi}_{A'} \\ &= \pi_{A'} + i\, T_{AA'}\, \omega^A \\ &= e_{\alpha A'}\, Z^\alpha + i\, T_{AA'}\, e_\alpha^{\ A}\, Z^\alpha \ , \end{aligned}$$
whence
$$\hat{e}_{\alpha A'} = e_{\alpha A'} + i\, T_{AA'}\, e_\alpha^{\ A} \ . \tag{5.1.4b}$$

Applying the same argument to $\bar{Z}_\alpha$ yields

$$\hat{e}^{\alpha A'} = e^{\alpha A'} \,, \qquad (5.1.4c)$$

$$\hat{e}^{\alpha}_{A} = e^{\alpha}_{A} - i\, T_{AA'}\, e^{\alpha A'} \,.$$

The twistor covariant derivative operator will be denoted by a symbol such as ∇^{σ}_{ρ}. Since ∇^{σ}_{ρ} is linear, satisfies the Leibnitz property and commutes with (twistor) contraction, it follows that

$$\nabla^{\sigma}_{\rho}\, Z^{\alpha} = e^{\sigma S'}\, e^{R}_{\rho}\, \nabla_{RS'}\, (e^{\alpha}_{A}\, \omega^{A} + e^{\alpha A'}\, \pi_{A'})$$

$$= e^{\sigma S'}\, e^{R}_{\rho}\, (e^{\alpha}_{A}\, \nabla_{RS'}\, \omega^{A} + \omega^{A}\, \nabla_{RS'}\, e^{\alpha}_{A} + e^{\alpha A'}\, \nabla_{RS'}\, \pi_{A'} + \pi_{A'}\, \nabla_{RS'}\, e^{\alpha A'}) \,. \qquad (5.1.5)$$

Under a conformal rescaling, (5.1.5) becomes, using (2.3.11),

$$\hat{\nabla}^{\sigma}_{\rho}\, \hat{Z}^{\alpha} = \hat{\nabla}^{\alpha}_{\rho} Z^{\alpha}$$

$$= e^{\sigma S'}\, e^{R}_{\rho}\, \big(e^{\alpha}_{A}\, (\nabla_{RS'}\, \omega^{A} + \varepsilon_{R}{}^{A}\, T_{BS'}\, \omega^{B}) + \omega^{A}\, \hat{\nabla}_{RS'}\, e^{\alpha}_{A} + e^{\alpha A'} (\nabla_{RS'}\, \pi_{A'} - T_{RA'}\, \pi_{S'}) + \pi_{A'}\, \hat{\nabla}_{RS'}\, e^{\alpha A'}\big) \,. \qquad (5.1.6)$$

Conformal invariance is required (i.e. $\hat{\nabla}^{\sigma}_{\rho}\, \hat{Z}^{\alpha} = \nabla^{\sigma}_{\rho}\, Z^{\alpha}$) and so comparison of (5.1.5) and (5.1.6) yields

$$\hat{\nabla}_{RS'}\, e^{\alpha}_{A} = \nabla_{RS'}\, e^{\alpha}_{A} - e^{\alpha}_{R}\, T_{AS'} \,, \qquad (5.1.7a)$$

$$\hat{\nabla}_{RS'}\, e^{\alpha A'} = \nabla_{RS'}\, e^{\alpha A'} + e^{\alpha B'}\, T_{RB'}\, \varepsilon_{S'}{}^{A'} \,; \qquad (5.1.7b)$$

notice the close similarity between (5.1.7) and (2.3.11).

The other desirable requirement is that a <u>constant</u> local twistor (i.e. one for which $\nabla^{\sigma}_{\rho}\, Z^{\alpha} = 0$) in <u>Minkowski</u> space-time, should correspond to a flat-space twistor of the type discussed in chapter 3. The object ω^{A} corresponds to the spinor field $\tilde{\omega}^{A}$ of section 3.4, and it satisfies

$$\nabla_{RS'}\, \omega^{A} = -\, i\, \pi_{S'}\, \varepsilon_{R}{}^{A} \,, \qquad (5.1.8a)$$

while the spinor $\pi_{A'}$ represents a constant spinor field, so that

$$\nabla_{RS'}\, \pi_{A'} = 0 \,. \qquad (5.1.8b)$$

Substitution of (5.1.8) into (5.1.5) then yields

$$0 = -i\, e^{\alpha}_{R}\, \pi_{S'} + \omega^{A}\, \nabla_{RS'}\, e^{\alpha}_{A} + \pi_{A'}\, \nabla_{RS'}\, e^{\alpha A'} ,$$

which must then hold identically, irrespective of the values of ω^{A} and $\pi_{A'}$; this means that

$$\begin{aligned} \nabla_{RS'}\, e^{\alpha}_{A} &= 0 , \\ \nabla_{RS'}\, e^{\alpha A'} &= i\, \varepsilon_{S'}{}^{A'}\, e^{\alpha}_{R} . \end{aligned} \qquad (5.1.9)$$

Reverting now to general space-time, suppose that

$$\nabla_{RS'}\, e^{\alpha A'} = i\, \varepsilon_{S'}{}^{A'}\, e^{\alpha}_{R} , \qquad (5.1.10a)$$

$$\nabla_{RS'}\, e^{\alpha}_{A} = i\, P_{RAS'A'}\, e^{\alpha A'} . \qquad (5.1.10b)$$

Then (5.1.9) is satisfied (since $P_{RAS'A'}$ vanishes conformally flat space-time) and it only remains to check (5.1.7): this will now be done, using (5.1.4) and (2.3.19).

From (5.1.10a) it follows that

$$\begin{aligned} \hat{\nabla}_{RS'}\, e^{\alpha A'} &= \hat{\nabla}_{RS'}\, \hat{e}^{\alpha A'} \\ &= i\, \hat{\varepsilon}_{S'}{}^{A'}\, \hat{e}^{\alpha}_{R} \\ &= i\, \varepsilon_{S'}{}^{A'}\, (e^{\alpha}_{R} - i\, T_{RA'}\, e^{\alpha A'}) \\ &= \nabla_{RS'}\, e^{\alpha A'} + e^{\alpha A'}\, T_{RA'}\, \varepsilon_{S'}{}^{A'} , \end{aligned}$$

which agrees with (5.1.7b).

Equation (5.1.10b) gives

$$\begin{aligned} \hat{\nabla}_{RS'}\, e^{\alpha}_{A} &= \hat{\nabla}_{RS'}(\hat{e}^{\alpha}_{A} + i\, T_{AA'}\, e^{\alpha A'}) \\ &= i\, \hat{P}_{RAS'A'}\, e^{\alpha A'} + i\, e^{\alpha A'}\, \hat{\nabla}_{RS'}\, T_{AA'} + i\, T_{AA'}\, i\, \varepsilon_{S'}{}^{A'}(e^{\alpha}_{R} - iT_{RB'}\, e^{\alpha B'}) \\ &= i\, e^{\alpha A'}(P_{RAS'A'} - \nabla_{RS'}\, T_{AA'} + T_{RA'}\, T_{AS'}) \\ &\quad + i\, e^{\alpha A'}(\nabla_{RS'}\, T_{AA'} - T_{AS'}\, T_{RA'} - T_{RA'}\, T_{AS'}) - T_{AS'}\, e^{\alpha}_{R} \\ &\quad + i\, T_{AS'}\, T_{RA'}\, e^{\alpha A'} \\ &= \nabla_{RS'}\, e^{\alpha}_{A} - T_{AS'}\, e^{\alpha}_{R} , \end{aligned}$$

as in (5.1.7a).

The complex conjugates of (5.1.10a) and (5.1.10b) read

$$\nabla_{RS'}\, e_{\alpha}^{A} = -\, i\, \varepsilon_{R}{}^{A}\, e_{\alpha S'} \,, \tag{5.1.10c}$$

$$\nabla_{RS'}\, e_{\alpha A'} = -\, i\, P_{S'A'RA}\, e_{\alpha}^{A} \quad . \tag{5.1.10d}$$

By using (5.1.10), equation (5.1.5) and its complex conjugate can be rewritten as

$$\nabla_{\rho}^{\sigma} Z^{\alpha} = e^{\sigma S'} e_{\rho}^{R} \{ e_{A}^{\alpha} (\nabla_{RS'} \omega^{A} + i\, \varepsilon_{R}{}^{A} \pi_{S'}) + e^{\alpha A'} (\nabla_{RS'} \pi_{A'} + i\, P_{RBS'A'} \omega^{B}) \}, \tag{5.1.11a}$$

$$\overline{\nabla_{\rho}^{\sigma} Z^{\alpha}} = e_{\sigma}^{R} e^{\rho S'} \{ e_{\alpha A'} (\nabla_{RS'} \bar{\omega}^{A'} - i\, \varepsilon_{S'}{}^{A'} \bar{\pi}_{R}) + e_{\alpha}^{A} (\nabla_{RS'} \bar{\pi}_{A} - i\, P_{RAS'B'} \bar{\omega}^{B'}) \} \quad ; \tag{5.1.11b}$$

and it follows immediately from (5.1.11) that

$$\overline{\nabla_{\rho}^{\sigma} Z^{\alpha}} = \nabla_{\sigma}^{\rho} \bar{Z}_{\alpha} \quad ,$$

i.e. that the operator ∇_{ρ}^{σ} commutes with complex conjugation.

Notice that although the choice (5.1.10) satisfies all the requirements for a covariant derivative, it has not been shown that this is the only possible choice; it remains an open question as to whether or not the suppositions (5.1.10) are necessary.

This section is concluded by remarking that local twistors, being closely tied to points in space-time, cannot be used as a basis for a formalism in which points are derived objects. However, local twistors are used in formulating the concepts of twistors relative to a hypersurface and, in particular, of asymptotic twistors.

§ 5.2 Asymptotic Twistors

Throughout this section, let (M, g_{ab}) be an asymptotically flat space-time and $(\hat{M}, \hat{g}_{ab})$ its compactification, in the sense discussed at the end of section 2.2.

The only part of global twistor theory that will be necessary here is summed up in the following definition: a null global twistor Z^{α} is a null geodesic Z in (M, g_{ab}) and a spinor $\pi_{A'}$, parallelly propagated along Z .

The problem with the space of global twistors is that it possesses too little structure. From the work of Roger Penrose it appears that although the concept of a global twistor is useful, other directions will be more fruitful in extending twistor theory to curved space-times. For instance it would be desirable to have a twistor space which possessed a complex analytic structure, since in that case contour integrals (cf. chapter 4) could be defined.

For a space-time which is asymptotically flat, it is in fact possible to define this sort of twistor space, relative to the hull hypersurfaces at infinity (namely $\mathscr{I}^-$ and $\mathscr{I}^+$). The twistors in this space will be called asymptotic twistors; the relevant twistor structure will be different from that in Minkowski space-time, since there is enough "residual" curvature at infinity to "curve" the twistor space. The hypersurface $\mathscr{I}^+$ will be used below, but the same arguments apply to asymptotic twistors constructed on $\mathscr{I}^-$.

The hypersurface $\mathscr{I}^+$, being null, is generated by null geodesics and these null geodesics can be labelled by a complex parameter ζ , the topology of $\mathscr{I}^+$ being $R^1 \times S^2$ (Hawking and Ellis (1)). Through each point of $\mathscr{I}^+$ passes exactly one of these generators, and so at each point of $\mathscr{I}^+$ it is possible to choose a spinor ι^A such that the flagpole of

ι^A (cf. section 2.3) points along the generator through that point. Since the generators are null geodesics, the ι^A can be specified along each generator by parallel propagation (Pirani(1), p.343), so that

$$\iota^B \iota^{B'} \nabla_{BB'} \iota^A = 0 \ .^{1} \qquad (5.2.1)$$

Consider local twistors Z^α on $\mathscr{I}^+$ which have a representation of the form

$$(\xi \iota^A, \pi_{A'}) \ , \qquad (5.2.2)$$

where ξ is a complex number, at each point of $\mathscr{I}^+$. In order to equate local twistors at different points on the same generator γ, local twistor transport is used; this transport is defined by

$$\Lambda^\alpha := \iota^B \iota^{B'} e^\rho_B e_{\sigma B'} \nabla^\sigma_\rho Z^\alpha = 0 \ ,$$

which holds if and only if

$$\Lambda^\alpha e^A_\alpha = 0 \ , \quad \Lambda^\alpha e_{\alpha A'} = 0 \ , \qquad (5.2.3)$$

in view of (5.1.3e). Using equations (5.1.3), (5.1.11) and (5.2.2) it is seen that the condition (5.2.3) is equivalent to

$$\begin{aligned} \iota^B \iota^{B'} \nabla_{BB'} (\xi \iota^A) &= - i \iota^A \iota^{B'} \pi_{B'} \ , \\ \iota^B \iota^{B'} \nabla_{BB'} \pi_{A'} &= - i \xi P_{BCB'A'} \iota^B \iota^{B'} \iota^C \ . \end{aligned} \qquad (5.2.4)$$

Writing

$$\pi_{1'} := \iota^{A'} \pi_{A'} \ ,$$

$$\nabla_{11'} := \iota^A \iota^{A'} \nabla_{AA'} \ ,$$

$$P_{111'A'} := P_{BAB'A'} \iota^B \iota^A \iota^{B'} \ ,$$

and using equation (5.2.1), the equaitons (5.2.4) can be expressed as

$$\nabla_{11'} \xi = - i \pi_{1'} \ , \qquad (5.2.5)$$

$$\nabla_{11'} \pi_{A'} = - i \xi P_{111'A'} \ . \qquad (5.2.6)$$

---

1 Strictly speaking, $\iota^{B'}$ should be written $\bar{\iota}^{B'}$ and $\nabla_{BB'}$ should be written $\hat{\nabla}_{BB'}$ (cf. section 2.2), but the "bar" and "hat" will both be dropped in this section.

Notice that the form (5.2.2) is preserved by equations (5.2.5) and (5.2.6). Since the generator γ is simply connected and one-dimensional, no integrability problems can arise. Consequently, the local twistors of the form (5.2.2) on γ form a three-complex-dimensional vector space, parameterized at any point of γ by ξ and the two complex components of $\pi_{A'}$. Considering all the generators of $\mathscr{I}^+$ (labelled by ζ) gives a four-complex-dimensional space of local twistors, called the space of <u>asymptotic twistors</u> relative to $\mathscr{I}^+$.

Let Z^α be null global twistor in M, consisting of a null geodesic Z in M and a spinor $\pi_{A'}$ parallelly propagated along Z. The line Z will also be a null geodesic in the space-time $(\hat{M},\hat{g}_{ab})$, since (cf. section 2.2) the mapping

$$\phi : (M,g_{ab}) \to (\hat{M},\hat{g}_{ab})$$

is conformal. Since (M,g_{ab}) is asymptotically simple, Z intersects $\mathscr{I}^+$ at some point, say P. The spinor $\pi_{A'}$ is also parallelly propagated with respect to the metric $\hat{g}_{ab}$, since, by (2.3.11),

$$\pi^{B'}\bar{\pi}^{B}\hat{\nabla}_{BB'}\pi_{A'} = \pi^{B'}\bar{\pi}^{B}(\nabla_{BB'}\pi_{A'} - T_{BA'}\pi_{B'})$$
$$= \pi^{B'}\bar{\pi}^{B}\nabla_{BB'}\pi_{A'} .$$

It follows that Z^α determines a local twistor $(0,\pi_{A'})$ at P. Conversely, a point $P \in \mathscr{I}^+$ and a local twistor $(0,\pi_{A'})$ at P determine a unique null global twistor Z^α in M.

The local twistor $(0,\pi_{A'})$ is of the form (5.2.2) and so, if γ is the generator through P, local twistor transport along γ produces a local twistor $(\xi\iota^A,\pi_{A'})$ at another point (say Q) on γ. Conversely, given a local twistor $(\xi\iota^A, \pi_{A'})$ at $Q \in \mathscr{I}^+$, one can work back to find a point P at which $\xi = 0$ (provided such a point exists) and hence find the corresponding global twistor Z^α. It is quite possible that such a point P will <u>not</u> exist, since ξ has two degrees of freedom and γ is only one-dimensional. This question is partially resolved by the following

Theorem: Suppose that $(\xi\,\iota^A, \pi_{A'})$ is a local twistor at $Q \in \mathcal{I}^+$ for which $\pi_{1'} \neq 0$, and let γ be the generator through Q. Then a point P on γ, such that $\xi = 0$ at P, exists if and only if, at Q,

$$\mathrm{Re}\,(\bar{\xi}\,\pi_{1'}) = 0\ . \qquad (5.2.7)$$

Proof:
Let u be a (real) parameter along γ, so that the operator $\nabla_{11'}$ is identically equal to the operator $d|du$ when applied to scalar functions on γ. Then equation (5.2.5) can be written as

$$d\xi|du = -\,i\,\pi_{1'}\ . \qquad (5.2.8)$$

In Penrose (3) it is shown that the conformal factor Ω can be chosen so that the divergence of the generators of $\mathcal{I}^+$ vanishes, and that this choice of Ω implies that

$$P_{111'A'} = 0\ . \qquad (5.2.9)$$

Combining (5.2.6) and (5.2.9) yields

$$\nabla_{11'}\,\pi_{A'} = 0\ , \qquad (5.2.10)$$

which, with equation (5.2.1), leads to

$$d\,\pi_{1'}|du = 0\ . \qquad (5.2.11)$$

Equation (5.2.11) implies that $\pi_{1'}$ is constant along γ. Equation (5.2.8) can then be solved, giving

$$\xi\big(u(P)\big) = -\,i\,\pi_{1'}\,u(P) + \xi\big(u(Q)\big) + i\,\pi_{1'}\,u(Q)\ ,$$

which is zero if and only if

$$u(P) = u(Q) - i\,\xi/\pi_{1'}\ , \qquad (5.2.12)$$

where $\xi = \xi\big(u(Q)\big)$ is the original value of ξ. The condition (5.2.7) implies that $i\,\xi\,|\,\pi_{1'}$ is <u>real</u>, so that (5.2.12) gives a real point P on γ at which $\xi = 0$. Conversely, if (5.2.7) does <u>not</u> hold, then no such point exists.

Consider now the case $\pi_{1'} = 0$, i.e.

$$\pi_{A'} = \eta\,\iota_{A'}\ ,$$

for some complex number η. Then the form (5.2.2) becomes

$$(\xi \iota^A, \eta \iota_{A'}) \quad , \tag{5.2.13}$$

and it follows from (5.2.8), (5.2.10) and (5.2.1) that both ξ and η are constant along γ. This situation can be interpreted in Minkowski space-time as follows. The hypersurface $\mathscr{I}^+$ becomes a null cone having the point I as vertex. As $\pi_{1'}$ tends to zero in (5.2.12), the point P tends to the point I. Thus the null geodesic Z determined by a local twistor of the form (5.2.13) on $\mathscr{I}^+$, passes through the point I at infinity. But all null geodesics through I are generators of the null cone of I, i.e. generators of $\mathscr{I}^+$, and therefore Z is a generator of $\mathscr{I}^+$. In Minkowski space-time, therfore, asymptotic twistors for which $\pi_{1'}$ is zero can be identified with null global twistors lying entirely on $\mathscr{I}^+$.

It may also be possible to carry out this sort of identification in space-times which are not Minkowski, but are still asymptotically flat. It is stated in Penrose and MacCallum (1) that, with a further specialization of the conformal factor, a type of local twistor transport over <u>any</u> curve in $\mathscr{I}^+$ (not necessarily a generator) may be defined, and that this transport is <u>integrable</u>. This proposition is not proved there, but it will be assumed here.

Given that this transport is integrable, it is meaningful to talk about equivalence between local twistors of the form (5.2.13) at any two points of $\mathscr{I}^+$, not necessarily on the same generator; the resulting equivalence classes will be called <u>asymptotic twistors entirely on $\mathscr{I}^+$</u>. With local twistors of the form (5.2.2), having $\pi_{1'} \neq 0$, equivalence can only be defined along generators of $\mathscr{I}^+$; the corresponding equivalence classes are called <u>asymptotic twistors not entirely on $\mathscr{I}^+$</u>. The collection of both these two types of asymptotic twistors forms the space of asymptotic twistors (relative to $\mathscr{I}^+$).

An asymptotic twistor which satisfies equation (5.2.7), is said to be null. From the above results it then follows that the null asymptotic twistors relative to $\mathscr{I}^+$ are in one-to-one correspondence with the null global twistors in $\hat{M}$.

The vector space operations of addition and scalar multiplication are defined in the usual way, but it should be noted that two asymptotic twistors can in general only be added if they are associated with the same generator of $\mathscr{I}^+$.

The next matter to be considered is that of defining a complex analytic structure on asymptotic twistor space. As a first step, the hypersurface $\mathscr{I}^+$ will be complexified: for this to be possible, it is necessary for $\mathscr{I}^+$ to be analytic in the real sense.

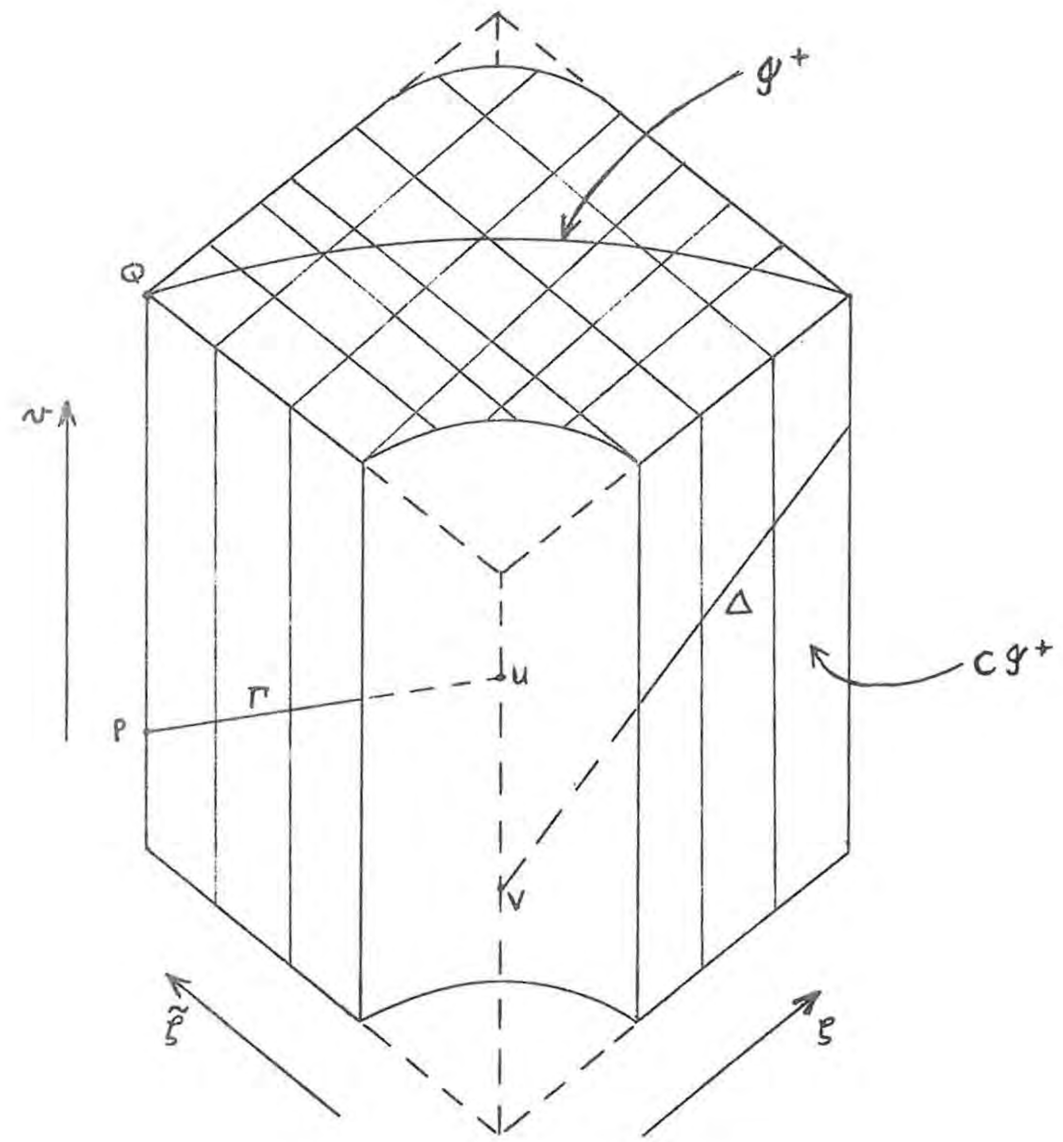
The coordinates u, ζ are used for $\mathscr{I}^+$: the complex number ζ to label generators of $\mathscr{I}^+$ and the real number u as a parameter along the generators. Complexify $\mathscr{I}^+$ by replacing u by a complex parameter v , and replacing ζ and its complex conjugate by two independent complex numbers ζ and $\tilde{\zeta}$ respectively. Denote the complexified hypersurface by $C\mathscr{I}^+$; then the original hypersurface $\mathscr{I}^+$ can be retrieved by letting v be real and letting $\tilde{\zeta} = \bar{\zeta}$. The hypersurface $\mathscr{I}^+$ has the topology $R^1 \times S^2$ and the choice of coordinates on $C\mathscr{I}^+$ ensures that $C\mathscr{I}^+$ has the topology $R^2 \times S^2 \times S^2$. By restricting $C\mathscr{I}^+$ to a sufficiently small neighbourhood of $\mathscr{I}^+$ in $R^2 \times S^2 \times S^2$, it can be ensured that $C\mathscr{I}^+$ is singularity-free (see figure 6).

Another proposition stated, but not proved, in Penrose and MacCallum (1), is that it is possible to choose the conformal factor and the scaling of ι^A so that

$$\nabla_{BB'} \iota^A = 0 \quad . \tag{5.2.14}$$

This proposition is not difficult to justify if the conformal structure of $\mathscr{I}^+$ is considered in detail, as for instance in Penrose (2).

Fig. 6. The Complex Hypersurface C$\mathcal{J}^+$.



It will be assumed here that such a choice has been made, and that it is consistent with the specialization of the conformal factor made earlier. Notice that the condition (5.2.1) is not violated.

The generalisation of (5.2.4), specifying parallel transport in an arbirtrary direction $t^{AA'}$, will be needed; it is defined by

$$t^{BB'} \nabla_{BB'} (\xi \iota^{A}) = - i\, t^{AB'} \pi_{B'} ,$$
$$t^{BB'} \nabla_{BB'} \pi_{A'} = - i\, \xi\, P_{BCB'A'} t^{BB'} \iota^{C} . \qquad (5.2.15)$$

Now suppose that a local twistor of the form $(\xi \iota^{A}, \pi_{A'})$, with $\pi_{1'} \neq 0$, is given at $Q \in \mathscr{I}^{+}$ and let γ be the generator of $\mathscr{I}^{+}$ through Q. Since v is complex, equation (5.2.12) can be used to find a point P on $C\gamma$ in $C\mathscr{I}^{+}$ at which the local twistor takes the form

$$(0, \pi_{A'}) . \qquad (5.2.16)$$

Given a local twistor of the form (5.2.16) at a point P in $C\mathscr{I}^{+}$, define Γ to be the null line through P with tangent $t^{AA'}$ given by

$$t^{AA'} = \iota^{A} \pi^{A'} , \qquad (5.2.17)$$

where ι^{A} and $\pi^{A'}$ are determined by (5.2.14) and (5.2.15) respectively. Because of the presence of the factor ι^{A}, the line Γ will lie in $C\mathscr{I}^{+}$. Now carry the local twistor (5.2.16) along Γ by local twistor transport; the transport equations are (5.2.15), which, using (5.2.14) and (5.2.16), become

$$\iota^{B} \pi^{B'} \nabla_{BB'} \xi = 0 ,$$
$$\iota^{B} \pi^{B'} \nabla_{BB'} \pi_{A'} = 0 . \qquad (5.2.18)$$

Equations (5.2.14) and (5.2.18) may be combined to give

$$\iota^{B} \pi^{B'} \nabla_{BB'} (\iota^{A} \pi^{A'}) = 0 , \qquad (5.2.19)$$

i.e.

$$t^{BB'} \nabla_{BB'} t^{AA'} = 0 ,$$

which means that Γ is a null geodesic (Pirani (1), p.333).

Suppose now that the coordinates u and ζ on $\mathscr{I}^+$ are chosen so that, on $C\mathscr{I}^+$, the surfaces ζ = constant have tangent vectors of the form $\iota^A \kappa^{A'}$ and the surfaces $\tilde{\zeta}$ = constant have tangent vectors of the form $\tau^A \iota^{A'}$, for some spinors $\kappa^{A'}$ and τ^A respectively. It is stated in Penrose and MacCallum (1) that such a choice can be made and it will be assumed here. As a consequence, equation (5.2.17) shows that the null geodesic Γ lies entirely in one ζ = constant surface (see figure 6).

So a local twistor $(\xi\,\iota^A, \pi_{A'})$ at Q in $\mathscr{I}^+$ determines a unique null geodesic Γ in some ζ = constant surface in $C\mathscr{I}^+$, with a spinor $\pi_{A'}$ defined along Γ. Conversely, given a null geodesic Γ lying in a ζ = constant surface, the tangent to Γ has the form $\iota^A \xi^{A'}$ for some spinor $\xi^{A'}$; comparison with (5.2.17) yields $\xi^{A'} = \pi^{A'}$, i.e. a spinor $\pi_{A'}$ is defined along Γ. Equations (5.2.14) and (5.2.19) (Γ being a null geodesic) imply equation (5.2.18), showing that the interpretation of $\pi_{A'}$ is consistent. Let P be the point on Γ where $\tilde{\zeta} = \bar{\zeta}$ (provided such a point exists) and let $C\gamma$ be the (complex) generator of $C\mathscr{I}^+$ which passes through P. Starting with $(0, \pi_{A'})$ at P, local twistor transport along $C\gamma$ produces a local twistor of the form $(\xi\,\iota^A, \pi_{A'})$ at each point of $C\gamma$; this local twistor on $\gamma = \mathrm{Re}\,(C\gamma)$ gives a unique <u>asymptotic twistor</u> not entirely on $\mathscr{I}^+$.

The existence and uniqueness of the point P is ensured if the coordinate $\tilde{\zeta}$ can be used to label different points of Γ. If $\tilde{\zeta}$ <u>cannot</u> be so used, it means (since the tangent to Γ has the form (5.2.17) and the tangents to $\tilde{\zeta}$ = constant surfaces have the form $\tau^A \iota^{A'}$) that $\pi^{A'}$ is proportional to $\iota^{A'}$, whence $\pi_{1'} = 0$. In this case, therefore, Γ will determine an asymptotic twistor entirely on $\mathscr{I}^+$.

By a similar procedure to that given above, a null geodesic Δ lying in the surface $\tilde{\zeta}$ = constant can be used to derive a <u>dual</u> asymptotic twistor not entirely on $\mathscr{I}^+$. If Δ lies in both a $\tilde{\zeta}$ = constant surface and a ζ = constant surface, then, as above, the asymptotic twistor determined by Δ will have $\pi_{1'} = 0$, i.e. will be entirely on $\mathscr{I}^+$.

Returning to the case where a null geodesic Γ lies in a ζ = constant surface, and determines an asymptotic twistor not entirely on $\mathscr{I}^+$, it may turn out that the point P in C $\mathscr{I}^+$ also lies in $\mathscr{I}^+$. Then the theorem proved above implies that condition (5.2.7) is satisfied, and hence (by definition) that the asymptotic twistor is null and determines a null global twistor.

Two null geodesics in a ζ = constant surface are said to be equivalent if they are equal as point sets, and if their tangents are proportional at every point along them. The space C is defined by requiring the points of C to be equivalence classes of null geodesics in ζ = constant surfaces. If Γ and Γ' are equivalent null geodesics and if Γ determines a null asymptotic twistor, then so does Γ' ; thus it is consistent to define N to be that subspace of C whose points determine (equivalence classes of) null asymptotic twistors. Finally, let I be the subspace of C whose points determine (equivalence classes of) asymptotic twistors entirely on $\mathscr{I}^+$; this definition is also consistent. Notice that the points of N are in one-to-one correspondence with the null geodesics in $\hat{M}$.

The "complexified" theory of asymptotic twistors has been described without introducing the idea of complex conjugate; this means that the space C has a complex structure. This complex structure is however only defined locally, in the sense that the hypersurface C $\mathscr{I}^+$ only exists near $\mathscr{I}^+$.

The asymptotic analogue of the Kerr theorem (cf. section 3.5) will now be considered. Let S be a complex analytic surface in C − I ; then S is a complex analytic congruence of (equivalence classes of) null geodesics in C $\mathscr{I}^+$. The intersection of S with N defines a field of spinors

$\pi_{A'}$ (up to proportionality) on $\mathscr{I}^+$; this spinor field must satisfy (5.2.18). If the spinors are defined away from $\mathscr{I}^+$ by parallel propagation along the null geodesics λ in $\hat{M}$ which intersect $\mathscr{I}^+$ in the directions determined by the $\pi_{A'}$ field, then the equation

$$\bar{\pi}^B \pi^{B'} \nabla_{BB'} \pi_{A'} = 0 \qquad (5.2.20)$$

is also satisfied. The spinors ι^B and $\bar{\pi}^B$ form a spinor basis (since points of I have been excluded so that $\pi_{1'} \neq 0$) and thus equations (5.2.18) and (5.2.20) can be combined to yield

$$\pi^{B'} \nabla_{BB'} \pi_{A'} = 0 \ ,$$

whence

$$\pi^{A'} \pi^{B'} \nabla_{BB'} \pi_{A'} = 0 \qquad (5.2.21)$$

at $\mathscr{I}^+$. Equation (5.2.21) is the condition for the congruence λ to be asymptotically shear-free (Newman and Penrose (1)).

The final concept to be defined is that of a scalar product for asymptotic twistors. Let Z^α denote an asymptotic twistor and W_α a dual asymptotic twistor; suppose that neither Z^α nor W_α is entirely on $\mathscr{I}^+$. Let Z^α correspond to the null geodesic Γ in the surface $\zeta = \zeta_1$ (constant) and let W_α correspond to Δ in the surface $\tilde{\zeta} = \tilde{\zeta}_2$ (constant). These two surfaces will intersect in a complex line L . Let U and V be the points where Γ and Δ , respectively, intersect L . If the segment of L between U and V lies outside C $\mathscr{I}^+$ (as in figure 6), the scalar product of Z^α and W_α is not defined, so suppose that this segment lies in C $\mathscr{I}^+$. Then W_α can be propagated along L from V to U by local twistor transport and the scalar product of Z^α and W_α can be defined to be the scalar product $Z^\alpha W_\alpha$ of the local twistors Z^α and W_α at the point U . Since the scalar product of two local twistors is preserved by local transport, one could also propagate Z^α along L to V and form the product at V , without altering its numerical value.

Chapter 6. Hypersurface Twistors

In chapter 4 the concept of asymptotic twistors (i.e. twistors defined relative to the null hypersurface at infinity) was introduced. This concept will now be generalised by considering twistors relative to hypersurfaces other than $\mathscr{I}$.

6.1 The Basic Concept

A real-valued function f defined on some region A of R^n is said to be analytic if, for every point x in A , there exists a neighbourhood N of x in A such that f can be written in the form of a power series which converges in N . A space-time (M,g_{ab}) is said to be analytic if the differentiable functions associated with M are analytic (for details, see Kobayashi and Nomizu (1), p.2). Finally, a hypersurface S in M is said to be analytic if it is given (locally at least) by an equation of the form

$$S(x^a) = \Sigma \quad ,$$

where (x^0, x^1, x^2, x^3) are local coordinates on M , Σ is a constant, and S is a real-valued analytic function on a region of R^4 .

Let (M,η_{ab}) represent Minkowski space-time and let S be an analytic, spacelike hypersurface in M . Let L be a null straight line (i.e. a null geodesic) in M and suppose that L intersects S in a point Q ; suppose further that the origin of the (Minkowski) coordinate system is translated, if necessary, so that it coincides with Q . Then, if Z^α is a twistor corresponding to L , Z^α has a spinor representation of the form

$$\underset{\sim}{Z}^\alpha = (0, \pi_{A'})^1 .$$

---

1 Notice that $\pi_{A'} \neq 0$, since L is not the null line at infinity. This condition will be assumed to hold throughout the chapter.

Now complexify the coordinates (x^0, x^1, x^2, x^3) on M to yield complexified Minkowski space-time, which will be denoted by M^* (for details of complexification, see Trautman (1), p.47). Since S is analytic by assumption and L is analytic by virtue of its being straight, these two structures can be complexified as well, yielding an ∞^6 (i.e. six-real-dimensional hypersurface S^* and an ∞^4 linear subspace L^* respectively. The possibility of global problems arising from this complexification can be eliminated by considering only those points lying in a sufficiently small (complex) neighbourhood R^* of the origin Q . Let R denote the real part of R^* .

In the real space-time, L intersects S only in the point Q ; but in M^* , the set $S^* \cap L^*$ will contain other points as well (these points being complex). The condition for a point $x^{AA'}$ to lie on the twistor Z^α is given by equation (3.2.2) with $\omega^A = 0$; in other words, by

$$x^{AA'} \pi_{A'} = 0 \quad . \tag{6.1.1}$$

This equation has the (in general complex) solution

$$x^{AA'} = \pi^{A'} \eta^A \quad , \tag{6.1.2}$$

where η^A is some spinor.

If $t^a = t^{AA'}$ is the vector[1] normal to S^* at Q , then the condition for $x^{AA'}$ to be a vector tangent to S^* is

$$t^{AA'} x_{AA'} = 0 \quad . \tag{6.1.3}$$

Combining (6.1.2) and (6.1.3) gives

$$t^{AA'} \pi_{A'} \eta_A = 0 \quad ,$$

whence

$$\eta^A \propto t^{AA'} \pi_{A'} \quad .$$

---

[1] From now on, all vectors will be complex, unless otherwise stated.

Thus the vector

$$x^{AA'} = t^{AB'}\pi_{B'}\pi^{A'} \tag{6.1.4}$$

is tangent both to S^{*} and to L^{*}. Equation (6.1.4) defines a tangent field on S^{*} which can be integrated to yield a curve V, lying in S^{*} and passing through Q. The curve V is an ∞^2 subset of S^{*} and, considered as a point set, equals the intersection of S^{*} and L^{*}. Any global problems that might arise in the integration can be avoided by choosing the neighbourhood R^{*} to be small enough; thus it can be assumed that the curve V does not intersect itself. Notice that V is a null curve, since the vector $x^{AA'}$ of (6.1.4) is null.

The spinor $\pi_{A'}$ in (6.1.4) is regarded as a constant spinor field on M^{*}, i.e. the components of $\pi_{A'}$ are numerically the same at every point of M^{*}. However, $\pi_{A'}$ can also be regarded as a spinor which is parallelly propagated along V by

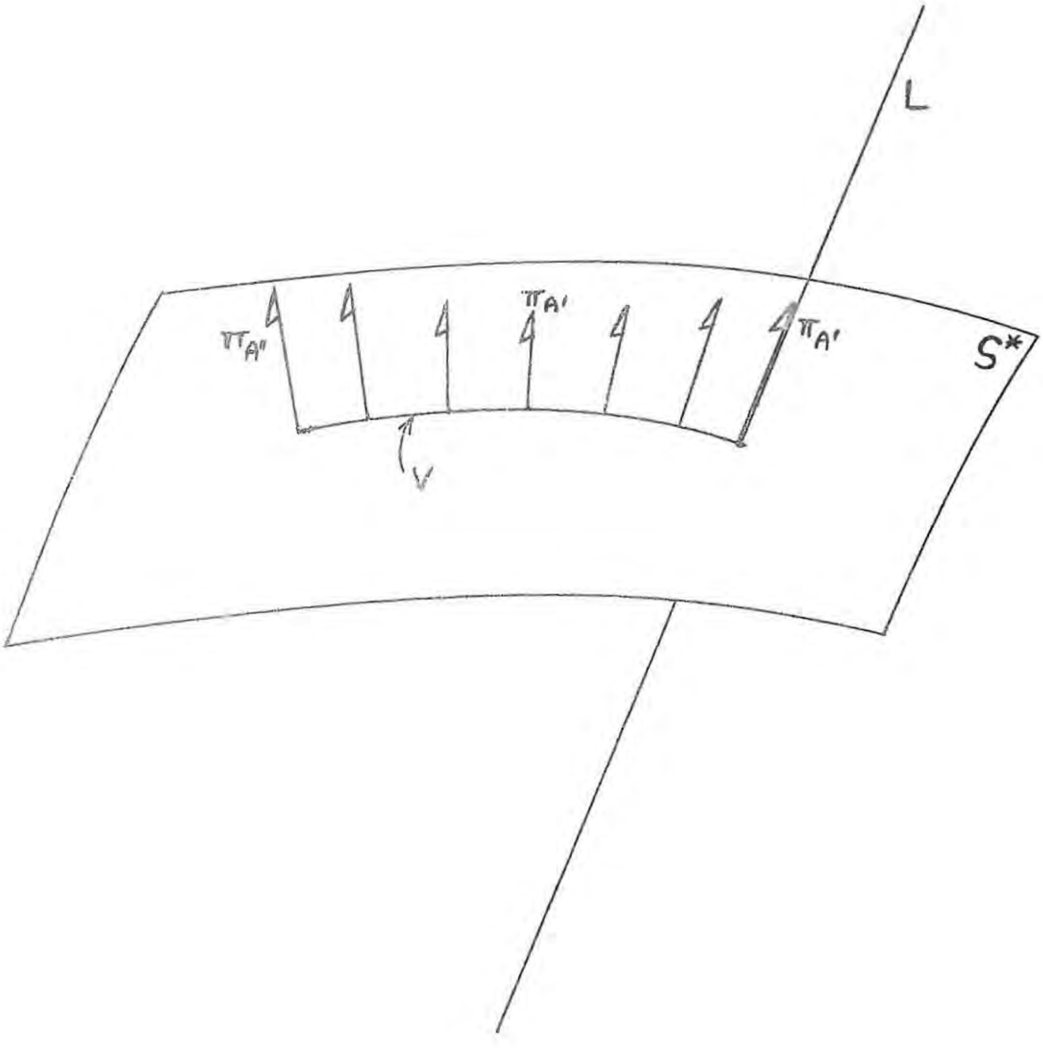
$$t^{AB'}\pi_{B'}\pi^{A'}\nabla_{AA'}\pi_{C'} = 0 , \tag{6.1.5}$$

since this means the same thing in Minkowski space-time. The curve V together with the parallelly propagated spinor $\pi_{A'}$ is an example of a hypersurface twistor.

Now consider the more general case where (M,g_{ab}) is an analytic space-time, not necessarily flat. Let S be an analytic spacelike hypersurface and L a null geodesic intersecting S at Q. A null global twistor Z^{α} is chosen to represent L, the spinor $\pi_{A'}$ being parallelly propagated along L. Then the above procedure can be carried out to obtain a curve V and a spinor $\pi_{A'}$ defined along V, using equations (6.1.4) and (6.1.5). See Figure 7.

It will now be shown that equation (6.1.5) is conformally invariant. Under a conformal rescaling, the spinor $\pi_{A'}$ is invariant; this follows from equation (3.4.4) with $\omega^{A} = 0$. The normal vector $t_{AB'}$ is the

Fig. 7. A Hypersurface Twistor.



gradient of a scalar function and so is invariant by (2.3.10); application of (2.3.9) then yields

$$\hat{t}^{AB'} = \Omega^{-2}\, t^{AB'} \quad .$$

Now (2.3.11) can be applied to equation (6.1.5) to give

$$\begin{aligned} \hat{t}^{AB'}\, \hat{\pi}_{B'}\, \hat{\pi}^{A'}\, \hat{\nabla}_{AA'}\, \hat{\pi}_{C'} &= \Omega^{-2}\, t^{AB'}\, \pi_{B'}\, \pi^{A'}\, (\nabla_{AA'}\, \pi_{C'} - T_{AC'}\, \pi_{A'}) \\ &= \Omega^{-2}\, t^{AB'}\, \pi_{B'}\, \pi^{A'}\, \nabla_{AA'}\, \pi_{C'} \\ &= 0 \ , \end{aligned}$$

which means that equation (6.1.5) is conformally invariant.

6.2 Hypersurface Twistor Space

A formal definition of hypersurface twistors may be given as follows. Let (M, g_{ab}) be an analytic space-time and R a region of M which is small enough to admit

(a) the complexification R^{*} of R, where R^{*} need only be defined near R;

(b) the complexification S^{*} of an analytic spacelike hypersurface S in R; and

(c) a complex curve V in S^{*} satisfying (6.1.4), with $\pi_{A'}$ a spinor parallelly propagated along V by (6.1.5); in addition, V is required not to intersect itself. Then the curve V, together with the spinor $\pi_{A'}$ defined along it, is by definition a __hypersurface twistor__.

Now given S^{*}, a point $\xi^{AA'}$ can be chosen on it (∞^6 possibilities) and a spinor $\kappa_{A'}$ at $\xi^{AA'}$ (∞^4 possibilities). This will determine a unique hypersurface twistor $(V, \pi_{A'})$ as follows: the curve V is represented by

$$V : x^{AA'} = x^{AA'}(\sigma) ,$$

where σ is a complex parameter along V and where the function $x^{AA'}(\sigma)$ satisfies (cf. (6.1.4))

$$dx^{AA'}(\sigma) \mid d\sigma = t^{AB'}\left(x^{CC'}(\sigma)\right) \pi_{B'}\left(x^{CC'}(\sigma)\right) \pi^{A'}\left(x^{CC'}(\sigma)\right) , \tag{6.2.1}$$

$$x^{AA'}(0) = \xi^{AA'} .$$

Notice that $x^{AA'}$ represents a position vector, while $dx^{AA'}/d\sigma$ represents a tangent vector, in fact the tangent vector $x^{AA'}$ of (6.1.4). The spinor $\pi_{A'}\left(x^{CC'}(\sigma)\right)$ is determined by parallel propagation along V, i.e. by (6.1.5) with the initial condition

$$\pi_{A'}\left(x^{CC'}(0)\right) = \pi_{A'}(\xi^{CC'}) = \kappa_{A'} .$$

Choosing any other point on the ∞^2 curve V and the corresponding spinor $\pi_{A'}$ at that point, to replace $\xi^{AA'}$ and $\kappa_{A'}$ respectively, will give exactly the same hypersurface twistor $(V, \pi_{A'})$. So the space of hypersurface twistors (which will be denoted by C) is an $\infty^{6+4-2} = \infty^8$ system.

If the curve V happens to contain a real point $p^{AA'}$, then V determines a unique null global twistor, namely that global twistor passing through $p^{AA'}$ with its π-spinor at $p^{AA'}$ equal to $\pi_{A'}(p^{CC'})$. Conversely, this global twistor determines the hypersurface twistor as in section 6.1. The uniqueness mentioned above can be ensured by choosing R small enough: thus V can contain at most one real point. Denote by N the collection of hypersurace twistors in C whose V-curve contain a real point. Since the points of N are in one-to-one correspondence with the null global twistors whose corresponding null geodesics intersect R , and since these null global twistors form an ∞^7 system, it follows that N is an ∞^7 subsystem of the ∞^8 system C. The hypersurface twistors in N will be called null.

Suppose that $(V, \pi_{A'})$ is a null hypersurface twistor, determining a null global twistor Z^α and hence a null geodesic Z . Let λ be a non-zero complex constant. The the null hypersurface twistor $(V, \lambda\,\pi_{A'})$ determines exactly the same null geodesic Z . Consequently, one could also regard C as a projective space, being an ∞^6 factor space of the whole ∞^8 space C ; the subspace N would become an ∞^5 system whose points were in one-to-one correspondence with the null geodesics intersecting R .

6.3 Hypersurface Twistor Intersections

Suppose that $\pi_{A'}$ is a spinor field defined on some real region R . Complexify R , yielding a complex region R^* , and extend the spinor field $\pi_{A'}$ to one on R^* . Then $\pi_{A'}$ becomes (in a sense) separated from its complex conjugate; thus there are two spinor fields on R^* , one of which is $\pi_{A'}$, and the other of which may be denoted by $\tilde{\pi}_A$. The field $\tilde{\pi}_{A'}$ has the property that, <u>on R</u> ,

$$\tilde{\pi}_A = \bar{\pi}_A ,$$

but that this equation is not true on $R^* - R$.

If the (real) null global twistor Z^α consists of the null geodesic Z and the spinor $\pi_{A'}$ defined along Z , let the <u>conjugate</u> global twistor, denoted by $\bar{Z}_\alpha$, consist of the <u>same</u> null geodesic Z with the spinor $\bar{\pi}_A$ parallelly propagated along it. It was seen in section 6.1 that Z^α determines a null hypersurface twistor $(V, \pi_{A'})$, via equations (6.1.4) and (6.1.5). The conjugate global twistor $\bar{Z}_\alpha$ may be used to define a <u>dual</u> hypersurface twistor, if equations (6.1.4) and (6.1.5) are replaced

$$x^{AA'} = t^{A'B}\, \tilde{\pi}_B\, \tilde{\pi}^A , \qquad (6.3.1a)$$

$$t^{A'B}\, \tilde{\pi}_B\, \tilde{\pi}^A\, \nabla_{AA'}\, \tilde{\pi}_C = 0 , \qquad (6.3.1b)$$

where the spinor field $\tilde{\pi}_A$ equals $\bar{\pi}_A$ <u>on Z</u> . Denote the curve derived from (6.3.1) by W ; then the dual hypersurface twistor may be denoted by $(W, \tilde{\pi}_A)$. In general, an element of the dual hypersurface twistor space C_* consists of a pair (W, η_A), where the curve W and the spinor η_A along W satisfy

$$x^{AA'} = t^{A'B}\, \eta_B\, \eta^A , \qquad (6.3.2a)$$

$$t^{A'B}\, \eta_B\, \eta^A\, \nabla_{AA'}\, \eta_C = 0 . \qquad (6.3.2b)$$

In Minkowski space-time, the spinor field $\pi_{A'}$ defined by (6.1.5) is <u>constant</u>, as is the field $\tilde{\pi}_A$ defined by (6.3.1b). Therefore, since

$\tilde{\pi}_A = \bar{\pi}_A$ at the point Q where Z intersects S , it follows that $\tilde{\pi}_A = \bar{\pi}_A$ at every point on W . However, this result does not hold (in general) in curved space-times.

Suppose now that $(V, \pi_{A'})$ is a hypersurface twistor and (W, η_A) a dual hypersurface twistor. If the curves V and W intersect, it is said (by definition) that the scalar product of $(V, \pi_{A'})$ and (W, η_A) exists and is zero. To define a scalar product when V and W do not intersect is certainly more difficult, and may be impossible; this remains an open question.

If $(V, \pi_{A'})$ is null, then it determines a null global twistor Z^α ; the conjugate global twistor $\bar{Z}_\alpha$ determines (as seen at the beginning of this section) a dual hypersurface twistor $(W, \tilde{\pi}_A)$. In addition, the curves V and W intersect (namely at the point Q) and so the scalar product of $(V, \pi_{A'})$ and $(W, \tilde{\pi}_A)$ is zero, which is not an unexpected result.

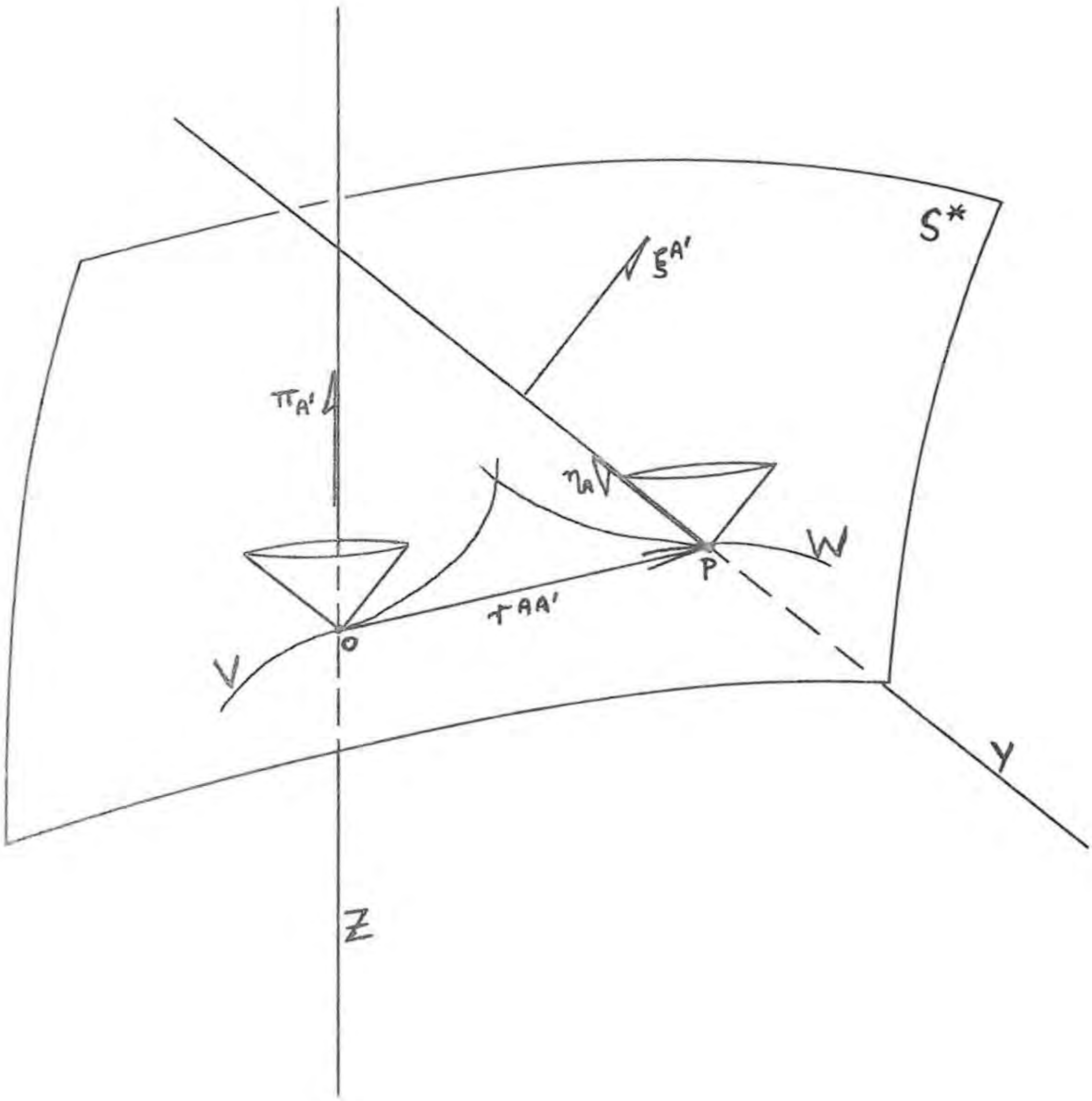
The final result of this section is the following

Theorem: Let the space-time (M, η_{ab}) be Minkowski and let S^* be a (complex) hypersurface containing two null hypersurface twistors $(V, \pi_{A'}) \in C$ and $(W, \eta_A) \in C_*$. Then the intersection of V and W in S^* implies the intersection of the two associated null geodesics in M .

Proof: Suppose that the null twistors associated with $(V, \pi_{A'})$ and (W, η_A) are Z^α (represented by $(\omega^A, \pi_{A'})$) and Y_α (represented by $(\eta_A, \xi^{A'})$) respectively. Without loss of generality it can be assumed that the null line (geodesic) Z represented by Z^α passes through the origin O (if it does not, shift the origin accordingly). Thus $\omega^A = 0$. Let P be the point where the null line Y associated with Y_α intersects S (see figure 8). Denote the vector $\vec{OP}$ by $r^{AA'}$.

Suppose the curve V is given by $x^{AA'} = x^{AA'}(\sigma)$. Equation (6.1.5)

Fig. 8. Hypersurface Twistor Intersection.



has as its solution, $\pi_{A'}$ constant. Equation (6.2.1) becomes

$$dx^{AA'}(\sigma) \mid d\sigma \equiv \dot{x}^{AA'}(\sigma) = t^{AB'}\left(x^{CC'}(\sigma)\right) \pi_{B'} \pi^{A'} \quad , \qquad x^{AA'}(0) = 0 \ . \tag{6.3.3}$$

Because of analyticity, the solution $x^{AA'}(\sigma)$ can be expanded in a power series as follows:

$$x^{AA'}(\sigma) = \sigma \, \dot{x}^{AA'}(0) + \tfrac{1}{2}\sigma^2 \, \ddot{x}^{AA'}(0) + \ldots \ .$$

But from (6.3.3) it is seen that each of $\dot{x}^{AA'}(0)$, $\ddot{x}^{AA'}(0)$, etc. can be written as

$$d^n x^{AA'} / d\sigma^n = \theta^{A}_{(n)} \, \pi^{A'} \quad ,$$

where $\theta^{A}_{(n)}$ is some spinor. Consequently, $x^{AA'}(\sigma)$ can be expressed as

$$x^{AA'}(\sigma) = \Psi^{A}(\sigma) \, \pi^{A'} \quad , \tag{6.3.4}$$

where $\Psi^{A}(\sigma)$ is some spinor function of σ .

Now suppose that the curve W is given by $y^{AA'} = y^{AA'}(\tau)$, τ being a complex parameter. By reasoning similar to that above, it is seen that $y^{AA'}(\tau)$ can be written as

$$y^{AA'}(\tau) = r^{AA'} + \Phi^{A'}(\tau) \, \eta^{A} \quad , \tag{6.3.5}$$

where $\Phi^{A'}(\tau)$ is a spinor function of τ .

By hypothesis, V and W intersect. Thus there exist numbers σ_1 and τ_1 such that

$$x^{AA'}(\sigma_1) = y^{AA'}(\tau_1) \quad . \tag{6.3.6}$$

The two cases $\xi^{A'} \bar{\eta}_{A'} \neq 0$ and $\xi^{A'} \bar{\eta}_{A'} = 0$ will now be considered separately. In the first case, the intersection of Y with the null cone of 0 is given by equation (3.2.5) as

$$- i \, (\xi^{A'} \bar{\eta}_{A'})^{-1} \, \bar{\xi}^{A} \xi^{A'} \ ,$$

and therefore $r^{AA'}$ can be expressed as

$$r^{AA'} = - i(\xi^{A'} \bar{\eta}_{A'})^{-1} \, \bar{\xi}^{A} \xi^{A'} + d \, \eta^{A} \bar{\eta}^{A'} \quad , \tag{6.3.7}$$

where d is a real constant.

Combining equations (6.3.4) to (6.3.7) gives

$$\Psi^{A}(\sigma_1)\pi^{A'} = -i\,(\xi^{A'}\bar{\eta}_{A'})^{-1}\,\bar{\xi}^{A}\,\xi^{A'} + d\,\eta^{A}\,\bar{\eta}^{A'} + \Phi^{A'}(\tau_1)\,\eta^{A}\,. \quad (6.3.8)$$

Transvection of (6.3.8) with η_A yields

$$(\Psi^{A}(\sigma_1)\,\eta_A)\,\pi^{A'} = -i(\xi^{A'}\,\bar{\eta}_{A'})^{-1}(\bar{\xi}^{A}\eta_A)\,\xi^{A'} \quad ,$$

whence

$$\pi_{A'}\,\xi^{A'} = 0\,.$$

In the second case ($\xi^{A'}\,\bar{\eta}_{A'}=0$), there exists a complex number λ such that

$$\xi^{A'} = \lambda\,\bar{\eta}^{A'}\,.$$

Let κ^A be a spinor for which

$$\eta^{A}\,\kappa_{A} = 1\,,$$

and put

$$u^{AA'} = i\,\lambda\kappa^{A}\,\bar{\eta}^{A'} - i\,\bar{\lambda}\,\bar{\kappa}^{A'}\,\eta^{A}\quad .$$

Then it follows that

$$\begin{aligned} i\,u^{AA'}\,\bar{\eta}_{A'} &= \bar{\lambda}\,\eta^{A} \\ &= \bar{\xi}^{A}\quad ; \end{aligned}$$

furthermore, $u^{AA'}$ represents a <u>real</u> vector (since $u^{AA'} - \bar{u}^{AA'} = 0$) and therefore (cf. (3.2.7)) the point given by $u^{AA'}$ lies on Y. Thus $r^{AA'}$ can be expressed as

$$r^{AA'} = u^{AA'} + c\eta^{A}\,\bar{\eta}^{A'}\quad , \qquad (6.3.9)$$

where c is some constant. Now combine equations (6.3.4),(6.3.5),(6.3.6) and (6.3.9) and transvect with η_A. This yields

$$\begin{aligned} (\Psi^{A}(\sigma_1)\,\eta_A)\,\pi^{A'} &= i\,\lambda\,\bar{\eta}^{A'} \\ &= i\,\xi^{A'}\quad , \end{aligned}$$

whence

$$\pi_{A'}\,\xi^{A'} = 0\,.$$

Thus in both cases, $\pi_{A'}\,\xi^{A'} = 0$, i.e. $Z^{\alpha}Y_{\alpha} = 0$. This means that Z and Y intersect.

This theorem is clearly valid in any conformally flat space-time, since the concepts of null twistor, null hypersurface twistor and their intersections are conformally invariant. However, the theorem fails when the space-time is conformally curved, for then anything could happen off the hypersurface, so as to prevent the two null geodesics from intersecting.

§ 6.4. The ω-Spinor.

In this section it will be shown how a pair of spinors (ω^A, $\pi_{A'}$), defined relative to some point in S^*, may have an interpretation as a hypersurface twistor.

Let P be a point of S^* and suppose that ω^A and $\pi_{A'}$ are two nonzero spinors. Let $U^{AA'}$ represent the null direction at P, tangent to S^*, given by

$$U^{AA'} := t^{BA'} \omega_B \omega^A \quad . \qquad (6.4.1)$$

A null curve U in S^* can be defined (locally at least) by imposing the conditions

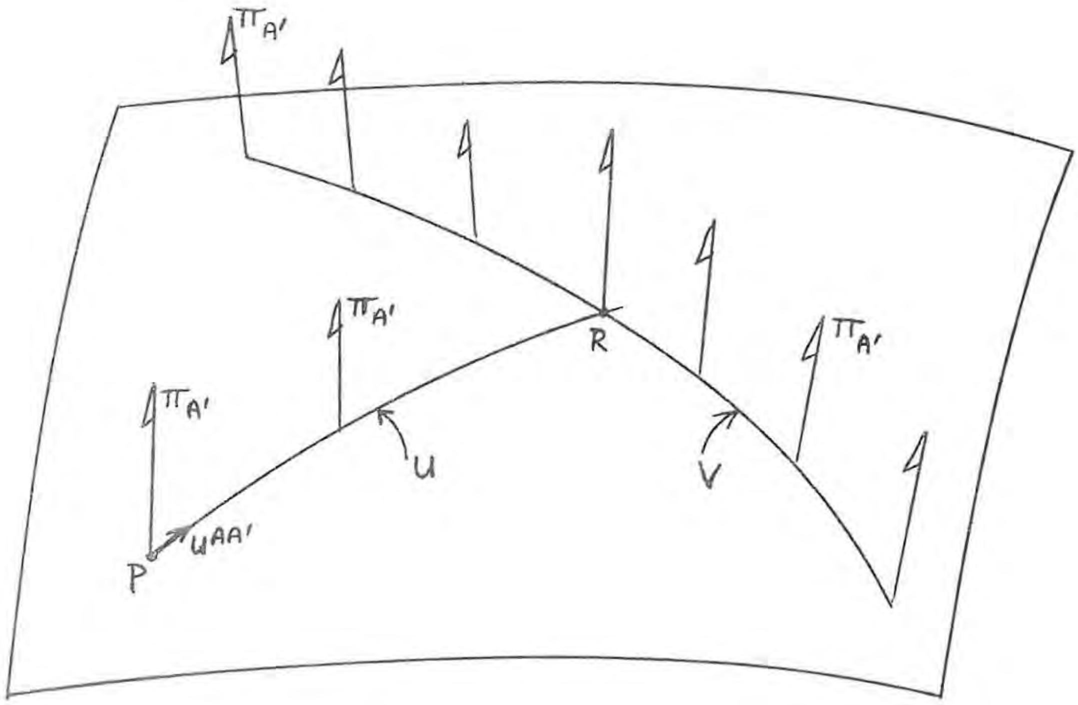
$$U^{BB'} \nabla_{BB'} \pi_{A'} = 0 \quad , \qquad (6.4.2)$$

$$U^{BB'} (\nabla_{BB'} \omega^A + i \, \varepsilon_B{}^A \pi_{B'}) = 0, \qquad (6.4.3)$$

as transport conditions for the spinors $\pi_{A'}$ and ω^A respectively, where $U^{BB'}$ is the tangent vector to U and $U^{BB'}$ is required to satisfy (6.4.1) at all points along U.

If at some point R on U, the ω-spinor is zero (i.e. $\omega^A = 0$), the point R and the spinor $\pi_{A'}$ at R can be used to construct a hypersurface twistor (V, $\pi_{A'}$) as in section 6.2 (see figure 9). Two open questions arising from this possibility are firstly, given a triple (P, ω^A, $\pi_{A'}$), does this triple determine a unique hypersurface twistor, and, secondly, is this hypersurface twistor (if it exists) null?

Fig. 9. The U-Curve.



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