Lattice-Valued Uniform Convergence Spaces: the Case of Enriched Lattices

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Abstract

Using a pseudo-bisymmetric enriched *cl*-premonoid as the underlying lattice, we examine different categories of lattice-valued spaces. Lattice-valued topological spaces, uniform spaces and limit spaces are described, and we produce a new definition of stratified lattice-valued uniform convergence spaces in this generalised lattice context. We show that the category of stratified L-uniform convergence spaces is topological, and that the forgetful functor preserves initial constructions for the underlying stratified L-limit space. For the case of L a complete Heyting algebra, it is shown that the category of stratified L-uniform convergence spaces is cartesian closed.

Contents

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Introduction

This work begins with an introduction to definitions and concepts from category theory, as categorical properties provide much of the motivation for our investigation into the chosen lattice-valued spaces. After looking at what defines a category, and some of the properties of categories, we introduce our first concrete category: the convergence spaces. Convergence as a primitive notion in topology was first suggested by Fréchet in 1906 [11]. Moore and Smith [37] investigated convergence in 1922, and this was followed by research in the 1950's by Kelley [28] and Arnold [2], with both of these works looking at convergence in terms of nets on directed sets. Filters were used to investigate convergence by both Choquet [7] and Sonner [43]. We have taken this same filter based approach to convergence and thus present important properties and definitions relating to filters and their images under mappings. From the category of convergence spaces, $CONV$, it is possible to define a number of interesting subcategories by enriching the set of axioms that characterise $CONV$.

From convergence spaces we move to the familiar category of uniform spaces, first proposed by Weil [44] in 1937, although we follow the convention of Bourbaki [4] in the definition that we use. Uniform convergence spaces were defined in 1967 by Cook and Fischer [8], where the uniform convergence structure is formed by a collection of filters on the product space $X \times X$. Every uniform structure on a set X generates a uniform convergence structure on the same set, and in turn, a uniform convergence structure can be used to define a limit structure on the set X. The category of uniform spaces can be shown to be a reflective subcategory of the category of uniform convergence spaces, and in 1976, Lee [33] showed that this category (when using the 1974 definition of Wyler [45]) is cartesian closed.

The second part of this thesis considers the lattice-valued equivalents of the categories already mentioned. This of course begins with an exploration of basic lattice theory. The lattice context that we consider, that of enriched lattices, consists of lattices equipped with two additional algebraic operations. Much of the background here comes from Höhle and Sostak [22], where they describe what are called enriched *cl*-premonoids. Chapter 4 deals extensively with properties and examples of enriched lattices.

As we did in the classical spaces, some time is spent on the lattice-valued equivalents of the filters, the L-filters. Chapter 5 is dedicated to examining L-sets and L-filters, specifically the images, products and compositions of L-filters. In encountering the product L-filters, we must add the condition that L be pseudo-bisymmetric in order to guarantee existence. We present in Proposition 5.8.5 a condition for the existence of the composition of two stratified L-filters in this generalised lattice setting.

Using the concept of an L-filter on a product space $X \times X$, lattice-valued uniform spaces (the category $SL-UNIF$) was defined and studied by Gutiérez García in his doctoral thesis [14] published in 2000. Here he used an enriched cl-premonoid as the underlying lattice, and produced a work that unified the different approaches of L-uniformities used by Hutton [23], Höhle [16], [18] and Lowen [35]. The major focus of our work, as the title suggests, is to extend the already established definition of lattice-valued uniform convergence spaces of Jäger and Burton [27] by using an enriched cl-premonoid as our lattice. In their 2005 paper, a definition of the category of stratified L-uniform convergence spaces ($SL-UCS$) was presented, using a complete Heyting algebra as the underlying lattice. In Chapter 7 we present a new definition of these spaces for our more general lattice context.

It was also shown in [27], mirroring the classical spaces, that $SL-UNIF$ is a reflective subcategory of $SL-UCS$ and this super category is cartesian closed. We attempted to gain these results for the case of a generalised enriched cl-premonoid, but have had to restrict this generalisation at different stages. Showing that the lattice-valued uniform spaces form a reflective subcategory of the lattice-valued uniform convergence spaces requires a further restriction on the enriched lattice, but again includes more examples than the original work, most notably that of the monoidal mean operator.

We use our generalised definition of $SL-UCS$ to induce a stratified L-limit space on the set X as defined by Jäger [24]. Our approach involves the definition and use of a specific filter on the product space, while still covering the work done by Jäger and Burton [27].

The investigation into the function spaces of the lattice-valued uniform convergence spaces was not very fruitful, with most of the results having to be restricted to the already proven case of L a complete Heyting algebra. Although our lattice context guarantees existence of product L-filters, many results in this section require being able to evaluate the product L-filters which we were not able to do. This presents a challenge for future research.

Part I

Categories

Chapter 1

Category Theory: Definitions and Results

Our approach to the investigation of the lattice-valued uniform convergence spaces is largely motivated by the categorical properties of this category as well as those closely related to it. In order to make this meaningful, we introduce here a collection of basic concepts and definitions from category theory. Except where otherwise stated, we have used Adámek *et al* [1] as a basis for the definitions given below.

1.1 Concrete categories

While the subject of category theory includes both abstract and concrete categories, we will focus our attention on the properties of concrete categories. $\mathbb{A}$ is a concrete category if it has a class of objects where each object is a set with a certain structure, and it has a class, $mor(\mathbb{A})$, of structure preserving mappings between the objects known as morphisms. In other words:

- objects are sets with structures,
- morphisms are mappings between sets which "preserve structures".

This is precisely defined below.

Definition 1.1.1. [1] A *concrete category*, $\mathbb{A}$, is a triple, where $\mathbb{A} = \big(ob(\mathbb{A}), |\cdot|, mor(\mathbb{A})\big)$ and

(i) $ob(\mathbb{A})$ is a class whose members are called $\mathbb{A}$-objects,

(ii) $|\cdot| : ob(\mathbb{A}) \longrightarrow$ *class of all sets*, where for each X that is an $\mathbb{A}$-object, $|X|$ denotes the underlying set of X,

(iii) $mor(\mathbb{A})$ is a class of mappings between $\mathbb{A}$-objects that satisfy the following properties:

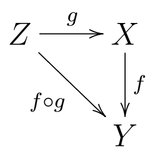
- for all $X \in ob(\mathbb{A})$, $id_X \in mor(\mathbb{A})$,
- $f \circ g \in mor(\mathbb{A})$ whenever $f, g \in mor(\mathbb{A})$.

Examples 1.1.2.

- SET: Objects are sets, morphisms are functions between sets.
- TOP: Objects are topological spaces, morphisms are the continuous mappings.
- GRP: Objects are groups, morphisms are group homomorphisms.
- $UNIF$: Objects are uniform spaces, morphisms are uniformly continuous mappings (see section 3.1).

Definition 1.1.3. [1] Let $\mathbb{A}$ be a concrete category and let $X, Y, Z \in ob(\mathbb{A})$. A morphism $f : X \longrightarrow Y$ is called *initial* if and only if for every mapping $g : |Z| \longrightarrow |X|$ (g a morphism in SET) we have that $g \in mor\mathbb{A} \iff f \circ g \in mor\mathbb{A}$.

That is, the following diagram commutes:



Definition 1.1.4. [1]

A *source* is a class of morphisms with common domain:

$$\{f_i : X \longrightarrow X_i, i \in I\}$$

where I may be a class and X, $X_i \in ob(\mathbb{A})$.

Definition 1.1.5. [1] Let X be any set, I a class, and $\{X_i : i \in I\}$ be a family of $\mathbb{A}$-objects with $\{f_i : X \longrightarrow |X_i| : i \in I\}$ being any family of mappings.

A category $\mathbb{A}$ is said to have *initial structures* if there exists a unique $\mathbb{A}$-structure, ξ, on X which is *initial* with respect to the source $\{f_i : X \longrightarrow X_i \mid i \in I\}$. That is, X endowed with the $\mathbb{A}$-structure ξ will make each of the f_i an initial mapping.

Definition 1.1.6. [39] A concrete category $\mathbb{A}$ is *topological over SET* if it satisfies the following three properties:

(TC1) $\mathbb{A}$ has initial structures,

(TC2) for any set X, the class of all objects $Y \in ob(\mathbb{A})$ with $|Y| = X$, is a set,

(TC3) for any singleton set $X = \{x\}$, there is exactly one $\mathbb{A}$-structure on X.

The above definition was proposed by Preuss [39]. A slightly weaker definition is suggested by Adámek *et al* [1]. Their definition of a topological category requires only the existence of initial structures. The class of all $\mathbb{A}$-structures on X is also known as the $\mathbb{A}$-fibre of X. The property **(TC2)** is referred to as a separate property know as *fibre-smallness*. **(TC3)** is known as the *terminal separator property* and is again considered by these authors to be a distinct property of a category.

1.2 Cartesian closedness

In this section, let $\mathbb{A}$ be a category, and let $A, B, C \in ob(\mathbb{A})$, $f \in mor(\mathbb{A})$. We denote by C^B the set of all morphisms from B to C.

Definition 1.2.1. [1] A category $\mathbb{A}$ is said to be *cartesian closed* if the following conditions are satisfied:

- finite products exist in $\mathbb{A}$, i.e. for any $A, B \in \mathbb{A}$ we have $A \times B \in ob(\mathbb{A})$,
- the structure of $\mathbb{A}$ can be given to C^B, i.e. $C^B \in ob(\mathbb{A})$,
- there exists a morphism $ev : C^B \times B \longrightarrow C$ such that:
  for every $f : A \times B \longrightarrow C$ there exists a unique morphism $\hat{f} : A \longrightarrow C^B$ such that $ev \circ (\hat{f} \times id_B) = f$.

As a diagram:

$$\begin{array}{ccc} A \times B & \xrightarrow{f} & C \\ {\scriptstyle \hat{f}\times id_B}\downarrow & \nearrow_{ev} & \\ C^B \times B & & \end{array}$$

The set C^B is called a *power object*, $ev : C^B \times B \longrightarrow C$ is the *evaluation map* and $\hat{f} : A \longrightarrow C^B$ the *exponential map*.

Proposition 1.2.2. *The category SET is cartesian closed.*

Proof: Let $A, B, C \in |SET|$ and $f : A \times B \longrightarrow C$. The evaluation map is defined:

$$ev : \begin{cases} C^B \times B \longrightarrow C \\ (f, b) \longmapsto f(b). \end{cases}$$

The proposed exponential map will be as follows:

$$\hat{f} : \begin{cases} A \longrightarrow C^B \\ a \longmapsto f(a, \cdot), \end{cases}$$

$$f(a, \cdot) : \begin{cases} B \longrightarrow C \\ b \longmapsto f(a, b). \end{cases}$$

First we check that $ev \circ (\hat{f} \times id_B) = f$. Let $(a, b) \in A \times B$. Then

$$ev \circ (\hat{f} \times id_B)(a, b) = ev\big(\hat{f}(a), b\big) = \big(\hat{f}(a)\big)(b) = \big(f(a, \cdot)\big)(b) = f(a, b).$$

Next we must check that this $\hat{f}$ is a unique mapping. Let $\psi : A \longrightarrow C^B$ such that $ev \circ (\psi \times id_B) = f$. Then we can see that for all $(a, b) \in A \times B$ we will obtain $\big(ev \circ (\psi \times id_B)\big)(a, b) = \big(\psi(a)\big)(b) = f(a, b)$.

Therefore for any $b \in B$, $\big(\psi(a)\big)(b) = f(a, b)$ and so $\psi(a) = f(a, \cdot) = \hat{f}$.

■

An important consequence of the above result is the relation: $C^{A \times B} \simeq (C^B)^A$, also known as the *exponential law*. We now provide the proof of this isomorphic relationship by proposing a mapping ϕ and showing that it is indeed an isomorphism between the two sets.

Define

$$\phi : \begin{cases} C^{A \times B} \longrightarrow (C^B)^A \\ f \longmapsto \phi(f) = \hat{f} \end{cases}$$

1. We show ϕ is 1-to-1:

Suppose that $\phi(f) = \phi(g)$, then clearly $\phi(f) \times id_B = \phi(g) \times id_B$. Now we get $ev \circ \big(\phi(f) \times id_B\big) = ev \circ \big(\phi(g) \times id_B\big)$ and so $ev \circ \big(\hat{f} \times id_B\big) = ev \circ \big(\hat{g} \times id_B\big)$. Thus $f = g$.

2. We show ϕ is onto:

Let $\lambda : A \longrightarrow C^B$ and then define: $f(a, b) = ev \circ (\lambda \times id_B)(a, b)$.

Now, from above we have that $\hat{f}$ is the only mapping with this property, and therefore $\lambda = \hat{f} = \phi(f)$. Therefore for any $\lambda \in (C^B)^A$ there exists $f \in C^{A \times B}$ as defined above and $\phi(f) = \lambda$.

■

1.3 Subcategories

Let $\mathbb{A}, \mathbb{B}$ be categories.

Definition 1.3.1. [1] A category $\mathbb{A}$ is said to be a *subcategory* of $\mathbb{B}$ if their relationship satisfies the following properties:

(i) $ob(\mathbb{A}) \subset ob(\mathbb{B})$,

(ii) $(\mathbb{A}) \subset mor(\mathbb{B})$,

(iii) composition of morphisms in $\mathbb{A}$ is the same composition of morphisms in $\mathbb{B}$,

(iv) identities in $\mathbb{A}$ are identities in $\mathbb{B}$.

Examples 1.3.2.

- The category of Hausdorff spaces is a subcategory of TOP.
- The category of finite sets is a subcategory of SET.

Definition 1.3.3. [1] A category $\mathbb{A}$ is a *full subcategory* of $\mathbb{B}$ if:

(i) $\mathbb{A}$ subcategory of $\mathbb{B}$,

(ii) for $A, A' \in |\mathbb{A}|$, $mor_{\mathbb{A}}(A, A') = mor_{\mathbb{B}}(A, A')$. That is, $f : A \longrightarrow A'$ is a morphism in $\mathbb{A}$ if and only if $f : A \longrightarrow A'$ is a morphism in $\mathbb{B}$.

Examples 1.3.4.

- For any category $\mathbb{A}$, $\mathbb{A}$ is a full subcategory of $\mathbb{A}$.
- SET_i with only the injective mappings as morphims is *not* a full subcategory of SET.

Definition 1.3.5. [1] Let $\mathbb{A}$ and $\mathbb{B}$ be categories. Then $F : \mathbb{A} \longrightarrow \mathbb{B}$ is a *functor* if for each $A \in ob(\mathbb{A})$ we have $F(A) \in ob(\mathbb{B})$, and for every $f \in mor(\mathbb{A})$ with $f : A \longrightarrow A'$ we get $F(f) \in mor(\mathbb{B})$ and $F(f) : F(A) \longrightarrow F(A')$. In addition, F must satisfy the following:

(i) F preserves composition of morphisms: $F(f \circ g) = F(f) \circ F(g)$,

(ii) F preserves identity morphisms: $F(id_A) = id_{F(A)}$.

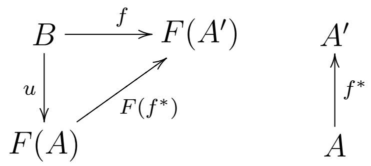
Definition 1.3.6. [1] Let $\mathbb{A}$ be a subcategory of $\mathbb{B}$. We define the *embedding functor*, $E : \mathbb{A} \longrightarrow \mathbb{B}$:

$$E : \begin{cases} \mathbb{A} \hookrightarrow \mathbb{B} \\ \begin{cases} A \longmapsto A \\ f \longmapsto f \end{cases} \end{cases}$$

Definition 1.3.7. [1] Let $\mathbb{A}$ and $\mathbb{B}$ be categories, with $A \in \mathbb{A}, B \in \mathbb{B}$. Further, let $F : \mathbb{A} \longrightarrow \mathbb{B}$ be a functor between the two categories.

Now consider the pair (u, A) where $u \in mor(\mathbb{B})$ and $u : B \longrightarrow F(A)$. This pair (u, A) is called a *universal map* for B with respect to F if:

For every $A' \in \mathbb{A}$ and for every $f : B \longrightarrow F(A')$ there exists a unique $\mathbb{A}$-morphism $f^* : A \longrightarrow A'$ such that the diagram below commutes:



Definition 1.3.8. [1] Let $\mathbb{A}$ be a subcategory of $\mathbb{B}$ and let $E : \mathbb{A} \longrightarrow \mathbb{B}$ be the embedding functor. The category $\mathbb{A}$ is a *reflective subcategory* of $\mathbb{B}$ if for each $B \in \mathbb{B}$ there exists a universal map with respect to E.

Given $\mathbb{A}$ a subcategory of $\mathbb{B}$, if we have for $B \in \mathbb{B}$ a universal map with respect to E, we call this an *E-universal map*, or an $\mathbb{A}$-reflection of B.

Chapter 2

The Category of Convergence Spaces

Here we introduce convergence spaces and their associated morphisms, the limit preserving maps. Convergence is described in terms of filters, and so we begin by exhibiting the major definitions and results from filter theory that will be relevant to the study of this category.

2.1 Classical Filter Theory

Filters are a common tool for investigating convergence in many different types of spaces. The idea of a convergence space, or limit space was first proposed by Fréchet in 1906 [11]. He relied on sequences to describe convergence, but his ideas have since been translated into the language of filters [43] to define convergence spaces.

Work on filters dates back as far as 1937 with the papers of Cartan [5], [6]. Further reading can also be found in Bourbaki [4].

Definition 2.1.1. [4] A *filter*, $\mathfrak{F}$, on a set X, is a collection of subsets of X that satisfies the following properties:

(F0) $\mathfrak{F} \neq \emptyset, \emptyset \notin \mathfrak{F}$,

(F1) $F \in \mathfrak{F}, F \subset G \Longrightarrow G \in \mathfrak{F}$,

(F2) $F, G \in \mathfrak{F} \Longrightarrow F \cap G \in \mathfrak{F}$.

Let X be a set. We denote the set of all filters on X by $\mathfrak{F}(X)$.

Definition 2.1.2. [4] A collection, $\mathfrak{B}$, of subsets of X is a *filter base* for some filter on X if:

(B0) $\mathfrak{B} \neq \emptyset, \emptyset \notin \mathfrak{B}$,

(B1) whenever $B_1, B_2 \in \mathfrak{B}$, then there exists $B_3 \in \mathfrak{B}$ such that $B_3 \subset B_1 \cap B_2$.

Another approach is to see that a subcollection, $\mathfrak{B}$, of a filter, $\mathfrak{F}$, will form a filter base for $\mathfrak{F}$ if and only if each element of $\mathfrak{F}$ contains some element of $\mathfrak{B}$. In other words, if:

$$\mathfrak{F} = \{F \subset X \mid B \subset F \text{ for some } B \in \mathfrak{B}\} = [\mathfrak{B}].$$

Examples 2.1.3.

- $\mathfrak{F} = \{X\}$ where $X \neq \emptyset$ is a filter on X.
- Let $A \subset X$. Then the collection $\{F \subset X \mid A \subset F\}$ is a filter on X. We can denote the filter generated by $A \subset X$ by $[A]$. Here a filter base is given by $\{A\}$.
- Consider the singleton set $\{x\}$, where $x \in X$. This will generate a filter $[\{x\}] = \{F \subset X \mid x \in F\}$, the point filter of x. We write $[x]$ for $[\{x\}]$.

The set $\mathfrak{F}(X)$ of all filters on X can be ordered by the following relation (see section 4.1). For any $\mathfrak{F}, \mathfrak{G} \in \mathfrak{F}(X)$,

$$\mathfrak{F} \leq \mathfrak{G} \iff \mathfrak{F} \subset \mathfrak{G}.$$

In this case, $\mathfrak{F}$ is said to be *coarser* than $\mathfrak{G}$, while $\mathfrak{G}$ is *finer* than $\mathfrak{F}$. If $\{\mathfrak{F}_i \mid i \in I\}$ is a non-empty family of filters on a set X, it can be shown that

$$\mathfrak{F} = \bigcap_{i \in I} \mathfrak{F}_i$$

will satisfy the axioms **(F0)**, **(F1)** and **(F2)**, and is thus a filter. This filter, $\mathfrak{F}$, is clearly the greatest lower bound for the family of filters $(\mathfrak{F}_i)$ in the ordered set of all filters on X, $\big(\mathfrak{F}(X), \leq\big)$.

The least upper bound of two filters $\mathfrak{F}$ and $\mathfrak{G}$, denoted by $\mathfrak{F} \vee \mathfrak{G}$, is defined as

$$\mathfrak{F} \vee \mathfrak{G} = \big[\{F \cap G \mid F \in \mathfrak{F}, G \in \mathfrak{G}\}\big]$$

whenever $\{F \cap G \mid F \in \mathfrak{F}, G \in \mathfrak{G}\}$ is a filter base. This will be satisfied if and only if for all $F \in \mathfrak{F}, G \in \mathfrak{G}, F \cap G \neq \emptyset$. The upper bound does, for example, not exist when $\mathfrak{F} = [x], \mathfrak{G} = [y]$ and $x \neq y$.

Definition 2.1.4. [4] Let X and Y be sets and $f : X \longrightarrow Y$. For $\mathfrak{F} \in \mathfrak{F}(X)$ the *image filter* of $\mathfrak{F}$, $f(\mathfrak{F})$, on Y is defined:

$$f(\mathfrak{F}) = \big[\{f(F) \mid F \in \mathfrak{F}\}\big].$$

Further, if X, Y and Z are sets and $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ such that $g \circ f : X \longrightarrow Z$, then for $\mathfrak{F} \in \mathfrak{F}(X)$,

$$(g \circ f)(\mathfrak{F}) = g\big(f(\mathfrak{F})\big).$$

Let $f : X \longrightarrow Y$ and let $\mathfrak{F} \in \mathfrak{F}(Y)$. The inverse image of $\mathfrak{F}$:

$$f^{\leftarrow}(\mathfrak{F}) = \{f^{\leftarrow}(F) \mid F \in \mathfrak{F}\}$$

is a filter on X if and only if $f^{\leftarrow}(F) \neq \emptyset$ for each $F \in \mathfrak{F}$. This is a consequence of the fact that $f^{\leftarrow}(F \cap G) = f^{\leftarrow}(F) \cap f^{\leftarrow}(G)$.

Definition 2.1.5. [4] Let $\{X_i \mid i \in I\}$ be a family of sets and $\mathfrak{F}_i$ a filter on X_i for each $i \in I$. The *product filter*, $\mathfrak{F}$, of the $\mathfrak{F}_i$ on the product space $X = \prod\limits_{i \in I} X_i$ is defined as a filter on X having as a base, subsets of X which are of the form $\prod\limits_{i \in I} F_i$ where $F_i \in \mathfrak{F}_i$ for each $i \in I$, and $F_i = X_i$ for all but a finite number of $i \in I$.

If we consider the projection mappings: $P_j : \prod\limits_{i \in I} X_i \longrightarrow X_j$, then it can be shown that the product of the filters $\mathfrak{F}_i$ can be defined as the coarsest filter $\mathfrak{G}$ on X such that $P_i(\mathfrak{G}) = \mathfrak{F}_i$ for all $i \in I$, or as $\prod\limits_{i \in I} \mathfrak{F}_i = \bigvee\limits_{i \in I} P_i^{\leftarrow}(\mathfrak{F}_i)$.

In the examples that we will deal with later, we often have to deal with only two spaces, X_1 and X_2. When the index set I described above is finite, that is, $I = \{1, 2, 3, \ldots, n\}$, then we write $\mathfrak{F} = \mathfrak{F}_1 \times \mathfrak{F}_2 \times \ldots \mathfrak{F}_n$ for $\prod\limits_{i=1}^{n} \mathfrak{F}_i$. Here, a base for $\mathfrak{F}$ is formed by all of the product sets $\prod\limits_{i \in I} F_i$ where $F_i \in \mathfrak{F}_i$ for all $i \in I$.

Definition 2.1.6. [4] Let X be a set. A filter $\mathfrak{F} \in \mathfrak{F}(X)$ is an *ultrafilter* if for all $\mathfrak{G} \in \mathfrak{F}(X)$ such that $\mathfrak{F} \leq \mathfrak{G}$, we have that $\mathfrak{G} = \mathfrak{F}$.

Examples of ultrafilters on a set X are the point filters $[x]$ where $x \in X$. Ultrafilters will be used later in the definition of Choquet limit spaces.

Let X, Y be sets and $M \subset Y^X$, where $Y^X = \{f : X \longrightarrow Y\}$, the set of all mappings from X to Y. We define the evaluation map ev by:

$$ev : \begin{cases} M \times X \longrightarrow Y \\ (f, x) \longmapsto f(x). \end{cases}$$

Now consider $\mathfrak{F} \in \mathfrak{F}(M)$, $\mathfrak{G} \in \mathfrak{F}(X)$. To make it clear, consider $F \in \mathfrak{F}$. Then F is a set of mappings from X to Y.

Definition 2.1.7. The image of the product filter, $\mathfrak{F} \times \mathfrak{G}$, under the evaluation mapping is given by:

$$ev(\mathfrak{F} \times \mathfrak{G}) = \big[\{F(G) \mid F \in \mathfrak{F}, G \in \mathfrak{G}\}\big],$$

where $F(G) = \{f(x) \mid f \in F, x \in G\}$ with $F \subset Y^X, G \subset X$.

2.2 Convergence spaces

Let X be a set, and $\mathfrak{F}(X)$ the set of all filters on X. We denote the power set of X by $\wp(X) = \{A \mid A \subset X\}$. Consider the following mapping:

$$\lim : \mathfrak{F}(X) \longrightarrow \wp(X).$$

Here $\lim \mathfrak{F}$ is the set of all points to which $\mathfrak{F}$ converges. This mapping can be interpreted as $x \in \lim \mathfrak{F}$ if and only if $\mathfrak{F}$ converges to x. From here on, we will write $\mathfrak{F} \longrightarrow x$ to denote "$\mathfrak{F}$ converges to x". Frechet's original requirements [11] for convergence spaces in terms of sequences are translated to the axioms shown below.

Definition 2.2.1. A *convergence space* is a set X, with a mapping $\lim : \mathfrak{F}(X) \longrightarrow \wp(X)$ that satisfies the following axioms:

(L1) for all $x \in X$, $x \in \lim[x]$,

(L2) $\mathfrak{F} \leq \mathfrak{G} \Longrightarrow \lim \mathfrak{F} \subset \lim \mathfrak{G}$.

A convergence space is often defined using one of the following two approaches. The first of these is a convergence relation R between $\mathfrak{F} \in \mathfrak{F}(X)$ and points of X:

$$R \subset \mathfrak{F}(X) \times \mathrm{X}$$

Interpretation: $(\mathfrak{F}, x) \in R \Longleftrightarrow \mathfrak{F} \longrightarrow x$.

The other approach is that of a convergence mapping, where each point $x \in X$ is mapped to the collection of all filters which converge to x.

$$\tau : X \longrightarrow \wp\big(\mathfrak{F}(X)\big)$$

Interpretation: $\mathfrak{F} \in \tau(x) \Longleftrightarrow \mathfrak{F} \longrightarrow x$.

These three approaches are in fact all the same if we identify $\wp(X)$ with $\{0,1\}^X$. This identification associates to each $A \subset X$, the *characteristic function* 1_A:

$$1_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

Using the limit mapping we have $\lim : \mathfrak{F}(X) \longrightarrow \{0,1\}^X$ and so $\lim \in \left(\{0,1\}^X\right)^{\mathfrak{F}(X)}$. The second approach using the relation has $R \subset \mathfrak{F}(X) \times X$ which requires that $R \in \{0,1\}^{\mathfrak{F}(X)\times X}$. The final approach has that $\tau : X \longrightarrow \{0,1\}^{\mathfrak{F}(X)}$ and so $\tau \in \left(\{0,1\}^{\mathfrak{F}(X)}\right)^X$. In order to prove that these approaches are the same we need to show:

$$\left(\{0,1\}^X\right)^{\mathfrak{F}(X)} \simeq \{0,1\}^{\mathfrak{F}(X)\times X} \simeq \left(\{0,1\}^{\mathfrak{F}(X)}\right)^X.$$

This is a specific case of the exponential law from the category SET:

$$\left(A^B\right)^C \simeq A^{B\times C} \simeq \left(A^C\right)^B.$$

This isomorphic relationship is a result of the fact that $\{0,1\}, \mathfrak{F}(X)$ and X are all elements of SET and that SET is cartesian closed (see Proposition 1.2.2).

2.3 Convergence Spaces as a Category

In order to show that the class of all convergence spaces forms a category, the objects of the category $CONV$ are defined to be convergence spaces and the morphisms are limit preserving mappings.

Definition 2.3.1. Let $(X, \lim_X)$ and $(Y, \lim_Y)$ be convergence spaces. A mapping $\varphi : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ is *continuous* (or *limit preserving*) if for all $x \in X$, if $\mathcal{F} \longrightarrow x$, then $\varphi(\mathcal{F}) \longrightarrow \varphi(x)$.

This definition can also be restated as:

- for all $x \in X$ if $x \in \lim_X \mathcal{F}$, then $\varphi(x) \in \lim_Y \varphi(\mathcal{F})$.
- $\varphi(\lim_X \mathcal{F}) \subset \lim_Y \varphi(\mathcal{F})$.

Proposition 2.3.2. *Let $(X, \lim_X), (Y, \lim_Y)$ and $(Z, \lim_Z)$ be convergence spaces. Then:*

(i) *The mapping $id_X : (X, \lim_X) \longrightarrow (X, \lim_X)$ is continuous.*

(ii) *If $\varphi : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ and $\psi : (Y, \lim_Y) \longrightarrow (Z, \lim_Z)$ are continuous, then $\psi \circ \varphi : (X, \lim_X) \longrightarrow (Z, \lim_Z)$ is continuous.*

PROOF:

(i) Let $x \in \lim \mathcal{F}$. Since $id_X(\mathcal{F}) = \mathcal{F}$ and $id_X(x) = x$ we get $id_X(\mathcal{F}) \in \lim id_X(x)$ and thus id_X is continuous.

(ii) Let $x \in \lim_X \mathcal{F}$. Since the mapping φ is continuous $\varphi(x) \in \lim_Y \varphi(\mathcal{F})$. In addition, since ψ is continuous $\psi\big(\varphi(x)\big) \in \lim_Z \psi\big(\varphi(\mathcal{F})\big)$. Thus we have that $(\psi \circ \varphi)(x) \in \lim_Z (\psi \circ \varphi)(\mathcal{F})$ and so $\psi \circ \varphi$ is continuous.

■

RESULT: The convergence spaces with the limit preserving mappings as morphisms form a concrete category.

We now show that the category $CONV$ has initial structures [10]. Suppose we have a set X, and a class of convergence spaces $\{(X_i \lim_i) \ : \ i \in I\}$ along with the following mappings:

$$f_i : X \longrightarrow (X_i, \lim_i), \text{ for all } i \in I$$

We want a convergence structure lim on X making all of the f_i continuous. Let $\mathfrak{F} \in \mathfrak{F}(X)$. We then require that $f_i(\lim \mathfrak{F}) \subset \lim_i f_i(\mathfrak{F})$ for all $i \in I$. This is equivalent to $\mathfrak{F} \subset f_i^{\leftarrow}\big(\lim_i f_i(\mathfrak{F})\big)$ for all $i \in I$, which, in turn, means $\lim \mathfrak{F} \subset \bigcap_{i \in I} f_i^{\leftarrow}\big(\lim_i f_i(\mathfrak{F})\big)$. So we define the coarsest of all possible convergence structures on X by:

$$\lim \mathfrak{F} = \bigcap_{i \in I} f_i^{\leftarrow}\big(\lim_i f_i(\mathfrak{F})\big)$$

for $\mathfrak{F} \in \mathfrak{F}(X)$. Now $(X, \lim)$ is the initial object for the source

$$\{f_i : X \longrightarrow (X_i, \lim_i) \mid i \in I\}.$$

Further, it can be shown that $CONV$ is fibre small and that it fulfils the terminal separator property. Thus the category $CONV$ is topological over SET as described in Definition 1.1.6.

2.4 Limit mappings for sets of functions

Just as a convergence function can be defined on a set, such a function can also be defined on the set of continuous mappings from one convergence space to another. Earlier, we defined cartesian closedness and demonstrated that SET is a category with this property. Once we have detailed the notion of continuous convergence, the same can be shown for $CONV$.

Let $(X, \lim_X), (Y, \lim_Y) \in |CONV|$ and denote by:

$$\mathcal{C}(X,Y) = \{\varphi : X \longrightarrow Y \text{ continuous}\}.$$

Definition 2.4.1. For $\varphi \in \mathcal{C}(X,Y)$, $\mathfrak{F} \in \mathfrak{F}\big(\mathcal{C}(X,Y)\big)$ define:

$$\varphi \ \in \ \text{c-}\lim \mathfrak{F} \qquad (\text{or } \mathfrak{F} \xrightarrow{c} \varphi)$$

if for all $x \in X$ and for all $\mathfrak{G} \in \mathfrak{F}(X)$, if $x \in \lim_X \mathfrak{G}$ then $\varphi(x) \in \lim_Y ev(\mathfrak{F} \times \mathfrak{G})$.

Proposition 2.4.2. *[38] Let $(X, \lim_X)$ and $(Y, \lim_Y)$ be convergence spaces. If we denote by $\mathcal{C}(X,Y)$ the class of all morphisms from X to Y, and by $c - \lim$ the continuous convergence function as defined above, then $\big(\mathcal{C}(X,Y),\ c\text{-}\lim\big) \in |CONV|$.*

Proposition 2.4.3. *[38], [40] The category $CONV$ is cartesian closed.*

2.5 Subcategories of CONV

We can add further axioms to those of the convergence spaces (see Definition 2.2.1) in order to give rise to subcategories of $CONV$. It will be stated later that these subcategories of $CONV$ are in fact reflective subcategories.

2.5.1 Additional axioms

From each additional convergence axiom, we get a more specialised category of convergence space.

L3W: For each $x \in X$, $x \in \lim \mathfrak{F} \Longrightarrow x \in \lim(\mathfrak{F} \cap [x])$.

> This axiom is also sometimes referred to as the *Kent axiom* and can be restated as: for each $x \in X$, $\lim \mathfrak{F} \subset \lim(\mathfrak{F} \cap [x])$.

L3: For all $x \in X$, $x \in \lim \mathfrak{F}, x \in \lim \mathfrak{G} \Longrightarrow x \in \lim(\mathfrak{F} \cap \mathfrak{G})$.

> In other words: $\lim \mathfrak{F} \cap \lim \mathfrak{G} \ \subset \ \lim(\mathfrak{F} \cap \mathfrak{G})$.

LC: If $x \in \lim \mathfrak{U}$ for every ultrafilter $\mathfrak{U}$ such that $\mathfrak{U} \geq \mathfrak{F}$, then $x \in \lim \mathfrak{F}$.

> Choquet axiom [7]: $\bigcap\{\lim \mathfrak{U} \mid \mathfrak{U} \geq \mathfrak{F} \text{ ultra }\} \subset \lim \mathfrak{F}$.

Note: **(LC)** can be stated as

$$\bigcap_{\mathfrak{U}\text{ ultra}} \lim \mathfrak{U} = \lim \Big(\bigcap_{\mathfrak{U}\text{ ultra}} \mathfrak{U} \Big).$$

Lp: For each $x \in X$, there exists $\mathfrak{N}^x \in \mathfrak{F}(X)$ such that $x \in \lim \mathfrak{F} \Longleftrightarrow \mathfrak{F} \geq \mathfrak{N}^x$.

Note: with **(L2)**, this axiom is equivalent to :

$$\mathfrak{N}^x = \bigcap_{x \in \lim \mathfrak{F}} \mathfrak{F} \qquad \text{and} \qquad x \in \lim \mathfrak{N}^x.$$

The axiom **(Lp)** is also equivalent to: $\lim \bigcap \mathfrak{F}_i = \bigcap \lim \mathfrak{F}_i$.

Lt: The convergence space $(X, \lim)$ satisfies **(Lp)** and for $\mathfrak{N}^x$ the axiom:

$U \in \mathfrak{N}^x \Longrightarrow$ there exists $V \in \mathfrak{N}^x$ such that $U \in \mathfrak{N}^y$ for all $y \in V$, holds.

Having presented the additional axioms we now display which combinations of axioms are used to define the various subcategories of $CONV$. For some of these subcategories there is more than one name in use, with the alternative options also being provided below.

Axioms	Name	Category
L1, L2, L3W	Kent convergence spaces [29]	$KCONV$
L1, L2, L3	Limit spaces (Fischer [10], Kowalsky [31])	LIM
L1, LC	Choquet limit spaces (pseudo-topological spaces) [7], [8]	$CLIM$
L1, Lp	principal convergence spaces (pre-topological spaces) [10], [31]	$PLIM$
L1, Lp, Lt	topological convergence spaces [10], [31]	$TCONV \simeq TOP$

Now we show that some axioms are stronger than others, and so requiring a particular axiom may necessarily result in the limit function of that subcategory also satisfying some of the weaker axioms.

Proposition 2.5.1. *The following relationships exist among the axioms:*

(i) ***Lp*** $\Longrightarrow$ ***LC****,*

(ii) ***LC*** $\Longrightarrow$ ***L2****,*

(iii) $\boldsymbol{LC} \Longrightarrow \boldsymbol{L3}$,

(iv) $\boldsymbol{L1}$,$\boldsymbol{L3} \Longrightarrow \boldsymbol{L3W}$.

PROOF:

(i) Let $x \in \lim \mathfrak{U}$, for all $\mathfrak{U} \geq \mathfrak{F}$ ultra. Thus for all $\mathfrak{U}$ ultra with $\mathfrak{U} \geq \mathcal{F}$ we have that $\mathfrak{U} \geq \mathfrak{N}^x$. Therefore, $\mathfrak{F} = \bigcap\{\mathfrak{U} \text{ ultra} \mid \mathfrak{U} \geq \mathfrak{F}\} \geq \mathfrak{N}^x$ and so $x \in \lim \mathfrak{F}$.

(ii) Suppose $\mathfrak{F} \leq \mathfrak{G}$ and $x \in \lim \mathfrak{F}$. From **(LC)**, $x \in \lim \mathfrak{U}$ for all $\mathfrak{U} \geq \mathfrak{F}$ ultra. Now take any $\mathfrak{W}$ ultra such that $\mathfrak{W} \geq \mathfrak{G}$, and we have that $\mathfrak{W} \geq \mathfrak{F} \Longrightarrow x \in \lim \mathfrak{W}$. Again from **(LC)** we get $x \in \lim \mathfrak{G}$.

(iii) Let $\mathfrak{W} \geq \mathfrak{F} \cap \mathfrak{G}$ be an ultrafilter. Without loss of generality suppose that $\mathfrak{W} \geq \mathfrak{F}$ [38]. Then let $x \in \lim \mathfrak{F} \cap \lim \mathfrak{G}$ and by **(L2)** we get $x \in \lim \mathfrak{W}$. Hence $x \in \lim \mathfrak{W}$, for all $\mathfrak{W}$ ultra and $\mathfrak{W} \geq \mathfrak{F} \cap \mathfrak{G}$. By **(LC)** $x \in \lim(\mathfrak{F} \cap \mathfrak{G})$.

(iv) Let $x \in \lim \mathfrak{F}$. By **(L1)** we have that $x \in \lim[x]$ and then from **(L3)** we get $x \in \lim \mathfrak{F} \cap \lim[x] = \lim(\mathfrak{F} \cap [x])$.

■

The morphisms between objects in each of the categories are always defined as the limit-preserving maps from one limit space to another. Thus a morphism in a smaller category will necessarily also be a morphism in the super category. From this fact and the use of the axioms above, we obtain the following hierarchy:

$$TOP \simeq TCONV \subset PLIM \subset CLIM \subset LIM \subset KCONV \subset CONV.$$

2.6 Reflective Subcategories of CONV

In order to show that one category is a reflective subcategory of another, we must produce a modification of the lim function that will satisfy the axioms for the lim function for the subcategory. We provide a basic outline of these modifications below.

1. Let $(X, \lim) \in |CONV|$. For $\mathfrak{F} \in \mathfrak{F}(X)$ define:

$x \in \lim^* \mathfrak{F} \Longleftrightarrow$ there exists $\mathfrak{G} \in \mathfrak{F}$ such that $x \in \lim \mathfrak{G}$ and $\mathfrak{G} \cap [x] \leq \mathfrak{F}$. Then $(X, \lim^*) \in |KCONV|$.

2. Let $(X, \lim) \in |KCONV|$. For $\mathfrak{F} \in \mathfrak{F}(X)$ define:

$$x \in \lim^* \mathfrak{F} \iff \text{there exist } \mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n \text{ such that } \bigcap_{i=1}^{n} \mathfrak{F}_i \leq \mathfrak{F} \text{ and } x \in \bigcap_{i=1}^{n} \mathfrak{F}_i.$$

Then $(X, \lim^*) \in |LIM|$.

3. Let $(X, \lim) \in |LIM|$. For $\mathfrak{F} \in \mathfrak{F}(X)$ define:

$x \in \lim^* \mathfrak{F} \iff x \in \lim \mathfrak{U}$ for all $\mathfrak{U}$ ultra with $\mathfrak{U} \geq \mathfrak{F}$. Then $(X, \lim^*) \in |CLIM|$.

4. Let $(X, \lim) \in |CLIM|$. For $\mathfrak{F} \in \mathfrak{F}(X)$ we define first: $\mathfrak{N}^x = \bigcap \{\mathfrak{F} : x \in \lim \mathfrak{F}\}$.

Then $x \in \lim^* \mathfrak{F} \iff \mathfrak{F} \geq \mathfrak{N}^x$, and $(X, \lim^*) \in |PLIM|$.

5. Let $(X, \lim) \in |PLIM|$. Now since $TCONV \simeq TOP$ we define a topology from the limit function. We say $x \in A_{\lim} \iff A \in \mathfrak{N}^x$ and $G \in \tau_{\lim} \iff G \subset G_{\lim}$.

Then $(X, \tau_{\lim}) \in |TOP|$.

The modifications outlined above will be used with the identity morphism to produce the E-universal map required in showing that one category is a reflective subcategory of another. For example, to show $KCONV$ a reflective subcategory of $CONV$, consider $(X, \lim) \in |CONV|$. The E-universal map will be $\big(id_X, (X, \lim^*)\big)$ where $\lim^*$ is as defined in **(1)** above.

With r representing a reflective subcategory and c a coreflective subcategory, we can display the following relationship [40]:

$$TOP \xrightarrow{r} PLIM \xrightarrow{r} CLIM \xrightarrow{r} LIM \xrightarrow{r} KCONV \xrightarrow[c]{r} CONV.$$

Chapter 3

Uniform Spaces and Uniform Convergence Spaces

In this chapter we introduce two further categories, and present the relationship that exists between them. While the category of uniform spaces is not cartesian closed, the category of uniform convergence spaces is, with the former forming a reflective subcategory of the latter.

3.1 Uniform Spaces

The idea of a uniform structure on a set was first proposed in 1937 by Weil [44] as a generalization of a metric on a set X, although we follow the definition of Bourbaki [4]. In order to properly define a uniform structure on a set X, we must first define the inverse of a subset of $X \times X$, and the composition of two subsets of $X \times X$. For $U, V \subset X \times X$,

$$U^{-1} = \{(y, x) \mid (x, y) \in U\}, \text{ and}$$

$$U \circ V = \{(x, y) \mid \exists\, z \in X \text{ such that } (x, z) \in U, (z, y) \in V\}.$$

The diagonal of the product space $X \times X$ is denoted Δ, where $\Delta = \{(x, x) \mid x \in X\}$.

Definition 3.1.1. [4] Let X be a set. A non-empty collection $\mathfrak{U}$ of subsets of $X \times X$ is called a *uniform structure* (or *uniformity*) on X if it satisfies the following axioms:

(U1) $U \in \mathfrak{U}$, $U \subset V \Longrightarrow V \in \mathfrak{U}$,

(U2) $U, V \in \mathfrak{U} \Longrightarrow U \cap V \in \mathfrak{U}$,

(U3) $U \in \mathfrak{U} \Longrightarrow \Delta \subset U$,

(U4) for all $U \in \mathfrak{U}$ there exists $V \in \mathfrak{U}$ such that $V \circ V \subset U$,

(U5) for all $U \in \mathfrak{U}$ there exists $V \in \mathfrak{U}$ such that $V^{-1} \subset U$.

A set X with such a structure $\mathfrak{U}$ is referred to as the *uniform space* $(X, \mathfrak{U})$.

From **(U3)** it is clear that $\emptyset \notin \mathfrak{U}$. Using then **(U1)** and **(U2)** we can see that $\mathfrak{U}$ is a filter on $X \times X$. A set $U \in \mathfrak{U}$ is referred to as an *entourage* or a *surrounding.* If we have $U \in \mathfrak{U}$ and $(x, y) \in U$ then x and y are called U*-close.*

Every uniformity on a set X generates a topology on X, although different uniformities can generate the same topology. From this it can be concluded that a uniform structure on X is in fact a richer structure than a topology.

Let $x \in X$ and $U \in \mathfrak{U}$. Then we define:

$$U^x = \{y \in X \mid (x, y) \in U\}.$$

Lemma 3.1.2. *[4] For each $x \in X$, the collection of sets $\mathfrak{U}^x = \{U^x \mid U \in \mathfrak{U}\}$ forms a basis of a neighbourhood filter of a topology for x.*

Definition 3.1.3. Suppose X, Y, W and Z are sets and consider the mappings $\varphi : X \longrightarrow W$ and $\psi : Y \longrightarrow Z$. Then the *product mapping* is defined:

$$\varphi \times \psi : \begin{cases} X \times Y \longrightarrow W \times Z \\ (x, y) \longmapsto \big(\varphi(x), \psi(y)\big). \end{cases}$$

Definition 3.1.4. [4] Let $(X, \mathfrak{U})$ and $(Y, \mathfrak{W})$ be uniform spaces. A mapping $\varphi : X \longrightarrow Y$ is said to be *uniformly continuous* if $\mathfrak{W} \subset (\varphi \times \varphi)(\mathfrak{U})$. This can be restated by saying that for each $V \in \mathfrak{W}$ there must exist some $U \in \mathfrak{U}$ such that $(\varphi \times \varphi)(U) \subset V$.

Lemma 3.1.5. *[4] Let $(X, \mathfrak{U}), (Y, \mathfrak{W})$ and $(Z, \mathfrak{V})$ be uniform spaces. Then the following are true:*

(i) *The identity map, $id_X : (X, \mathfrak{U}) \longrightarrow (X, \mathfrak{U})$ is uniformly continuous.*

(ii) *If $\varphi : (X, \mathfrak{U}) \longrightarrow (Y, \mathfrak{W})$ and $\psi : (Y, \mathfrak{W}) \longrightarrow (Z, \mathfrak{V})$ are uniformly continuous, then $\psi \circ \varphi : (X, \mathfrak{U}) \longrightarrow (Z, \mathfrak{V})$ is uniformly continuous.*

RESULT: We have a concrete category $UNIF$ where the objects are uniform spaces and the morphisms are the uniformly continuous mappings.

3.2 Uniform Convergence Spaces

The category of uniform spaces ($UNIF$) is not cartesian closed [40]. However, it is possible to embed this category into the category of uniform convergence spaces and it has been shown by Lee [33] that this category is cartesian closed.

Let X be a set. In order to properly describe uniform convergence spaces, we need to define two constructions with filters on the product space $X \times X$. Let $\mathfrak{F}, \mathfrak{G} \in \mathfrak{F}(X \times X)$. Then

$$\mathfrak{F}^{-1} = \{F^{-1} \mid F \in \mathfrak{F}\}, \text{ and}$$

$$\mathfrak{F} \circ \mathfrak{G} = \big[\{F \circ G \mid F \in \mathfrak{F}, G \in \mathfrak{G}\}\big].$$

Note that $\mathfrak{F}^{-1}$ is always a filter on $X \times X$, while $\mathfrak{F} \circ \mathfrak{G}$ will be a filter if and only if $F \circ G \neq \emptyset$ for all $F \in \mathfrak{F}, G \in \mathfrak{G}$. In this case we say that $\mathfrak{F} \circ \mathfrak{G}$ exists.

Using this notation, some of the axioms for a uniformity on X can be restated as:

(U3) $\mathfrak{U} \leq [\Delta]$,

(U4) $\mathfrak{U} \leq \mathfrak{U}^{-1}$,

(U5) $\mathfrak{U} \leq \mathfrak{U} \circ \mathfrak{U}$.

The definition below was proposed in 1974 by Wyler [45]. Here Λ is a collection of filters on the product space $X \times X$.

Definition 3.2.1. Let X be a non-empty set, and consider $\Lambda \subset \mathfrak{F}(X \times X)$. Then Λ is a *uniform convergence structure* on X if it satisfies:

(UC1) for all $x \in X$, $[x] \times [x] \in \Lambda$,

(UC2) $\mathfrak{F} \in \Lambda, \mathfrak{F} \leq \mathfrak{G} \Longrightarrow \mathfrak{G} \in \Lambda$,

(UC3) $\mathfrak{F} \in \Lambda \Longrightarrow \mathfrak{F}^{-1} \in \Lambda$,

(UC4) $\mathfrak{F}, \mathfrak{G} \in \Lambda \Longrightarrow \mathfrak{F} \wedge \mathfrak{G} \in \Lambda$,

(UC5) $\mathfrak{F}, \mathfrak{G} \in \Lambda, \mathfrak{F} \circ \mathfrak{G}$ exists $\Longrightarrow \mathfrak{F} \circ \mathfrak{G} \in \Lambda$.

The pair (X, Λ) is called a *uniform convergence space*. Wyler's definition altered that published in 1967 by Cook and Fischer [8]. Their work had the first axiom as follows:

(UC1′) $[\Delta] \in \Lambda$

The original axiom is in fact a stronger condition as for all $x \in X$ we have that $[\Delta] \leq [x] \times [x]$. The reason for making this change is to enable one to define a suitable uniform convergence structure on the set of uniformly continuous functions (defined below). Once we have defined the category of uniform convergence spaces, this structure on the set of continuous functions will be used to show that the category is cartesian closed.

Definition 3.2.2. [8] Let $(X, \Lambda), (Y, \Sigma)$ be uniform convergence spaces. A mapping $\varphi : X \longrightarrow Y$ is *uniformly continuous* if:

$$(\varphi \times \varphi)(\Lambda) \subset \Sigma.$$

That is, for each $\mathfrak{F} \in \Lambda$, we have $(\varphi \times \varphi)(\mathfrak{F}) \in \Sigma$.

Lemma 3.2.3. *[8] Let $(X, \Lambda), (Y, \Sigma)$ and (Z, Γ) be uniform convergence spaces. Then the following are true:*

(i) *The identity map, $id_X : (X, \Lambda) \longrightarrow (X, \Lambda)$ is uniformly continuous.*

(ii) *If $\varphi : (X, \Lambda) \longrightarrow (Y, \Sigma)$ and $\psi : (Y, \Sigma) \longrightarrow (Z, \Gamma)$ are uniformly continuous, then $\psi \circ \varphi : (X, \Lambda) \longrightarrow (Z, \Gamma)$ is uniformly continuous.*

RESULT: We have a concrete category UCS where the objects are uniform convergence spaces and the morphisms are the uniformly continuous maps.

3.3 Relationship between $UNIF$ and UCS

Let X be a set and let $\mathfrak{F}$ be a filter on $X \times X$, that is, $\mathfrak{F} \in \mathfrak{F}(X \times X)$. The collection of all filters on $X \times X$ that are finer than $\mathfrak{F}$ is referred to as the collection *generated* by $\mathfrak{F}$, and denoted $[\mathfrak{F}]$. That is,

$$[\mathfrak{F}] = \{\mathfrak{G} \mid \mathfrak{G} \in \mathfrak{F}(X \times X), \mathfrak{G} \geq \mathfrak{F}\}.$$

A uniform convergence structure that is generated by a filter $\mathfrak{F}$ is called a *principal uniform convergence structure*. Using the uniformly continuous mappings as morphisms it is possible to define the category of principal uniform convergence spaces.

Lemma 3.3.1. *[8] If Λ is a uniform convergence structure on X that is generated by the filter $\mathfrak{F}$ (ie $\Lambda = [\mathfrak{F}]$), then $\mathfrak{F}$ is a uniform structure on X. Conversely, if $\mathfrak{F}$ is a uniform structure on X, then $[\mathfrak{F}]$ is a uniform convergence structure on X.*

Lemma 3.3.2. *[8] Let* $(X,\mathfrak{U})$ *and* $(Y,\mathfrak{W})$ *be uniform spaces and let the mapping* $\varphi : (X,\mathfrak{U}) \longrightarrow (Y,\mathfrak{W})$ *be uniformly continuous. Then* $\varphi : (X,[\mathfrak{U}]) \longrightarrow (Y,[\mathfrak{W}])$ *is also uniformly continuous.*

Let X be a set, (X_i,Λ_i), $i \in I$, a collection of uniform convergence spaces and let $\{\varphi_i : X \longrightarrow X_i \,|\, i \in I\}$ be a collection of mappings. The initial uniform convergence structure on X is defined by:

$$\Lambda = \bigcap_{i\in I} \big\{\mathfrak{F} \in \mathfrak{F}(X \times X) \,|\, (\varphi_i \times \varphi_i)(\mathfrak{F}) \in \Lambda_i\big\}.$$

Lemma 3.3.3. *[8] The category* UCS *is a topological category.*

Given $(X,\Lambda) \in |UCS|$, we can define a uniformity on X by: $\mathfrak{U}_\Lambda = \bigcap\big\{\mathfrak{F} : \mathfrak{F} \in \Lambda\big\}$. Thus we can use $\big(id_X, (X,\mathfrak{U}_\Lambda)\big)$ as our E-universal map. Hence we are able to obtain the following result.

Lemma 3.3.4. *[40] The category of uniform spaces,* $UNIF$*, is a reflective subcategory of* UCS*.*

3.4 Induced Limit Space

Any uniform convergence structure on X can be used to define a convergence structure on X as shown below.

Lemma 3.4.1. *[8] Let* Λ *be a uniform convergence structure on a set* X *and define for* $\mathfrak{F} \in \mathfrak{F}(X)$*:*

$$x \in \lim(\Lambda)\mathfrak{F} \Longleftrightarrow \mathfrak{F} \times [x] \in \Lambda.$$

Then $\lim(\Lambda)$ *is a convergence structure on* X*, and* $\big(X, \lim(\Lambda)\big)$ *is a limit space.*

Proof:

L1: Let $x \in X$. By **(UC1)**, $[x] \times [x] \in \Lambda$ and hence $x \in \lim(\Lambda)[x]$.

L2: Let $\mathfrak{F} \leq \mathfrak{G}$ and $x \in \lim(\Lambda)\mathfrak{F}$. We therefore have that $\mathfrak{F} \times [x] \in \Lambda$ and, since $\mathfrak{F} \leq \mathfrak{G}$, we get that $\mathfrak{F} \times [x] \leq \mathfrak{G} \times [x]$. Using **(UC2)** we can say $\mathfrak{G} \times [x] \in \Lambda$ and, by the definition of $\lim(\Lambda)$, we now get $x \in \lim(\Lambda)\mathfrak{G}$. We can thus conclude that $\lim(\Lambda)\mathfrak{F} \subset \lim(\Lambda)\mathfrak{G}$.

L3: Suppose $x \in \lim(\Lambda)\mathfrak{F}$ and $x \in \lim(\Lambda)\mathfrak{G}$. From the definition of $\lim(\Lambda)$ we get that $\mathfrak{F} \times [x], \mathfrak{G} \times [x] \in \Lambda$ and using **(UC4)** we have that $(\mathfrak{F} \times [x]) \wedge (\mathfrak{G} \times [x]) \in \Lambda$.

$(\mathfrak{F} \times [x]) \wedge (\mathfrak{G} \times [x]) = \big[\{(F \times A) \cup (G \times B) \ : \ F \in \mathfrak{F}, G \in \mathfrak{G}, A, B \in [x]\}\big]$ and $\mathfrak{F} \wedge \mathfrak{G} = \big[\{F \cup G \ : \ F \in \mathfrak{F}, G \in \mathfrak{G}\}\big]$. Therefore we have:

$$(\mathfrak{F} \wedge \mathfrak{G}) \times [x] = \big[\{(F \cup G) \times A \ : \ F \in \mathfrak{F}, G \in \mathfrak{G}, A \in [x]\}\big].$$

Since $(F \cup G) \times A = (F \times A) \cup (G \times A)$ we get $(\mathfrak{F} \times [x]) \wedge (\mathfrak{G} \times [x]) = (\mathfrak{F} \wedge \mathfrak{G}) \times [x]$. Thus $(\mathfrak{F} \times [x]) \wedge (\mathfrak{G} \times [x]) \in \Lambda$ implies $(\mathfrak{F} \wedge \mathfrak{G}) \times [x] \in \Lambda$ and so $x \in \lim(\Lambda)(\mathfrak{F} \wedge \mathfrak{G})$.

■

3.5 Function Spaces of UCS

The possibility of a uniform convergence structure on the set of continuous functions UCS was investigated by Lee [33]. Here he uses the fact that UCS is a topological category and hence that product spaces exist. Using Wyler's definition [45] of a uniform convergence structure he was able to show that such a function space structure exists.

Let (X, Λ) and (Y, Σ) be uniform convergence spaces. Now define:

$$UC(X, Y) = \{f : X \longrightarrow Y, \ f \text{ uniformly continuous}\}$$

A uniform convergence structure on $UC(X, Y)$ would have to be a collection of filters on $UC(X, Y) \times UC(X, Y)$.

Let $\mathfrak{F} \in \mathfrak{F}\big(UC(X, Y) \times UC(X, Y)\big)$ and let $\mathfrak{G} \in \mathfrak{F}(X \times X)$. Now for $F \subset UC(X, Y) \times UC(X, Y)$ and $G \subset X \times X$, define:

$$F(G) = \{\big(f(x), g(y)\big) \ : \ (f, g) \in F, (x, y) \in G\}.$$

Lemma 3.5.1. *[33] Let (X, Λ) and (Y, Σ) be uniform convergence spaces. Also, let $\mathfrak{F} \in \mathfrak{F}\big(UC(X, Y) \times UC(X, Y)\big)$ and $\mathfrak{G} \in \mathfrak{F}(X \times X)$. Then $\mathfrak{B} = \{F(G) | F \in \mathfrak{F}, G \in \mathfrak{G}\}$ is a filter basis on $Y \times Y$.*

From the definition of a filter basis, it is then easy to see that

$$\mathfrak{F}(\mathfrak{G}) = \big[\{F(G) \mid F \in \mathfrak{F}, G \in \mathfrak{G}\}\big]$$

is a filter on $Y \times Y$.

The definitions used above are well suited to the case of classical uniform convergence spaces, but are not useful when trying to generalize them to the lattice-valued case. In order to make this generalisation, Jäger and Burton [27] propose a different approach. We will show that this approach and that of Lee [33] are identical in the classical case.

First, we define a mapping: η. Let $(X,\Lambda),(Y,\Sigma) \in |UCS|$ and consider $UC(X,Y)$ as defined above. Then

$$\eta : \begin{cases} \Big(UC(X,Y) \times UC(X,Y)\Big) \times \Big(X \times X\Big) \longrightarrow \Big(UC(X,Y) \times X\Big) \times \Big(UC(X,Y) \times X\Big) \\ \big((f,g),(x,y)\big) \longmapsto \big((f,x),(g,y)\big). \end{cases}$$

Further, consider the evaluation mapping, ev, defined as follows:

$$ev : \begin{cases} UC(X,Y) \times X \longrightarrow Y \\ (f,x) \longmapsto f(x). \end{cases}$$

Finally we can consider the composite mapping:

$$(ev \times ev) \circ \eta : \begin{cases} \Big(UC(X,Y) \times UC(X,Y)\Big) \times \Big(X \times X\Big) \longrightarrow Y \times Y \\ \big((f,g),(x_1,x_2)\big) \longmapsto \big(f(x_1),g(x_2)\big). \end{cases}$$

Lemma 3.5.2. *Let* $\mathfrak{F} \in \mathfrak{F}\big(UC(X,Y) \times UC(X,Y)\big)$ *and* $\mathfrak{G} \in \mathfrak{F}(X \times X)$. *Then*

$$\mathfrak{F}(\mathfrak{G}) = (ev \times ev) \circ \eta(\mathfrak{F} \times \mathfrak{G}).$$

Proof: First we show that if $F \subset UC(X,Y) \times UC(X,Y)$ and $G \subset X \times X$ then

$$F(G) = (ev \times ev) \circ \eta(F \times G) \quad \text{for } F(G) \text{ as defined above.}$$

Let $(f,g) \in F, (x,y) \in G$. By the definition of the mapping η we get that

$$(ev \times ev) \circ \eta\big((f,g),(x,y)\big) = (ev \times ev)\big((f,x),(g,y)\big).$$

From the definition of the product evaluation mapping we get $\big(ev(f,x), ev(g,y)\big)$ and so clearly $(ev \times ev) \circ \eta\big((f,g),(x,y)\big) = \big(f(x),g(y)\big)$.

With this in mind, and taking F and G as defined above we get

$$\begin{aligned}(ev \times ev) \circ \eta(F \times G) &= \{(ev \times ev) \circ \eta\big((f,g),(x,y)\big) \mid (f,g) \in F, (x,y) \in G\} \\ &= \{\big(f(x),g(y)\big) \mid (f,g) \in F, (x,y) \in G\} \\ &= F(G).\end{aligned}$$

We can now proceed towards the main result. From the definition of the product filter in 2.1.5, we have $\mathfrak{F} \times \mathfrak{G} = \big[\{F \times G \mid F \in \mathfrak{F}, G \in \mathfrak{G}\}\big]$.

Now we use definitions from 2.1.4 and 2.1.7 to get
$(ev \times ev) \circ \eta(\mathfrak{F} \times \mathfrak{G}) = \big[\{(ev \times ev) \circ \eta(H) \mid H \in \mathfrak{F} \times \mathfrak{G}\}\big]$.

From our definition of a product filter, we know $H \in \mathfrak{F} \times \mathfrak{G}$ if and only if there exists $F \in \mathfrak{F}$ and $G \in \mathfrak{G}$ such that $H \supset F \times G$. This implies that

$$\begin{aligned}(ev \times ev) \circ \eta(\mathfrak{F} \times \mathfrak{G}) &= \Big[\big[\{(ev \times ev) \circ \eta(F \times G) \mid F \in \mathfrak{F}, G \in \mathfrak{G}\}\big]\Big] \\ &= \big[\{(ev \times ev) \circ \eta(F \times G) \mid F \in \mathfrak{F}, G \in \mathfrak{G}\}\big] \\ &= \big[\{F(G) \mid F \in \mathfrak{F}, G \in \mathfrak{G}\}\big] \\ &= \mathfrak{F}(\mathfrak{G}).\end{aligned}$$

■

Definition 3.5.3. [33] Let $(X, \Lambda), (Y, \Sigma) \in |UCS|$ and $\mathfrak{F} \in \mathfrak{F}\big(UC(X,Y) \times UC(X,Y)\big)$. The *uniform convergence structure*,Ψ, on $UC(X,Y)$ is defined:

$$\mathfrak{F} \in \Psi \iff \text{ for all } \mathfrak{G} \in \Lambda,\ \mathfrak{F}(\mathfrak{G}) \in \Sigma.$$

Using this uniform convergence structure, Ψ, on $UC(X,Y)$ the following result was shown:

Proposition 3.5.4. *[33] The category UCS is cartesian closed.*

Part II

Lattice-Valued Spaces

Chapter 4

Lattice Theory

In 1940, Birkhoff published the first edition of his major work describing lattices [3]. Since then, lattice theory has been the subject of much development, and here, after providing an introduction to the basic lattice concepts, we turn our attention to enriched lattices.

4.1 Ordered Sets

Definition 4.1.1. [3] A *partially ordered* set $(L, \leq)$ is a set L with an order relation $\leq$ that satisfies the following axioms for all $\alpha, \beta, \delta \in L$:

(P1) Reflexivity: $\alpha \leq \alpha$,

(P2) Transitivity: $\alpha \leq \beta,\ \beta \leq \delta \implies \alpha \leq \delta$,

(P3) Anti-Symmetry: $\alpha \leq \beta,\ \beta \leq \alpha \implies \alpha = \beta$.

Definition 4.1.2. [3] A *linearly ordered set* is a partially ordered set, $(L, \leq)$, where for all $\alpha, \beta \in L$ either $\alpha \leq \beta$ or $\beta \leq \alpha$. Such a set is also known as a *chain*.

Examples 4.1.3.

- $(\mathbb{R}, \leq)$ is a linearly ordered set.
- $(\wp(X), \subset)$ is not linearly ordered.
- $\mathbb{R}^X$ with $f \leq g \iff f(x) \leq g(x)$ for all $x \in X$ is not linearly ordered.
- $\mathfrak{F}(X)$ the set of all filters on X is ordered by:
  $\mathfrak{F} \leq \mathcal{G} \iff \mathfrak{F} \subset \mathfrak{G} \iff$ for all $F, F \in \mathfrak{F}$ implies $F \in \mathfrak{G}$.

4.2 Lattices

Definition 4.2.1. [3] A *lattice* is a partially ordered set $(L, \leq)$, with two operations $\vee$, $\wedge$ such that for all $\alpha, \beta \in L$, there exists $\alpha \wedge \beta \in L$ and $\alpha \vee \beta \in L$ where:

$\alpha \vee \beta = \sup\{\alpha, \beta\} =$ *least upper bound* of $\{\alpha, \beta\}$, or the *join* of α and β i.e.

- $\alpha, \beta \leq \alpha \vee \beta$ (upper bound)
- $\alpha, \beta \leq \gamma \Longrightarrow \alpha \vee \beta \leq \gamma$ (smallest upper bound)

$\alpha \wedge \beta = \inf\{\alpha, \beta\} =$ *greatest lower bound* of $\{\alpha, \beta\}$, or the *meet* of α and β i.e.

- $\alpha \wedge \beta \leq \alpha, \beta$ (lower bound)
- $\delta \leq \alpha, \beta \Longrightarrow \delta \leq \alpha \wedge \beta$ (greatest lower bound)

Examples 4.2.2.

- $(\wp(X), \subset, \cup, \cap)$ with $A \vee B = A \cup B, A \wedge B = A \cap B$.
- $\big(\mathfrak{F}(X), \leq\big)$ is not a lattice, since $\mathfrak{F} \vee \mathcal{G}$ does not exist in general (see page 9).
- $\mathcal{T}$, a topology on X, is a lattice where for all $U, V \in \mathcal{T}$

$$U \vee V = U \cup V \quad \text{and} \quad U \wedge V = U \cap V.$$

- Any linearly ordered set (chain) is a lattice, where

$$\alpha \wedge \beta = \begin{cases} \alpha & \text{if } \alpha \leq \beta \\ \beta & \text{else,} \end{cases}$$

$$\alpha \vee \beta = \begin{cases} \beta & \text{if } \alpha \leq \beta \\ \alpha & \text{else.} \end{cases}$$

- A special case of this is: $(\mathbb{R}, \leq \ \vee, \wedge)$ where we define

$$\alpha \wedge \beta = \min\{\alpha, \beta\} = \begin{cases} \alpha & \text{if } \alpha \leq \beta \\ \beta & \text{else,} \end{cases}$$

$$\alpha \vee \beta = \max\{\alpha, \beta\} = \begin{cases} \alpha & \text{if } \alpha \geq \beta \\ \beta & \text{else.} \end{cases}$$

Consider an arbitrary subset S of a lattice L. We define:

$$\alpha \in L \text{ a lower bound for } S \iff \alpha \leq \delta, \text{ for all } \delta \in S.$$

$$\beta \in L \text{ an upper bound for } S \iff \beta \geq \gamma, \text{ for all } \gamma \in S.$$

Definition 4.2.3. [3] A lattice $(L, \leq)$ is said to be a *complete lattice* if for any non-empty $S \subset L$ we have that the greatest lower bound of S, denoted $\bigwedge S$, and the least upper bound $\bigvee S$ both exist and are elements of L.

A complete lattice L will have a greatest element, $\top$, and a least element, $\bot$, where:

$$\bigvee L = \top \qquad \text{and} \qquad \bigwedge L = \bot.$$

Examples 4.2.4.

- $(\wp(X), \subset, \cup, \cap)$ is a complete lattice when we define:

$$\bigwedge G_i = \bigcap G_i \text{ and } \bigvee G_i = \bigcup G_i.$$

- $\mathcal{T}$, a topology on X, is a complete lattice when we define:

$$\bigvee \{G_i : i \in I\} = \bigcup \{G_i : i \in I\}$$

  and

$$\bigwedge \{G_i : i \in I\} = \left(\bigcap \{G_i : i \in I\}\right)^o$$

  where for $G \subset X$, G^o denotes the interior of G.

- $\big((0,1), \leq\big)$ is not a complete lattice. Clearly there can not exist an upper bound for $(0,1)$ that is an element of $(0,1)$.

In a complete lattice we define:

$$\bigvee \emptyset = \bot \qquad \text{and} \qquad \bigwedge \emptyset = \top.$$

This can be explained by considering that α is an upper bound for $\emptyset$ if and only if $\alpha \geq \beta$, for all $\beta \in \emptyset$. From this we can see that for any $\alpha \in L$, α is an upper bound for $\emptyset$, and the set of all upper bounds for $\emptyset$ is the entire set L. Hence the least upper bound for $\emptyset$, or $\bigvee \emptyset$ will be the least element of L, which is $\bot$. Similarly, it can be argued that $\bigwedge \emptyset = \top$.

4.3 GL-monoids

Let $(L, \leq)$ be a lattice. Here we will introduce two additional binary "algebraic" operations on L, $\otimes : L \times L \longrightarrow L$ and $* : L \times L \longrightarrow L$, that will give rise to more complex lattice structures: GL-monoids and cl-premonoids.

Definition 4.3.1. [41] Let $(L, \leq)$ be a complete lattice. Then the triple $(L, \leq, *)$ is called a *quantale* if

(Q1) $(L, *)$ is a semigroup, i.e.

$\alpha * \beta \in L$ for all $\alpha, \beta \in L$ (closure),

$(\alpha * \beta) * \delta = \alpha * (\beta * \delta)$, for all $\alpha, \beta, \delta \in L$ (associativity),

(Q2) $*$ is distributive over arbitrary joins, i.e.

$$\left(\bigvee_{i \in J} \alpha_i\right) * \beta = \bigvee_{i \in J} (\alpha_i * \beta) \quad \text{and} \quad \beta * \left(\bigvee_{i \in J} \alpha_i\right) = \bigvee_{i \in J} (\beta * \alpha_i).$$

As consequence of the distributivity we have that when $\alpha, \beta \in L$ are such that $\alpha \leq \beta$, then for any $\gamma \in L$ we will have $\alpha * \gamma \leq \beta * \gamma$.

Proof: Let $\alpha \leq \beta$. Then $\beta * \gamma = (\alpha \vee \beta) * \gamma = (\alpha * \gamma) \vee (\beta * \gamma) \geq \alpha * \gamma$.

■

When we have that $*$ is $\wedge$, the quantale $(L, \leq, \wedge)$ is also called a *complete Heyting algebra.*

Example: A topology, $L = \mathcal{T}$, the collection of open sets, with sups and infs as defined before, is a quantale (complete Heyting algebra). For $H \in \mathcal{T}$ and for $G_i \in \mathcal{T}$ for all $i \in I$, we have:

$$\left(\bigvee_{i \in I} G_i\right) \wedge H = \bigvee_{i \in I} (G_i \wedge H).$$

Definition 4.3.2. [41] A quantale, $(L, \leq, *)$, is *strictly two-sided* if the greatest element, $\top$, is the unit with respect to $*$. That is, if $\alpha * \top = \top * \alpha = \alpha$ for all $\alpha \in L$.

Examples 4.3.3.

- $([0,1], \leq, \cdot)$, where "$\cdot$" is the usual multiplication, is a strictly two-sided quantale since for all $\alpha, \alpha_i, \beta \in L$ we have

$1 \cdot \alpha = \alpha \cdot 1 = \alpha$ and $\big(\bigvee_{i\in I} \alpha_i\big) \cdot \beta = \bigvee_{i\in I}(\alpha_i \cdot \beta)$

PROOF: Clearly $\alpha_j \cdot \beta \leq \big(\bigvee_{i\in I} \alpha_i\big) \cdot \beta$ for all $j \in I$. This then gives us $\bigvee_{i\in I}(\alpha_i \cdot \beta) \leq \big(\bigvee_{i\in I} \alpha_i\big) \cdot \beta$.

For the reverse inequality, consider an increasing sequence δ_n, that converges to $\bigvee_{i\in I} \alpha_i$, and whose terms are taken from the set $\{\alpha_i | i \in I\}$. Clearly then for any $n \in \mathbb{N}$ we have $(\delta_n \cdot \beta) \leq \bigvee_{i\in I}(\alpha_i \cdot \beta)$. By continuity of the multiplication we have that $\delta_n \cdot \beta$ is an increasing sequence that converges to $\big(\bigvee_{i\in I} \alpha_i\big) \cdot \beta$ and so $\big(\bigvee_{i\in I} \alpha_i\big) \cdot \beta \leq \bigvee_{i\in I}(\alpha_i \cdot \beta)$.

- $([0, 1], \leq, \vee)$ is not strictly two-sided since $\alpha \vee 1 = 1$ for all $\alpha \in [0, 1]$.
- A Heyting algebra, $(L, \leq \wedge)$, is a strictly two-sided quantale since we have $\alpha \wedge \top = \top \wedge \alpha = \alpha$ for all $\alpha \in L$.

Definition 4.3.4. [41] A commutative quantale, $(L, \leq, *)$, is *divisible* if for every inequality $\beta \leq \alpha$ there exists $\delta \in L$ such that $\beta = \alpha * \delta$.

Examples 4.3.5.

- A Heyting algebra $(L, \leq, \wedge)$ is divisible. Suppose $\beta \leq \alpha$, we can choose $\delta = \beta$, since $\beta = \alpha \wedge \beta$.
- $\big([0, 1], \leq, \cdot\big)$ is divisible. Here for $\beta \leq \alpha$ we take $\delta = \frac{\beta}{\alpha}$ since $0 \leq \frac{\beta}{\alpha} \leq 1$.
- $([0, 1], \leq, \vee)$ is not divisible. Consider $0 \leq 1$ and suppose that there exists a δ such that $0 = 1 \vee \delta$. This would imply that $1 \leq 1 \vee \delta = 0$ and hence that $1 \leq 0$, a contradiction.

Definition 4.3.6. [19] A quantale $(L, \leq, *)$ is called a *GL-monoid* if it is commutative, strictly two-sided and divisible.

Here *GL* stands for "generalised logics".

In a commutative quantale we have the *implication operator*:

$$\alpha \to \beta = \bigvee \big\{\lambda \in L \ : \ \alpha * \lambda \leq \beta\big\}.$$

Lemma 4.3.7. *Let $(L, \leq, *)$ be a GL-monoid and let $\alpha, \beta, \delta \in L$. Then*

$$\delta \leq \alpha \to \beta \iff \delta * \alpha \leq \beta.$$

PROOF: Suppose $\delta * \alpha \leq \beta$. Then $\delta \in \{\lambda \in L \,:\, \alpha * \lambda \leq \beta\}$ and so we have $\delta \leq \bigvee \{\lambda \in L \,:\, \alpha * \lambda \leq \beta\}$. Therefore $\delta \leq \alpha \to \beta$.

Conversely suppose that $\delta \leq \alpha \to \beta$. Then we get:

$$\delta * \alpha \leq \alpha * \bigvee \{\lambda \mid \alpha * \lambda \leq \beta\} = \bigvee \{\alpha * \lambda \mid \alpha * \lambda \leq \beta\} \leq \beta.$$

■

Lemma 4.3.8. *[22] Let $(L, \leq, *)$ be a* GL*-monoid and $\alpha, \beta, \delta, \alpha_i, \beta_i \in L$. Then the following properties hold:*

(i) $\alpha \to \beta = \top \Longleftrightarrow \alpha \leq \beta$

(ii) $\alpha \to \Big(\bigwedge_{i \in I} \beta_i\Big) = \bigwedge_{i \in I} (\alpha \to \beta_i)$

(iii) $\Big(\bigvee_{i \in I} \alpha_i\Big) \to \beta = \bigwedge_{i \in I} (\alpha_i \to \beta)$

(iv) $\alpha * \Big(\bigwedge_{i \in I} \beta_i\Big) = \bigwedge_{i \in I} (\alpha * \beta_i)$

(v) $(\alpha \to \delta) * (\delta \to \beta) \leq (\alpha \to \beta)$

(vi) $\alpha \leq \beta \Longrightarrow \delta \to \alpha \leq \delta \to \beta$

(vii) $\alpha \leq \beta \Longrightarrow \beta \to \delta \leq \alpha \to \beta$

(viii) $\alpha \to (\beta \to \delta) = (\alpha * \beta) \to \delta$

(ix) $\alpha * (\alpha \to \beta) = \alpha \wedge \beta$

The properties above are given in [22], but we will need two further properties that are required when proving results in chapter 7.

Lemma 4.3.9. *Let $(L, \leq, *)$ be a GL-monoid and let $\alpha_i, \beta_i \in L$ for each $i \in I$. Then*

$$\bigwedge_{i \in I} (\alpha_i * \beta_i) \geq \big(\bigwedge_{i \in I} \alpha_i\big) * \big(\bigwedge_{i \in I} \beta_i\big).$$

PROOF: Clearly, $\bigwedge_{i \in I} \alpha_i \leq \alpha_j$ and $\bigwedge_{i \in I} \beta_i \leq \beta_j$ for all $j \in I$. Since the $*$ operation is order-preserving, we have that $\big(\bigwedge_{i \in I} \alpha_i\big) * \big(\bigwedge_{i \in I} \beta_i\big) \leq \alpha_j * \beta_j$ for all $j \in I$, and therefore $\big(\bigwedge_{i \in I} \alpha_i\big) * \big(\bigwedge_{i \in I} \beta_i\big) \leq \bigwedge_{j \in I} (\alpha_j * \beta_j)$.

■

Lemma 4.3.10. *Let $(L, \leq, *)$ be a GL-monoid and let $\alpha, \beta, \delta, \gamma \in L$. Then*

$$(\alpha \to \beta) * (\delta \to \gamma) \leq (\alpha * \delta) \to (\beta * \gamma).$$

PROOF: Since $\alpha \to \beta \leq \alpha \to \beta$ we have from Lemma 4.3.7 that $\alpha * (\alpha \to \beta) \leq \beta$. Using this fact, as well as the associativity and the commutativity of the $*$ we have that

$$\begin{aligned}(\alpha * \delta) * (\alpha \to \beta) * (\delta \to \gamma) &= \delta * \big(\alpha * (\alpha \to \beta)\big) * (\delta \to \gamma) \\ &\leq \delta * \beta * (\delta \to \gamma) \\ &= \beta * \big(\delta * (\delta \to \gamma)\big) \\ &\leq \beta * \gamma.\end{aligned}$$

With yet another application of Lemma 4.3.7 we have that

$$(\alpha \to \beta) * (\delta \to \gamma) \leq (\alpha * \delta) \to (\beta * \gamma).$$

■

If we restrict the lattice L to $\{0, 1\}$, the two point chain, then we obtain the classical logical implication:

$$\begin{aligned}0 \to 0 &= 1, \\ 0 \to 1 &= 0, \\ 1 \to 0 &= 1, \\ 1 \to 1 &= 1.\end{aligned}$$

4.4 Examples of Quantales : T-norms on $[0, 1]$

Triangular norms were first introduced by Karl Menger in 1942 [36]. They were used to generalise the triangle inequality from ordinary metric spaces to probabilistic metric spaces, formerly known as statistical metric spaces. The set of axioms that are currently used to describe t-norms are due to Schweizer and Sklar [42], and are far stricter than those originally used. Previously no associativity was required, meaning that t-norms did not always form semi-group operations.

Definition 4.4.1. [42] A *triangular norm* or *t-norm* is a binary operation T on the unit interval $[0, 1]$:

$$T : [0, 1] \times [0, 1] \longrightarrow [0, 1]$$

such that for all $\alpha, \beta, \delta \in [0, 1]$ the four axioms **(T1) - (T4)** are satisfied:

(T1) $T(\alpha, \beta) = T(\beta, \alpha)$ (commutativity)

(T2) $T\big(\alpha, T(\beta, \delta)\big) = T\big(T(\alpha, \beta), \delta)\big)$ (associativity)

(T3) $T(\alpha, \beta) \le T(\alpha, \delta)$ whenever $\beta \le \delta$ (monotonicity)

(T4) $T(\alpha, 1) = \alpha$ (boundary condition)

One can naturally replace the notation $T(\alpha, \beta)$ with $\alpha * \beta$ when we consider $([0,1], T)$ as a quantale.

There are uncountably many t-norms, but there are four basic ones that are commonly used:

$$
\begin{aligned}
T_M(\alpha, \beta) &= \min(\alpha, \beta) && \text{(minimum)}\\
T_P(\alpha, \beta) &= \alpha \cdot \beta && \text{(product)}\\
T_L(\alpha, \beta) &= \max(\alpha + \beta - 1, 0) && \text{(Lukasiewicz t-norm)}\\
T_D(\alpha, \beta) &= \begin{cases} 0 \text{ if } (\alpha, \beta) \in [0,1)^2 \\ \min(\alpha, \beta) \text{ otherwise} \end{cases} && \text{(drastic product)}
\end{aligned}
$$

Further, from the definition it is possible to deduce an additional boundary condition [30]:

$$T(0, \alpha) = T(\alpha, 0) = 0.$$

Properties **(T1)** and **(T3)** can be combined to give the following equivalent property:

$$T(\alpha_1, \beta_1) \le T(\alpha_2, \beta_2) \text{ whenever } \alpha_1 \le \alpha_2 \text{ and } \beta_1 \le \beta_2.$$

T-norms can be compared pointwise. That is, we say T_1 is *weaker* than T_2 if for all $\alpha, \beta \in [0,1], T_1(\alpha, \beta) \le T_2(\alpha, \beta)$. Equivalently we could express this by saying that T_2 is stronger than T_1. In the case $T_1 \le T_2$ and for some $(\alpha_0, \beta_0) \in [0,1]^2$, we have $T_1(\alpha_0, \beta_0) < T_2(\alpha_0, \beta_0)$ we then write $T_1 < T_2$.

The drastic product is the weakest t-norm, while the minimum is the strongest t-norm. That is, for any t-norm, T, $T_D \le T \le T_M$. Further it can be established (see [30]) that: $T_D < T_L < T_P < T_M$.

The minimum is also the only t-norm for which every $\alpha \in [0,1]$ is an idempotent element, i.e. $T_M(\alpha, \alpha) = \alpha$.

4.5 Square Roots

Definition 4.5.1. [19] A GL-monoid $(L, \le, *)$ is said to have *square roots* if there exists a unary operator $S : L \longrightarrow L$ with the following properties:

(S1) for all $\alpha \in L$, $S(\alpha) * S(\alpha) = \alpha$,

(S2) $\beta * \beta \leq \alpha \Longrightarrow \beta \leq S(\alpha)$.

Proposition 4.5.2. *When they exist, square roots are unique.*

Proof: Suppose there exist two unary operators S_1 and S_2 both of which satisfy the conditions listed above. Now let $\alpha \in L$. Since S_1 is a square root, by **(S1)** we get $S_1(\alpha) * S_1(\alpha) = \alpha$ and hence $S_1(\alpha) * S_1(\alpha) \leq \alpha$. Now since S_2 satisfies **(S2)** we have $S_1(\alpha) \leq S_2(\alpha)$.

Similarly, we can get $S_2(\alpha) \leq S_1(\alpha)$ and therefore $S_1(\alpha) = S_2(\alpha)$.

■

Examples 4.5.3.

- T_M : for all $\alpha \in L$ we have $S(\alpha) = \alpha$.
- T_P : for all $\alpha \in L$, $S(\alpha) = \sqrt{\alpha}$.
- T_L : for all $\alpha \in L$, $S(\alpha) = \frac{\alpha+1}{2}$.
- When $\wedge$ is used as the quantale operator to form a complete Heyting algebra, we again have square roots: for all $\alpha \in L$, $S(\alpha) = \alpha$. This can be seen from property **(S1)** where $S(\alpha) \wedge S(\alpha) = \alpha$ and so $S(\alpha) = \alpha$.

Not all t-norms have square roots. Consider the drastic product T_D and let $\alpha = \frac{1}{2}$. If $S(\alpha) \neq 1$, then we have $S(\alpha) * S(\alpha) = 0 \neq \alpha$. If we have $S(\alpha) = 1$ then we get $S(\alpha) * S(\alpha) = 1 \neq \alpha$.

Lemma 4.5.4. *Let $(L, \leq, *)$ be a GL-monoid with square roots. Then for all $\alpha, \alpha_i, \beta, \delta, \gamma \in L$ the following hold:*

(i) $\alpha \leq \beta \Longrightarrow S(\alpha) \leq S(\beta)$,

(ii) $\bigwedge_{i \in I} S(\alpha_i) = S\big(\bigwedge_{i \in I} \alpha_i\big)$,

(iii) $(\alpha \to \beta) * (\delta \to \gamma) \leq S\big((\alpha * \delta) \to (\beta * \gamma)\big)$,

(iv) $\alpha \leq S(\alpha)$,

(v) $S(\top) = \top$.

PROOF:

(i) From **(S1)** we get that $S(\alpha) * S(\alpha) = \alpha \leq \beta$. Now **(S2)** gives us $S(\alpha) \leq S(\beta)$.

(ii) Clearly we have that for all $j \in I$, $\bigwedge_{i \in I} \alpha_i \leq \alpha_j$ and as a consequence of **(i)** we get that for all $j \in I$, $S\big(\bigwedge_{i \in I} \alpha_i\big) \leq S(\alpha_j)$, and so $S\big(\bigwedge_{i \in I} \alpha_i\big) \leq \bigwedge_{i \in I} S(\alpha_i)$.
For the reverse inequality, consider that for all $j \in I$,

$$\bigwedge_{i \in I} S(\alpha_i) * \bigwedge_{i \in I} S(\alpha_i) \leq S(\alpha_j) * S(\alpha_j).$$

Then by **(S1)** we get for all $j \in I$, $\bigwedge_{i \in I} S(\alpha_i) * \bigwedge_{i \in I} S(\alpha_i) \leq \alpha_j$ and this implies

$$\bigwedge_{i \in I} S(\alpha_i) * \bigwedge_{i \in I} S(\alpha_i) \leq \bigwedge_{i \in I} \alpha_i.$$

Finally, **(S2)** gives us $\bigwedge_{i \in I} S(\alpha_i) \leq S\big(\bigwedge_{i \in I} \alpha_i\big)$.

(iii) If we let $\epsilon \leq (\alpha \to \beta) * (\delta \to \gamma)$, then clearly $\epsilon \leq (\alpha \to \beta)$ and $\epsilon \leq (\delta \to \gamma)$. From Lemma 4.3.7 we get $\epsilon * \alpha \leq \beta$ and $\epsilon * \delta \leq \gamma$. Now we have that

$$(\epsilon * \alpha) * (\epsilon * \delta) \ \leq \ \beta * \gamma,$$

and by the associativity of the $*$ operation we get $(\epsilon * \epsilon) * (\alpha * \delta) \ \leq \ \beta * \gamma$. Using Lemma 4.3.7 gives us $\epsilon * \epsilon \ \leq \ (\alpha * \delta) \to (\beta * \gamma)$ and finally, by **(S2)**, we have $\epsilon \ \leq \ S\big((\alpha * \delta) \to (\beta * \gamma)\big)$.

(iv) It is clear that $\alpha * \alpha \leq \alpha * \top \leq \alpha$ and so by **(S2)** we have $\alpha \leq S(\alpha)$.

(v) From **(iv)** we get $\top \leq S(\top)$ and so $S(\top) = \top$.

■

4.6 *cl*-premonoids

Having discussed the quantale operation, $*$, we now introduce a second possible algebraic operation, $\otimes$, on a lattice L. If this operation satisfies the properties given below, L will be a known as a *completely lattice ordered premonoid*, where "*completely lattice ordered*" is commonly abbreviated by "*cl*".

Definition 4.6.1. [22] Let $(L, \leq)$ be a lattice. The triple $(L, \leq, \otimes)$ is a *cl-premonoid* if:

(CL1) $(L, \leq)$ is a complete lattice,

(CL2) the binary operation $\otimes$ on L satisfies the *isotonicity axiom* :

$$\alpha_1 \leq \alpha_2 \ , \ \beta_1 \leq \beta_2 \implies \alpha_1 \otimes \beta_1 \leq \alpha_2 \otimes \beta_2,$$

(CL3) for each $\alpha \in L$, $\alpha \leq \alpha \otimes \top$ and $\alpha \leq \top \otimes \alpha$,

(CL4) the operation $\otimes$ is distributive over *non-empty* joins, ie: for $J \neq \emptyset$,

$$\left(\bigvee_{i\in J} \alpha_i\right) \otimes \beta = \bigvee_{i\in J} (\alpha_i \otimes \beta), \quad \beta \otimes \left(\bigvee_{i\in J} \alpha_i\right) = \bigvee_{i\in J} (\beta \otimes \alpha_i).$$

When both the quantale operation, $*$, and the *cl*-premonoid operation, $\otimes$, are attached to complete lattice L, we then have what is known as an *enriched cl-premonoid*. This is the lattice that will be used in most of the investigations conducted in the chapters that follow.

Definition 4.6.2. [22] An *enriched cl-premonoid* is a quadruple $(L, \leq, \otimes, *)$ such that:

(E1) $(L, \leq, \otimes)$ is a *cl*-premonoid,

(E2) $(L, \leq, *)$ is a GL-monoid,

(E3) the operation $*$ is dominated by $\otimes$. That is, for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in L$:

$$(\alpha_1 \otimes \beta_1) * (\alpha_2 \otimes \beta_2) \leq (\alpha_1 * \alpha_2) \otimes (\beta_1 * \beta_2).$$

If we take $\alpha_1 = \alpha, \alpha_2 = \top, \beta_1 = \top, \beta_2 = \beta$ we get:

$$\alpha * \beta \overset{CL3}{\leq} (\alpha \otimes \top) * (\top \otimes \beta) \overset{E3}{\leq} (\alpha * \top) \otimes (\top * \beta) = \alpha \otimes \beta.$$

In other words: $\alpha * \beta \leq \alpha \otimes \beta$.

Examples 4.6.3.

- The complete Heyting algebra, $(L, \leq, \wedge, \wedge)$, is an enriched *cl*-premonoid.
- The lattice $(L, \leq, \wedge, *)$ is an enriched *cl*-premonoid.

Definition 4.6.4. [22] Let $(L, \leq, *)$ be a GL-monoid that has square roots. The *monoidal mean operator*, $\circledast : L \times L \longrightarrow L$, is defined by:

$$\alpha \circledast \beta = S(\alpha) * S(\beta).$$

Examples 4.6.5.

- The monoidal mean operator for T_L becomes the arithmetic mean:

$$\alpha \circledast \beta = \frac{\alpha + \beta}{2}.$$

- For $([0,1], \leq, T_P)$ and $\alpha, \beta \in [0,1]$ we get $\alpha \circledast \beta = \sqrt{\alpha} \cdot \sqrt{\beta}$.
- For $([0,1], \leq, T_M)$, $\alpha, \beta \in [0,1]$ we get $\alpha \circledast \beta = \min\{\alpha, \beta\}$.
- For $(L, \leq, \wedge)$, $\alpha, \beta \in L$ we get $\alpha \circledast \beta = \alpha \wedge \beta$.

Lemma 4.6.6. *[22] Let $(L, \leq, *)$ be a GL-monoid with square roots. If $(L, \leq, *)$ also satisfies:*

(S3) $S(\alpha * \beta) = \big(S(\alpha) * S(\beta)\big) \vee S(\perp)$ *for all* $\alpha, \beta \in L$

then if we use the monoidal mean operator as the cl-*premonoid operation we get an enriched cl-premonoid:* $(L, \leq, \circledast, *)$.

Examples 4.6.7.

- $(L, \leq, \circledast, \wedge)$ is an enriched *cl*-premonoid.
- $\big([0,1], \leq, \circledast, T_P\big)$ is an enriched *cl*-premonoid.
- $\big([0,1], \leq, \circledast, T_M\big)$ is an enriched *cl*-premonoid.
- $\big([0,1], \leq, \circledast, T_L\big)$ is an enriched *cl*-premonoid.

For T_L we have that:

$$\alpha \circledast \beta = \frac{\alpha + \beta}{2}.$$

Now consider $\alpha \circledast 0 = \frac{\alpha+0}{2} = \frac{\alpha}{2}$ and for $\alpha \neq 0$ we have $\frac{\alpha}{2} \neq 0$ and therefore $\alpha \circledast 0 \neq 0$ in general. Further, $\alpha \circledast 1 = \frac{\alpha+1}{2} \geq \alpha$ and so for say $\alpha = 0$ we have $\alpha \circledast 1 \neq \alpha$. These two results show that $\perp$ is not the zero with respect to $\circledast$, and $\top$ is not the unit with respect to $\circledast$.

For the case $J = \emptyset$, we can see that by specifying the distributivity in **(CL4)** as being over *non-empty* joins, this allows inclusion of examples such as the one just mentioned. For $J = \emptyset$ we get

$$\Big(\bigvee_{j \in J} \alpha_j\Big) \otimes \beta = \perp \otimes \beta \quad \text{and} \quad \bigvee_{j \in J} (\alpha_j \otimes \beta) = \perp.$$

If empty joins were allowed, this result would necessitate that $\bot$ be the zero with respect to $\otimes$.

The following lemma is presented because it provides a result that is needed in Lemma 7.3.7.

Lemma 4.6.8. *Let $(L, \leq, \otimes, *)$ be an enriched cl-premonoid and let $\alpha, \beta, \delta \in L$. If $\alpha \leq \alpha \otimes \alpha$, then*

$$(\alpha \to \beta) \otimes (\alpha \to \delta) \ \leq \ \alpha \to (\beta \otimes \delta).$$

Proof: Here we will use Lemma 4.3.8 (ix) and the fact that the $*$ operation is dominated by the $\otimes$ to show

$$\begin{aligned}\big((\alpha \to \beta) \otimes (\alpha \to \delta)\big) * \alpha \ &\leq \ \big((\alpha \to \beta) \otimes (\alpha \to \delta)\big) * (\alpha \otimes \alpha) \\ &\leq \ \big((\alpha \to \beta) * \alpha\big) \otimes \big((\alpha \to \delta) * \alpha\big) \\ &= \ (\alpha \wedge \beta) \otimes (\alpha \wedge \delta) \\ &\leq \ \beta \otimes \delta.\end{aligned}$$

Using Lemma 4.3.7 we can then see the desired result.

■

Some important examples where we will have $\alpha \leq \alpha \otimes \alpha$ for all $\alpha \in L$ are:

- $(L, \leq, \circledast, T_M)$, where $\alpha \circledast \alpha = \min\{S(\alpha), S(\alpha)\} = \min\{\alpha, \alpha\} = \alpha$.
- $(L, \leq, \circledast, T_P)$, where $\alpha \circledast \alpha = \sqrt{\alpha} \cdot \sqrt{\alpha} = \alpha$.
- $(L, \leq, \circledast, T_L)$, where $\alpha \circledast \alpha = \frac{\alpha + \alpha}{2} = \alpha$.
- $(L, \leq, \wedge, \wedge)$, where $\alpha \wedge \alpha = \alpha$.

Below we present an interesting property of enriched cl-premonoids, that of *pseudo-bisymmetry.* This is required when using the least upper bound of two L-filters, as will be done extensively in chapter 5.

Definition 4.6.9. [22] Let $(L, \leq, \otimes, *)$ be an enriched cl-premonoid and let $S \subseteq L$ be non-empty. Then S is *pseudo-bisymmetric* in $(L, \leq, \otimes, *)$ if it satisfies:

$$(\alpha_1 * \beta_1) \otimes (\alpha_2 * \beta_2) = \big((\alpha_1 \otimes \alpha_2) * (\beta_1 \otimes \beta_2)\big) \vee \big((\alpha_1 \otimes \bot) * (\beta_1 \otimes \top)\big) \vee \big((\bot \otimes \alpha_2) * (\top \otimes \beta_2)\big)$$

for all $\alpha_1, \alpha_2 \in S$ and $\beta_1, \beta_2 \in L$.

If the above equality is satisfied for any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in L$ then we say that $(L, \leq, \otimes, *)$ is *pseudo-bisymmetric.*

Examples:

- If the *cl*-premonoid operation, $\otimes$, is identical to the quantale operation, $*$, then $(L, \leq, \otimes, *)$ is pseudo-bisymmetric.
- A quantale $(L, \leq, *)$ with square roots satisfying **(S3)**, and using the monoidal mean operator $\circledast$ as the $\otimes$ operation, will give $(L, \leq, \circledast, *)$ to be pseudo-bisymmetric (see [22]).
- The enriched *cl*-premonoid, $(L, \leq, \wedge, *)$, is pseudo-bisymmetric.
- The complete Heyting algebra, $(L, \leq, \wedge, \wedge)$, is pseudo-bisymmetric.

Chapter 5

L-sets and L-filters

The concept of a fuzzy set was first suggested by Zadeh [46] in 1965. He used a mapping from a set X to the unit interval, $[0,1]$, to assign a "grade of membership" of an element of X to the fuzzy set. This idea was generalised in 1967 to "L-fuzzy sets" by Goguen [12] where the unit interval was replaced by a general lattice L. There are many different approaches to lattice-valued filters [13] and we include a brief discussion of these on page 43.

5.1 L-sets (L-fuzzy sets)

Let X be a set and $(L,\leq)$ be a lattice. (From here on, we will use L, rather than $(L,\leq)$ to denote a lattice.) We call a map $a : X \longrightarrow L$ an L-set. These L-sets form a lattice $(L^X,\leq)$ under pointwise ordering:

$$a \leq b \iff \text{for all } x \in X, a(x) \leq b(x).$$

Examples 5.1.1.

- Let $A \subseteq X$ and $\alpha \in L$. Define the following function:

$$\alpha_A(x) = \begin{cases} \alpha \text{ if } x \in A \\ \bot \text{ if } x \notin A. \end{cases}$$

- Let $A \subseteq X$. For $\alpha = \top$ we obtain the characteristic function:

$$\top_A(x) = \begin{cases} \top \text{ if } x \in A \\ \bot \text{ if } x \notin A. \end{cases}$$

- $\top_X(x) = \top$, for all $x \in X$, the constant $\top$ function.

- $\bot_X(x) = \bot$, for all $x \in X$, the constant $\bot$ function.

- Let $\alpha \in L$. Define the constant function:

$$\alpha_X : X \longrightarrow L \ , \ x \longmapsto \alpha \text{ for all } x \in X.$$

If $L = \{0, 1\}$ is used as the lattice, then there is a one-to-one correspondence between subsets of X and L-sets:

$$\psi : \begin{cases} \wp(X) \longrightarrow \{0,1\}^X \\ A \longmapsto \top_A. \end{cases}$$

If $(L, \leq)$ is a complete lattice, then $(L^X, \leq)$ will also form a complete lattice where all of the operations are defined pointwise as follows:

$$\left[\bigvee_{i \in I} a_i\right](x) = \bigvee_{i \in I} a_i(x) \quad \text{and} \quad \left[\bigwedge_{i \in I} a_i\right](x) = \bigwedge_{i \in I} a_i(x).$$

In the case of $(L, \leq, \otimes, *)$, the algebraic operations $*$ and $\otimes$ will also be extended pointwise:

$$\left[a \otimes b\right](x) = a(x) \otimes b(x), \text{ and}$$

$$\left[a * b\right](x) = a(x) * b(x).$$

5.2 Images and inverse images of L-sets

Let X and Y be sets. Also, let $\varphi : X \longrightarrow Y$ and let $a \in L^X$. We then consider the image of a under φ, which will be an L-set on Y. That is, $\varphi^{\rightarrow}(a) \in L^Y$. For $b \in L^Y$, we can likewise consider the L-set on X, $\varphi^{\leftarrow}(b)$. These two mappings are defined as follows for $x \in X$ and $y \in Y$:

$$\left[\varphi^{\rightarrow}(a)\right](y) = \bigvee\{a(x) \ : \ \varphi(x) = y\} \qquad \text{and} \qquad \left[\varphi^{\leftarrow}(b)\right](x) = (b \circ \varphi)(x)$$

Note that it is possible that for some $y \in Y$, there will not exist any $x \in X$ such that $\varphi(x) = y$. In this case $[\varphi^{\rightarrow}(a)](y) = \bigvee \emptyset = \bot$ for every $a \in L^X$.

The following properties of images and inverse images are described by Kubiak [32].

Lemma 5.2.1. *[32] Let $(L, \leq, *)$ be a GL-monoid, $\varphi : X \longrightarrow Y, \psi : Y \longrightarrow Z$, $x \in X, \alpha \in L, a \in L^X, b \in L^Y$ and $c \in L^Z$. Then the following hold:*

(i) *$\varphi^{\leftarrow}$ preserves arbitrary sups and arbitrary infs; $\varphi^{\rightarrow}$ preserves arbitrary sups (both mappings are order preserving)*

(ii) *$a \leq \varphi^{\leftarrow}\big(\varphi^{\rightarrow}(a)\big)$ and $\varphi^{\leftarrow}\big(\varphi^{\rightarrow}(a)\big) = a$ if φ is injective*

(iii) $\varphi^{\rightarrow}\big(\varphi^{\leftarrow}(b)\big) \leq b$ *and* $\varphi^{\rightarrow}\big(\varphi^{\leftarrow}(b)\big) = b$ *if* φ *is surjective*

(iv) $(\psi \circ \varphi)^{\rightarrow}(a) = \psi^{\rightarrow}\big(\varphi^{\rightarrow}(a)\big)$

(v) $(\psi \circ \varphi)^{\leftarrow}(c) = \varphi^{\leftarrow}\big(\psi^{\leftarrow}(c)\big)$

(vi) $\varphi^{\leftarrow}(\alpha * b) = \alpha * \varphi^{\leftarrow}(b)$

(vii) $\varphi^{\rightarrow}(\alpha * a) = \alpha * \varphi^{\rightarrow}(a)$.

5.3 L-filters

As the classical filters are used in the description of classical uniform spaces, convergence spaces and uniform convergence spaces, so we will use the L-filters to describe these spaces' lattice-valued equivalents. One of the earliest notions of an L-filter is from Lowen [34] where he considers $L = [0,1]$ and uses prefilters as filters in the lattice $[0,1]^X$. Höhle meanwhile considered 1-filters [17], [18] before going on to define [20] for MV-algebras what are now commonly referred to as L-filters and stratified L-filters [21], [22]. Earlier, Gähler and Eklund [9] developed what are now known as tight L-filters. We use the definition of Höhle and Sostak [22].

Definition 5.3.1. [22] Let X be a set and $(L, \leq, \otimes, *)$ an enriched *cl*-premonoid. A map $\mathcal{F} : L^X \longrightarrow L$ is an *L-filter* on X if $\mathcal{F}$ satisfies:

(LF0) $\mathcal{F}(\top_X) = \top, \quad \mathcal{F}(\bot_X) = \bot$,

(LF1) $a_1 \leq a_2 \in L^X \Longrightarrow \mathcal{F}(a_1) \leq \mathcal{F}(a_2)$,

(LF2) $\mathcal{F}(a_1) \otimes \mathcal{F}(a_2) \leq \mathcal{F}(a_1 \otimes a_2)$ for all $a_1, a_2 \in L^X$.

We will also require an additional property, that of *stratification*:

(LFS) for all $\alpha \in L$, for all $a \in L^X$, $\quad \alpha * \mathcal{F}(a) \leq \mathcal{F}(\alpha_X * a)$.

If the L-filter $\mathcal{F}$ satisfies **(LFS)**, it is said to be a *stratified L-filter* on X. The set of all stratified L-filters on X is denoted by $\mathcal{F}^S_L(X)$.

In the classical definition (see section 2.1.1) $\mathfrak{F} \subset \wp(X)$ is a filter if:

(F0) $\mathfrak{F} \neq \emptyset, \emptyset \notin \mathfrak{F}$,

(F1) $F \in \mathfrak{F}, F \subset G \Longrightarrow G \in \mathfrak{F}$,

(F2) $F, G \in \mathfrak{F} \Longrightarrow F \cap G \in \mathfrak{F}$.

We show that for $L = \{\bot, \top\}$, and with $\top$ as the unit with respect to $\otimes$, there is a 1-to-1 correspondence between L-filters and classical filters. (Note: if $\top$ is the unit with respect to $\otimes$, then $\bot$ is necessarily the zero with respect to $\otimes$, as $\top \otimes \bot = \bot$ and $\bot \otimes \bot \leq \top \otimes \bot$.) For $\mathfrak{F} \in \mathfrak{F}(X)$ we shall define a $\{\bot, \top\}$-filter, $\mathcal{F}_{\mathfrak{F}}$, and show that the mapping $\psi : \mathfrak{F}(X) \longrightarrow \mathcal{F}_L^S(X)$, $\mathfrak{F} \longmapsto \mathcal{F}_{\mathfrak{F}}$ is 1-to-1.

For $\mathfrak{F} \in \mathfrak{F}(X)$ (a classical filter) we define $\mathcal{F}_{\mathfrak{F}} : \{\bot, \top\}^X \longrightarrow \{\bot, \top\}$ as follows:

Let $a \in \{\bot, \top\}^X$. We further define $S_a \subseteq X$, $S_a = \{x \in X \; : \; a(x) = \top\}$.

Now, we define the $\{\bot, \top\}$-filter, $\mathcal{F}_{\mathfrak{F}}$, by:

$$\mathcal{F}_{\mathfrak{F}}(a) = \begin{cases} \top \text{ if } S_a \in \mathfrak{F} \\ \bot \text{ else} \end{cases}$$

Proposition 5.3.2. *Let X be a set and $\mathfrak{F} \in \mathfrak{F}(X)$. Then $\mathcal{F}_{\mathfrak{F}}$, as defined above, is a $\{\bot, \top\}$-filter.*

Proof:

LF0: For $a = \top_X$ we get $S_{(\top_X)} = X$ and since $X \in \mathfrak{F}$ we have that $\mathcal{F}_{\mathfrak{F}}(\top_X) = \top$. Similarly, $S_{(\bot_X)} = \emptyset$ and $\emptyset \notin \mathfrak{F}$ gives us $\mathcal{F}_{\mathfrak{F}}(\bot_X) = \bot$.

LF1: Suppose $a, b \in \{\bot, \top\}^X$, If $a \leq b$, then $S_a \subseteq S_b$.

If $\mathcal{F}_{\mathfrak{F}}(a) = \bot$ then clearly $\mathcal{F}_{\mathfrak{F}}(a) \leq \mathcal{F}_{\mathfrak{F}}(b)$. Suppose that $\mathcal{F}_{\mathfrak{F}}(a) = \top$. This implies that $S_a \in \mathfrak{F}$, and then by **(F1)** we have $S_b \in \mathfrak{F}$, and therefore $\mathcal{F}_{\mathfrak{F}}(b) = \top$. So, $\mathcal{F}_{\mathfrak{F}}(a) \leq \mathcal{F}_{\mathfrak{F}}(b)$.

LF2: Suppose for at least one of a or b, $\mathcal{F}_{\mathfrak{F}}(a) = \bot$ or $\mathcal{F}_{\mathfrak{F}}(b) = \bot$. Since $\bot$ is the zero with respect to $\otimes$, $\mathcal{F}_{\mathfrak{F}}(a) \otimes \mathcal{F}_{\mathfrak{F}}(b) = \bot \leq \mathcal{F}_{\mathfrak{F}}(a \otimes b)$.

Else suppose that both $\mathcal{F}_{\mathfrak{F}}(a) = \top$ and $\mathcal{F}_{\mathfrak{F}}(b) = \top$. This gives us $S_a, S_b \in \mathfrak{F}$ and $\mathcal{F}_{\mathfrak{F}}(a) \otimes \mathcal{F}_{\mathfrak{F}}(b) = \top$. By **(F2)** we have $S_a \cap S_b \in \mathfrak{F}$.

Since 0 is the zero with respect to $\otimes$ we have that for all $\alpha, \beta \in \{\bot, \top\}$, $\alpha \otimes \beta = \alpha \wedge \beta$. From this we can show the following:

$$\begin{aligned} S_a \cap S_b &= \{x \in X \; : \; a(x) = \top, b(x) = \top\} \\ &= \{x \; : \; a(x) \wedge b(x) = \top\} \\ &= S_{a \wedge b} = S_{a \otimes b}. \end{aligned}$$

Now we already know from **(F2)** that $S_a \cap S_b \in \mathfrak{F}$, and therefore we have $S_{a \wedge b} \in \mathfrak{F}$ and hence $S_{a \otimes b} \in \mathfrak{F}$. This gives us $\mathcal{F}_{\mathfrak{F}}(a \otimes b) = \top$.

LFS: Let $\alpha = \bot$ and $a \in \{\bot, \top\}^X$. Then $\alpha * \mathfrak{F}(a) = \bot * \mathfrak{F}(a) = \bot \le \mathfrak{F}(\bot_X * a)$. If $\alpha = \top$ then since $\top_X * a = a$ we have that $\mathfrak{F}(\top_X * a) = \mathfrak{F}(a)$. Therefore $\top * \mathfrak{F}(a) = \mathfrak{F}(a) = \mathfrak{F}(\top_X * a)$.

■

For the opposite side of the 1-to-1 correspondence we must show that for any $\{\bot, \top\}$-filter $\mathcal{F}$, there exists a corresponding classical filter $\mathfrak{F}_{\mathcal{F}}$.

Proposition 5.3.3. *Let X be a set and $\mathcal{F}$ a $\{\bot, \top\}$-filter. Define $\mathfrak{F}_{\mathcal{F}}$ by:*

$$\mathfrak{F}_{\mathcal{F}} = \{Y \subseteq X \ : \ \mathcal{F}(\top_Y) = \top\}.$$

Then $\mathfrak{F}_{\mathcal{F}} \in \mathfrak{F}(X)$.

Proof:

F0: $\mathcal{F}(\top_\emptyset) = \mathcal{F}(\bot_X) = \bot$ implies that $\emptyset \notin \mathfrak{F}_{\mathcal{F}}$. Also, $\mathcal{F}(\top_X) = \top$ means that $X \in \mathfrak{F}_{\mathcal{F}}$ and so $\mathfrak{F}_{\mathcal{F}} \neq \emptyset$.

F1: Suppose $F \in \mathfrak{F}_{\mathcal{F}}$ and $G \subset X$ such that $F \subset G$. Since $F \in \mathfrak{F}_{\mathcal{F}}$ we have that $\mathcal{F}(\top_F) = \top$. Also, $F \subset G \Longrightarrow \top_F \le \top_G$ and by **(LF1)** this give us that $\mathcal{F}(\top_F) \le \mathcal{F}(\top_G)$. So, $\mathcal{F}(\top_G) = \top$ and therefore $G \in \mathfrak{F}_{\mathcal{F}}$.

F2: Let $F, G \in \mathfrak{F}_{\mathcal{F}}$. Then $\mathcal{F}(\top_F) = \top$ and $\mathcal{F}(\top_G) = \top$. By **(LF2)** we get that $\mathcal{F}(\top_F \otimes \top_G) = \top$. Using a similar argument as before, it can be shown that $\top_F \otimes \top_G = \top_{F \cap G}$. Therefore $\mathcal{F}(\top_{F\cap G}) = \top$ and so $F \cap G \in \mathfrak{F}_{\mathcal{F}}$.

■

Proposition 5.3.4. *$\psi : \mathfrak{F}(X) \longrightarrow \mathcal{F}^S_L(X)$, $\mathfrak{F} \longmapsto \mathcal{F}_{\mathfrak{F}}$ is bijective.*

Proof:

Injectivity: Suppose $\mathfrak{F}, \mathfrak{G} \in \mathfrak{F}(X)$, but $\mathfrak{F} \neq \mathfrak{G}$. That is, there exists $Y \subset X$ such that either $Y \in \mathfrak{F}$ but $Y \notin \mathfrak{G}$ or $Y \in \mathfrak{G}, Y \notin \mathfrak{F}$. Without loss of generality we assume that $Y \in \mathfrak{F}$, but $Y \notin \mathfrak{G}$.

Now consider $\top_Y \in \{\bot, \top\}^X$. Clearly $S_{(\top_Y)} = \{x \in X \ : \ \top_Y(x) = \top\} = Y$. Therefore we have that $\mathcal{F}_{\mathfrak{F}}(\top_Y) = \top$ and $\mathcal{F}_{\mathfrak{G}}(\top_Y) = \bot$, and hence $\mathcal{F}_{\mathfrak{F}} \neq \mathcal{F}_{\mathfrak{G}}$.

Surjectivity: Let $\mathcal{G}$ be a $\{\bot, \top\}$-filter on X. We must show that there exists a classical filter on $\mathfrak{G}$ on X such that $\mathcal{F}_{\mathfrak{G}} = \mathcal{G}$. We claim that such a filter exists and that in fact $\mathfrak{G} = \mathfrak{F}_{\mathcal{G}}$.

Let $a \in \{\bot, \top\}^X$. Suppose that $\mathcal{G}(a) = \top$. Now since $a = \top_{(S_a)}$, we can say that $\mathcal{G}(\top_{(S_a)}) = \top$ and therefore that $S_a \in \mathfrak{F}_{\mathcal{G}}$. Finally, from the definition of $\mathcal{F}_{\mathfrak{F}_{\mathcal{G}}}$ we have that $\mathcal{F}_{\mathfrak{F}_{\mathcal{G}}}(a) = \top$. Similarly, if $\mathcal{G}(a) = \bot$, we can show that $\mathcal{F}_{\mathfrak{F}_{\mathcal{G}}}(a) = \bot$ and hence we have $\mathcal{G} = \mathcal{F}_{\mathfrak{F}_{\mathcal{G}}}$.

■

Examples 5.3.5.

- The point filter $[x] : L^X \longrightarrow L, a \longmapsto a(x)$ is a stratified L-filter for every $x \in X$.
- $[A] : a \longmapsto \bigwedge_{x \in A} a(x)$ is a stratified L-filter, for $A \subset X$.
- $[X]$ is the coarsest stratified L-filter on X.

5.4 Ordering of L-filters

Let L be a lattice and X a set. A partial ordering can be defined on the set of all stratified L-filters on X. The partial ordering is defined by:

$$\mathcal{F} \leq \mathcal{G} \quad \Longleftrightarrow \quad \mathcal{F}(a) \leq \mathcal{G}(a) \quad \forall\, a \in L^X.$$

Here we say that $\mathcal{F}$ is coarser than $\mathcal{G}$, or $\mathcal{G}$ is finer than $\mathcal{F}$.

Definition 5.4.1. [14] For a collection of stratified L-filters on X, $\{\mathcal{F}_i \,:\, i \in I\}$, the greatest lower bound is defined for $a \in L^X$:

$$\Big(\bigwedge_{i \in I} \mathcal{F}_i\Big)(a) = \bigwedge_{i \in I} \mathcal{F}_i(a)$$

and $\bigwedge_{i \in I} \mathcal{F}_i \in \mathcal{F}_L^S(X)$.

The least upper bound of two stratified L-filters does not always exist but it has been shown [15] that an upper bound for two L-filters will exist when they satisfy certain conditions.

Proposition 5.4.2. *[15] Let $(L, \leq, \otimes, *)$ be an enriched cl-premonoid such that $(L, \leq, \otimes, *)$ is pseudo-bisymmetric. Further let $\mathcal{F}$ and $\mathcal{G}$ be two stratified L-filters on X. If $\mathcal{F}(a_1) * \mathcal{G}(a_2) = \bot$ for all $a_1, a_2 \in L^X$ such that $a_1 * a_2 = \bot_X$, then there exists an upper bound for both $\mathcal{F}$ and $\mathcal{G}$.*

Corollary 5.4.3. *[15] In the case of an enriched cl-premonoid where the $\otimes$ and $*$ operations are the same, it can be concluded that given two stratified L-filters $\mathcal{F}$ and $\mathcal{G}$ on X, then the supremum $\mathcal{F} \vee \mathcal{G}$ exists if and only if $\mathcal{F}(a_1) * \mathcal{G}(a_2) = \bot$ for all $a_1, a_2 \in L^X$ such that $a_1 * a_2 = \bot_X$. In particular, the supremum is the stratified L-filter given, for each $a \in L^X$, by:*

$$[\mathcal{F} \vee \mathcal{G}](a) = \bigvee\{\mathcal{F}(a_1) * \mathcal{G}(a_2) \mid a_1, a_2 \in L^X \text{ and } a_1 * a_2 \leq a\}.$$

5.5 Images of stratified L-filters

Definition 5.5.1. [22] Let X and Y be sets, $\varphi : X \longrightarrow Y$ and $\mathcal{F} \in \mathcal{F}_L^S(X)$. The *image of* $\mathcal{F}$ *under* φ, $\varphi(\mathcal{F}) : L^Y \longrightarrow L$, is always a stratified L-filter on Y and is defined for $a \in L^Y$:

$$[\varphi(\mathcal{F})](a) = \mathcal{F}\big(\varphi^{\leftarrow}(a)\big) = \mathcal{F}(a \circ \varphi).$$

Definition 5.5.2. [22] Let X and Y be sets. Suppose $\varphi : X \longrightarrow Y$ and let $\mathcal{F} \in \mathcal{F}_L^S(Y)$. For $a \in L^X$ define $\varphi^{\leftarrow}(\mathcal{F}) : L^X \longrightarrow L$ by

$$[\varphi^{\leftarrow}(\mathcal{F})](a) = \bigvee\{\mathcal{F}(b) \mid \varphi^{\leftarrow}(b) \leq a\}.$$

The mapping $\varphi^{\leftarrow}(\mathcal{F})$ is a stratified L-filter on X if and only if, for $b \in L^Y$, $\mathcal{F}(b) = \bot$ whenever $\varphi^{\leftarrow}(b) = b \circ \varphi = \bot_X$. If φ is surjective, then $\varphi^{\leftarrow}(\mathcal{F})$ will always be a stratified L-filter and $\varphi\big(\varphi^{\leftarrow}(\mathcal{F})\big) = \mathcal{F}$. Also, $\varphi^{\leftarrow}\big(\varphi(\mathcal{F})\big)$ will always be a a stratified L-filter and if φ is injective then we have $\varphi^{\leftarrow}\big(\varphi(\mathcal{F})\big) = \mathcal{F}$.

We now present two small lemmas relating to the images of stratified L-filters.

Lemma 5.5.3. *Let X, Y and Z be sets. Further, let $\mathcal{F} \in \mathcal{F}_L^S(X)$, $\varphi : X \longrightarrow Y$ and $\psi : Y \longrightarrow Z$. Then*

$$\varphi \circ \psi(\mathcal{F}) = \varphi\big(\psi(\mathcal{F})\big).$$

Proof: Let $a \in L^Z$ and let $x \in X$. Now $\varphi \circ \psi(\mathcal{F})(a) = \mathcal{F}\big((\varphi \circ \psi)^{\leftarrow}(a)\big)$. We have that

$$(\varphi \circ \psi)^{\leftarrow}(a)(x) = a\big(\varphi \circ \psi(x)\big) = a\Big(\varphi\big(\psi(x)\big)\Big) = \varphi^{\leftarrow}\Big(a\big(\psi(x)\big)\Big) = \psi^{\leftarrow}\big(\varphi^{\leftarrow}(a)\big)(x).$$

Therefore,

$$(\varphi \circ \psi)(\mathcal{F})(a) = \mathcal{F}\Big(\psi^{\leftarrow}\big(\varphi^{\leftarrow}(a)\big)\Big) = \psi(\mathcal{F})\big(\varphi^{\leftarrow}(a)\big) = \varphi\big(\psi(\mathcal{F})\big)(a).$$

■

Lemma 5.5.4. *Let X and Y be sets. If $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X)$ and $\varphi : X \longrightarrow Y$, then*

$$\varphi(\mathcal{F}) \wedge \varphi(\mathcal{G}) = \varphi(\mathcal{F} \wedge \mathcal{G}).$$

Proof: Let $a \in L^Y$. Now,

$$\begin{aligned}\big(\varphi(\mathcal{F}) \wedge \varphi(\mathcal{G})\big)(a) &= [\varphi(\mathcal{F})](a) \wedge [\varphi(\mathcal{G})](a)\\ &= \mathcal{F}(a \circ \varphi) \wedge \mathcal{G}(a \circ \varphi)\\ &= (\mathcal{F} \wedge \mathcal{G})(a \circ \varphi)\\ &= \big[\varphi(\mathcal{F} \wedge \mathcal{G})\big](a).\end{aligned}$$

■

Consequence: If we have φ and $\mathcal{F}, \mathcal{G}$ as above with $\mathcal{F} \leq \mathcal{G}$, then $\varphi(\mathcal{F}) \leq \varphi(\mathcal{G})$.

So, we have defined a stratified L-filter on a set X, and will use this definition as well as the various properties listed above in our subsequent investigations of lattice-valued spaces.

5.6 L-filters on Products

In this section, we consider our lattice L to be an enriched *cl*-premonoid $(L, \leq \otimes, *)$ that is pseudo-bisymmetric. The pseudo-bisymmetry is required as it will guarantee the existence of upper bounds (see Proposition 5.4.2), and this is required in the definition of a product L-filter.

Definition 5.6.1. Let X and Y be sets and consider the product space $X \times Y$. We define the *projection mappings*:

$$P_1 : \begin{cases} X \times Y \longrightarrow X \\ (x, y) \longmapsto x \end{cases} \qquad \text{and} \qquad P_2 : \begin{cases} X \times Y \longrightarrow Y \\ (x, y) \longmapsto y. \end{cases}$$

Definition 5.6.2. Let X and Y be sets and let $\mathcal{F} \in \mathcal{F}_L^S(X)$ and $\mathcal{G} \in \mathcal{F}_L^S(Y)$. We define their product $\mathcal{F} \times \mathcal{G}$ by:

$$\mathcal{F} \times \mathcal{G} = P_1^{\leftarrow}(\mathcal{F}) \vee P_2^{\leftarrow}(\mathcal{G}).$$

Proposition 5.6.3. *The mapping $\mathcal{F} \times \mathcal{G}$ is a stratified L-filter on $X \times Y$.*

Proof: Here we use the result of Proposition 5.4.2. That is, we must show that $P_1^{\leftarrow}(\mathcal{F})(a) * P_2^{\leftarrow}(\mathcal{G})(b) = \bot$ for all $a, b \in L^{X \times Y}$ such that $a * b = \bot$. This will show that there exists an upper bound for $P_1^{\leftarrow}(\mathcal{F}) \vee P_2^{\leftarrow}(\mathcal{G})$, and hence there must exist a least upper bound, and both of these will be stratified L-filters.

Suppose that $a, b \in L^{X \times Y}$ are such that $a * b = \bot$. Then

$$\begin{aligned}&P_1^{\leftarrow}(\mathcal{F})(a) * P_2^{\leftarrow}(\mathcal{G})(b)\\ &= \bigvee \big\{\mathcal{F}(c) \mid c \in L^X, P_1^{\leftarrow}(c) \leq a\big\} * \bigvee \big\{\mathcal{G}(d) \mid d \in L^Y, P_2^{\leftarrow}(d) \leq b\big\}\\ &\leq \bigvee \big\{\mathcal{F}(c) * \mathcal{G}(d) \mid c \in L^X, d \in L^Y, P_1^{\leftarrow}(c) * P_2^{\leftarrow}(d) \leq a * b\big\}.\end{aligned}$$

Now, $P_1^{\leftarrow}(c)(x,y) = c \circ P_1(x,y) = c(x)$. Similarly $P_2^{\leftarrow}(d)(x,y) = d(y)$. Therefore

$$P_1^{\leftarrow}(\mathcal{F})(a) * P_Y^{\leftarrow}(\mathcal{G})(b) \le \bigvee \big\{\mathcal{F}(c) * \mathcal{G}(d) \mid c(x) * d(y) = \bot,\ \forall\, x \in X,\ \forall\, y \in Y\big\}.$$

Now we note that if $c(x) * d(y) = \bot$ for all $x \in X, y \in Y$, then since $*$ is a quantale operation, and using **(Q2)** from Definition 4.3.1 we get

$$\bot = \bigvee_{x\in X} \bigvee_{y \in Y} \big(c(x) * d(y)\big) = \big(\bigvee_{x\in X} c(x)\big) * \big(\bigvee_{y\in Y} d(y)\big).$$

Together with **(LF1)**, this yields

$$P_1^{\leftarrow}(\mathcal{F})(a) * P_2^{\leftarrow}(\mathcal{G})(b) \le \bigvee \big\{\mathcal{F}(\alpha_X) * \mathcal{G}(\beta_Y) \mid \alpha * \beta = \bot\big\}.$$

Now, using **(LFS)**, the stratification of the L-filters $\mathcal{F}$ and $\mathcal{G}$, we have that for all $\alpha, \beta \in L$,

$$\mathcal{F}(\alpha_X) * \mathcal{G}(\beta_Y) \le \mathcal{F}\Big(\alpha_X * \big(\mathcal{G}(\beta_Y)\big)_X\Big).$$

Then consider

$$\begin{aligned}\big[\alpha_X * \big(\mathcal{G}(\beta_Y)\big)_X\big](x) &= \alpha_X(x) * \big(\mathcal{G}(\beta_Y)\big)_X(x)\\ &= \alpha * \mathcal{G}(\beta_Y)\\ &\le \mathcal{G}(\alpha_Y * \beta_Y)\\ &= \mathcal{G}(\bot_Y) = \bot.\end{aligned}$$

From this we get that $\mathcal{F}(\alpha_X) * \mathcal{G}(\beta_Y) \le \mathcal{F}(\bot_X) = \bot$. Therefore, since

$$P_1^{\leftarrow}(\mathcal{F})(a) * P_2^{\leftarrow}(\mathcal{G})(b) \le \bigvee \big\{\mathcal{F}(\alpha_X) * \mathcal{G}(\beta_Y) \mid \alpha * \beta = \bot\big\}$$

we get that $P_1^{\leftarrow}(\mathcal{F})(a) * P_2^{\leftarrow}(\mathcal{G})(b) = \bot$.

■

Note 5.6.4. For X and Y sets, $\mathcal{F} \in \mathcal{F}_L^S(X)$ and $\mathcal{G} \in \mathcal{F}_L^S(Y)$ and for L a complete Heyting algebra [24] we have for $a \in L^{X\times Y}$:

$$(\mathcal{F} \times \mathcal{G})(a) = \bigvee_{\substack{f\in L^X, g \in L^Y\\ f\times g \le a}} \mathcal{F}(f) \wedge \mathcal{G}(g),$$

where $(f \times g)(x,y) = f(x) \wedge g(y)$ for $(x,y) \in X \times Y$.

Many of the L-filter results contained in this chapter have already been shown for the case of L a complete Heyting algebra [27]. For the enriched *cl*-premonoid the definition of a stratified L-filter differs because of the different additional operations used. Thus in most cases the proofs are not the same, although we still reference the original as in many cases a very similar approach is used.

Lemma 5.6.5. *[27] Let X and Y be sets and let $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X), \mathcal{H}, \mathcal{K} \in \mathcal{F}_L^S(Y)$. If $\mathcal{F} \leq \mathcal{G}$ and $\mathcal{H} \leq \mathcal{K}$ then $\mathcal{F} \times \mathcal{H} \leq \mathcal{G} \times \mathcal{K}$.*

Proof: Let $a \in L^{X\times Y}$ and consider any $b \in L^X$ such that $P_1^{\leftarrow}(b) \leq a$. From $\mathcal{F} \leq \mathcal{G}$ we have that $\mathcal{F}(b) \leq \mathcal{G}(b)$ and therefore for $c \in L^X$ we have

$$\bigvee \{\mathcal{F}(b) \; : \; P_1^{\leftarrow}(b) \leq a\} \;\leq\; \bigvee\{\mathcal{G}(c) \; : \; P_1^{\leftarrow}(c) \leq a\}.$$

The two expressions above are the definitions of the inverse images of $\mathcal{F}$ and $\mathcal{G}$ under P_1. Therefore we have $P_1^{\leftarrow}(\mathcal{F})(a) \leq P_1^{\leftarrow}(\mathcal{G})(a)$, and so $P_1^{\leftarrow}(\mathcal{F}) \leq P_1^{\leftarrow}(\mathcal{G})$ since a was arbitrary.

Similarly, it can be shown that $P_2^{\leftarrow}(\mathcal{H}) \leq P_2^{\leftarrow}(\mathcal{K})$. From these two results we get that $P_1^{\leftarrow}(\mathcal{F}) \vee P_2^{\leftarrow}(\mathcal{H}) \leq P_1^{\leftarrow}(\mathcal{G}) \vee P_2^{\leftarrow}(\mathcal{K})$ and so

$$\mathcal{F} \times \mathcal{H} \leq \mathcal{G} \times \mathcal{K}.$$

■

Lemma 5.6.6. *[27] Let X and Y be sets and $(L, \leq, \wedge, \wedge)$ a complete Heyting algebra. Further, let $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X)$ and let $\mathcal{H} \in \mathcal{F}_L^S(Y)$. Then*

$$(\mathcal{F} \wedge \mathcal{G}) \times \mathcal{H} = (\mathcal{F} \times \mathcal{H}) \wedge (\mathcal{G} \times \mathcal{H}).$$

Proof: The inequality $(\mathcal{F} \wedge \mathcal{G}) \times \mathcal{H} \leq (\mathcal{F} \times \mathcal{H}) \wedge (\mathcal{G} \times \mathcal{H})$ follows from the result of Lemma 5.6.5. For the reverse inequality, we have that for all $a \in L^{X\times Y}$

$$\begin{aligned}
\big((\mathcal{F} \times \mathcal{H}) \wedge (\mathcal{G} \times \mathcal{H})\big)(a) &= (\mathcal{F} \times \mathcal{H})(a) \wedge (\mathcal{G} \times \mathcal{H})(a) \\
&= \left(\bigvee_{a_1 \times a_2 \leq a} \big(\mathcal{F}(a_1) \wedge \mathcal{H}(a_2)\big) \right) \wedge \left(\bigvee_{b_1 \times b_2 \leq a} \big(\mathcal{G}(b_1) \wedge \mathcal{H}(b_2)\big) \right) \\
&= \bigvee_{\substack{a_1 \times a_2 \leq a \\ b_1 \times b_2 \leq a}} \big(\mathcal{F}(a_1) \wedge \mathcal{G}(b_1) \wedge \mathcal{H}(a_2) \wedge \mathcal{H}(b_2)\big) \\
&\leq \bigvee_{\substack{a_1 \times a_2 \leq a \\ b_1 \times b_2 \leq a}} \big(\mathcal{F}(a_1) \wedge \mathcal{G}(b_1) \wedge \mathcal{H}(a_2 \wedge b_2)\big) \\
&\leq \bigvee_{\substack{a_1 \times c_2 \leq a \\ b_1 \times c_2 \leq a}} \big(\mathcal{F}(a_1) \wedge \mathcal{G}(b_1) \wedge \mathcal{H}(c_2)\big) \\
&\leq \bigvee_{(a_1 \vee b_1) \times c_2 \leq a} \big(\mathcal{F}(a_1 \vee b_1) \wedge \mathcal{G}(a_1 \vee b_1) \wedge \mathcal{H}(c_2)\big) \\
&\leq \bigvee_{c_1 \times c_2 \leq a} \big((\mathcal{F} \wedge \mathcal{G})(c_1) \wedge \mathcal{H}(c_2)\big) \\
&= \big((\mathcal{F} \wedge \mathcal{G}) \times \mathcal{H}\big)(a).
\end{aligned}$$

■

REMARK: For the case of L an enriched cl-premonoid we were not able to prove the inequality $(\mathcal{F} \times \mathcal{H}) \wedge (\mathcal{G} \times \mathcal{H}) \leq (\mathcal{F} \wedge \mathcal{G}) \times \mathcal{H}$.

Let X and Y be sets. We then present the following two lemmas regarding stratified L-filters on the product space $X \times Y$.

Lemma 5.6.7. *[25] Let $\mathcal{F} \in \mathcal{F}_L^S(X \times Y)$. The projections are defined as being $P_1 : X \times Y \longrightarrow X$ and $P_2 : X \times Y \longrightarrow Y$. Then $P_1(\mathcal{F}) \times P_2(\mathcal{F}) \leq \mathcal{F}$.*

PROOF: By the definition of the product L-filter, we have that

$$P_1(\mathcal{F}) \times P_2(\mathcal{F}) = P_1^{\leftarrow}\big(P_1(\mathcal{F})\big) \vee P_2^{\leftarrow}\big(P_2(\mathcal{F})\big).$$

Further, we know from Definition 5.5.2 that for $a \in L^{X \times Y}$

$$\begin{aligned} P_1^{\leftarrow}\big(P_1(\mathcal{F})\big)(a) &= \bigvee\{P_1(\mathcal{F})(b) \mid P_1^{\leftarrow}(b) \leq a\} \\ &= \bigvee\{\mathcal{F}\big(P_1^{\leftarrow}(b)\big) \mid P_1^{\leftarrow}(b) \leq a\} \\ &\leq \mathcal{F}(a). \end{aligned}$$

It can be similarly shown that $P_2^{\leftarrow}\big(P_2(\mathcal{F})\big) \leq \mathcal{F}$ and thus we have

$$P_1(\mathcal{F}) \times P_2(\mathcal{F}) = P_1^{\leftarrow}\big(P_1(\mathcal{F})\big) \vee P_2^{\leftarrow}\big(P_2(\mathcal{F})\big) \leq \mathcal{F} \vee \mathcal{F} = \mathcal{F}.$$

■

Lemma 5.6.8. *[25] Let $\mathcal{F} \in \mathcal{F}_L^S(X)$ and $\mathcal{G} \in \mathcal{F}_L^S(Y)$, and define the projection mappings as before. Then $P_1(\mathcal{F} \times \mathcal{G}) \geq \mathcal{F}$ and $P_2(\mathcal{F} \times \mathcal{G}) \geq \mathcal{G}$.*

PROOF: Since the projections are surjective,

$$P_1(\mathcal{F} \times \mathcal{G}) = P_1\big(P_1^{\leftarrow}(\mathcal{F}) \vee P_2^{\leftarrow}(\mathcal{G})\big) \geq P_1\big(P_1^{\leftarrow}(\mathcal{F})\big) = \mathcal{F}.$$

Similarly, $P_2(\mathcal{F} \times \mathcal{G}) \geq \mathcal{G}$.

■

Now we propose the definition of a new mapping from $L^{X \times X} \longrightarrow L$, and will then show that it is in fact a stratified L-filter on the product space $X \times X$. This is later used when inducing a stratified L-limit space from a stratified L-uniform convergence space in section 7.4.

Definition 5.6.9. Let X be a set, $\mathcal{F} \in \mathcal{F}_L^S(X), x \in X$ and $d \in L^{X \times X}$. We define $\mathcal{F}_x : L^{X \times X} \longrightarrow L$ by

$$\mathcal{F}_x(d) = \mathcal{F}\big(d(\cdot, x)\big).$$

Proposition 5.6.10. *Let X be a set, $\mathcal{F} \in \mathcal{F}_L^S(X)$ and $x \in X$. Then $\mathcal{F}_x \in \mathcal{F}_L^S(X \times X)$.*

PROOF:

LF0: By definition we have that $\mathcal{F}_x(\top_{X\times X}) = \mathcal{F}\big(\top_{X\times X}(\cdot, x)\big)$. It is clear that $\top_{X\times X}(\cdot, x) = \top_X$. Therefore $\mathcal{F}\big(\top_{X\times X}(\cdot, x)\big) = \mathcal{F}(\top_X)$ and since $\mathcal{F}$ is a stratified L-filter on X, we get that $\mathcal{F}(\top_X) = \top = \mathcal{F}_x(\top_{X\times X})$. Similarly, it can be shown that $\mathcal{F}_x(\bot_{X\times X}) = \mathcal{F}(\bot_X) = \bot$.

LF1: Suppose $a, b \in L^{X\times X}$ and $a \leq b$. Clearly, for any $y \in X$ we will have that $a(y, x) \leq b(y, x)$ and hence $a(\cdot, x) \leq b(\cdot, x)$. Since $\mathcal{F}$ is a stratified L-filter on X we get

$$\mathcal{F}_x(a) = \mathcal{F}\big(a(\cdot, x)\big) \leq \mathcal{F}\big(b(\cdot, x)\big) = \mathcal{F}_x(b).$$

LF2: Let $a, b \in L^{X\times X}$. First we show that $a(\cdot, x) \otimes b(\cdot, x) = a \otimes b(\cdot, x)$. For $y \in X$,

$$\begin{aligned}
[a(\cdot, x) \otimes b(\cdot, x)](y) &= a(\cdot, x)(y) \otimes b(\cdot, x)(y) \\
&= a(y, x) \otimes b(y, x) \\
&= a \otimes b(y, x) \\
&= [a \otimes b(\cdot, x)](y).
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathcal{F}_x(a) \otimes \mathcal{F}_x(b) &= \mathcal{F}\big(a(\cdot, x)\big) \otimes \mathcal{F}\big(b(\cdot, x)\big) \\
&\leq \mathcal{F}\big(a(\cdot, x) \otimes b(\cdot, x)\big) \\
&= \mathcal{F}\big(a \otimes b(\cdot, x)\big) \\
&= \mathcal{F}_x(a \otimes b).
\end{aligned}$$

LFS: Let $\alpha \in L$ and $a \in L^{X\times X}$.

We must first show that $\alpha_X * a(\cdot, x) = (\alpha_{X\times X} * a)(\cdot, x)$. If we let $y \in X$, then

$$\begin{aligned}
[\alpha_X * a(\cdot, x)](y) &= \alpha_X(y) * a(y, x) \\
&= \alpha * a(y, x) \\
&= \alpha_{X\times X}(y, x) * a(y, x) \\
&= (\alpha_{X\times X} * a)(y, x) \\
&= \big((\alpha_{X\times X} * a)(\cdot, x)\big)(y).
\end{aligned}$$

Now we can use this fact to show the desired stratification:

$$\begin{aligned}
\alpha * \mathcal{F}_x(a) &= \alpha * \mathcal{F}\big(a(\cdot, x)\big) \\
&\leq \mathcal{F}\big(\alpha_X * a(\cdot, x)\big) \\
&= \mathcal{F}\big((\alpha_{X\times X} * a)(\cdot, x)\big) \\
&= \mathcal{F}_x(\alpha_{X\times X} * a).
\end{aligned}$$

■

Lemma 5.6.11. *[27] Let X be set and $(L, \leq, \wedge, \wedge)$ a complete Heyting algebra. If $\mathcal{F} \in \mathcal{F}_L^S(X)$ and $x \in X$, then*

$$\mathcal{F}_x = \mathcal{F} \times [x].$$

Proof: Let $d \in L^{X \times X}$. We then have

$$\begin{aligned}
\big(\mathcal{F} \times [x]\big)(d) &= \bigvee_{\substack{d_1, d_2 \in L^X: \\ d_1 \times d_2 \leq d}} \big(\mathcal{F}(d_1) \wedge [x](d_2)\big) \\
&= \bigvee_{\substack{d_1, d_2 \in L^X: \\ d_1 \times d_2 \leq d}} \big(\mathcal{F}(d_1) \wedge d_2(x)\big) \\
&\leq \bigvee_{\substack{d_1, d_2 \in L^X: \\ d_1 \times d_2 \leq d}} \mathcal{F}\big(d_1 \wedge d_2(x)\big) \\
&\leq \mathcal{F}\big(d(\cdot, x)\big) \\
&= \mathcal{F}_x(d).
\end{aligned}$$

For the reverse inequality, we consider $u, v \in X$:

$$\begin{aligned}
\big(d(\cdot, x) \times \top_x\big)(u, v) &= d(u, x) \wedge \top_x(v) \\
&= \begin{cases} d(u, v) & x = v \\ \bot & x \neq v \end{cases} \\
&\leq d(u, v).
\end{aligned}$$

Therefore $\mathcal{F}_x(d) = \mathcal{F}\big(d(\cdot, x)\big) = \mathcal{F}\big(d(\cdot, x)\big) \wedge \big([x](\top_x)\big) \leq \big(\mathcal{F} \times [x]\big)(d)$.

■

Corollary 5.6.12. *Let X be a set, L a complete Heyting algebra, and $x \in X$. Then $[(x, x)] = [x] \times [x]$.*

Proof: From above, $[x] \times [x] = [x]_x$. For $d \in L^{X \times X}$ we have

$$[x]_x(d) = [x]\big(d(\cdot, x)\big) = d(x, x) = [(x, x)](d).$$

■

The following two lemmas relating to the filter $\mathcal{F}_x$ are used later on when proving that it is possible to use a stratified L-uniform convergence structure to induce a stratified L-limit structure on a set X. This is done in section 7.4.3.

Lemma 5.6.13. *Let X be a set and $x, y \in X$. If we consider $[x] \in \mathcal{F}_L^S(X)$ and $[(x, y)] \in \mathcal{F}_L^S(X \times X)$, then $[x]_y = [(x, y)]$.*

Proof: Let $d \in L^{X\times X}$. Then

$$\begin{aligned}[x]_y(d) &= [x]\big(d(\cdot, y)\big)\\ &= d(x,y)\\ &= [(x,y)](d).\end{aligned}$$

■

Lemma 5.6.14. *Let X be a set and $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X)$. Then for $x \in X$*

$$\mathcal{F}_x \wedge \mathcal{G}_x = (\mathcal{F} \wedge \mathcal{G})_x.$$

Proof: Let $d \in L^{X\times X}$. We have

$$\begin{aligned}(\mathcal{F}_x \wedge \mathcal{G}_x)(d) &= \mathcal{F}_x(d) \wedge \mathcal{G}_x(d)\\ &= \mathcal{F}\big(d(\cdot, x)\big) \wedge \mathcal{G}\big(d(\cdot, x)\big)\\ &= (\mathcal{F} \wedge \mathcal{G})\big(d(\cdot, x)\big)\\ &= (\mathcal{F} \wedge \mathcal{G})_x(d).\end{aligned}$$

■

5.7 Inverses of L-filters on Products

For a stratified L-filter $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$ and $d \in L^{X\times X}$, define:

$$\mathcal{F}^{-1}(d) = \mathcal{F}(d^{-1}), \text{ where } d^{-1}(x,y) = d(y,x) \text{ for } (x,y) \in X \times X.$$

Proposition 5.7.1. *[27] Let X be a set and $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$. Then*

$$\mathcal{F}^{-1} \in \mathcal{F}_L^S(X \times X).$$

Proof: Let $(x,y) \in X \times X$. Then

$$\begin{aligned}\top_{X\times X}(x,y) &= \top\\ &= \top_{X\times X}(y,x)\\ &= (\top_{X\times X})^{-1}(x,y).\end{aligned}$$

Therefore we have $\top_{X\times X} = (\top_{X\times X})^{-1}$. Similarly, $\bot_{X\times X} = (\bot_{X\times X})^{-1}$.

LF0: Using what we have shown above, we get that

$$\mathcal{F}^{-1}(\top_{X\times X}) = \mathcal{F}\big((\top_{X\times X})^{-1}\big) = \mathcal{F}(\top_{X\times X}) = \top,$$

and

$$\mathcal{F}^{-1}(\bot_{X\times X}) = \mathcal{F}\big((\bot_{X\times X})^{-1}\big) = \mathcal{F}(\bot_{X\times X}) = \bot.$$

For **(LF1)**,**(LF2)** and **(LFS)** we consider $a, b \in L^{X\times X}$ and $\alpha \in L$.

LF1: By definition, $a \leq b$ implies that $a(x,y) \leq b(x,y)$ for all $(x,y) \in X \times X$. Hence $a^{-1}(y,x) = a(x,y) \leq b(x,y) = b^{-1}(y,x)$ for all $(x,y) \in X \times X$, and therefore $a^{-1} \leq b^{-1}$. This then gives us that

$$\mathcal{F}^{-1}(a) = \mathcal{F}(a^{-1}) \leq \mathcal{F}(b^{-1}) = \mathcal{F}^{-1}(b).$$

LF2: We first show that $a^{-1} \otimes b^{-1} = (a \otimes b)^{-1}$. Let $(x,y) \in X \times X$.

$$\begin{aligned}(a^{-1} \otimes b^{-1})(x,y) &= a^{-1}(x,y) \otimes b^{-1}(x,y) \\ &= a(y,x) \otimes b(y,x) \\ &= (a \otimes b)(y,x) \\ &= (a \otimes b)^{-1}(x,y).\end{aligned}$$

Using the above result:

$$\begin{aligned}\mathcal{F}^{-1}(a) \otimes \mathcal{F}^{-1}(b) &= \mathcal{F}(a^{-1}) \otimes \mathcal{F}(b^{-1}) \\ &\leq \mathcal{F}(a^{-1} \otimes b^{-1}) \\ &= \mathcal{F}\big((a \otimes b)^{-1}\big) \\ &= \mathcal{F}^{-1}(a \otimes b).\end{aligned}$$

LFS: We show $(\alpha_X * a)^{-1} = \alpha_X * a^{-1}$. With $(x,y) \in X \times X$ we get

$$\begin{aligned}(\alpha_X * a)^{-1}(x,y) &= (\alpha_X * a)(y,x) \\ &= \alpha * \big(a(y,x)\big) \\ &= \alpha * \big(a^{-1}(x,y)\big) \\ &= (\alpha_X * a^{-1})(x,y).\end{aligned}$$

With this we have

$$\begin{aligned}\alpha * \mathcal{F}^{-1}(a) &= \alpha * \mathcal{F}(a^{-1}) \\ &\leq \mathcal{F}(\alpha_X * a^{-1}) \\ &= \mathcal{F}\big((\alpha_X * a)^{-1}\big) \\ &= \mathcal{F}^{-1}(\alpha_X * a).\end{aligned}$$

■

Lemma 5.7.2. *[27] Let X be a set and $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$. Then $(\mathcal{F}^{-1})^{-1} = \mathcal{F}$.*

Proof: We show that for any $a \in L^{X \times X}$ we will have $(a^{-1})^{-1} = a$. If we let $(x, y) \in X \times X$ we get $(a^{-1})^{-1}(x, y) = a^{-1}(y, x) = a(x, y)$.

Now $(\mathcal{F}^{-1})^{-1}(a) = (\mathcal{F}^{-1})(a^{-1}) = \mathcal{F}\big((a^{-1})^{-1}\big) = \mathcal{F}(a)$.

■

Lemma 5.7.3. *[27] Let X be a set. If $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X \times X)$ and $\mathcal{F} \leq \mathcal{G}$, then $\mathcal{F}^{-1} \leq \mathcal{G}^{-1}$.*

Proof: For any $a \in L^{X \times X}$ we get

$$\mathcal{F}^{-1}(a) = \mathcal{F}(a^{-1}) \leq \mathcal{G}(a^{-1}) = \mathcal{G}^{-1}(a).$$

■

Here we collect some results about the images of stratified L-filters on product spaces.

Lemma 5.7.4. *Let X and Y be sets, $\varphi : X \longrightarrow Y$ and $x \in X$. Then*

$$(\varphi \times \varphi)\big([(x, x)]\big) = \big[\big(\varphi(x), \varphi(x)\big)\big].$$

Proof: For $a \in L^{Y \times Y}$ we have

$$\begin{aligned}(\varphi \times \varphi)\big([(x, x)]\big)(a) &= \big([(x, x)]\big)\big((\varphi \times \varphi)^{\leftarrow}(a)\big) \\ &= \big([(x, x)]\big)\big(a \circ (\varphi \times \varphi)\big) \\ &= \big(a \circ (\varphi \times \varphi)\big)(x, x) \\ &= a\big(\varphi(x), \varphi(x)\big) \\ &= \big[\big(\varphi(x), \varphi(x)\big)\big](a).\end{aligned}$$

■

Lemma 5.7.5. *Let X and Y be sets, $\varphi : X \longrightarrow Y$ and $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$. Then*

$$(\varphi \times \varphi)(\mathcal{F}^{-1}) = \big((\varphi \times \varphi)(\mathcal{F})\big)^{-1}.$$

Proof: Let $a \in L^{Y \times Y}$ and let $(x_1, x_2) \in X \times X$.

We first show that $\big(a \circ (\varphi \times \varphi)\big)^{-1} = a^{-1} \circ (\varphi \times \varphi)$.

$$\begin{aligned}\big(a \circ (\varphi \times \varphi)\big)^{-1}(x_1, x_2) &= \big(a \circ (\varphi \times \varphi)\big)(x_2, x_1) \\ &= a\big(\varphi(x_2), \varphi(x_1)\big) \\ &= a^{-1}\big(\varphi(x_1), \varphi(x_2)\big) \\ &= \big(a^{-1} \circ (\varphi \times \varphi)\big)(x_1, x_2).\end{aligned}$$

Now we consider

$$\begin{aligned}\big[(\varphi\times\varphi)(\mathcal{F}^{-1})\big](a) &= \mathcal{F}^{-1}\big(a\circ(\varphi\times\varphi)\big)\\ &= \mathcal{F}\Big(\big(a\circ(\varphi\times\varphi)\big)^{-1}\Big)\\ &= \mathcal{F}\big(a^{-1}\circ(\varphi\times\varphi)\big)\\ &= \big[(\varphi\times\varphi)(\mathcal{F})\big](a^{-1})\\ &= \big((\varphi\times\varphi)(\mathcal{F})\big)^{-1}(a).\end{aligned}$$

■

Lemma 5.7.6. *Let X, Y and Z be sets, $\varphi : X \longrightarrow Y$ and $\psi : Y \longrightarrow Z$. Then, for all $\mathcal{F} \in \mathcal{F}_L^S(X\times X)$,*

$$(\psi\times\psi)\circ\big((\varphi\times\varphi)(\mathcal{F})\big) = \big((\psi\circ\varphi)\times(\psi\circ\varphi)\big)(\mathcal{F}).$$

PROOF: Let $a \in L^{Z\times Z}$. Then

$$\begin{aligned}\big[(\psi\times\psi)\circ\big((\varphi\times\varphi)(\mathcal{F})\big)\big](a) &= \big[(\varphi\times\varphi)(\mathcal{F})\big]\big((\psi\times\psi)^{\leftarrow}(a)\big)\\ &= \big[(\varphi\times\varphi)(\mathcal{F})\big]\big(a\circ(\psi\times\psi)\big)\\ &= \mathcal{F}\Big(\big(a\circ(\psi\times\psi)\big)\circ(\varphi\times\varphi)\Big).\end{aligned}$$

Now let $(x_1, x_2) \in X\times X$ and consider:

$$\begin{aligned}\Big(\big(a\circ(\psi\times\psi)\big)\circ(\varphi\times\varphi)\Big)(x_1,x_2) &= \big(a\circ(\psi\times\psi)\big)\big(\varphi(x_1),\varphi(x_2)\big)\\ &= a\Big(\psi\big(\varphi(x_1)\big),\psi\big(\varphi(x_2)\big)\Big)\\ &= a\big(\psi\circ\varphi(x_1),\psi\circ\varphi(x_2)\big)\\ &= \Big(a\big((\psi\circ\varphi)\times(\psi\circ\varphi)\big)\Big)(x_1,x_2).\end{aligned}$$

Therefore

$$\begin{aligned}\big[(\psi\times\psi)\circ\big((\varphi\times\varphi)(\mathcal{F})\big)\big](a) &= \mathcal{F}\Big(\big(a\circ(\psi\times\psi)\big)\circ(\varphi\times\varphi)\Big)\\ &= \mathcal{F}\Big(a\circ\big((\psi\circ\varphi)\times(\psi\circ\varphi)\big)\Big)\\ &= \big[\big((\psi\circ\varphi)\times(\psi\circ\varphi)\big)(\mathcal{F})\big](a).\end{aligned}$$

■

Lemma 5.7.7. *Let X be a set, $x \in X$ and $\mathcal{F} \in \mathcal{F}_L^S(X)$. Further suppose that $\varphi : X \longrightarrow Y$. Then $(\varphi \times \varphi)(\mathcal{F}_x) = \varphi(\mathcal{F})_{\varphi(x)}$.*

PROOF: Let $a \in L^{Y \times Y}$. Then

$$\begin{aligned}
(\varphi \times \varphi)(\mathcal{F}_x)(a) &= \mathcal{F}_x\big(a \circ (\varphi \times \varphi)\big) \\
&= \mathcal{F}\Big(\big(a \circ (\varphi \times \varphi)\big)(\cdot, x)\Big) \\
&= \mathcal{F}\Big(a\big(\varphi(\cdot), \varphi(x)\big)\Big) \\
&= \mathcal{F}\Big(a\big(\cdot, \varphi(x)\big) \circ \varphi\Big) \\
&= \mathcal{F}\Big(\varphi^{\leftarrow}\Big(a\big(\cdot, \varphi(x)\big)\Big)\Big) \\
&= \varphi(\mathcal{F})\Big(a\big(\cdot, \varphi(x)\big)\Big) \\
&= \varphi(\mathcal{F})_{\varphi(x)}(a).
\end{aligned}$$

■

5.8 Composition of L-filters:

Let $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X \times X)$. We then define the mapping $\mathcal{F} \circ \mathcal{G} : L^{X \times X} \longrightarrow L$ by:

$$\mathcal{F} \circ \mathcal{G}(d) = \bigvee \{\mathcal{F}(a) * \mathcal{G}(b) : a, b \in L^{X \times X}, a \circ b \leq d\}$$

with $a \circ b(x, y) = \bigvee_{z \in X} a(x, z) * b(z, y)$.

Below we will give a condition that, when satisfied, will give us $\mathcal{F} \circ \mathcal{G} \in \mathcal{F}_L^S(X \times X)$. In order to prove the main condition for the existence of the composition of two filters, we must first prove some results about the composition of L-sets on $X \times X$.

In these results we consider $f, g, \bar{f}, \bar{g} \in L^{X \times X}$ and $a, b \in L^{X \times X}$.

Lemma 5.8.1. *Let $(L, \leq, \otimes, *)$ be an enriched cl-premonoid. If $f \circ g \leq a$ and $\bar{f} \circ \bar{g} \leq b$, then*

$$(f \otimes \bar{f}) \circ (g \otimes \bar{g}) \leq a \otimes b.$$

PROOF: For $(x, y) \in X \times X$

$$\begin{aligned}
(f \otimes \bar{f}) \circ (g \otimes \bar{g})(x, y) &= \bigvee_{z \in X} (f \otimes \bar{f})(x, z) * (g \otimes \bar{g})(z, y) \\
&= \bigvee_{z \in X} \Big(f(x, z) \otimes \bar{f}(x, z)\Big) * \Big(g(z, y) \otimes \bar{g}(z, y)\Big) \\
&\overset{E3}{\leq} \bigvee_{z \in X} \Big(f(x, z) * g(z, y)\Big) \otimes \Big(\bar{f}(x, z) * \bar{g}(z, y)\Big).
\end{aligned}$$

Now for all $z \in X$, $f(x,z) * g(z,y) \leq \bigvee_{z \in X} f(x,z) * g(z,y) = f \circ g(x,y)$.

Likewise for every $z \in X$, $\bar{f}(x,z) * \bar{g}(z,y) \leq \bigvee_{z \in X} \bar{f}(x,z) * \bar{g}(z,y) = \bar{f} \circ \bar{g}(x,y)$.

Therefore for all $z \in X$

$$\Big(f(x,z) * g(z,y)\Big) \otimes \Big(\bar{f}(x,z) * \bar{g}(z,y)\Big) \leq f \circ g(x,y) \otimes \bar{f} \circ \bar{g}(x,y).$$

This gives us the following inequality:

$$\begin{aligned}\bigvee_{z \in X} \Big(f(x,z) * g(z,y)\Big) \otimes \Big(\bar{f}(x,z) * \bar{g}(z,y)\Big) &\leq f \circ g(x,y) \otimes \bar{f} \circ \bar{g}(x,y) \\ &\leq a(x,y) \otimes b(x,y) \\ &= (a \otimes b)(x,y).\end{aligned}$$

Lemma 5.8.2. *Let $(L\ \leq, \otimes, *)$ be an enriched cl-premonoid. If $f \circ g \leq a$ and $\bar{f} \circ \bar{g} \leq b$, then*

$$(f \otimes \bot_{X \times X}) \circ (g \otimes \top_{X \times X}) \leq a \otimes b.$$

PROOF: Let $(x,y) \in X \times X$. Then

$$\begin{aligned}(f \otimes \bot_{X \times X}) \circ (g \otimes \top_{X \times X})(x,y) &= \bigvee_{z \in X} (f \otimes \bot_{X \times X})(x,z) * (g \otimes \top_{X \times X})(z,y) \\ &= \bigvee_{z \in X} \Big(f(x,z) \otimes \bot_{X \times X}(x,z)\Big) * \Big(g(z,y) \otimes \top_{X \times X}(z,y)\Big) \\ &= \bigvee_{z \in X} \Big(f(x,z) \otimes \bot\Big) * \Big(g(z,y) \otimes \top\Big) \\ &\overset{E3}{\leq} \bigvee_{z \in X} \big(f(x,z) * g(z,y)\big) \otimes (\bot * \top) \\ &= \Big(\bigvee_{z \in X} f(x,z) * g(z,y)\Big) \otimes \bot \\ &= f \circ g(x,y) \otimes \bot \\ &\leq f \circ g(x,y) \otimes b(x,y) \\ &\leq a(x,y) \otimes b(x,y) \\ &= (a \otimes b)(x,y).\end{aligned}$$

■

Lemma 5.8.3. *Let $(L, \leq, \otimes, *)$ be an enriched cl-premonoid. If $f \circ g \leq a$ and $\bar{f} \circ \bar{g} \leq b$, then*

$$(\bot_{X \times X} \otimes \bar{f}) \circ (\top_{X \times X} \otimes \bar{g}) \leq a \otimes b.$$

PROOF:

$$
\begin{aligned}
(\bot_{X\times X}\otimes\bar f)\circ(\top_{X\times X}\otimes\bar g)(x,y) &= \bigvee_{z\in X}(\bot_{X\times X}\otimes\bar f)(x,z)*(\top_{X\times X}\otimes\bar g)(z,y)\\
&= \bigvee_{z\in X}\Big(\bot_{X\times X}(x,z)\otimes\bar f(x,z)\Big)*\Big(\top_{X\times X}(z,y)\otimes\bar g(z,y)\Big)\\
&= \bigvee_{z\in X}\Big(\bot\otimes\bar f(x,z)\Big)*\Big(\top\otimes\bar g(z,y)\Big)\\
&\overset{E3}{\leq} \bigvee_{z\in X}(\bot*\top)\otimes\big(\bar f(x,z)*\bar g(z,y)\big)\\
&=\bot\otimes\Big(\bigvee_{z\in X}\bar f(x,z)*\bar g(z,y)\Big)\\
&=\bot\otimes\bar f\circ\bar g(x,y)\\
&\leq a(x,y)\otimes\bar f\circ\bar g(x,y)\\
&\leq a(x,y)\otimes b(x,y)\\
&=(a\otimes b)(x,y).
\end{aligned}
$$

■

Lemma 5.8.4. *Let $\alpha\in L$ and $f,g,a\in L^{X\times X}$. If $f\circ g\leq a$, then*

$$(\alpha_{X\times X}*f)\circ g\leq\alpha_{X\times X}*a.$$

PROOF: Let $(x,y)\in X\times X$. Then

$$
\begin{aligned}
\big[(\alpha_{X\times X}*f)\circ g\big](x,y) &= \bigvee_{z\in X}(\alpha_{X\times X}*f)(x,z)*g(z,y)\\
&= \bigvee_{z\in X}\alpha*f(x,z)*g(z,y)\\
&= \alpha*\bigvee_{z\in X}f(x,z)*g(z,y)\\
&= \alpha*\big(f\circ g(x,y)\big)\\
&\leq \alpha*a(x,y)\\
&= (\alpha_{X\times X}*a)(x,y).
\end{aligned}
$$

■

The equivalence below was proved by Jäger and Burton [27], using a complete Heyting algebra as the underlying lattice. We now do it using an enriched *cl*-premonoid that is pseudo-bisymmetric. Besides requiring the additional lemmas above, the proof we present makes use of the pseudo-bisymmetry of L to prove **(LF2)**.

Proposition 5.8.5. *Let $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X \times X)$ and let $(L, \leq, \otimes, *)$ be a pseudo-bisymmetric enriched cl-premonoid. For any $f, g \in L^{X \times X}$, the following are equivalent:*

(i) *the mapping $\mathcal{F} \circ \mathcal{G} \in \mathcal{F}_L^S(X \times X)$,*

(ii) *if $f \circ g = \perp_{X \times X}$, then $\mathcal{F}(f) * \mathcal{G}(g) = \perp$.*

PROOF:

Suppose **(i)** and let $f \circ g = \perp_{X \times X}$. From **(LF0)** we have $\mathcal{F} \circ \mathcal{G}(\perp_{X \times X}) = \perp$. This will only be the case if

$$\perp = \bigvee_{\substack{h,k \in L^{X \times X} \\ h \circ k = \perp_{X \times X}}} \mathcal{F}(h) * \mathcal{G}(k).$$

Since $\mathcal{F}(f) * \mathcal{G}(g) \leq \bigvee_{\substack{h,k \in L^{X \times X} \\ h \circ k = \perp_{X \times X}}} \mathcal{F}(h) * \mathcal{G}(k)$, we get $\mathcal{F}(f) * \mathcal{G}(g) = \perp$.

Conversely, suppose **(ii)**.

LF0: We have for $(x, y) \in X \times X$,

$$\begin{aligned} \top_{X \times X} \circ \top_{X \times X}(x, y) &= \bigvee_{z \in X} \top_{X \times X}(x, z) * \top_{X \times X}(z, y) \\ &= \bigvee_{z \in X} \top * \top \\ &= \top_{X \times X}(x, y). \end{aligned}$$

Therefore $\top_{X \times X} \circ \top_{X \times X} \leq \top_{X \times X}$. With this we conclude that

$$\begin{aligned} \mathcal{F} \circ \mathcal{G}(\top_{X \times X}) &= \bigvee_{f \circ g \leq \top_{X \times X}} \mathcal{F}(f) * \mathcal{G}(g) \\ &\geq \mathcal{F}(\top_{X \times X}) * \mathcal{G}(\top_{X \times X}) \\ &= \top * \top \\ &= \top. \end{aligned}$$

$$\begin{aligned} \mathcal{F} \circ \mathcal{G}(\perp_{X \times X}) &= \bigvee_{f \circ g = \perp_{X \times X}} \mathcal{F}(f) * \mathcal{G}(g) \\ &= \bigvee_{f \circ g = \perp_{X \times X}} \perp * \perp \qquad \text{by } \textbf{(ii)} \\ &= \perp. \end{aligned}$$

LF1: Consider $a, b \in L^{X\times X}$ with $a \leq b$. This will give us the containment:

$$\big\{\mathcal{F}(f) * \mathcal{G}(g) \;:\; f \circ g \leq a\big\} \subset \big\{\mathcal{F}(\bar{f}) * \mathcal{G}(\bar{g}) \;:\; \bar{f} \circ \bar{g} \leq b\big\}$$

which in turn results in the inequality:

$$\bigvee_{f\circ g\leq a} \mathcal{F}(f) * \mathcal{G}(g) \leq \bigvee_{\bar{f}\circ\bar{g}\leq b} \mathcal{F}(\bar{f}) * \mathcal{G}(\bar{g}),$$

and this implies $\mathcal{F} \circ \mathcal{G}(a) \leq \mathcal{F} \circ \mathcal{G}(b)$.

LF2: Let $a, b \in L^{X\times X}$. Then

$$\begin{aligned}\mathcal{F} \circ \mathcal{G}(a) \otimes \mathcal{F} \circ \mathcal{G}(b) =& \Big(\bigvee_{f\circ g\leq a} \mathcal{F}(f) * \mathcal{G}(g)\Big) \otimes \Big(\bigvee_{\bar{f}\circ\bar{g}\leq b} \mathcal{F}(\bar{f}) * \mathcal{G}(\bar{g})\Big)\\ =& \bigvee_{\substack{f\circ g\leq a,\\ \bar{f}\circ\bar{g}\leq b}} \Big(\big(\mathcal{F}(f) * \mathcal{G}(g)\big) \otimes \big(\mathcal{F}(\bar{f}) * \mathcal{G}(\bar{g})\big)\Big) = P.\end{aligned}$$

The above equality is as a result of the distributivity of the $\otimes$ operation over non-empty joins. We now use the property of a pseudo-bisymmetric subset to produce the following inequality.

$$\begin{aligned}P \leq& \bigvee_{f\circ g\leq a, \bar{f}\circ\bar{g}\leq b} \Big(\big[\mathcal{F}(f) \otimes \mathcal{F}(\bar{f})\big] * \big[\mathcal{G}(g) \otimes \mathcal{G}(\bar{g})\big]\Big)\\ &\vee \Big(\big[\mathcal{F}(f) \otimes \bot\big] * \big[\mathcal{G}(g) \otimes \top\big]\Big)\\ &\vee \Big(\big[\bot \otimes \mathcal{F}(\bar{f})\big] * \big[\top \otimes \mathcal{G}(\bar{g})\big]\Big)\\ &= \bigvee_{f\circ g\leq a, \bar{f}\circ\bar{g}\leq b} \Big(\big[\mathcal{F}(f) \otimes \mathcal{F}(\bar{f})\big] * \big[\mathcal{G}(g) \otimes \mathcal{G}(\bar{g})\big]\Big)\\ &\vee \Big(\big[\mathcal{F}(f) \otimes \mathcal{F}(\bot_{X\times X})\big] * \big[\mathcal{G}(g) \otimes \mathcal{G}(\top_{X\times X})\big]\Big)\\ &\vee \Big(\big[\mathcal{F}(\bot_{X\times X}) \otimes \mathcal{F}(\bar{f})\big] * \big[\mathcal{G}(\top_{X\times X}) \otimes \mathcal{G}(\bar{g})\big]\Big)\\ &= Q.\end{aligned}$$

The equality above comes as a result of the fact that both $\mathcal{F}$ and $\mathcal{G}$ are stratified L-filters and from property **(LF0)** described earlier. We can further produce another inequality by, instead of taking the join over a single small set, we take the join of the joins of three larger sets:

$$\begin{aligned} Q \leq & \bigvee_{f\circ g\leq a, \bar{f}\circ\bar{g}\leq b} \big[\mathcal{F}(f)\otimes\mathcal{F}(\bar{f})\big] * \big[\mathcal{G}(g)\otimes\mathcal{G}(\bar{g})\big] \\ & \vee \bigvee_{f\circ g\leq a, \bar{f}\circ\bar{g}\leq b} \big[\mathcal{F}(f)\otimes\mathcal{F}(\bot_{X\times X})\big] * \big[\mathcal{G}(g)\otimes\mathcal{G}(\top_{X\times X})\big] \\ & \vee \bigvee_{f\circ g\leq a, \bar{f}\circ\bar{g}\leq b} \big[\mathcal{F}(\bot_{X\times X})\otimes\mathcal{F}(\bar{f})\big] * \big[\mathcal{G}(\top_{X\times X})\otimes\mathcal{G}(\bar{g})\big] \qquad = R. \end{aligned}$$

Now we use the results from 5.8.1, 5.8.2 and 5.8.3 to choose larger sets for each of the joins shown above:

$$\begin{aligned} R \leq & \bigvee_{(f\otimes\bar{f})\circ(g\otimes\bar{g})\leq a\otimes b} \big[\mathcal{F}(f)\otimes\mathcal{F}(\bar{f})\big] * \big[\mathcal{G}(g)\otimes\mathcal{G}(\bar{g})\big] \\ & \vee \bigvee_{(f\otimes\bot_{X\times X})\circ(g\otimes\top_{X\times X})\leq a\otimes b} \big[\mathcal{F}(f)\otimes\mathcal{F}(\bot_{X\times X})\big] * \big[\mathcal{G}(g)\otimes\mathcal{G}(\top_{X\times X})\big] \\ & \vee \bigvee_{(\bot_{X\times X}\otimes\bar{f})\circ(\top_{X\times X}\otimes\bar{g})\leq a\otimes b} \big[\mathcal{F}(\bot_{X\times X})\otimes\mathcal{F}(\bar{f})\big] * \big[\mathcal{G}(\top_{X\times X})\otimes\mathcal{G}(\bar{g})\big] \\ & = S. \end{aligned}$$

Now we again choose larger sets by allowing any $f, g, \bar{f}$ or $\bar{g}$ instead of $\bot_{X\times X}$ and $\top_{X\times X}$.

$$\begin{aligned} S \leq & \bigvee_{(f\otimes\bar{f})\circ(g\otimes\bar{g})\leq a\otimes b} \Big(\big[\mathcal{F}(f)\otimes\mathcal{F}(\bar{f})\big] * \big[\mathcal{G}(g)\otimes\mathcal{G}(\bar{g})\big]\Big) \\ & \vee \bigvee_{(f\otimes\bar{f})\circ(g\otimes\bar{g})\leq a\otimes b} \Big(\big[\mathcal{F}(f)\otimes\mathcal{F}(\bar{f})\big] * \big[\mathcal{G}(g)\otimes\mathcal{G}(\bar{g})\big]\Big) \\ & \vee \bigvee_{(f\otimes\bar{f})\circ(g\otimes\bar{g})\leq a\otimes b} \Big(\big[\mathcal{F}(f)\otimes\mathcal{F}(\bar{f})\big] * \big[\mathcal{G}(g)\otimes\mathcal{G}(\bar{g})\big]\Big). \end{aligned}$$

Since each of these sups is the same, we have:

$$\begin{aligned} S = & \bigvee_{(f\otimes\bar{f})\circ(g\otimes\bar{g})\leq a\otimes b} \Big(\big[\mathcal{F}(f)\otimes\mathcal{F}(\bar{f})\big] * \big[\mathcal{G}(g)\otimes\mathcal{G}(\bar{g})\big]\Big) \\ \leq & \bigvee_{(f\otimes\bar{f})\circ(g\otimes\bar{g})\leq a\otimes b} \Big[\mathcal{F}(f\otimes\bar{f}) * \mathcal{G}(g\otimes\bar{g})\Big] \\ \leq & \bigvee_{h\circ k\leq a\otimes b} \mathcal{F}(h) * \mathcal{G}(k) \\ = & \mathcal{F}\circ\mathcal{G}(a\otimes b). \end{aligned}$$

LFS: Here we use Lemma 5.8.4. For $\alpha \in L$ and $a \in L^{X \times X}$ we get

$$\begin{aligned}
\alpha * \mathcal{F} \circ \mathcal{G}(a) &= \alpha * \bigvee_{f \circ g \leq a} \mathcal{F}(f) * \mathcal{G}(g) \\
&= \bigvee_{f \circ g \leq a} \alpha * \mathcal{F}(f) * \mathcal{G}(g) \\
&\leq \bigvee_{f \circ g \leq a} \mathcal{F}(\alpha_{X \times X} * f) * \mathcal{G}(g) \qquad \text{(by } \textbf{(LFS)} \text{ of } \mathcal{F}) \\
&\leq \bigvee_{(\alpha_{X \times X} * f) \circ g \leq \alpha * a} \mathcal{F}(\alpha_{X \times X} * f) * \mathcal{G}(g) \\
&\leq \bigvee_{\bar{f} \circ \bar{g} \leq \alpha_{X \times X} * a} \mathcal{F}(\bar{f}) * \mathcal{G}(\bar{g}) \\
&= \mathcal{F} \circ \mathcal{G}(\alpha_{X \times X} * a).
\end{aligned}$$

■

We have thus provided a condition that, if satisfied, will guarantee that the composition of two stratified L-filters will again be a stratified L-filter. We will now show some further results relating to the composition of L-filters that will be needed later on.

Lemma 5.8.6. *[27] Let X be a set and let $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{K} \in \mathcal{F}_L^S(X \times X)$ such that $\mathcal{F} \leq \mathcal{H}$ and $\mathcal{G} \leq \mathcal{K}$. If $\mathcal{H} \circ \mathcal{K}$ exists, then $\mathcal{F} \circ \mathcal{G}$ exists and $\mathcal{F} \circ \mathcal{G} \leq \mathcal{H} \circ \mathcal{K}$.*

Proof: Suppose that $\mathcal{H} \circ \mathcal{K}$ is a stratified L-filter on $X \times X$, and consider $f, g \in L^{X \times X}$ such that $f \circ g = \perp_{X \times X}$. Now

$$\begin{aligned}
\mathcal{F}(f) * \mathcal{G}(g) &\leq \mathcal{H}(f) * \mathcal{K}(g) \\
&\leq \bigvee_{\bar{f} \circ \bar{g} \leq \perp_{X \times X}} \mathcal{H}(\bar{f}) * \mathcal{K}(\bar{g}) \\
&= \mathcal{H} \circ \mathcal{K}(\perp_{X \times X}) \\
&= \perp.
\end{aligned}$$

Therefore, $\mathcal{F} \circ \mathcal{G} \in \mathcal{F}_L^S(X \times X)$. Now let $a \in L^{X \times X}$. For $b, c \in L^{X \times X}$ we have by the assumptions that $\mathcal{F}(b) * \mathcal{G}(c) \leq \mathcal{H}(b) * \mathcal{K}(c)$. Hence

$$\begin{aligned}
\mathcal{F} \circ \mathcal{G}(a) &= \bigvee_{b \circ c \leq a} \mathcal{F}(b) * \mathcal{G}(c) \\
&\leq \bigvee_{b \circ c \leq a} \mathcal{H}(b) * \mathcal{K}(c) \\
&= \mathcal{H} \circ \mathcal{K}(a).
\end{aligned}$$

■

Lemma 5.8.7. *Let X and Y be sets, $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X \times X)$ and $\varphi : X \longrightarrow Y$. Then*

$$(\varphi \times \varphi)(\mathcal{F}) \circ (\varphi \times \varphi)(\mathcal{G}) \leq (\varphi \times \varphi)(\mathcal{F} \circ \mathcal{G}).$$

PROOF: We first must show that for $d_1, d_2, a \in L^{X \times X}$, if $d_1 \circ d_2 \leq a$, then

$$\big(d_1 \circ (\varphi \times \varphi)\big) \circ \big(d_2 \circ (\varphi \times \varphi)\big) \leq a \circ (\varphi \times \varphi).$$

Suppose $d_1 \circ d_2 \leq a$ and let $(x, y) \in X \times X$. Then

$$\begin{aligned}
\big(d_1 \circ (\varphi \times \varphi)\big) \circ \big(d_2 \circ (\varphi \times \varphi)\big)(x, y) &= \bigvee_{z \in X} d_1 \circ (\varphi \times \varphi)(x, z) * d_2 \circ (\varphi \times \varphi)(z, y) \\
&= \bigvee_{z \in X} d_1\big(\varphi(x), \varphi(z)\big) * d_2\big(\varphi(z), \varphi(y)\big) \\
&\leq \bigvee_{w \in Y} d_1\big(\varphi(x), w\big) * d_2\big(w, \varphi(y)\big) \\
&= (d_1 \circ d_2)\big(\varphi(x), \varphi(y)\big) \\
&\leq a\big(\varphi(x), \varphi(y)\big) \\
&= a \circ (\varphi \times \varphi)(x, y).
\end{aligned}$$

Now we let $b \in L^{Y \times Y}$ and show that

$$\begin{aligned}
\big((\varphi \times \varphi)(\mathcal{F}) \circ (\varphi \times \varphi)(\mathcal{G})\big)(b) &= \bigvee_{\substack{d_1, d_2 \in L^{Y \times Y} \\ d_1 \circ d_2 \leq b}} \big((\varphi \times \varphi)(\mathcal{F})\big)(d_1) * \big((\varphi \times \varphi)(\mathcal{G})\big)(d_2) \\
&= \bigvee_{\substack{d_1, d_2 \in L^{Y \times Y} \\ d_1 \circ d_2 \leq b}} \mathcal{F}\big(d_1 \circ (\varphi \times \varphi)\big) * \mathcal{G}\big(d_2 \circ (\varphi \times \varphi)\big) \\
&\leq \bigvee_{\substack{c_1, c_2 \in L^{X \times X} \\ c_1 \circ c_2 \leq b \circ (\varphi \times \varphi)}} \mathcal{F}(c_1) * \mathcal{G}(c_2) \\
&= (\mathcal{F} \circ \mathcal{G})\big(b \circ (\varphi \times \varphi)\big) \\
&= \big((\varphi \times \varphi)(\mathcal{F} \circ \mathcal{G})\big)(b).
\end{aligned}$$

■

In this chapter it has been necessary to present a large number of technical results related to stratified L-filters. These results will be used during various stages when proving results relating to stratified L-uniform convergence spaces in Chapter 7.

Chapter 6

L-Uniformities and L-Topologies

Here we present the definition of a stratified L-uniform space, from Gutiérez García [14] and show how it is a generalisation of a classical uniform space. In addition, we look at stratified L-neighbourhood spaces and stratified L-topological spaces. As with their classical equivalents, one can generate a stratified L-neighbourhood space from a stratified L-uniform space, and any stratified L-neighbourhood space will be equivalent to a stratified L-topological space.

6.1 Stratified L-Uniform Spaces

Let $(L, \leq, \otimes, *)$ be an enriched *cl*-premonoid.

Definition 6.1.1. [15] Let X be a non-empty set and $\mathcal{U}$ a stratified L-filter on $X \times X$. If $\mathcal{U}$ satisfies the properties below it is a called *stratified L-uniformity* on X.

(LU1) $\mathcal{U}(d) \leq \bigwedge\{d(x,x) \,:\, x \in X\} \quad \forall\, d \in L^{X\times X}$,

(LU2) $\mathcal{U}(d) \leq \mathcal{U}(d^{-1}) \quad \forall\, d \in L^{X\times X}$,

(LU3) $\mathcal{U}(d) \leq \bigvee\{\mathcal{U}(d_1) * \mathcal{U}(d_2) \,:\, d_1 \circ d_2 \leq d\} \quad \forall\, d \in L^{X\times X}$.

Here $d_1 \circ d_2 \in L^{X\times X}$ is defined: $d_1 \circ d_2(x,y) = \bigvee\{d_1(x,z) * d_2(z,y) \mid z \in X\}$. The pair $(X, \mathcal{U})$ is called a *stratified L-uniform space.* In [15], the case of non-stratified L-uniformities is also considered, and the **(LU3)** axiom then states that $\mathcal{U}(d) \leq \bigvee\{\alpha * \mathcal{U}(d_1) * \mathcal{U}(d_2) \,:\, \alpha \in L, \alpha * (d_1 \circ d_2) \leq d\}$.

The diagonal of a product space $X \times X$ is defined as $\Delta = \{(x,x) \,|\, x \in X\}$. From this we can define the diagonal L-filter on $X \times X$. Let $a \in L^{X\times X}$:

$$[\Delta](a) = \bigwedge_{x\in X} a(x,x) = \bigwedge_{x\in X} [(x,x)](a), \text{ so } [\Delta] = \bigwedge_{x\in X} [(x,x)]$$

With this in mind, and using the definitions from sections 5.7 and 5.8, the axioms above can also be expressed as follows:

$$\textbf{(LU1)} \Longleftrightarrow \mathcal{U} \leq [\Delta]$$
$$\textbf{(LU2)} \Longleftrightarrow \mathcal{U} \leq \mathcal{U}^{-1}$$
$$\textbf{(LU3)} \Longleftrightarrow \mathcal{U} \leq \mathcal{U} \circ \mathcal{U}$$

Definition 6.1.2. [14] Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be stratified L-uniform spaces and $\varphi : X \longrightarrow Y$. Then φ is *L-uniformly continuous* if for all $d \in L^{Y\times Y}$,

$$\mathcal{U}\big((\varphi\times\varphi)^{\leftarrow}(d)\big) \geq \mathcal{V}(d).$$

In other words, $(\varphi\times\varphi)(\mathcal{U}) \geq \mathcal{V}$.

Result: We have the category $SL-UNIF$, where the objects are stratified L-uniform spaces, and the morphisms are the L-uniformly continuous maps.

A classical uniform space $(X,\mathfrak{U})$ is a set X with filter $\mathfrak{U}$ of subsets of $X\times X$ as shown in Definition 3.1.1. It will now be shown that when $L = \{\bot,\top\}$, and $\top$ is the unit with respect to $\otimes$, that a stratified L-uniformity, $\mathcal{U}$ as defined above will satisfy the properties of a classical uniformity.

For $\mathfrak{U} \in \mathfrak{F}(X\times X)$ we will define a stratified $\{\bot,\top\}$-filter $\mathcal{U}_{\mathfrak{U}}$ and show that the mapping $\phi : \mathfrak{F}(X\times X) \longrightarrow \mathcal{F}^S_L(X\times X)$, $\mathfrak{U} \longmapsto \mathcal{U}_{\mathfrak{U}}$ is 1-to-1.

For a classical uniformity $\mathfrak{U} \in \mathfrak{F}(X\times X)$ we define $\mathcal{U}_{\mathfrak{U}} : \{\bot,\top\}^X \longrightarrow \{\bot,\top\}$ as follows:

Let $d \in L^{X\times X} = \{\bot,\top\}^{X\times X}$. We further define $S_d \subseteq X\times X$ by

$$S_d = \{(x,y) \in X\times X \;:\; d(x,y) = \top\}.$$

Now, we propose a definition of the stratified $\{\bot,\top\}$-uniformity, $\mathcal{U}_{\mathfrak{U}}$, by:

$$\mathcal{U}_{\mathfrak{U}}(d) = \begin{cases} \top \text{ if } S_d \in \mathfrak{U} \\ \bot \text{ else} \end{cases}$$

From Proposition 5.3.2 it is clear that $\mathcal{U}_{\mathfrak{U}}$ is a $\{\bot,\top\}$-filter on $X\times X$.

Proposition 6.1.3. *Let X be a set, and $\mathfrak{U} \in \mathfrak{F}(X\times X)$ such that $(X,\mathfrak{U})$ is a uniform space. Then $\mathcal{U}_{\mathfrak{U}}$ is stratified $\{\bot,\top\}$-uniformity on $X\times X$.*

PROOF:

LU1: Consider $d \in L^{X \times X}$ and S_d. If $S_d \notin \mathfrak{U}$ then $\mathcal{U}(d) = \bot$ and so clearly $\mathcal{U}(d) \leq \bigwedge_{x \in X} d(x,x)$. On the other hand, if $S_d \in \mathfrak{U}$ then $\mathcal{U}(d) = \top$ and by **(U3)**, $\Delta \subset S_d$.

This gives us that for all $x \in X, d(x,x) = \top$, and so $\bigwedge_{x \in X} d(x,x) = \top$. Hence we have that $\mathcal{U}(d) \leq \bigwedge_{x \in X} d(x,x)$.

LU2: Again we let $d \in L^{X \times X}$ and consider S_d. If $S_d \notin \mathfrak{U}$ we then have that $\mathcal{U}(d) = \bot \leq \mathcal{U}(d^{-1})$. If $S_d \in \mathfrak{U}$ then $\mathcal{U}(d) = \top$. In addition, if $S_d \in \mathfrak{U}$ then $(S_d)^{-1} = S_{d^{-1}} \in \mathfrak{U}$ and so $\mathcal{U}(d^{-1}) = \top$. Thus $\mathcal{U}(d) \leq \mathcal{U}(d^{-1})$.

LU3: Using $d \in L^{X \times X}$ it is obvious that if $S_d \notin \mathfrak{U}$, then $\mathcal{U}(d) = \bot$ and

$$\mathcal{U}(d) \leq \bigvee \{\mathcal{U}(d_1) * \mathcal{U}(d_2) \ : \ (d_1 \circ d_2) \leq d\}.$$

If we suppose that $S_d \in \mathfrak{U}$ then $\mathcal{U}(d) = \top$ and we must show that there exists e_1, e_2 with $(e_1 \circ e_2) \leq d$ such that $\mathcal{U}(e_1) * \mathcal{U}(e_2) = \top$. Now from **(U4)** there exists $E \in \mathfrak{U}$ such that $E \circ E \subset S_d$. Using the notation given in the examples from chapter 5 , we have that $\top_{E \circ E} \leq d$. Further, $\top_{E \circ E} = \top_E \circ \top_E$. Since $E \in \mathfrak{U}, \mathcal{U}(\top_E) = \top$.

Since $\top_E \circ \top_E = \top_{E \circ E} \leq d$ we have that $(\top_E \circ \top_E) \leq d$. Finally, we get $\mathcal{U}(\top_E) * \mathcal{U}(\top_E) = \top * \top = \top$.

■

In order to complete the demonstration of φ being a 1-to-1 correspondence we must show that for any stratified $\{\bot, \top\}$-uniformity $\mathcal{U}$, there exists a corresponding classical uniformity $\mathfrak{U}_{\mathcal{U}}$.

Proposition 6.1.4. *Let X be a set and $\mathcal{U}$ a stratified $\{\bot, \top\}$-uniformity on $X \times X$. Define $\mathfrak{U}_{\mathcal{U}}$ by:*

$$\mathfrak{U}_{\mathcal{U}} = \{Y \subseteq X \times X \ : \mathcal{U}(\top_Y) = \top\}.$$

Then $\mathfrak{U}_{\mathcal{U}}$ is a uniformity on X.

PROOF: We have shown in Proposition 5.3.3 that since $\mathcal{U}$ is a stratified L-filter on $X \times X$ that $\mathfrak{U}_{\mathcal{U}}$ will be a filter on $X \times X$.

U3: Let $U \in \mathfrak{U}$. Thus there exists $d \in L^{X \times X}$ such that $\mathcal{U}(d) = \top$ and $S_d = U$. By **(LU3)** we have then that $\bigwedge_{x \in X} d(x,x) = \top$. From this we can see that $d(x,x) = \top$

for all $x \in X$. Thus $\Delta = \{(x,x) : x \in X\} \subset S_d = U$.

U4: Let $U \in \mathfrak{U}$ and we will first show that $U^{-1} \in \mathfrak{U}$. We use the $d \in L^{X\times X}$ such that $\mathcal{U}(d) = \top$ and $S_d = U$. From **(LU2)** we get that $\mathcal{U}(d^{-1}) = \top$ and so $S_{d^{-1}} \in \mathfrak{U}$. Now since $U = S_d$, we get that $S_{d^{-1}} = (S_d)^{-1} = U^{-1}$ and $U^{-1} \in \mathfrak{U}$. By **(U2)** we get $U \cap U^{-1} \in \mathfrak{U}$. Since $U \cap U^{-1}$ is a symmetric set, we have $(U \cap U^{-1}) = (U \cap U^{-1})^{-1} \subset U$.

U5: For $U \in \mathfrak{U}$ there exists a $d \in L^{X\times X}$, such that $\mathcal{U}(d) = \top$ and so

$$\top = \mathcal{U}(d) \leq \bigvee\{\mathcal{U}(d_1) * \mathcal{U}(d_2) : (d_1 \circ d_2) \leq d\}.$$

Hence there must exist $e_1, e_2 \in L^{X\times X}$ with $(e_1 \circ e_2) \leq d$ such that $\mathcal{U}(e_1) * \mathcal{U}(e_2) = \top$. Clearly we must have $\mathcal{U}(e_1) = \mathcal{U}(e_2) = \top$ and so $S_{e_1}, S_{e_2} \in \mathfrak{U}$. Now by **(U2)**, $S_{e_1} \cap S_{e_2} \in \mathfrak{U}$ and since $S_{e_1 \wedge e_2} = S_{e_1} \cap S_{e_2}$ we get that $S_{e_1 \wedge e_2} \in \mathfrak{U}$.

Since $e_1 \wedge e_2 \leq e_1$ and $e_1 \wedge e_2 \leq e_2$ it is clear that $S_{e_1 \wedge e_2} \subset S_{e_1}$ and $S_{e_1 \wedge e_2} \subset S_{e_2}$.

We now show $S_{e_1 \circ e_2} = S_{e_1} \circ S_{e_2}$. We have $S_{e_1 \circ e_2} = \{(x,y) \in X \times X \,|\, e_1 \circ e_2(x,y) = \top\}$. Consider now $(x,y) \in S_{e_1 \circ e_2}$ if and only if $e_1 \circ e_2(x,y) = \top$, and this is true if and only if there exists $z \in X$ such that $e_1(x,z) * e_2(z,y) = \top$. Since $\perp$ is the zero with respect to $*$ this will only be the case if $e_1(x,z) = \top$ and $e_2(z,y) = \top$ for this z. That is, $(x,z) \in S_{e_1}$ and $(z,y) \in S_{e_2}$ which implies that $(x,y) \in S_{e_1} \circ S_{e_2}$. All of this gives $S_{e_1 \circ e_2} \subset S_{e_1} \circ S_{e_2}$.

If we take $(x,y) \in S_{e_1} \circ S_{e_2}$ then there exists a $z \in X$ such that $(x,z) \in S_{e_1}$ and $(z,y) \in S_{e_2}$. That is, $e_1(x,z) = \top$ and $e_2(z,y) = \top$ which gives $e_1(x,z) * e_2(z,y) = \top$ and therefore $e_1 \circ e_2(x,y) = \top$ and so $(x,y) \in S_{e_1 \circ e_2}$. Hence $S_{e_1} \circ S_{e_2} \subset S_{e_1 \circ e_2}$.

We now consider $S_{e_1 \wedge e_2}$ from above and show

$$\begin{aligned} S_{e_1 \wedge e_2} \circ S_{e_1 \wedge e_2} &\subset S_{e_1} \wedge S_{e_2} \\ &= S_{e_1 \circ e_2} \\ &\subset S_d \quad \text{(since } e_1 \circ e_2 \leq d\text{)} \\ &= U. \end{aligned}$$

So finally we can conclude that there exists $E = S_{e_1 \wedge e_2} \in \mathfrak{U}$ such that $E \circ E \subset U$.

■

Proposition 6.1.5. *The mapping* $\phi : \mathfrak{F}(X \times X) \longrightarrow \mathcal{F}_L^S(X \times X)$, $\mathfrak{U} \longmapsto \mathcal{U}_{\mathfrak{U}}$ *is a bijection.*

Proof: From Proposition 5.3.4 it is clear that for $\mathfrak{U} \neq \mathfrak{W}$ we will have that $\mathcal{U}_{\mathfrak{U}} \neq \mathcal{U}_{\mathfrak{W}}$. Also, for $\mathcal{W}$ a $\{0,1\}$-uniformity on $X \times X$, we will have a classical uniformity $\mathfrak{W}$ on $X \times X$ such that $\mathcal{U}_{\mathfrak{W}} = \mathcal{W}$. As before we will use $\mathfrak{W} = \mathfrak{U}_{\mathcal{W}}$.

■

6.2 L-neighbourhood systems

Definition 6.2.1. [22] Let $\mathcal{N} : X \longrightarrow L^{(L^X)}$ be a map. We denote for every $x \in X$, $\mathcal{N}(x) = \mathcal{N}^x$. So $\mathcal{N}^x : L^X \longrightarrow L$. Then $\mathcal{N}$ is said to be a *stratified L-neighbourhood system* on X if $\mathcal{N}$ satisfies for every $x \in X$:

(LN0) $\mathcal{N}^x(\top_X) = \top, \mathcal{N}^x(\bot_X) = \bot$

(LN1) $a \leq b \Longrightarrow \mathcal{N}^x(a) \leq \mathcal{N}^x(b)$

(LN2) $\mathcal{N}^x(a) \otimes \mathcal{N}^x(b) \leq \mathcal{N}^x(a \otimes b)$

(LN3) $\mathcal{N}^x(a) \leq a(x) \qquad \forall\, a \in L^X$

(LN4) $\mathcal{N}^x(a) \leq \bigvee\{\mathcal{N}^x(b) \,:\, b(y) \leq \mathcal{N}^y(a)\}$

(LNS) $\alpha * \mathcal{N}^x(a) \leq \mathcal{N}^x(\alpha_X * a) \qquad \forall \alpha \in L, a \in L^X$

A set X with a stratified L-neighbourhood system $\mathcal{N}$ is called the *stratified L-neighbourhood space* $\big(X, (\mathcal{N}^x)_{x \in X}\big)$.

Clearly if we choose any $x \in X$, then $\mathcal{N}^x$ is an L-filter on X.

Let us fix $a \in L^X$. We can then define an L-set:

$$\mathcal{N}\_(a) : X \longrightarrow L \;,\; [\mathcal{N}\_(a)](x) = \mathcal{N}^x(a)$$

We will now show [14] that $\mathcal{N}^x\big(\mathcal{N}\_(a)\big) = \bigvee\{\mathcal{N}^x(b) \,:\, b(y) \leq \mathcal{N}^y(a)\, \forall\, y \in X\}$.

By definition we have that $[\mathcal{N}\_(a)](x) \leq \mathcal{N}^x(a)$ and therefore we can see that $\mathcal{N}^x\big(\mathcal{N}\_(a)\big) \in \{\mathcal{N}^x(b) \,:\, b(y) \leq \mathcal{N}^y(a)\, \forall\, y \in X\}$ and so of course

$$\mathcal{N}^x\big(\mathcal{N}\_(a)\big) \leq \bigvee\{\mathcal{N}^x(b) \,:\, b(y) \leq \mathcal{N}^y(a)\, \forall\, y \in X\}.$$

For the reverse inequality, consider $b \in L^X$ such that $b(y) \leq \mathcal{N}^y(a)$ for all $y \in X$. This clearly implies that $b(y) \leq [\mathcal{N}\_(a)](y)$ for all $y \in X$, and thus $\mathcal{N}\_(a)$ is an upper bound for $\{b \,:\, b(y) \leq \mathcal{N}^y(a)\, \forall\, y \in X\}$. That is, for any b an element of $\{b \,:\, b(y) \leq \mathcal{N}^y(a)\, \forall\, y \in X\}$, we have $b \leq \mathcal{N}\_(a)$.

Now **(LN1)** implies that $\mathcal{N}^x(b) \leq \mathcal{N}^x\big(\mathcal{N}\_(a)\big)$ and so we have that $\mathcal{N}^x\big(\mathcal{N}\_(a)\big)$ is an upper bound for $\{\mathcal{N}^x(b) \,:\, b(y) \leq \mathcal{N}^y(a)\, \forall\, y \in X\}$. Since it is not the least upper bound

$$\bigvee\{\mathcal{N}^x(b) \,:\, b(y) \leq \mathcal{N}^y(a)\, \forall\, y \in X\} \leq \mathcal{N}^x\big(\mathcal{N}\_(a)\big).$$

And so, we get $\bigvee\{\mathcal{N}^x(b) \,:\, b(y) \leq \mathcal{N}^y(a)\, \forall\, y \in X\} = \mathcal{N}^x\big(\mathcal{N}\_(a)\big)$.

Thus we can replace axiom **(LN4)** by:

(LN4′) $\mathcal{N}^x(a) \leq \mathcal{N}^x\big(\mathcal{N}_{\_}(a)\big)$.

Definition 6.2.2. [22] Let X be a set and $(L, \leq, \otimes, *)$ be an enriched *cl*-premonoid. A subset, τ, of L^X is a *stratified L-topology* on X if it satisfies:

(LT1) $\top_X, \bot_X \in \tau$,

(LT2) $a, b \in \tau \Longrightarrow a \otimes b \in \tau$,

(LT3) for any collection $a_i \in \tau$, $\forall i \in I \Longrightarrow \bigvee_{i \in I} a_i \in \tau$,

(LTS) if $\alpha \in L$ and $a \in \tau$, then $\alpha_X * a \in \tau$.

The pair (X, τ) is called a stratified L-topological space.

Definition 6.2.3. [22] Let (X, τ_1) and (Y, τ_2) be stratified L-topological spaces, and $\varphi : X \longrightarrow Y$. Then φ is *L-continuous* if

$$\{a \circ \varphi \mid a \in \tau_2\} \subset \tau_1.$$

Result: [22] The stratified L-topological spaces and the L-continuous mappings form a category, $SL-TOP$.

There is a one-to-one correspondence between an L-neighbourhood system on a set X and an L-topology on X. Let $(X, \tau) \in |SL-TOP|$, $x \in X$ and $a \in L^X$. Then define:

$$\mathcal{N}_\tau^x(a) = \bigvee \big\{ b(x) \ : \ b \in \tau, b \leq a \big\}$$

and $\mathcal{N}$ is a stratified L-neighbourhood system [22].

Given a stratified L-neighbourhood system $(X, \mathcal{N})$, we say for an L-set a,

$$a \in \tau_{\mathcal{N}} \Longleftrightarrow \text{ for all } x \in X, a(x) \leq \mathcal{N}^X(a).$$

Lemma 6.2.4. *[22] Let $(X, \tau) \in |SL-TOP|$. Then $\tau_{(\mathcal{N}_\tau)} = \tau$. Further, for $(X, \mathcal{N}) \in |SL-NEIGH|$, $\mathcal{N}_{(\tau_{\mathcal{N}})}$.*

That is, there is a one-to-one correspondence between stratified L-topological spaces and stratified L-neighbourhood spaces. The fact that these two categories are isomorphic will be used later, in section 7.6.

Chapter 7

Lattice-Valued Uniform Convergence Spaces

Here we propose a new definition of a lattice-valued uniform convergence structure on a set X, generalising the work of Jäger and Burton [27]. That work was the first lattice-valued generalisation of the concept of a uniform convergence space as introduced in 3.2 of Part I. We show that our category is topological and (with an additional restriction on the underlying lattice) that the category of stratified L-uniform spaces is a reflective subcategory of this new category. Finally we present the induced stratified L-limit structure and show that here the initial structures are preserved. Unless otherwise stated, our lattice L will be a pseudo-bisymmetric enriched *cl*-premonoid.

7.1 Stratified L-Uniform Convergence Spaces

Definition 7.1.1. Let X be a non-empty set, and $(L, \leq, \otimes, *)$ a pseudo-bisymmetric enriched *cl*-premonoid. A mapping $\Lambda : \mathcal{F}_L^S(X \times X) \longrightarrow L$ is called a *stratified L-uniform convergence structure* if Λ satisfies the following:

(LUC1) for all $x \in X$, $\Lambda([(x, x)]) = \top$,

(LUC2) $\mathcal{F} \leq \mathcal{G} \Longrightarrow \Lambda(\mathcal{F}) \leq \Lambda(\mathcal{G})$,

(LUC3) $\Lambda(\mathcal{F}) \leq \Lambda(\mathcal{F}^{-1})$,

(LUC4) $\Lambda(\mathcal{F}) \wedge \Lambda(\mathcal{G}) \leq \Lambda(\mathcal{F} \wedge \mathcal{G})$,

(LUC5) $\Lambda(\mathcal{F}) * \Lambda(\mathcal{G}) \leq \Lambda(\mathcal{F} \circ \mathcal{G})$ whenever $\mathcal{F} \circ \mathcal{G}$ exists.

The pair (X, Λ) is called a *stratified L-uniform convergence space.*

The original definition proposed by Jäger and Burton [27] was for the case where L is a complete Heyting algebra. Their **(LUC1)** stated that for all $x \in X$,

$\Lambda([x] \times [x]) = \top$. For the case of L a complete Heyting algebra, it is shown in Corollary 5.6.12 that $[(x, x)] = [x] \times [x]$ for all $x \in X$, and so we see how the new definition is a generalisation of the previous one. In addition, the **(LUC5)** given in [27] stated $\Lambda(\mathcal{F}) \wedge \Lambda(\mathcal{G}) \leq \Lambda(\mathcal{F} \circ \mathcal{G})$. For the Heyting algebra case, $* = \wedge$, and so the above definition is thus a useful generalisation as it includes the specific case that was investigated in that work.

Definition 7.1.2. [27] Let (X, Λ) and (Y, Σ) be stratified L-uniform convergence spaces. A mapping $\varphi : (X, \Lambda) \longrightarrow (Y, \Sigma)$ is *uniformly continuous* if for all $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$, we have:

$$\Lambda(\mathcal{F}) \leq \Sigma\big((\varphi \times \varphi)(\mathcal{F})\big).$$

Proposition 7.1.3. *[27] Let $(X, \Lambda), (Y, \Sigma)$ be stratified L-uniform convergence spaces. Then:*

(i) *The mapping $id_X : (X, \Lambda) \longrightarrow (X, \Lambda)$ is uniformly continuous.*

(ii) *If $\varphi : (X, \Lambda) \longrightarrow (Y, \Sigma)$ and $\psi : (Y, \Sigma) \longrightarrow (Z, \Gamma)$ are uniformly continuous, then $\psi \circ \varphi : (X, \Lambda) \longrightarrow (Z, \Gamma)$ is uniformly continuous.*

PROOF:

(i) By definition of the image filter

$$(id \times id)(\mathcal{F})(a) = \mathcal{F}\big((id \times id)^{\leftarrow}(a)\big) = \mathcal{F}\big(a \circ (id \times id)\big) = \mathcal{F}(a).$$

and therefore $(id \times id)(\mathcal{F}) = \mathcal{F}$ for all $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$. Now since $\Lambda(\mathcal{F}) \leq \Lambda(\mathcal{F})$ we have $\Lambda(\mathcal{F}) \leq \Lambda\big((id \times id)(\mathcal{F})\big)$.

(ii) Let $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$. Since φ uniformly continuous, we have that $\Lambda(\mathcal{F}) \leq \Sigma\big((\varphi \times \varphi)(\mathcal{F})\big)$. Also, since ψ uniformly continuous, $\Sigma\big((\varphi \times \varphi)(\mathcal{F})\big) \leq \Gamma\big((\psi \times \psi) \circ (\varphi \times \varphi)(\mathcal{F})\big)$.

We now use Lemma 5.7.6 to conclude

$$\Lambda(\mathcal{F}) \leq \Gamma\big((\psi \times \psi) \circ (\varphi \times \varphi)(\mathcal{F})\big) = \Gamma\big((\psi \circ \varphi) \times (\psi \circ \varphi)(\mathcal{F})\big).$$

■

RESULT: We have the category $SL\!-\!UCS$, where the objects are stratified L-uniform convergence spaces, and the morphisms are the uniformly continuous maps.

Lemma 7.1.4. *[27] Let $(X, \Lambda), (Y, \Sigma) \in |SL - UCS|$ and $\varphi : X \longrightarrow Y, \varphi(x) \equiv y_0$ be a constant map. Then φ is uniformly continuous.*

PROOF: Let $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$. We show that $(\varphi \times \varphi)(\mathcal{F}) \geq [(y_0, y_0)]$. With $a \in L^{Y \times Y}$ we have

$$\big((\varphi \times \varphi)(\mathcal{F})\big)(a) = \mathcal{F}\big((\varphi \times \varphi)^{\leftarrow}(a)\big) = \mathcal{F}\big(a \circ (\varphi \times \varphi)\big).$$

Let $(x_1, x_2) \in X \times X$ and consider the two L-sets $a \circ (\varphi \times \varphi)$ and $a(y_0, y_0)_{X \times X} * \top_{X \times X}$. We get

$$\begin{aligned}
\big(a \circ (\varphi \times \varphi)\big)(x_1, x_2) &= a(y_0, y_0) \\
&= a(y_0, y_0) * \top \\
&= a(y_0, y_0) * \top_{X \times X}(x_1, x_2) \\
&= \big(a(y_0, y_0)_{X \times X} * \top_{X \times X}\big)(x_1, x_2).
\end{aligned}$$

Using the above we have

$$\begin{aligned}
\big((\varphi \times \varphi)(\mathcal{F})\big)(a) &= \mathcal{F}\big(a \circ (\varphi \times \varphi)\big) \\
&= \mathcal{F}\big(a(y_0, y_0)_{X \times X} * \top_{X \times X}\big) \\
&\geq a(y_0, y_0) * \mathcal{F}(\top_{X \times X}) \\
&\geq a(y_0, y_0) * \top \\
&= a(y_0, y_0) \\
&= [(y_0, y_0)](a).
\end{aligned}$$

Now by **(LUC2)** and **(LUC1)** we have

$$\Sigma\big((\varphi \times \varphi)(\mathcal{F})\big) \;\geq\; \Sigma\big([(y_0, y_0)]\big) \;=\; \top \;\geq \Lambda(\mathcal{F}).$$

■

If we have two different stratified L-uniform convergence structures, Λ and Λ', on a set X, we can order them in the following manner:

$(X, \Lambda) \leq (X, \Lambda')$ if and only if $id_X : (X, \Lambda') \longrightarrow (X, \Lambda)$ is uniformly continuous.

From the definition of uniform continuity of a mapping, this can be expressed as:

$(X, \Lambda) \leq (X, \Lambda')$ if and only if, for all $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$, $\Lambda'(\mathcal{F}) \leq \Lambda(\mathcal{F})$.

We now present two examples of stratified L-uniform convergence spaces. These two examples, like many of the lemmas in this chapter are proposed in Jäger and Burton's work [27] for the case of L a complete Heyting algebra. These examples and lemmas differ from the originals because of our new definition of a stratified

L-uniform convergence structure. However, we still reference the original paper as in most cases the structure of the proof has been followed.

Example 7.1.5. [27] Let X be a set. The *indiscrete stratified L-uniform convergence structure* Λ_i is defined:

$$\Lambda_i(\mathcal{F}) = \top \quad \text{for all } \mathcal{F} \in \mathcal{F}_L^S(X \times X).$$

We show that this defines a stratified L uniform convergence structure.

LUC1: Clearly $\Lambda_i\big([(x,x)]\big) = \top$.

LUC2: Let $\mathcal{F} \leq \mathcal{G}$ and we see that $\Lambda_i(\mathcal{F}) = \top \leq \top = \Lambda_i(\mathcal{G})$.

LUC3: It is obvious that for $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$ we have

$$\Lambda_i(\mathcal{F}) = \top \leq \top = \Lambda_i(\mathcal{F}^{-1}).$$

LUC4: For any $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X \times X)$ we have

$$\Lambda_i(\mathcal{F}) \wedge \Lambda_i(\mathcal{G}) = \top \wedge \top \leq \top = \Lambda_i(\mathcal{F} \wedge \mathcal{G}).$$

LUC5: Suppose $\mathcal{F} \circ \mathcal{G}$ exists. On the one hand, $\Lambda_i(\mathcal{F}) * \Lambda_i(\mathcal{G}) = \top * \top = \top$. On the other hand $\Lambda_i(\mathcal{F} \circ \mathcal{G}) = \top$. Therefore $\Lambda_i(\mathcal{F}) * \Lambda_i(\mathcal{G}) = \Lambda_i(\mathcal{F} \circ \mathcal{G})$.

■

Example 7.1.6. [27] Let X be a set. Then for $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$, we define the *discrete stratified L-uniform convergence structure* by:

$$\Lambda_d(\mathcal{F}) = \begin{cases} \top & \text{if } \mathcal{F} \geq \bigwedge_{x \in E} [(x,x)] \text{ for some finite } E \subset X \\ \bot & \text{else} \end{cases}$$

LUC1: Let $x \in X$ and $E = \{x\}$. Thus $[(x,x)] \geq \bigwedge_{y \in E} [(y,y)]$, and so $\Lambda_d\big([(x,x)]\big) = \top$ for all $x \in X$.

LUC2: Let $\mathcal{F} \leq \mathcal{G}$. If we suppose that $\Lambda_d(\mathcal{F}) = \bot$, then clearly $\Lambda_d(\mathcal{F}) \leq \Lambda_d(\mathcal{G})$. If however we suppose $\Lambda_d(\mathcal{F}) = \top$, then this implies that there exists $E \subset X$, a finite set, such that $\mathcal{F} \geq \bigwedge_{x \in E} [(x,x)]$.

But $\mathcal{G} \geq \mathcal{F} \geq \bigwedge_{x \in E} [(x,x)]$ and so $\Lambda_d(\mathcal{G}) = \top$.

LUC3: If $\Lambda_d(\mathcal{F}) = \bot$ then clearly $\Lambda_d(\mathcal{F}) \leq \Lambda_d(\mathcal{F}^{-1})$. Now if $\Lambda_d(\mathcal{F}) = \top$, there exists $E \subset X$ finite, such that $\mathcal{F} \geq \bigwedge\limits_{x \in E} [(x,x)]$.

For $d \in L^{X \times X}$ we get $\big(\bigwedge\limits_{x\in E}[(x,x)]\big)(d) = \bigwedge\limits_{x \in E} d(x,x)$, and thus

$$\begin{aligned}\mathcal{F}^{-1}(d) = \mathcal{F}(d^{-1}) &\geq \big(\bigwedge_{x\in E}[(x,x)]\big)(d^{-1})\\ &= \bigwedge_{x\in E} d^{-1}(x,x)\\ &= \bigwedge_{x\in E} d(x,x)\\ &= \big(\bigwedge_{x\in E}[(x,x)]\big)(d).\end{aligned}$$

Now, using the same $E \subset X$, we have:

$$\mathcal{F}^{-1} \geq \bigwedge_{x\in E}[(x,x)]$$

and therefore $\Lambda_d(\mathcal{F}^{-1}) = \top \geq \Lambda_d(\mathcal{F})$.

LUC4: Suppose for at least one of $\mathcal{F}, \mathcal{G}$ that $\Lambda_d(\mathcal{F}) = \bot$ or $\Lambda_d(\mathcal{G}) = \bot$. This would give us $\Lambda_d(\mathcal{F}) \wedge \Lambda_d(\mathcal{G}) = \bot$ and so $\Lambda_d(\mathcal{F}) \wedge \Lambda_d(\mathcal{G}) \leq \Lambda_d(\mathcal{F} \wedge \mathcal{G})$. Else if we suppose that $\Lambda_d(\mathcal{F}) = \top$ and $\Lambda_d(\mathcal{G}) = \top$ then we get that there exists $F \subset X$ and $G \subset X$ (both finite) such that:

$$\mathcal{F} \geq \bigwedge_{x\in F}[(x,x)] \quad \text{and} \quad \mathcal{G} \geq \bigwedge_{y\in G}[(y,y)].$$

This implies that $\mathcal{F} \wedge \mathcal{G} \;\geq\; \Big(\bigwedge\limits_{x\in F}[(x,x)]\Big) \wedge \Big(\bigwedge\limits_{y\in G}[(y,y)]\Big) \;\geq\; \bigwedge\limits_{z\in F\cup G}[(z,z)]$.

Since F and G are both finite, $F \cup G$ will be finite and therefore

$$\Lambda_d(\mathcal{F} \wedge \mathcal{G}) = \top \geq \Lambda_d(\mathcal{F}) \wedge \Lambda_d(\mathcal{G}).$$

LUC5: Suppose either $\Lambda_d(\mathcal{F}) = \bot$ or $\Lambda_d(\mathcal{G}) = \bot$. Since $\bot$ is the zero with respect to $*$, we have that $\Lambda_d(\mathcal{F}) * \Lambda_d(\mathcal{G}) = \bot$. If $\top = \Lambda_d(\mathcal{F}) * \Lambda_d(\mathcal{G})$ this implies that $\top = \Lambda_d(\mathcal{F})$ and $\top = \Lambda_d(\mathcal{G})$.

This in turn implies that there exists F and G finite such that

$$\mathcal{F} \geq \bigwedge_{x\in F}[(x,x)] \quad \text{and} \quad \mathcal{G} \geq \bigwedge_{y\in G}[(y,y)].$$

We now use the result from Lemma 5.8.6 that $\mathcal{F} \leq \mathcal{H}, \mathcal{G} \leq \mathcal{K} \Longrightarrow \mathcal{F} \circ \mathcal{G} \leq \mathcal{H} \circ \mathcal{K}$ to show

$$\begin{aligned}
\mathcal{F} \circ \mathcal{G}(a) &\geq \Big(\bigwedge_{x \in F} [(x,x)] \circ \bigwedge_{y \in G} [(y,y)] \Big)(a) \\
&\geq \Big(\bigwedge_{x \in F \cup G} [(x,x)] \circ \bigwedge_{y \in F \cup G} [(y,y)] \Big)(a) \\
&= \bigvee_{f \circ g \leq a} \Big[\big(\bigwedge_{x \in F \cup G} [(x,x)] \big)(f) * \big(\bigwedge_{y \in F \cup G} [(y,y)] \big)(g) \Big] \\
&= \bigvee_{f \circ g \leq a} \Big[\big(\bigwedge_{x \in F \cup G} f(x,x) \big) * \big(\bigwedge_{y \in F \cup G} g(y,y) \big) \Big] \\
&= Q.
\end{aligned}$$

Now, $a \circ \top_{\Delta}(x,y) = \bigvee_{z \in X} a(x,z) * \top_{\Delta}(z,y)$ with

$$a(x,z) * \top_{\Delta}(z,y) = \begin{cases} a(x,z) * \top & \text{for } z = y \\ a(x,z) * \bot & \text{for } z \neq y. \end{cases}$$

Therefore $a \circ \top_{\Delta}(x,y) = a(x,y)$ and so $a \circ \top_{\Delta} \leq a$. We then have

$$\begin{aligned}
\mathcal{F} \circ \mathcal{G}(a) &\geq Q \\
&\geq \Big(\bigwedge_{x \in F \cup G} a(x,x) \Big) * \Big(\bigwedge_{y \in F \cup G} \top_{\Delta}(y,y) \Big) \\
&= \Big(\bigwedge_{x \in F \cup G} a(x,x) \Big) * \top \\
&= \bigwedge_{x \in F \cup G} [(x,x)](a).
\end{aligned}$$

This implies that there exists a finite set $F \cup G$ such that $\mathcal{F} \circ \mathcal{G} \geq \bigwedge_{x \in F \cup G} [(x,x)]$. Hence we get that $\Lambda_d(\mathcal{F} \circ \mathcal{G}) = \top = \Lambda_d(\mathcal{F}) * \Lambda_d(\mathcal{G})$.

■

7.2 Initial structures

Cook and Fischer [8] showed that the classical uniform convergence spaces form a topological category, and Jäger and Burton [27] showed the analogous result for complete Heyting algebra-valued uniform convergence spaces. Now we show this same result for the case where L is a pseudo-bisymmetric enriched *cl*-premonoid.

Proposition 7.2.1. *[27] The category $SL-UCS$ is a topological category (in the sense of Definition 1.1.6).*

PROOF: First we will show the existence of initial structures. Consider a family $\{\varphi_i : i \in I\}$ such that $X \xrightarrow{\varphi_i} (X_i, \Lambda_i)$ for all $i \in I$.

For $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$, define:

$$\Lambda(\mathcal{F}) = \bigwedge_{i \in I} \Lambda_i\Big((\varphi_i \times \varphi_i)(\mathcal{F})\Big).$$

We show that $(X, \Lambda) \in |SL - UCS|$.

LUC1: We use Lemma 5.7.4 $(\varphi \times \varphi)\big([(x,x)]\big) = \big[\big(\varphi(x), \varphi(x)\big)\big]$.

Now,

$$\begin{aligned}
\Lambda\big([(x,x)]\big) &= \bigwedge_{i \in I} \Lambda_i\Big(\big(\varphi_i \times \varphi_i\big)\big([(x,x)]\big)\Big) \\
&= \bigwedge_{i \in I} \Lambda_i\big[\big(\varphi_i(x), \varphi_i(x)\big)\big] \\
&= \top.
\end{aligned}$$

LUC2: Suppose $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X \times X)$ with $\mathcal{F} \leq \mathcal{G}$. With the consequence of Lemma 5.5.4, we have for all $i \in I$ that $(\varphi_i \times \varphi_i)(\mathcal{F}) \leq (\varphi_i \times \varphi_i)(\mathcal{G})$. Now by **(LUC2)** for each of the Λ_i we get

$$\Lambda_i\big((\varphi_i \times \varphi_i)(\mathcal{F})\big) \leq \Lambda_i\big((\varphi_i \times \varphi_i)(\mathcal{G})\big),$$

for all $i \in I$ and therefore

$$\bigwedge_{i \in I} \Lambda_i\big((\varphi_i \times \varphi_i)(\mathcal{F})\big) \leq \bigwedge_{i \in I} \Lambda_i\big((\varphi_i \times \varphi_i)(\mathcal{G})\big).$$

Thus $\Lambda(\mathcal{F}) \leq \Lambda(\mathcal{G})$.

LUC3: We use Lemma 5.7.5: $(\varphi \times \varphi)(\mathcal{F}^{-1}) = \big((\varphi \times \varphi)(\mathcal{F})\big)^{-1}$.

$$\begin{aligned}
\Lambda(\mathcal{F}^{-1}) &= \bigwedge_{i \in I} \Lambda_i\big((\varphi_i \times \varphi_i)(\mathcal{F}^{-1})\big) \\
&= \bigwedge_{i \in I} \Lambda_i\Big(\big((\varphi_i \times \varphi_i)(\mathcal{F})\big)^{-1}\Big) \\
&\geq \bigwedge_{i \in I} \Lambda_i\big((\varphi_i \times \varphi_i)(\mathcal{F})\big) \\
&= \Lambda(\mathcal{F}).
\end{aligned}$$

LUC4: Here we will use Lemma 5.5.4: $\varphi(\mathcal{F}) \wedge \varphi(\mathcal{G}) = \varphi(\mathcal{F} \wedge \mathcal{G})$.

$$\begin{aligned}
\Lambda(\mathcal{F}) \wedge \Lambda(\mathcal{G}) &= \bigwedge_{i\in I} \Lambda_i\big((\varphi_i \times \varphi_i)(\mathcal{F})\big) \wedge \bigwedge_{i\in I} \Lambda_i\big((\varphi_i \times \varphi_i)(\mathcal{G})\big) \\
&\leq \bigwedge_{i\in I} \Big(\Lambda_i\big((\varphi_i \times \varphi_i)(\mathcal{F})\big) \wedge \Lambda_i\big((\varphi_i \times \varphi_i)(\mathcal{G})\big)\Big) \\
&\leq \bigwedge_{i\in I} \Big(\Lambda_i\big((\varphi_i \times \varphi_i)(\mathcal{F}) \wedge (\varphi_i \times \varphi_i)(\mathcal{G})\big)\Big) \\
&= \bigwedge_{i\in I} \Lambda_i\big((\varphi_i \times \varphi_i)(\mathcal{F} \wedge \mathcal{G})\big) \\
&= \Lambda(\mathcal{F} \wedge \mathcal{G}).
\end{aligned}$$

LUC5: From Proposition 5.8.7 we see that

$$(\varphi_i \times \varphi_i)(\mathcal{F} \circ \mathcal{G}) \geq (\varphi_i \times \varphi_i)(\mathcal{F}) \circ (\varphi_i \times \varphi_i)(\mathcal{G}).$$

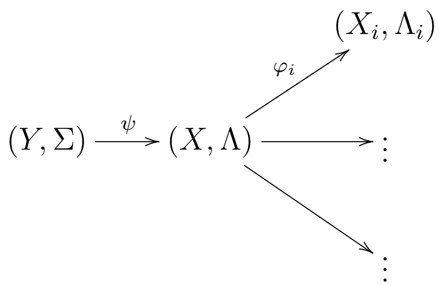
Now we use that result to show

$$\begin{aligned}
\Lambda(\mathcal{F} \circ \mathcal{G}) &= \bigwedge_{i\in I} \Lambda_i\big((\varphi_i \times \varphi_i)(\mathcal{F} \circ \mathcal{G})\big) \\
&\geq \bigwedge_{i\in I} \Lambda_i\Big((\varphi_i \times \varphi_i)(\mathcal{F}) \circ (\varphi_i \times \varphi_i)(\mathcal{G})\Big) \\
&\geq \bigwedge_{i\in I} \Big(\Lambda_i\big((\varphi_i \times \varphi_i)(\mathcal{F})\big) * \Lambda_i\big((\varphi_i \times \varphi_i)(\mathcal{G})\big)\Big) \\
&\geq \bigwedge_{i\in I} \Lambda_i\big((\varphi_i \times \varphi_i)(\mathcal{F})\big) * \bigwedge_{i\in I} \Lambda_i\big((\varphi_i \times \varphi_i)(\mathcal{G})\big) \\
&= \Lambda(\mathcal{F}) * \Lambda(\mathcal{G}).
\end{aligned}$$

Therefore we have that $(X, \Lambda) \in |SL - UCS|$.

Let $(Y, \Sigma) \in |SL - UCS|$ and let $\psi : Y \longrightarrow X$ such that for all $i \in I$,

$\varphi_i \circ \psi : (Y, \Sigma) \longrightarrow (X_i, \Lambda_i)$ is uniformly continuous.



Thus for all $i \in I$ we have

$$\begin{aligned}\Sigma(\mathcal{F}) &\leq \Lambda_i\Big(\big((\varphi_i \circ \psi) \times (\varphi_i \circ \psi)\big)(\mathcal{F})\Big) \\ &= \Lambda_i\big((\varphi_i \times \varphi_i) \circ (\psi \times \psi)(\mathcal{F})\big),\end{aligned}$$

and consequently

$$\begin{aligned}\Sigma(\mathcal{F}) &\leq \bigwedge_{i \in I} \Lambda_i\big((\varphi_i \times \varphi_i) \circ (\psi \times \psi)(\mathcal{F})\big) \\ &= \Lambda\big((\psi \times \psi)(\mathcal{F})\big).\end{aligned}$$

Therefore $\psi : Y \longrightarrow X$ is uniformly continuous. Now we have that $\psi : (Y, \Sigma) \longrightarrow (X, \Lambda)$ is uniformly continuous if and only if $\varphi_i \circ \psi : (Y, \Sigma) \longrightarrow (X, \Lambda_i)$ is uniformly continuous.

Further, we show the *fibre-smallness* of $SL - UCS$. Since each stratified L-uniform convergence structure is a mapping $\Lambda : \mathcal{F}_L^S(X \times X) \longrightarrow L$ we have that the class of all possible stratified L-uniform convergence structures on a set X, is a subset of $\{0, 1\}^{L^{\mathcal{F}_L^S(X \times X)}}$ and so it is a set.

Lastly, in order to show the *terminal separator* property, consider $X = \{x\}$ and hence $X \times X = \{(x, x)\}$. Since there is only one element in $X \times X$, the only L-sets that exist (ie elements of $L^{X \times X}$) are the constant maps $\alpha_{X \times X}(x, x) = \alpha$ for each $\alpha \in L$. We now show that for any $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$, we have $\mathcal{F} \geq [(x, x)]$.

Let $\alpha \in L$. $[(x, x)](\alpha_{X \times X}) = \alpha_{X \times X}(x, x) = \alpha$. Now let $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$ and consider the L-set $\top_{X \times X} \in L^{X \times X}$. From **(LFS)** we have that $\alpha * \mathcal{F}(\top_{X \times X}) \leq \mathcal{F}(\alpha_{X \times X} * \top_{X \times X})$. Since $\mathcal{F}(\top_{X \times X}) = \top$ and $\top$ is the unit with respect to $*$, we get from this equation that $\alpha \leq \mathcal{F}(\alpha_{X \times X} * \top_{X \times X}) = \mathcal{F}(\alpha_{X \times X})$. Therefore we have that $[(x, x)](\alpha_{X \times X}) \leq \mathcal{F}(\alpha_{X \times X})$. Since α was arbitrary and the only L-sets are the constant maps, we have that for any $\mathcal{F} \in \mathcal{F}_L^S(X \times X), \mathcal{F} \geq [(x, x)]$. Now by **(LUC1)** we must have that $\Lambda([(x, x)] = \top$ and by **(LUC2)** we must have that $[(x, x)] \leq \mathcal{F} \Longrightarrow \Lambda\big([(x, x)]\big) \leq \Lambda(\mathcal{F})$. Therefore the only permissible stratified L-uniform convergence structure is that which has $\Lambda(\mathcal{F}) = \top$ for all $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$.

■

Example 7.2.2. Let $(X, \Lambda), (Y, \Sigma) \in |SL - UCS|$ and consider the projection mappings $P_1 : X \times Y \longrightarrow X$ and $P_2 : X \times Y \longrightarrow Y$. Then the *product L-uniform convergence structure*, $\Lambda \times \Sigma$ on $X \times Y$ is defined by using the initial uniform convergence structure for the projection mappings. That is, for a stratified L-filter $\mathcal{F} \in \mathcal{F}_L^S\big((X \times Y) \times (X \times Y)\big)$:

$$(\Lambda \times \Sigma)(\mathcal{F}) = \Lambda\big((P_1 \times P_1)(\mathcal{F})\big) \wedge \Sigma\big((P_2 \times P_2)(\mathcal{F})\big).$$

7.3 $SL - UNIF$ as a subcategory of $SL - UCS$

Here we will show that $SL - UNIF$ is a reflective subcategory of $SL - UCS$. In order to do this we will first introduce the category of principal stratified L-uniform convergence spaces ($SL - PUCS$), a subcategory of $SL - UCS$. Then we will proceed by showing that $SL - UNIF$ is categorically isomorphic to $SL - PUCS$.

Definition 7.3.1. [27] The pair $(X, \Lambda) \in |SL - UCS|$ is a *principal stratified L-uniform convergence space* if there exists a stratified L-filter $\mathcal{U} \in \mathcal{F}_L^S(X \times X)$ such that:

(LUCP) $\Lambda(\mathcal{F}) = \bigwedge_{a \in L^{X \times X}} \big(\mathcal{U}(a) \to \mathcal{F}(a)\big) \quad$ for all $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$.

The following lemma shows that from any principal stratified L-uniform convergence space, we can get a stratified L-uniform space.

Lemma 7.3.2. *[27] Let $(X, \Lambda) \in |SL - PUCS|$ where:*

$$\Lambda(\mathcal{F}) = \bigwedge_{a \in L^{X \times X}} \big(\mathcal{U}(a) \to \mathcal{F}(a)\big).$$

Then $(X, \mathcal{U}) \in |SL - UNIF|$.

Proof:

LU1: Let $x \in X$. By **(LUC1)** we have $\Lambda\big([(x,x)]\big) = \top$ and so we get

$$\bigwedge_{a \in L^{X \times X}} \Big(\mathcal{U}(a) \to \big([(x,x)]\big)(a)\Big) \;=\; \top \;=\; \bigwedge_{a \in L^{X \times X}} \big(\mathcal{U}(a) \to a(x,x)\big).$$

Therefore $\mathcal{U}(a) \to (a)(x,x) = \top$ for all $a \in L^{X \times X}$, and thus from the property of the implication operator we see that

$$\mathcal{U}(a) \leq a(x,x) \text{ for all } a \in L^{X \times X}.$$

Since x is arbitrary,

$$\mathcal{U}(a) \leq \bigwedge_{x \in X} a(x,x) \quad \text{for all } a \in L^{X \times X}.$$

LU2: We have $\Lambda(\mathcal{U}) = \bigwedge_{a \in L^{X \times X}} \big(\mathcal{U}(a) \to \mathcal{U}(a)\big) = \top$. Now by **(LUC3)** we know that $\Lambda(\mathcal{U}) \leq \Lambda(\mathcal{U}^{-1})$ and therefore $\Lambda(\mathcal{U}^{-1}) = \top$. That gives

$$\bigwedge_{a \in L^{X \times X}} \big(\mathcal{U}(a) \to \mathcal{U}^{-1}(a)\big) = \top$$

which implies $\mathcal{U}(a) \leq \mathcal{U}^{-1}(a)$ for all $a \in L^{X\times X}$, and so $\mathcal{U} \leq \mathcal{U}^{-1}$.

LU3: From above we have that $\Lambda(\mathcal{U}) = \top$ and this implies $\Lambda(\mathcal{U}) * \Lambda(\mathcal{U}) = \top$. From property **(LUC5)** we have that $\Lambda(\mathcal{U}) * \Lambda(\mathcal{U}) \leq \Lambda(\mathcal{U} \circ \mathcal{U})$ and therefore $\Lambda(\mathcal{U} \circ \mathcal{U}) = \top$. That is,

$$\top = \bigwedge_{a \in L^{X\times X}} \big(\mathcal{U}(a) \to \mathcal{U} \circ \mathcal{U}(a)\big).$$

This is only the case if for all $a \in L^{X\times X}$, we have that $\mathcal{U}(a) \leq \mathcal{U} \circ \mathcal{U}(a)$, and hence we have $\mathcal{U} \leq \mathcal{U} \circ \mathcal{U}$.

■

Below we show that from any stratified L-uniform space we can generate a principal stratified L-uniform convergence space.

Lemma 7.3.3. *[27] Let $(X,\mathcal{U}) \in |SL - UNIF|$, and define:*

$$\Lambda^{\mathcal{U}}(\mathcal{F}) = \bigwedge_{a \in L^{X\times X}} \big(\mathcal{U}(a) \to \mathcal{F}(a)\big).$$

Then $(X, \Lambda^{\mathcal{U}}) \in |SL - PUCS|$.

Proof:

LUC1: From **(U1)** we know that $\mathcal{U}(a) \leq \bigwedge_{x \in X} a(x,x)$ for all $a \in L^{X\times X}$, and hence we see that for all $x \in X$ and for all $a \in L^{X\times X}$, $\mathcal{U}(a) \leq a(x,x)$. Therefore

$$\begin{aligned}
\Lambda^{\mathcal{U}}\big([(x,x)]\big) &= \bigwedge_{a \in L^{X\times X}} \big(\mathcal{U}(a) \to [(x,x)](a)\big) \\
&= \bigwedge_{a \in L^{X\times X}} \big(\mathcal{U}(a) \to a(x,x)\big) \\
&= \bigwedge \top \\
&= \top.
\end{aligned}$$

LUC2: Suppose $\mathcal{F} \leq \mathcal{G}$. Then $\mathcal{U}(a) \to \mathcal{F}(a) \ \leq \ \mathcal{U}(a) \to \mathcal{G}(a)$ for all $a \in L^{X\times X}$.

$$\begin{aligned}
\Lambda^{\mathcal{U}}(\mathcal{F}) &= \bigwedge_{a \in L^{X\times X}} \big(\mathcal{U}(a) \to \mathcal{F}(a)\big) \\
&\leq \mathcal{U}(a) \to \mathcal{F}(a) \text{ for all } a \in L^{X\times X} \\
&\leq \mathcal{U}(a) \to \mathcal{G}(a) \text{ for all } a \in L^{X\times X}.
\end{aligned}$$

Therefore we have that

$$\bigwedge_{a \in L^{X\times X}} \big(\mathcal{U}(a) \to \mathcal{F}(a)\big) \leq \bigwedge_{a \in L^{X\times X}} \big(\mathcal{U}(a) \to \mathcal{G}(a)\big),$$

and so

$$\Lambda^{\mathcal{U}}(\mathcal{F}) \leq \Lambda^{\mathcal{U}}(\mathcal{G}).$$

LUC3:

$$\begin{aligned}
\Lambda^{\mathcal{U}}(\mathcal{F}^{-1}) &= \bigwedge_{a \in L^{X \times X}} \big(\mathcal{U}(a) \to \mathcal{F}^{-1}(a)\big) \\
&= \bigwedge_{a \in L^{X \times X}} \big(\mathcal{U}(a) \to \mathcal{F}(a^{-1})\big) \\
&\geq \bigwedge_{a \in L^{X \times X}} \big(\mathcal{U}^{-1}(a) \to \mathcal{F}(a^{-1})\big) \\
&= \bigwedge_{a \in L^{X \times X}} \big(\mathcal{U}(a^{-1}) \to \mathcal{F}(a^{-1})\big) \\
&= \bigwedge_{a \in L^{X \times X}} \big(\mathcal{U}(a) \to \mathcal{F}(a)\big) \\
&= \Lambda^{\mathcal{U}}(\mathcal{F}).
\end{aligned}$$

LUC4:

$$\begin{aligned}
\Lambda^{\mathcal{U}}(\mathcal{F}) \wedge \Lambda^{\mathcal{U}}(\mathcal{G}) &= \Big(\bigwedge_{a \in L^{X \times X}} \mathcal{U}(a) \to \mathcal{F}(a)\Big) \wedge \Big(\bigwedge_{b \in L^{X \times X}} \mathcal{U}(b) \to \mathcal{G}(b)\Big) \\
&\leq \bigwedge_{c \in L^{X \times X}} \Big(\big(\mathcal{U}(c) \to \mathcal{F}(c)\big) \wedge \big(\mathcal{U}(c) \to \mathcal{G}(c)\big)\Big) \\
&= \bigwedge_{c \in L^{X \times X}} \Big(\mathcal{U}(c) \to \big(\mathcal{F}(c) \wedge \mathcal{G}(c)\big)\Big) \\
&= \bigwedge_{c \in L^{X \times X}} \big(\mathcal{U}(c) \to (\mathcal{F} \wedge \mathcal{G})(c)\big) \\
&= \Lambda^{\mathcal{U}}(\mathcal{F} \wedge \mathcal{G}).
\end{aligned}$$

LUC5: By definition we have that

$$\Lambda^{\mathcal{U}}(\mathcal{F} \circ \mathcal{G}) = \bigvee_{a \in L^{X \times X}} \big(\mathcal{U}(a) \to \mathcal{F} \circ \mathcal{G}(a)\big).$$

Now since $(X,\mathcal{U}) \in |SL - UNIF|$ we use **(LU3)** to give

$$\begin{aligned}
\Lambda^{\mathcal{U}}(\mathcal{F} \circ \mathcal{G}) &\geq \bigwedge_{a \in L^{X \times X}} \bigg(\Big(\bigvee_{d_1 \circ d_2 \leq a} \big(\mathcal{U}(d_1) * \mathcal{U}(d_2)\big)\Big) \to \mathcal{F} \circ \mathcal{G}(a)\bigg) \\
&= \bigwedge_{a \in L^{X \times X}} \bigwedge_{d_1 \circ d_2 \leq a} \Big(\mathcal{U}(d_1) * \mathcal{U}(d_2) \to \mathcal{F} \circ \mathcal{G}(a)\Big),
\end{aligned}$$

using property **(iii)** of Lemma 4.3.8 in the equality. From the definition of the stratified L-filter $\mathcal{F} \circ \mathcal{G}$ we have

$$\begin{aligned}
&\bigwedge_{a\in L^{X\times X}} \bigwedge_{d_1\circ d_2\le a} \Big(\mathcal{U}(d_1) * \mathcal{U}(d_2) \to \mathcal{F}\circ\mathcal{G}(a)\Big)\\
&\ge \bigwedge_{a\in L^{X\times X}} \bigwedge_{d_1\circ d_2\le a} \Big(\mathcal{U}(d_1) * \mathcal{U}(d_2) \to \mathcal{F}(d_1) * \mathcal{G}(d_2)\Big) = Q.
\end{aligned}$$

Here we use Lemma 4.3.10 in the first step, and properties of infima thereafter, to get

$$\begin{aligned}
Q &\ge \bigwedge_{a\in L^{X\times X}} \bigwedge_{d_1\circ d_2\le a} \Big(\big(\mathcal{U}(d_1) \to \mathcal{F}(d_1)\big) * \big(\mathcal{U}(d_2) \to \mathcal{G}(d_2)\big)\Big)\\
&\ge \bigwedge_{d_1\in L^{X\times X}} \Bigg(\bigwedge_{d_2\in L^{X\times X}} \Big(\big(\mathcal{U}(d_1) \to \mathcal{F}(d_1)\big) * \big(\mathcal{U}(d_2) \to \mathcal{G}(d_2)\big)\Big)\Bigg)\\
&\ge \bigwedge_{d_1\in L^{X\times X}} \big(\mathcal{U}(d_1) \to \mathcal{F}(d_1)\big) * \bigwedge_{d_2\in L^{X\times X}} \big(\mathcal{U}(d_2) \to \mathcal{G}(d_2)\big)\\
&= \Lambda^{\mathcal{U}}(\mathcal{F}) * \Lambda^{\mathcal{U}}(\mathcal{G}).
\end{aligned}$$

That is, we have $\Lambda^{\mathcal{U}}(\mathcal{F}\circ\mathcal{G}) \ge \Lambda^{\mathcal{U}}(\mathcal{F}) * \Lambda^{\mathcal{U}}(\mathcal{G})$.

■

The following proofs of these two lemmas are produced in [27]. The change of the lattice, L, from a Heyting algebra to an enriched lattice does not affect the workings of the proof and hence they remain unchanged.

Lemma 7.3.4. *[27] Let $(X,\mathcal{U}) \in |SL-UNIF|$. Then for $a \in L^{X\times X}$,*

$$\mathcal{U}(a) = \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \big(\Lambda^{\mathcal{U}}(\mathcal{F}) \to \mathcal{F}(a)\big).$$

Proof: Let $a \in L^{X\times X}$,

$$\begin{aligned}
\bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \big(\Lambda^{\mathcal{U}}(\mathcal{F}) \to \mathcal{F}(a)\big) &= \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Big(\big(\bigwedge_{b\in L^{X\times X}} \mathcal{U}(b) \to \mathcal{F}(b)\big) \to \mathcal{F}(a)\Big)\\
&\ge \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Big(\big(\mathcal{U}(a) \to \mathcal{F}(a)\big) \to \mathcal{F}(a)\Big)\\
&\ge \mathcal{U}(a)
\end{aligned}$$

Now, since $\Lambda^{\mathcal{U}}(\mathcal{U}) = \top$, we get

$$\begin{aligned}
\bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \big(\Lambda^{\mathcal{U}}(\mathcal{F}) \to \mathcal{F}(a)\big) &\le \Lambda^{\mathcal{U}}(\mathcal{U}) \to \mathcal{U}(a)\\
&= \top \to \mathcal{U}(a)\\
&= \mathcal{U}(a).
\end{aligned}$$

■

We have shown that there is an isomorphism between the class objects of $SL-PUCS$ and $SL-UNIF$, and now do the same for the class of morphisms.

Lemma 7.3.5. *[27] Let $(X,\mathcal{U}),(Y,\mathcal{V}) \in |SL-UNIF|$ and let $\varphi : X \longrightarrow Y$ be a mapping. Then the following are equivalent:*

(i) *$\varphi : (X,\mathcal{U}) \longrightarrow (Y,\mathcal{V})$ is L-uniformly continuous,*

(ii) *$\varphi : (X,\Lambda^{\mathcal{U}}) \longrightarrow (Y,\Lambda^{\mathcal{V}})$ is uniformly continuous.*

Proof: Let $\varphi : (X,\mathcal{U}) \longrightarrow (Y,\mathcal{V})$ be L-uniformly continuous. Then $(\varphi\times\varphi)(\mathcal{U}) \geq \mathcal{V}$. Now, let $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$:

$$\begin{aligned}
\Lambda^{\mathcal{V}}\big((\varphi\times\varphi)(\mathcal{F})\big) &= \bigwedge_{b\in L^{Y\times Y}} \big(\mathcal{V}(b) \to (\varphi\times\varphi)(\mathcal{F})(b)\big)\\
&= \bigwedge_{b\in L^{Y\times Y}} \Big(\mathcal{V}(b) \to \mathcal{F}\big((\varphi\times\varphi)^{\leftarrow}(b)\big)\Big)\\
&\geq \bigwedge_{b\in L^{Y\times Y}} \Big(\mathcal{U}\big(\varphi\times\varphi)^{\leftarrow}(b)\big) \to \mathcal{F}\big((\varphi\times\varphi)^{\leftarrow}(b)\big)\Big)\\
&\geq \bigwedge_{a\in L^{X\times X}} \mathcal{U}(a) \to \mathcal{F}(a)\\
&= \Lambda^{\mathcal{U}}(\mathcal{F}).
\end{aligned}$$

Hence $\varphi : (X,\Lambda^{\mathcal{U}}) \longrightarrow (Y,\Lambda^{\mathcal{V}})$ is uniformly continuous.

Now suppose that $\varphi : (X,\Lambda^{\mathcal{U}}) \longrightarrow (Y,\Lambda^{\mathcal{V}})$ is uniformly continuous. Then for all $\mathcal{F} \in \mathcal{F}_L^S(X\times X)$, we have $\Lambda^{\mathcal{V}}\big((\varphi\times\varphi)(\mathcal{F})\big) \geq \Lambda^{\mathcal{U}}(\mathcal{F})$.

For $b \in L^{Y\times Y}$,

$$\begin{aligned}
\mathcal{U}\big((\varphi\times\varphi)^{\leftarrow}(b)\big) &= \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Lambda^{\mathcal{U}}(\mathcal{F}) \to \mathcal{F}\big((\varphi\times\varphi)^{\leftarrow}(b)\big)\\
&\geq \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Lambda^{\mathcal{V}}\big((\varphi\times\varphi)(\mathcal{F})\big) \to \mathcal{F}\big((\varphi\times\varphi)^{\leftarrow}(b)\big)\\
&= \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Lambda^{\mathcal{V}}\big((\varphi\times\varphi)(\mathcal{F})\big) \to \big((\varphi\times\varphi)(\mathcal{F})\big)(b)\\
&\geq \bigwedge_{\mathcal{G}\in\mathcal{F}_L^S(Y\times Y)} \Lambda^{\mathcal{V}}(\mathcal{G}) \to \mathcal{G}(b)\\
&= \mathcal{V}(b).
\end{aligned}$$

From this we get that $\varphi : (X,\mathcal{U}) \longrightarrow (Y,\mathcal{V})$ is L-uniformly continuous.

■

Corollary 7.3.6. *[27] The categories $SL-PUCS$ and $SL-UNIF$ are categorically isomorphic.*

Having shown the above result, we now proceed to show that $SL-PUCS$ is a reflective subcategory of $SL-UCS$. This will then give us the corollary that $SL-UNIF$ is isomorphic to a reflective subcategory of $SL-UCS$.

In order to prove that $SL-PUCS$ is a reflective subcategory of $SL-UCS$ we have to, unfortunately, restrict the lattice context from that stated at the beginning of this chapter. We must now consider a pseudo-bisymmetric enriched *cl*-premonoid with the additional property that $\alpha \leq \alpha \otimes \alpha$ for all $\alpha \in L$. This restriction still includes the cases where $\otimes = \circledast$, the monoidal mean operator, as well as $(L, \leq, \wedge, *)$ and the complete Heyting algebra $(L, \leq, \wedge, \wedge)$.

The result where this restriction is required is the one below, where we will show that from any stratified L-uniform convergence space, we can define a stratified L-filter. This will then in turn be used to generate a principal stratified L-uniform convergence space. This fact will then give us that the stratified L-filter defined below is in fact a stratified L-uniformity.

Lemma 7.3.7. *Let $(L, \leq, \otimes, *)$ be a pseudo-bisymmetric enriched cl-premonoid such that $\alpha \leq \alpha \otimes \alpha$ for all $\alpha \in L$, and let $(X, \Lambda) \in |SL-UCS|$. We define the mapping $\mathcal{U}_\Lambda : L^{X\times X} \longrightarrow L$ for $a \in L^{X\times X}$ by:*

$$\mathcal{U}_\Lambda(a) = \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \big(\Lambda(\mathcal{F}) \to \mathcal{F}(a)\big).$$

Then $\mathcal{U}_\Lambda$ is a stratified L-filter on $X \times X$.

Proof:

LF0: $\mathcal{U}_\Lambda(\perp_{X\times X}) = \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Lambda(\mathcal{F}) \to \mathcal{F}(\perp_{X\times X})$. Now we can take $x \in X$ and consider the stratified L-filter $[(x,x)]$, where $\Lambda\big([(x,x)]\big) = \top$. In this case, $\Lambda\big([(x,x)]\big) \to [(x,x)](\perp_{X\times X}) = \top \to \perp = \perp$, and so we have $\mathcal{U}_\Lambda(\perp_{X\times X}) = \perp$.

Now consider $\mathcal{U}_\Lambda(\top_{X\times X}) = \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Lambda(\mathcal{F}) \to \mathcal{F}(\top_{X\times X})$. Clearly, for all $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$ we have $\mathcal{F}(\top_{X\times X}) = \top$. Now, for any $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$ we have that $\Lambda(\mathcal{F}) \to \top = \top$ and therefore $\mathcal{U}_\Lambda(\top_{X\times X}) = \top$.

LF1: Let $a, b \in L^{X\times X}$ with $a \leq b$. Since for all $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$ we have that $\mathcal{F}(a) \leq \mathcal{F}(b)$, we then get that $\Lambda(\mathcal{F}) \to \mathcal{F}(a) \leq \Lambda(\mathcal{F}) \to \mathcal{F}(b)$. This gives us

$$\mathcal{U}_\Lambda(a) = \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Lambda(\mathcal{F}) \to \mathcal{F}(a) \leq \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Lambda(\mathcal{F}) \to \mathcal{F}(b) = \mathcal{U}_\Lambda(b).$$

LF2: Let $a, b \in L^{X\times X}$.

$$\begin{aligned}\mathcal{U}_\Lambda(a) \otimes \mathcal{U}_\Lambda(b) &= \Big(\bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \big(\Lambda(\mathcal{F}) \to \mathcal{F}(a)\big)\Big) \otimes \Big(\bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \big(\Lambda(\mathcal{F}) \to \mathcal{F}(b)\big)\Big) \\ &\leq \bigwedge_{\mathcal{G}\in\mathcal{F}_L^S(X\times X)} \Big(\big(\Lambda(\mathcal{G}) \to \mathcal{G}(a)\big) \otimes \big(\Lambda(\mathcal{G}) \to \mathcal{G}(b)\big)\Big).\end{aligned}$$

Now with Lemma 4.6.8 we have that

$$\mathcal{U}_\Lambda(a) \otimes \mathcal{U}_\Lambda(b) \leq \bigwedge_{\mathcal{G}\in\mathcal{F}_L^S(X\times X)} \Big(\Lambda(\mathcal{G}) \to \big(\mathcal{G}(a) \otimes \mathcal{G}(b)\big)\Big).$$

We can then use **(LF2)** of the stratified L-filters to get

$$\begin{aligned}\mathcal{U}_\Lambda(a) \otimes \mathcal{U}_\Lambda(b) &\leq \bigwedge_{\mathcal{G}\in\mathcal{F}_L^S(X\times X)} \big(\Lambda(\mathcal{G}) \to \mathcal{G}(a \otimes b)\big) \\ &= \mathcal{U}_\Lambda(a \otimes b).\end{aligned}$$

LFS: Let $\alpha \in L$ and $a \in L^{X\times X}$. Then

$$\begin{aligned}\alpha * \mathcal{U}_\Lambda(a) &= \alpha * \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \big(\Lambda(\mathcal{F}) \to \mathcal{F}(a)\big) \\ &= \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \alpha * \big(\Lambda(\mathcal{F}) \to \mathcal{F}(a)\big) \\ &= \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \alpha * \bigvee \Big\{\lambda \in L \,:\, \Lambda(\mathcal{F}) * \lambda \leq \mathcal{F}(a)\Big\} \\ &= \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \bigvee \Big\{(\alpha * \lambda) \in L \,:\, \Lambda(\mathcal{F}) * \lambda \leq \mathcal{F}(a)\Big\}.\end{aligned}$$

Now, for the sup described above, we have that

$$\begin{aligned}&\bigvee \Big\{(\alpha * \lambda) \in L \,:\, \Lambda(\mathcal{F}) * \lambda \leq \mathcal{F}(a)\Big\} \\ &\leq \bigvee \Big\{(\alpha * \lambda) \in L \,:\, \alpha * \Lambda(\mathcal{F}) * \lambda \leq \alpha * \mathcal{F}(a)\Big\} \\ &\leq \bigvee \Big\{(\alpha * \lambda) \in L \,:\, (\alpha * \lambda) * \Lambda(\mathcal{F}) \leq \mathcal{F}(\alpha_{X\times X} * a)\Big\} \\ &\leq \Lambda(\mathcal{F}) \to \mathcal{F}(\alpha_{X\times X} * a).\end{aligned}$$

This therefore gives us that

$$\begin{aligned}\alpha * \mathcal{U}_\Lambda(a) &\leq \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Lambda(\mathcal{F}) \to \mathcal{F}(\alpha_{X\times X} * a) \\ &= \mathcal{U}_\Lambda(\alpha_{X\times X} * a).\end{aligned}$$

Thus **(LFS)** has been proved and we have that $\mathcal{U}_\Lambda \in \mathcal{F}_L^S(X \times X)$.

∎

Now that we have shown that $\mathcal{U}_\Lambda$ is a stratified L-filter on $X \times X$, we can proceed with the main result.

Lemma 7.3.8. *Let $(L, \leq, \otimes, *)$ be an enriched cl-premonoid such that $\alpha \leq \alpha \otimes \alpha$ for all $\alpha \in L$. Then $SL - PUCS$ is a reflective subcategory of $SL - UCS$.*

Proof: For any $(X, \Lambda) \in |SL - PUCS|$ it is clear that $(X, \Lambda) \in |SL - UCS|$. Therefore we can consider the embedding functor

$$E : \begin{cases} SL - PUCS \longrightarrow SL - UCS \\ (X, \Lambda) \longmapsto (X, \Lambda) \\ \varphi \longmapsto \varphi \end{cases}.$$

Now we let $(X, \Lambda) \in |SL - UCS|$ and define:

$$\mathcal{U}_\Lambda(a) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X \times X)} \big(\Lambda(\mathcal{F}) \to \mathcal{F}(a)\big).$$

From the previous lemma we have that $\mathcal{U}_\Lambda$ is a stratified L-filter on $X \times X$. We now use $\mathcal{U}_\Lambda$ to define:

$$\Lambda^*(\mathcal{F}) = \bigwedge_{a \in L^{X \times X}} \big(\mathcal{U}_\Lambda(a) \to \mathcal{F}(a)\big)$$

From the earlier definition we can see that Λ^* is a principal stratified L-uniform convergence structure, and from Lemma 7.3.2 we have that $(X, \mathcal{U}_\Lambda) \in |SL - UNIF|$. Further we have

$$\begin{aligned} \Lambda^*(\mathcal{F}) &= \bigwedge_{a \in L^{X \times X}} \big(\mathcal{U}_\Lambda(a) \to \mathcal{F}(a)\big) \\ &= \bigwedge_{a \in L^{X \times X}} \Big(\big(\bigwedge_{\mathcal{G} \in \mathcal{F}_L^S(X \times X)} \Lambda(\mathcal{G}) \to \mathcal{G}(a)\big) \to \mathcal{F}(a)\Big) \\ &\geq \bigwedge_{a \in L^{X \times X}} \Big(\big(\Lambda(\mathcal{F}) \to \mathcal{F}(a)\big) \to \mathcal{F}(a)\Big) \\ &\geq \Lambda(\mathcal{F}) \end{aligned}$$

By the order on the structures defined on page 74 we have that $\Lambda^* \leq \Lambda$. Now consider $\tilde{\Lambda} \leq \Lambda$ such that:

$$\tilde{\Lambda}(\mathcal{F}) = \bigwedge_{a \in L^{X \times X}} \big(\mathcal{U}(a) \to \mathcal{F}(a)\big)$$

for some stratified L-filter $\mathcal{U}$. Since $(X, \tilde{\Lambda}) \in |SL - PUCS|$ we get that:

$$\mathcal{U}(a) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X \times X)} \big(\tilde{\Lambda}(\mathcal{F}) \to \mathcal{F}(a)\big) \leq \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X \times X)} \big(\Lambda(\mathcal{F}) \to \mathcal{F}(a)\big) = \mathcal{U}_\Lambda(a).$$

Hence

$$\tilde{\Lambda}(\mathcal{F}) \geq \bigwedge_{a\in L^{X\times X}} \big(\mathcal{U}_\Lambda(a) \to \mathcal{F}(a)\big) = \Lambda^*(\mathcal{F}).$$

This serves to show that $\tilde{\Lambda} \leq \Lambda^*$, and so we conclude that Λ^* is the finest $SL-PUCS$ structure on X which is coarser than Λ.

Consider now $(Y,\Sigma) \in |SL-UCS|$ and let $\varphi : (X,\Lambda) \longrightarrow (Y,\Sigma)$ be a morphism in the category $SL-UCS$. We must show that $\varphi : (X,\Lambda^*) \longrightarrow (Y,\Sigma^*)$ is also uniformly continuous. For $b \in L^{Y\times Y}$ we have

$$\begin{aligned}
\mathcal{U}_\Lambda\big((\varphi\times\varphi)^{\leftarrow}(b)\big) &= \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Big(\Lambda(\mathcal{F}) \to \mathcal{F}\big((\varphi\times\varphi)^{\leftarrow}(b)\big)\Big)\\
&\geq \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Big(\Sigma\big((\varphi\times\varphi)(\mathcal{F})\big) \to \mathcal{F}\big((\varphi\times\varphi)^{\leftarrow}(b)\big)\Big)\\
&= \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Big(\Sigma\big((\varphi\times\varphi)(\mathcal{F})\big) \to \big((\varphi\times\varphi)(\mathcal{F})\big)(b)\Big)\\
&\geq \bigwedge_{\mathcal{G}\in\mathcal{F}_L^S(Y\times Y)} \big(\Sigma(\mathcal{G}) \to \mathcal{G}(b)\big)\\
&= \mathcal{U}_\Sigma(b).
\end{aligned}$$

From this we can see that:

$$\begin{aligned}
\Sigma^*\big((\varphi\times\varphi)(\mathcal{F})\big) &= \bigwedge_{b\in L^{Y\times Y}} \Big(\mathcal{U}_\Sigma(b) \to \big((\varphi\times\varphi)(\mathcal{F})\big)(b)\Big)\\
&\geq \bigwedge_{b\in L^{Y\times Y}} \Big(\mathcal{U}_\Lambda\big((\varphi\times\varphi)^{\leftarrow}(b)\big) \to \big((\varphi\times\varphi)(\mathcal{F})\big)(b)\Big)\\
&= \bigwedge_{b\in L^{Y\times Y}} \Big(\mathcal{U}_\Lambda\big((\varphi\times\varphi)^{\leftarrow}(b)\big) \to \mathcal{F}\big((\varphi\times\varphi)^{\leftarrow}(b)\big)\Big)\\
&\geq \bigwedge_{a\in L^{X\times X}} \big(\mathcal{U}_\Lambda(a) \to \mathcal{F}(a)\big)\\
&= \Lambda^*(\mathcal{F}).
\end{aligned}$$

Now that we know that the uniformly continuous mappings will remain morphisms, we can define a functor:

$$K : \begin{cases} SL-UCS \longrightarrow SL-PUCS\\ (X,\Lambda) \longmapsto (X,\Lambda^*)\\ \varphi \longmapsto \varphi \end{cases}$$

For $(X,\Lambda) \in |SL-PUCS|$ we have $K\big(E(X,\Lambda)\big) = (X,\Lambda)$ and since $\Lambda^* \leq \Lambda$ we know that $E\big(K(X,\Lambda)\big) \leq (X,\Lambda)$ for $(X,\Lambda) \in |SL-UCS|$.

This in turn means that $id_X : (X, \Lambda) \longrightarrow E(X, \Lambda^*)$ is continuous. Therefore, for $(X, \Lambda) \in |SL - UCS|$ we propose our E-universal map to be $\big(id_X, (X, \Lambda^*)\big)$.

We show now that this is an E-universal map for (X, Λ).

Let $(Z, \Gamma) \in |SL - PUCS|$ and $\psi : (X, \Lambda) \longrightarrow E\big((Z, \Gamma)\big)$. We require a unique $SL - PUCS$ morphism $\phi : (X, \Lambda^*) \longrightarrow (Z, \Gamma)$ such that the following diagram commutes:

$$\begin{array}{ccc} (X,\Lambda) & \xrightarrow{\psi} & E\big((Z,\Gamma)\big) \\ {\scriptstyle id_X}\big\downarrow & \nearrow{\scriptstyle E(\phi)} & \\ E\big((X,\Lambda^*)\big) & & \end{array} \qquad \begin{array}{c} (Z,\Gamma) \\ \big\uparrow{\scriptstyle \phi} \\ (X,\Lambda^*) \end{array}$$

It is clear that the mapping ϕ will be none other than $K(\psi) = \psi$.

■

Corollary 7.3.9. *Let $(L, \leq, \otimes, *)$ be a pseudo-bisymmetric enriched cl-premonoid with the additional property that $\alpha \leq \alpha \otimes \alpha$ for all $\alpha \in L$. Then $SL - UNIF$ is isomorphic to a reflective subcategory of $SL - UCS$.*

We are unfortunately unable to show the above reflective subcategory result except for the case where $\alpha \leq \alpha \otimes \alpha$ for all $\alpha \in L$. However, this minor restriction still includes many of our main examples: when the $\otimes$ operator is the monoidal mean operator, $\circledast$, and when $\otimes$ is the minimum, $\wedge$. In order to try and show this result for the general case of a pseudo-bisymmetric enriched *cl*-premonoid, one might have to show that $\alpha \leq \alpha \otimes \alpha$ for all $\alpha \in L$ or else an alternative approach may be attempted to define a stratified L-filter from a stratified L-uniform convergence space.

7.4 Lattice-valued Convergence

From the stratified L-uniform convergence structures described in the previous section, it is possible to generate stratified L-limit structures. The stratified L-limit spaces are then sets equipped with a map $\lim(\Lambda) : \mathcal{F}_L^S(X) \longrightarrow L^X$. That is, for each $x \in X$, $\lim(\Lambda)\mathcal{F}(x)$ is the degree to which $\mathcal{F}$ converges to x.

Definition 7.4.1. [24] Let X be a set and $\lim : \mathcal{F}_L^S(X) \longrightarrow L^X$. We call $(X, \lim)$ a *stratified L-limit space* if lim satisfies the following axioms:

(L1) for all $x \in X$, $\lim[x](x) = \top$,

(L2) $\mathcal{F} \leq \mathcal{G} \Longrightarrow \lim \mathcal{F} \leq \lim \mathcal{G}$,

(L3) $\lim \mathcal{F} \wedge \lim \mathcal{G} \leq \lim(\mathcal{F} \wedge \mathcal{G})$.

Definition 7.4.2. [24] Let $(X, \lim_X)$ and $(X, \lim_Y)$ be stratified L-limit spaces and consider a mapping $\varphi : X \longrightarrow Y$. The mapping φ is *continuous* if for all $x \in X$ and for all $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$,

$$\lim \mathcal{F}(x) \leq \lim_Y \varphi(\mathcal{F})\big(\varphi(x)\big)$$

RESULT: $SL - LIM$ is a concrete category, where the objects are stratified L-limit spaces and the morphisms are the continuous mappings defined above.

The original work [27] that used a complete Heyting algebra as the lattice, defines for a stratified L-uniform convergence structure Λ, the induced limit function to be:

$$\lim(\Lambda)\mathcal{F}(x) = \Lambda(\mathcal{F} \times [x])$$

However, since we are unable to work directly with the product filters when using a generalised enriched *cl*-premonoid, we produce here an alternative definition for the induced limit function. We use the stratified L-filter $\mathcal{F}_x$ as described earlier. It can be shown, for the case of a Heyting algebra, that $\mathcal{F} \times [x] = \mathcal{F}_x$ (see 5.6.11).

Lemma 7.4.3. *Let* $(X, \Lambda) \in |SL - UCS|$. *Define* $\lim(\Lambda) : \mathcal{F}_L^S(X) \longrightarrow L^X$ *by:*

$$\lim(\Lambda)\mathcal{F}(x) = \Lambda(\mathcal{F}_x)$$

Then $\big(X, \lim(\Lambda)\big)$ *is a stratified L-limit space.*

PROOF:

L1: Let $x \in X$ and consider $\lim(\Lambda)[x](x) = \Lambda([x]_x)$. By Lemma 5.6.13 we have this equal to $\Lambda\big([(x,x)]\big)$ and therefore $\lim(\Lambda)[x](x) = \top$ by **(LUC1)**.

L2: Suppose $\mathcal{F} \leq \mathcal{G}$ and let $x \in X$ and $d \in L^{X \times X}$. We first show that

$$\mathcal{F}_x(d) = \mathcal{F}\big(d(\cdot, x)\big) \leq \mathcal{G}\big(d(\cdot, x)\big) = \mathcal{G}_x(d).$$

Now we can use **(LUC2)** to show that $\lim(\Lambda)\mathcal{F}(x) = \Lambda(\mathcal{F}_x) \leq \Lambda(\mathcal{G}_x) = \lim(\Lambda)\mathcal{G}(x)$.

L3: Let $x \in X$. Now

$$\big(\lim(\Lambda)\mathcal{F} \wedge \lim(\Lambda)\mathcal{G}\big)(x) = \lim(\Lambda)\mathcal{F}(x) \wedge \lim(\Lambda)\mathcal{G}(x) = \Lambda(\mathcal{F}_x) \wedge \Lambda(\mathcal{G}_x).$$

We see by **(LUC4)** that $\Lambda(\mathcal{F}_x) \wedge \Lambda(\mathcal{G}_x) \leq \Lambda(\mathcal{F}_x \wedge \mathcal{G}_x)$ and then, by Lemma 5.6.14, we have $\Lambda(\mathcal{F}_x \wedge \mathcal{G}_x) = \Lambda\big((\mathcal{F} \wedge \mathcal{G})_x\big) = \lim(\Lambda)(\mathcal{F} \wedge \mathcal{G})(x)$. Thus we have

$$\big(\lim(\Lambda)\mathcal{F} \wedge \lim(\Lambda)\mathcal{G}\big)(x) \leq \lim(\Lambda)(\mathcal{F} \wedge \mathcal{G})(x).$$

■

Lemma 7.4.4. *Let $\varphi : (X, \Lambda) \longrightarrow (Y, \Sigma)$ be uniformly continuous. Then $\varphi : \big(X, \lim(\Lambda)\big) \longrightarrow \big(Y, \lim(\Sigma)\big)$ is continuous.*

Proof: Let $\mathcal{F} \in \mathcal{F}_L^S(X), a \in L^{X \times X}$ and let $x \in X$.

We use Lemma 5.7.7 which states $(\varphi \times \varphi)(\mathcal{F}_x) = \varphi(\mathcal{F})_{\varphi(x)}$. Now

$$\begin{aligned}\lim(\Lambda)\mathcal{F}(x) &= \Lambda(\mathcal{F}_x)\\ &\leq \Sigma\big((\varphi \times \varphi)\mathcal{F}_x\big)\\ &= \Sigma\big(\varphi(\mathcal{F})_{\varphi(x)}\big)\\ &= \lim(\Sigma)\varphi(\mathcal{F})\big(\varphi(x)\big).\end{aligned}$$

■

Result: We can define a forgetful functor

$$F : \begin{cases} SL-UCS \longrightarrow SL-LIM \\ (X, \Lambda) \longmapsto \big(X, \lim(\Lambda)\big) \\ \varphi \longmapsto \varphi \end{cases}$$

7.5 Initial Structures in $SL - LIM$

Definition 7.5.1. [24] Let X be a set and for all $i \in I$, let $(X_i, \lim_i) \in |SL-LIM|$. For an $SL - LIM$ source $\varphi_i : X \longrightarrow (X_i, \lim_i), i \in I$, the *initial stratified L-limit structure* on X for $\mathcal{F} \in \mathcal{F}_L^S(X)$ is defined by:

$$\lim \mathcal{F}(x) = \bigwedge_{i \in I} \lim_i \varphi_i(\mathcal{F})\big(\varphi_i(x)\big).$$

Lemma 7.5.2. *[27] Let $\varphi_i : X \longrightarrow (X_i, \Lambda_i)$ be a source in $SL - UCS$ and let Λ be the initial $SL - UCS$ structure on X. Then $\lim(\Lambda)$ is the initial $SL - LIM$ structure with respect to the source $\varphi_i : X \longrightarrow \big(X_i, \lim(\Lambda_i)\big), i \in I$.*

Proof: Let $\mathcal{F} \in \mathcal{F}_L^S(X), x \in X$.

$$\begin{aligned}\lim(\Lambda)\mathcal{F}(x) &= \Lambda(\mathcal{F}_x)\\ &= \bigwedge_{i \in I} \Lambda_i\big((\varphi_i \times \varphi_i)(\mathcal{F}_x)\big)\\ &= \bigwedge_{i \in I} \Lambda_i\big(\varphi_i(\mathcal{F})_{\varphi_i(x)}\big)\\ &= \bigwedge_{i \in I} \lim(\Lambda_i)\varphi_i(\mathcal{F})\big(\varphi_i(x)\big).\end{aligned}$$

■

Result: The forgetful functor F preserves initial structures.

7.6 Convergence in $SL-UNIF$

Now we show that for $(X,\mathcal{U}) \in |SL-UNIF|$ there are two ways of inducing a convergence function. What is remarkable is that these two pathways produce identical convergence structures.

It is shown in [14] that from a stratified L-uniformity we can define a stratified L-neighbourhood system for each $x \in X$:

$$\mathcal{N}_{\mathcal{U}}^{x}(a) = \bigvee \big\{\mathcal{U}(d) \mid d(\cdot,x) \leq a\big\}.$$

The stratified L-neighbourhood space $\big(X,(\mathcal{N}_{\mathcal{U}}^{x})_{x\in X}\big)$ is shown in section 6.2.4 to be equivalent to a stratified L-topological space. From this stratified L-topological space it is shown in [22] that we can induce a stratified L-limit space by:

$$\lim(\mathcal{U})\mathcal{F}(x) = \bigwedge_{a\in L^X} \big(\mathcal{N}_{\mathcal{U}}^{x}(a) \rightarrow \mathcal{F}(a)\big).$$

Using the results from section 7.3 we can also consider the stratified L-uniform space $(X,\mathcal{U})$ as a principal stratified L-uniform convergence space (X,Λ) where

$$\Lambda^{\mathcal{U}}(\mathcal{F}) = \bigwedge_{d\in L^{X\times X}} \big(\mathcal{U}(d) \rightarrow \mathcal{F}(d)\big).$$

From here we can consider the induced stratified L-limit structure that is shown in section 7.4.3:

$$\lim(\Lambda)\mathcal{F}(x) = \Lambda(\mathcal{F}_x) = \bigwedge_{d\in L^{X\times X}} \big(\mathcal{U}(d) \rightarrow (\mathcal{F}_x)(d)\big).$$

Using the two routes of obtaining a stratified L-limit space that are described above, we can show the following categorical relationship when L is a pseudo-bisymmetric enriched cl-premonoid:

$$\begin{array}{ccc}
 & (X,\mathcal{U}) \in |SL-UNIF| & \\
 \swarrow^{F_1} & & \searrow^{E_2} \\
\big(X,\tau(\mathcal{U})\big) \in |SL-TOP| & & (X,\Lambda^{\mathcal{U}}) \in |SL-UCS| \\
\downarrow E_1 & & \downarrow F_2 \\
\big(X,\lim(\mathcal{U})\big) \in |SL-LIM| & & \big(X,\lim(\Lambda^{\mathcal{U}})\big) \in |SL-LIM|
\end{array}$$

In the diagram above, the functors F_1 and F_2 are both forgetful functors, while E_1 and E_2 are embeddings.

It should be noted that in this section we do not have the requirement on L that $\alpha \leq \alpha \otimes \alpha$ for all $\alpha \in L$ as we do not need to produce a stratified L-filter from a stratified L-uniform convergence space. We merely require that we can produce a stratified L-uniform convergence structure from a stratified L-uniformity.

The result below was proved in [27] for the case of L a complete Heyting algebra. The proof presented below for the case of L a pseudo-bisymmetric enriched *cl*-premonoid uses the same procedure, except that when considering the induced L-limit space, we make use of the filter $\mathcal{F}_x$ on the product space.

Proposition 7.6.1. *Let* $(X,\mathcal{U}) \in |SL - UNIF|$. *Then*

$$\big(X, \lim(\mathcal{U})\big) = \big(X, \lim(\Lambda^{\mathcal{U}})\big).$$

Proof: We prove this result by showing that $\lim(\mathcal{U}) = \lim(\Lambda^{\mathcal{U}})$. Let $\mathcal{F} \in \mathcal{F}_L^S(X)$ and let $x \in X$.

$$\begin{aligned}\lim(\mathcal{U})\mathcal{F}(x) &= \bigwedge_{a \in L^X} \big(\mathcal{N}_{\mathcal{U}}^x(a) \to \mathcal{F}(a)\big) \\ &= \bigwedge_{a \in L^X} \Big(\big(\bigvee_{d(\cdot,x) \leq a} \mathcal{U}(d)\big) \to \mathcal{F}(a)\Big) \\ &= \bigwedge_{a \in L^X} \bigwedge_{d : d(\cdot,x) \leq a} \big(\mathcal{U}(d) \to \mathcal{F}(a)\big).\end{aligned}$$

If we have $a \in L^X, d \in L^{X \times X}$ such that $d(\cdot, x) \leq a$ we have:

$$\mathcal{U}(d) \to \mathcal{F}(a) \ \geq \mathcal{U}(d) \to \mathcal{F}\big(d(\cdot, x)\big)$$

and this implies

$$\begin{aligned}\bigwedge_{a \in L^X} \bigwedge_{d : d(\cdot,x) \leq a} \big(\mathcal{U}(d) \to \mathcal{F}(a)\big) &\geq \bigwedge_{d \in L^{X \times X}} \Big(\mathcal{U}(d) \to \mathcal{F}\big(d(\cdot, x)\big)\Big) \\ &= \bigwedge_{d \in L^{X \times X}} \big(\mathcal{U}(d) \to \mathcal{F}_x(d)\big) \\ &= \Lambda^{\mathcal{U}}(\mathcal{F}_x) \\ &= \lim(\Lambda^{\mathcal{U}})\mathcal{F}(x).\end{aligned}$$

Now, for $d \in L^{X \times X}$ we have $a_0 = d(\cdot, x) \in L^X$ and thus

$$\begin{aligned}\bigwedge_{a \in L^X} \bigwedge_{d : d(\cdot,x) \leq a} \big(\mathcal{U}(d) \to \mathcal{F}(a)\big) &\leq \bigwedge_{e : e(\cdot,x) \leq a_0} \big(\mathcal{U}(e) \to \mathcal{F}(a_0)\big) \\ &\leq \mathcal{U}(d) \to \mathcal{F}\big(d(\cdot, x)\big) \\ &= \mathcal{U}(d) \to \mathcal{F}_x(d).\end{aligned}$$

This is true for all $d \in L^{X \times X}$ and so

$$\begin{aligned} \lim(\mathcal{U})\mathcal{F}(x) &= \bigwedge_{a \in L^X} \bigwedge_{d : d(\cdot, x) \leq a} \big(\mathcal{U}(d) \to \mathcal{F}(a)\big) \\ &\leq \bigwedge_{d \in L^{X \times X}} \big(\mathcal{U}(d) \to \mathcal{F}_x(d)\big) \\ &= \Lambda^{\mathcal{U}}(\mathcal{F}_x) \\ &= \lim(\Lambda^{\mathcal{U}})\mathcal{F}(x). \end{aligned}$$

■

Many different properties of stratified L-uniform convergence spaces have been shown in this chapter. We have investigated the initial structures of $SL-UCS$, its role as a supercategory of $SL-UNIF$ as well as the induced lattice-valued convergence structure on the underlying set X. We will now make an attempt to show that this category is cartesian closed.

Chapter 8

Function Spaces of $SL - UCS$

Here we attempt to define a structure on the set of all continuous functions from one stratified L-uniform convergence space to another. Once equipped with this structure, it is possible to prove that $SL - UCS$ is a cartesian closed category.

Our attempts to prove the necessary results for the case of $(L, \leq, \otimes, *)$, an enriched *cl*-premonoid, have not been successful. When trying to generalise the results obtained in [27] there are some crucial lemmas that we have been unable to extend to the generalised lattice context. It may be that there is an alternative approach that could be taken, but our efforts have not yet yielded any positive results.

For this chapter, unless otherwise stated, we will always assume that $(L, \leq, \wedge, \wedge)$ is a complete Heyting algebra.

8.1 The mapping η

On page 24 we described the mapping used by Jäger and Burton [27] to generalise the concept of a uniform convergence structure on the set of uniformly continuous mappings. There are a number of results involving this mapping, some of which we have been able to generalise, and some that can only be completed for the case of L a complete Heyting algebra. For the case of L a complete Heyting algebra, we have $* = \otimes = \wedge$ and so we can use the result of Proposition 5.4.3 to evaluate the product of two L-filters. In the case of a pseudo-bisymmetric enriched *cl*-premonoid we can guarantee existence, but are not able to evaluate an L-set under the product L-filter.

Lemma 8.1.1. *[27] Let $a_1 \in L^{X\times X}, a_2 \in L^{Y\times Y}$. Then $\big(\eta(a_1 \times a_2)\big)^{-1} = \eta(a_1^{-1} \times a_2^{-1})$.*

This result will be used in the proof of the next lemma.

Lemma 8.1.2. *[27] Let $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$ and $\mathcal{G} \in \mathcal{F}_L^S(Y \times Y)$. Then*

$$\eta(\mathcal{F}^{-1} \times \mathcal{G}) = \big(\eta(\mathcal{F} \times \mathcal{G}^{-1})\big)^{-1}.$$

PROOF: Let $b \in L^{(X\times Y)\times(X\times Y)}$. Then by definition of the η mapping

$$\begin{aligned}\big(\eta(\mathcal{F}^{-1} \times \mathcal{G})\big)(b) &= (\mathcal{F}^{-1} \times \mathcal{G})\big(\eta^{\leftarrow}(b)\big)\\ &= \bigvee_{\substack{f\in L^{X\times X}, g\in L^{Y\times Y}:\\ \eta(f\times g)\le b}} \mathcal{F}^{-1}(f) \wedge \mathcal{G}(g)\\ &= \bigvee_{\substack{f\in L^{X\times X}, g\in L^{Y\times Y}:\\ \eta(f\times g)\le b}} \mathcal{F}(f^{-1}) \wedge \mathcal{G}^{-1}(g^{-1}).\end{aligned}$$

Now, using the previous lemma we have

$$\begin{aligned}\big(\eta(\mathcal{F}^{-1} \times \mathcal{G})\big)(b) &= \bigvee_{\substack{f\in L^{X\times X}, g\in L^{Y\times Y}:\\ \eta(f^{-1}\times g^{-1})\le b^{-1}}} \mathcal{F}(f^{-1}) \wedge \mathcal{G}^{-1}(g^{-1})\\ &= (\mathcal{F} \times \mathcal{G}^{-1})\big(\eta^{\leftarrow}(b^{-1})\big)\\ &= \big(\eta(\mathcal{F} \times \mathcal{G}^{-1})\big)^{-1}(b).\end{aligned}$$

■

The following result holds for $(L, \le, \otimes, *)$ a pseudo-bisymmetric enriched *cl*-premonoid. That fact is interesting in itself, as this same result, when shown for L a complete Heyting algebra [27], made use of the evaluation of a product L-filter on an L-set.

Lemma 8.1.3. *Let $\mathcal{F} \in \mathcal{F}_L^S(X\times X)$ and $\mathcal{G} \in \mathcal{F}_L^S(Y\times Y)$. Then let $P_1 : X\times Y \longrightarrow X$ and $P_2 : X \times Y \longrightarrow Y$ be projection mappings. Then*

$$(P_1 \times P_1)\big(\eta(\mathcal{F} \times \mathcal{G})\big) \ge \mathcal{F} \quad \text{and} \quad (P_2 \times P_2)\big(\eta(\mathcal{F} \times \mathcal{G})\big) \ge \mathcal{G}.$$

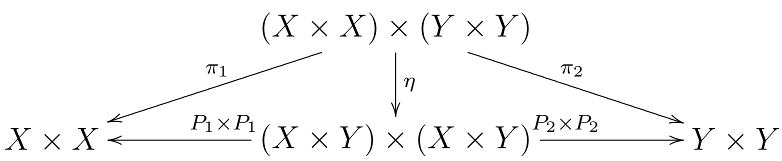
PROOF: We define the projections

$$\pi_1 : (X \times X) \times (Y \times Y) \longrightarrow X \times X \quad \text{and} \quad \pi_2 : (X \times X) \times (Y \times Y) \longrightarrow Y \times Y$$

That is, for $\big((x_1, x_2), (y_1, y_2)\big) \in (X \times X) \times (Y \times Y)$ we have

$$\pi_1\big((x_1, x_2), (y_1, y_2)\big) = (x_1, x_2) \quad \text{and} \quad \pi_2\big((x_1, x_2), (y_1, y_2)\big) = (y_1, y_2).$$

The role of these mappings can be illustrated as follows:



Now we can show the following:

$$\begin{aligned}(P_1 \times P_1)\Big(\eta\big((x_1, x_2), (y_1, y_2)\big)\Big) &= (P_1 \times P_1)\big((x_1, y_1), (x_2, y_2)\big) \\ &= (x_1, x_2) \\ &= \pi_1\big((x_1, x_2), (y_1, y_2)\big)\end{aligned}$$

Therefore we have that $(P_1 \times P_1) \circ \eta = \pi_1$. Similarly it can be shown that $(P_2 \times P_2) \circ \eta = \pi_2$.

In order to show the main result we make use of Lemma 5.6.8 to show that

$$(P_1 \times P_1)\big(\eta(\mathcal{F} \times \mathcal{G})\big) = \pi_1(\mathcal{F} \times \mathcal{G}) \geq \mathcal{F}.$$

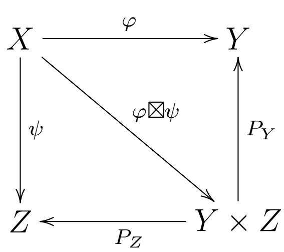
Again, it is easy to show the same result result for $P_2 \times P_2$.

■

Suppose $\varphi : X \longrightarrow Y$ and $\psi : X \longrightarrow Z$. Then we define the following mapping:

$$\varphi \boxtimes \psi : \begin{cases} X \longrightarrow Y \times Z \\ (\varphi \boxtimes \psi)(x) \longmapsto \big(\varphi(x), \psi(x)\big) \end{cases}$$

As a diagram:



Let $x \in X$. Then

$$P_Y \circ (\varphi \boxtimes \psi)(x) = P_Y\big(\varphi(x), \psi(x)\big) = \varphi(x)$$

and

$$P_Z \circ (\varphi \boxtimes \psi)(x) = P_Z\big(\varphi(x), \psi(x)\big) = \psi(x).$$

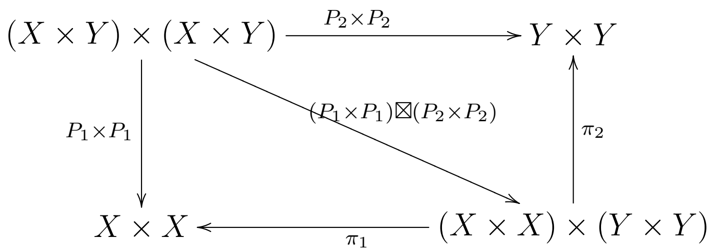
That is, we have that $P_Y \circ (\varphi \boxtimes \psi) = \varphi$ and $P_Z \circ (\varphi \boxtimes \psi) = \psi$.

Our next lemma again holds for $(L, \leq, \otimes, *)$ a pseudo-bisymmetric enriched *cl*-premonoid, and the proof again works without making use of the evaluation of the product L-filter.

Lemma 8.1.4. *Let X and Y be sets and $\mathcal{F} \in \mathcal{F}_L^S\big((X \times Y) \times (X \times Y)\big)$. Then*

$$\eta\Big((P_1 \times P_1)(\mathcal{F}) \times (P_2 \times P_2)(\mathcal{F})\Big) \leq \mathcal{F}.$$

Proof: Consider the situation illustrated below



First, we will show that $\pi_1\big((P_1 \times P_1) \boxtimes (P_2 \times P_2)\big) = P_1 \times P_1$. To do this we let $\big((x_1, y_1), (x_2, y_2)\big) \in (X \times Y) \times (X \times Y)$ and get

$$\begin{aligned}
&\pi_1\big((P_1 \times P_1) \boxtimes (P_2 \times P_2)\big)\big((x_1, y_1), (x_2, y_2)\big) \\
&= \pi_1\big((x_1, x_2), (y_1, y_2)\big) \\
&= (x_1, x_2) \\
&= (P_1 \times P_1)\big((x_1, y_1), (x_2, y_2)\big).
\end{aligned}$$

It can be shown in the same way that $\pi_2\big((P_1 \times P_1) \boxtimes (P_2 \times P_2)\big) = P_2 \times P_2$.

Now we also show that $\eta \circ \big((P_1 \times P_1) \boxtimes (P_2 \times P_2)\big) = id_{(X \times Y) \times (X \times Y)}$. Again, let $\big((x_1, y_1), (x_2, y_2)\big) \in (X \times Y) \times (X \times Y)$. Then

$$\begin{aligned}
&\big[\eta \circ \big((P_1 \times P_1) \boxtimes (P_2 \times P_2)\big)\big]\big((x_1, y_1), (x_2, y_2)\big) \\
&= \eta\big((x_1, x_2), (y_1, y_2)\big) \\
&= \big((x_1, y_1), (x_2, y_2)\big) \\
&= id_{(X \times Y) \times (X \times Y)}\big((x_1, y_1), (x_2, y_2)\big)
\end{aligned}$$

Therefore, for $\mathcal{F} \in \mathcal{F}_L^S\big((X \times Y) \times (X \times Y)\big)$ we have that

$$\mathcal{F} = \big[\eta \circ \big((P_1 \times P_1) \boxtimes (P_2 \times P_2)\big)\big](\mathcal{F}).$$

Further, from above we can say that $\big[\pi_1\big((P_1 \times P_1) \boxtimes (P_2 \times P_2)\big)\big](\mathcal{F}) = (P_1 \times P_1)(\mathcal{F})$.

Using Lemma 5.6.7, the following is true:

$$\begin{aligned}
&(P_1 \times P_1)(\mathcal{F}) \times (P_2 \times P_2)(\mathcal{F}) \\
&= \big[\pi_1\big((P_1 \times P_1) \boxtimes (P_2 \times P_2)\big)\big](\mathcal{F}) \times \big[\pi_2\big((P_1 \times P_1) \boxtimes (P_2 \times P_2)\big)\big](\mathcal{F}) \\
&\leq (P_1 \times P_1) \boxtimes (P_2 \times P_2)(\mathcal{F}).
\end{aligned}$$

Since the mapping η will respect the ordering of the filters

$$\eta\big((P_1 \times P_1)(\mathcal{F}) \times (P_2 \times P_2)(\mathcal{F})\big) \leq \eta\big((P_1 \times P_1) \boxtimes (P_2 \times P_2)(\mathcal{F})\big) = \mathcal{F}.$$

■

8.2 Stratified L-uniform convergence structure on $UC(X,Y)$

Now that we have shown the necessary preliminary results relating to the mapping η we can move towards defining a convergence structure on the morphisms of $SL-UCS$. If $(X,\Lambda),(Y,\Sigma) \in |SL-UCS|$ then we will denote the set of all continuous mappings between the two spaces by:

$$UC = UC(X,Y) = \{\varphi : (X,\Lambda) \longrightarrow (Y,\Sigma), \quad \varphi \text{ is uniformly continuous}\}$$

Before we attempt to define the proposed stratified L-uniform convergence structure, we consider the *evaluation map*:

$$ev : \begin{cases} UC(X,Y) \times X \longrightarrow Y \\ (\varphi, x) \longmapsto \varphi(x) \end{cases}$$

Also we consider the mapping η defined in this case as follows:

$$\eta : \begin{cases} (UC \times UC) \times (X \times X) \longrightarrow (UC \times X) \times (UC \times X) \\ \big((\varphi,\psi),(x_1,x_2)\big) \longmapsto \big((\varphi,x_1),(\psi,x_2)\big) \end{cases}$$

Suppose we have $\Phi \in \mathcal{F}_L^S(UC \times UC)$, then define:

$$\Lambda_C(\Phi) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X \times X)} \bigg(\Lambda(\mathcal{F}) \to \Sigma\Big(ev \times ev\big(\eta(\Phi \times \mathcal{F})\big)\Big)\bigg)$$

The work of Jäger and Burton [27] shows that the definition above will produce a stratified L-uniform convergence structure on $UC \times UC$. In this work, the lattice L was a complete Heyting algebra. Our attempted generalisation to the case of the enriched cl-premonoid fails already in the Lemma 8.2.3.

Proposition 8.2.1. *[27] Let $(L, \leq, \wedge, \wedge)$ be a complete Heyting algebra. Then*

$$\big(UC(X,Y), \Lambda_C\big) \in |SL-UCS|$$

PROOF: In order to prove the axioms **(LUC1)** to **(LUC5)** we will require some additional lemmas along the way. The two lemmas presented immediately below will be used in the proof of **(LUC1)**. The first of these can be proven for the case of $(L, \leq, \otimes, *)$ an enriched cl-premonoid (not necessarily pseudo-bisymmetric), but the second lemma requires evaluation of the product filter and hence we have to revert back to the case of L a complete Heyting algebra.

Lemma 8.2.2. *[27] Let $(L, \leq, \otimes, *)$ be a pseudo-bisymmetric enriched cl-premonoid, and let $\varphi : X \longrightarrow Y$ and $a \in L^{X \times X}$. Then*

$$(\varphi \times \varphi)(a) = (ev \times ev)\big(\eta(\top_{(\varphi,\varphi)} \times a)\big).$$

Proof: Clearly $(\varphi \times \varphi)(a) \in L^{Y \times Y}$, so we let $(y_1, y_2) \in Y \times Y$. Now

$$\begin{aligned}\big((\varphi \times \varphi)(a)\big)(y_1, y_2) &= \bigvee \big\{a(x_1, x_2) \mid (\varphi \times \varphi)(x_1, x_2) = (y_1, y_2)\big\} \\ &= \bigvee \big\{a(x_1, x_2) \mid \varphi(x_1) = y_1, \varphi(x_2) = y_2\big\}.\end{aligned}$$

Considering the other side of the equality, we get

$$\begin{aligned}&(ev \times ev)\big(\eta(\top_{(\varphi,\varphi)} \times a)\big)(y_1, y_2) \\ &= \bigvee \{\big(\eta(\top_{(\varphi,\varphi)} \times a)\big)\big((\psi_1, x_1), (\psi_2, x_2)\big) \mid \psi_1(x_1) = y_1, \psi_2(x_2) = y_2\} \\ &= \bigvee \{(\top_{(\varphi,\varphi)} \times a)\big((\psi_1, \psi_2), (x_1, x_2)\big) \mid \psi_1(x_1) = y_1, \psi_2(x_2) = y_2\} \\ &= \bigvee \{\top_{(\varphi,\varphi)}(\psi_1, \psi_2) * a(x_1, x_2) \mid \psi_1(x_1) = y_1, \psi_2(x_2) = y_2\} \\ &= H.\end{aligned}$$

Since $\top_{(\varphi,\varphi)}(\psi_1, \psi_2) = \begin{cases} \top & \text{if } (\psi_1, \psi_2) = (\varphi, \varphi) \\ \bot & \text{else} \end{cases}$ we have that

$$\begin{aligned}H &= \bigvee \{a(x_1, x_2) \mid \varphi(x_1) = y_1, \varphi(x_2) = y_2\} \\ &= \big((\varphi \times \varphi)(a)\big)(y_1, y_2).\end{aligned}$$

■

Lemma 8.2.3. *[27] Let X and Y be sets, $\varphi : X \longrightarrow Y$ and $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$. Then*

$$(ev \times ev)\Big(\eta\big(([\varphi] \times [\varphi]) \times \mathcal{F}\big)\Big) \geq (\varphi \times \varphi)(\mathcal{F}).$$

Proof: Let $b \in L^{Y \times Y}$. By definition of the image of a stratified L-filter we have:

$$\big((\varphi \times \varphi)(\mathcal{F})\big)(b) = \bigvee_{\substack{a \in L^{X \times X}: \\ (\varphi \times \varphi)(a) \leq b}} \mathcal{F}(a).$$

Now using Lemma 8.2.2 from above:

$$\begin{aligned}\big((\varphi \times \varphi)(\mathcal{F})\big)(b) &= \bigvee_{\substack{a \in L^{X \times X}: \\ ev \times ev\left(\eta(\top_{(\varphi,\varphi)} \times a)\right) \leq b}} \big([\varphi] \times [\varphi]\big)(\top_{(\varphi,\varphi)}) \wedge \mathcal{F}(a) \\ &\leq \bigvee_{\substack{a \in L^{X \times X}, \phi \in L^{UC \times UC}: \\ \phi \times a \leq \eta^{\leftarrow}\left((ev \times ev)^{\leftarrow}(b)\right)}} \big([\varphi] \times [\varphi]\big)(\phi) \wedge \mathcal{F}(a) \\ &= \big(([\varphi] \times [\varphi]) \times \mathcal{F}\big)\Big(\eta^{\leftarrow}\big((ev \times ev)^{\leftarrow}(b)\big)\Big) \\ &= ev \times ev\Big(\eta\big(([\varphi] \times [\varphi]) \times \mathcal{F}\big)\Big)(b).\end{aligned}$$

■

Using Lemmas 8.2.2 and 8.2.3 we are able to prove property **(LUC1)** by:

$$\begin{aligned}\Lambda_C\big([(\varphi,\varphi)]\big) &= \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Lambda(\mathcal{F}) \to \Sigma\Big(ev\times ev\Big(\eta\big(([x]\times[x])\times\mathcal{F}\big)\Big)\Big)\\ &\geq \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Big(\Lambda(\mathcal{F}) \to \Sigma\big((\varphi\times\varphi)(\mathcal{F})\big)\Big).\end{aligned}$$

Now, since $\varphi \in UC(X,Y)$, we have from the definition of a uniformly continuous map that $\Lambda(\mathcal{F}) \leq \Sigma\big((\varphi\times\varphi)(\mathcal{F})\big)$ and thus

$$\Lambda_C\big([(\varphi,\varphi)]\big) \geq \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Big(\Lambda(\mathcal{F}) \to \Sigma\big((\varphi\times\varphi)(\mathcal{F})\big)\Big) = \top.$$

To prove **(LUC2)** we use Lemma 5.6.5. Suppose that $\Phi,\Psi\in\mathcal{F}_L^S(UC\times UC)$, with $\Phi\leq\Psi$. From 5.6.5 we get that $\Phi\times\mathcal{F}\leq\Psi\times\mathcal{F}$ and so $\Sigma\Big(ev\times ev\big(\eta(\Phi\times\mathcal{F})\big)\Big) \leq \Sigma\Big(ev\times ev\big(\eta(\Psi\times\mathcal{F})\big)\Big)$. Finally this will give

$$\begin{aligned}\Lambda_C(\Phi) &= \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Lambda(\mathcal{F}) \to \Sigma\Big(ev\times ev\big(\eta(\Phi\times\mathcal{F})\big)\Big)\\ &\leq \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Lambda(\mathcal{F}) \to \Sigma\Big(ev\times ev\big(\eta(\Psi\times\mathcal{F})\big)\Big)\\ &= \Lambda_C(\Psi).\end{aligned}$$

We use the result of Lemma 8.1.2 to prove **(LUC3)**.

$$\begin{aligned}\Lambda_C(\Phi^{-1}) &= \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Lambda(\mathcal{F}) \to \Sigma\Big(ev\times ev\big(\eta(\Phi^{-1}\times\mathcal{F})\big)\Big)\\ &= \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Lambda(\mathcal{F}) \to \Sigma\Big(ev\times ev\Big(\big(\eta(\Phi\times\mathcal{F}^{-1})\big)^{-1}\Big)\Big)\\ &\geq \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Lambda(\mathcal{F}^{-1}) \to \Sigma\Big(ev\times ev\Big(\big(\eta(\Phi\times\mathcal{F}^{-1})\big)^{-1}\Big)\Big)\\ &= \bigwedge_{\mathcal{G}\in\mathcal{F}_L^S(X\times X)} \Lambda(\mathcal{G}) \to \Sigma\Big(ev\times ev\Big(\big(\eta(\Phi\times\mathcal{G})\big)^{-1}\Big)\Big)\\ &\geq \bigwedge_{\mathcal{G}\in\mathcal{F}_L^S(X\times X)} \Lambda(\mathcal{G}) \to \Sigma\Big(ev\times ev\big(\eta(\Phi\times\mathcal{G})\big)\Big)\\ &= \Lambda_C(\Phi).\end{aligned}$$

We use Lemma 5.6.6 to prove **(LUC4)**.

$$\begin{aligned}
&\Lambda_C(\Phi \wedge \Psi)\\
&= \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Lambda(\mathcal{F}) \to \Sigma\Big(ev \times ev\Big(\eta\big((\Phi\wedge\Psi)\times\mathcal{F}\big)\Big)\Big)\\
&= \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Lambda(\mathcal{F}) \to \Sigma\Big(ev \times ev\Big(\eta\big((\Phi\times\mathcal{F})\wedge(\Psi\times\mathcal{F})\big)\Big)\Big)\\
&= \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Lambda(\mathcal{F}) \to \Sigma\Big(ev \times ev\big(\eta(\Phi\times\mathcal{F})\big) \wedge ev\times ev\big(\eta(\Psi\times\mathcal{F})\big)\Big)\\
&\geq \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Lambda(\mathcal{F}) \to \Big(\Sigma\Big(ev \times ev\big(\eta(\Phi\times\mathcal{F})\big)\Big) \wedge \Sigma\Big(ev\times ev\big(\eta(\Psi\times\mathcal{F})\big)\Big)\Big)\\
&= \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Bigg[\Big(\Lambda(\mathcal{F}) \to \Sigma\Big(ev \times ev\big(\eta(\Phi\times\mathcal{F})\big)\Big)\Big)\\
&\qquad\qquad \wedge \Big(\Lambda(\mathcal{F}) \to \Sigma\Big(ev \times ev\big(\eta(\Psi\times\mathcal{F})\big)\Big)\Big)\Bigg]\\
&\geq \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Big(\Lambda(\mathcal{F}) \to \Sigma\Big(ev \times ev\big(\eta(\Phi\times\mathcal{F})\big)\Big)\Big)\\
&\qquad\qquad \wedge \bigwedge_{\mathcal{G}\in\mathcal{F}_L^S(X\times X)} \Big(\Lambda(\mathcal{G}) \to \Sigma\Big(ev \times ev\big(\eta(\Psi\times\mathcal{G})\big)\Big)\Big)\\
&= \Lambda_C(\Phi)\wedge\Lambda_C(\Psi).
\end{aligned}$$

Before being able to show **(LUC5)** we must equip ourselves with two further technical lemmas.

Lemma 8.2.4. *[27] Let $\phi, \theta \in L^{UC\times UC}$ and let $f \in L^{X\times X}$. Then*

$$(ev\times ev)\Big(\eta\big((\phi\circ\theta)\times f\big)\Big) \leq \Big(ev\times ev\big(\eta(\phi\times f)\big)\Big)\circ\Big(ev\times ev\Big(\eta\big(\theta\times(f^{-1}\circ f)\big)\Big)\Big).$$

Proof: We let $(u,v)\in Y\times Y$.

$$ev\times ev\Big(\eta\big((\phi\circ\theta)\times f\big)\Big)(u,v) = \bigvee_{\substack{(\varphi,x),(\sigma,y):\\ \varphi(x)=u,\sigma(y)=v}} \big((\phi\circ\theta)(\varphi,\sigma)\wedge f(x,y)\big).$$

Now we use the fact that $f(x,y) \leq (f^{-1}\circ f)(y,y)$ to get the right hand side above

$$= \bigvee_{\substack{(\varphi,x),(\sigma,y):\\ \varphi(x)=u,\sigma(y)=v}} \Big(\Big(\bigvee_{\rho\in UC}\phi(\varphi,\rho)\wedge\theta(\rho,\sigma)\Big)\wedge f(x,y)\wedge(f^{-1}\circ f)(y,y)\Big).$$

From the distributivity of the $\wedge$ operation over arbitrary joins, we get the above

$$= \bigvee_{\substack{(\varphi,x),(\sigma,y):\\ \varphi(x)=u,\sigma(y)=v}} \left(\bigvee_{\rho\in UC} \Big(\phi(\varphi,\rho) \wedge \theta(\rho,\sigma) \wedge f(x,y) \wedge (f^{-1}\circ f)(y,y)\Big)\right).$$

Now we fix $z \in X$ and define for $\rho \in UC(X,Y)$, $w_\rho = \rho(z)$ to give

$$\begin{aligned}&\phi(\varphi,\rho) \wedge \theta(\rho,\sigma) \wedge f(x,y) \wedge f^{-1}\circ f(y,y)\\ &\leq \bigvee_{\substack{\mu,\nu\in UC,s,t\in X:\\ \mu(s)=w_\rho,\nu(t)=w_\rho}} \phi(\varphi,\mu) \wedge \theta(\nu,\sigma) \wedge f(x,s) \wedge f^{-1}\circ f(t,y).\end{aligned}$$

From this we can show

$$\begin{aligned}&ev \times ev\Big(\eta\big((\phi\circ\theta)\times f\big)\Big)(u,v)\\ &\leq \bigvee_{\rho\in UC}\left(\left(\bigvee_{\substack{(\varphi,x),(\mu,s):\\ \varphi(x)=u,\mu(s)=w_\rho}} \big(\phi(\varphi,\mu)\wedge f(x,s)\big)\right) \wedge \left(\bigvee_{\substack{(\sigma,y),(\nu,t):\\ \sigma(y)=v,\nu(t)=w_\rho}} \big(\theta(\nu,\sigma)\wedge f^{-1}\circ f(t,y)\big)\right)\right)\\ &= \bigvee_{\rho\in UC}\left(\Big(ev\times ev\big(\eta(\phi\times f)\big)(u,w_\rho)\Big) \wedge \Big(ev\times ev\big(\eta(\theta\times(f^{-1}\circ f))\big)(w_\rho,v)\Big)\right)\\ &\leq \Big(ev\times ev\big(\eta(\phi\times f)\big)\Big)\circ\Big(ev\times ev\Big(\eta\big(\theta\times(f^{-1}\circ f)\big)\Big)\Big)(u,v).\end{aligned}$$

■

Lemma 8.2.5. *[27] Let $\Phi,\Psi \in \mathcal{F}_L^S(UC\times UC)$ and let $\mathcal{F}\in\mathcal{F}_L^S(X\times X)$. If $\Phi\circ\Psi$ exists then*

$$\Big(ev\times ev\big(\eta(\Phi\times\mathcal{F})\big)\Big)\circ\Big(ev\times ev\Big(\eta\big(\Psi\times(\mathcal{F}^{-1}\circ\mathcal{F})\big)\Big)\Big)$$

exists and

$$\Big(ev\times ev\big(\eta(\Phi\times\mathcal{F})\big)\Big)\circ\Big(ev\times ev\Big(\eta\big(\Psi\times(\mathcal{F}^{-1}\circ\mathcal{F})\big)\Big)\Big) \leq (ev\times ev)\Big(\eta\big((\Phi\circ\Psi)\times\mathcal{F}\big)\Big).$$

Proof: Let $d \in L^{Y\times Y}$ and let $b\circ c \leq d$.

$$\begin{aligned}&\Big(ev\times ev\big(\eta(\Phi\times\mathcal{F})\big)\Big)(b) \wedge \Big(ev\times ev\big(\eta(\Psi\times(\mathcal{F}^{-1}\circ\mathcal{F}))\big)\Big)(c)\\ &= \bigvee_{\substack{\phi,f,\psi,g:\\ ev\times ev\left(\eta(\phi\times f)\right)\leq b\\ ev\times ev\left(\eta(\psi\times g)\right)\leq c}} \Big(\Phi(\phi)\wedge\mathcal{F}(f)\wedge\Psi(\psi)\wedge(\mathcal{F}^{-1}\circ\mathcal{F})(g)\Big)\\ &= \bigvee_{\substack{\phi,f,\psi,g:\\ ev\times ev\left(\eta(\phi\times f)\right)\leq b\\ ev\times ev\left(\eta(\psi\times g)\right)\leq c}} \left(\Phi(\phi)\wedge\mathcal{F}(f)\wedge\Psi(\psi)\wedge\left(\bigvee_{\substack{k,j:\\ k^{-1}\circ j\leq g}}\big(\mathcal{F}(k)\wedge\mathcal{F}(j)\big)\right)\right)\end{aligned}$$

$$= \bigvee_{\substack{\phi,f,\psi,k,j:\\ ev\times ev\left(\eta(\phi\times f)\right)\leq b\\ ev\times ev\left(\eta(\psi\times(k^{-1}\circ j))\right)\leq c}} \Big(\Phi(\phi)\wedge\mathcal{F}(f)\wedge\Psi(\psi)\wedge\mathcal{F}(f\wedge k\wedge j)\Big)$$

$$= \bigvee_{\substack{\phi,\psi,h:\\ ev\times ev\left(\eta(\phi\times h)\right)\leq b\\ ev\times ev\left(\eta(\psi\times(h^{-1}\circ h))\right)\leq c}} \Big(\Phi(\phi)\wedge\mathcal{F}(f)\wedge\Psi(\psi)\wedge\mathcal{F}(h)\Big)$$

$$= Q.$$

Now we use Lemma 8.2.4 to show that

$$Q \leq \bigvee_{\substack{\phi,\psi,h:\\ ev\times ev\left(\eta\left((\phi\circ\psi)\times h\right)\right)\leq b}} \Big(\Phi(\phi)\wedge\Psi(\psi)\wedge\mathcal{F}(\mathcal{H})\Big)$$

$$= \bigvee_{\substack{\Gamma,h:\\ ev\times ev\left(\eta(\Gamma\times h)\right)\leq b\circ c}} \Bigg(\bigvee_{\substack{\phi,\psi:\\ \phi\circ\psi\leq\Gamma}} \big(\Phi(\phi)\wedge\Psi(\psi)\big)\Bigg)\wedge\mathcal{F}(h)$$

$$= \bigvee_{\substack{\Gamma,h:\\ ev\times ev\left(\eta(\Gamma\times h)\right)\leq b\circ c}} \Phi\circ\Psi(\Gamma)\wedge\mathcal{F}(h)$$

$$= \Big(ev\times ev\big(\eta(\Phi\circ\Psi\times\mathcal{F})\big)\Big)(b\circ c).$$

With the inequality that we have just produced for an arbitrary $d \in L^{Y\times Y}$ we show:

$$\Big(ev\times ev\big(\eta(\Phi\times\mathcal{F})\big)\Big)\circ\Big(ev\times ev\Big(\eta\big(\Psi\times(\mathcal{F}^{-1}\circ\mathcal{F})\big)\Big)\Big)(d)$$

$$= \bigvee_{b\circ c\leq d} ev\times ev\big(\eta(\Phi\times\mathcal{F})\big)(b)\wedge ev\times ev\Big(\eta\big(\Psi\times(\mathcal{F}^{-1}\circ\mathcal{F})\big)\Big)(c)$$

$$\leq \bigvee_{b\circ c\leq d} ev\times ev\big(\eta(\Phi\circ\Psi\times\mathcal{F})\big)(b\circ c)$$

$$=(ev\times ev)\Big(\eta\big((\Phi\circ\Psi)\times\mathcal{F}\big)\Big)(d).$$

Since $\Phi\circ\Psi$ is a stratified L-filter, we have that $ev\times ev\Big(\eta\big((\Phi\circ\Psi)\times\mathcal{F}\big)\Big)$ is a stratified L-filter. Now if we take $d=\perp_{Y\times Y}$ then since $ev\times ev\Big(\eta\big((\Phi\circ\Psi)\times\mathcal{F}\big)\Big)$ is a stratified L-filter, we have by Lemma 5.8.5 that $\Big(ev \times ev\big(\eta(\Phi \circ \Psi \times \mathcal{F})\big)\Big)(d) = \bot$. Therefore, from the inequality that we have just proven, we get that

$$\Big(ev \times ev\big(\eta(\Phi \times \mathcal{F})\big)\Big) \circ \Big(ev \times ev\Big(\eta\big(\Psi \times (\mathcal{F}^{-1} \circ \mathcal{F})\big)\Big)\Big)(d) = \bot,$$

and so this is also a stratified L-filter.

■

Finally, we are ready to prove **(LUC5)**. Let $\Phi \circ \Psi$ exist.

$$\begin{aligned}
&\Lambda_C(\Phi \circ \Psi) \\
&= \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X \times X)} \Lambda(\mathcal{F}) \to \Sigma\Big(ev \times ev\Big(\eta\big((\Phi \circ \Psi) \times \mathcal{F}\big)\Big)\Big) \\
&\geq \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X \times X)} \Lambda(\mathcal{F}) \to \Sigma\Big(\Big(ev \times ev\big(\eta(\Phi \circ \mathcal{F})\big)\Big) \circ \Big(ev \times ev\big(\eta(\Psi \times (\mathcal{F}^{-1} \circ \mathcal{F}))\big)\Big)\Big) \\
&= P.
\end{aligned}$$

Now since Σ is a stratified L-uniform convergence structure, by **(LUC5)** we get

$$\begin{aligned}
P \geq & \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X \times X)} \Lambda(\mathcal{F}) \to \Big[\Sigma\Big(ev \times ev\big(\eta(\Phi \circ \mathcal{F})\big)\Big) \\
& \qquad\qquad \wedge \Sigma\Big(ev \times ev\big(\eta(\Psi \times (\mathcal{F}^{-1} \circ \mathcal{F}))\big)\Big)\Big] \\
= & \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X \times X)} \Big[\Big(\Lambda(\mathcal{F}) \to \Sigma\Big(ev \times ev\big(\eta(\Phi \circ \mathcal{F})\big)\Big)\Big) \\
& \qquad\qquad \wedge \Big(\Lambda(\mathcal{F}) \to \Sigma\Big(ev \times ev\big(\eta(\Psi \times (\mathcal{F}^{-1} \circ \mathcal{F}))\big)\Big)\Big)\Big] \\
\geq & \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X \times X)} \Lambda(\mathcal{F}) \to \Sigma\Big(ev \times ev\big(\eta(\Phi \circ \mathcal{F})\big)\Big) \\
& \qquad\qquad \wedge \bigwedge_{\mathcal{G} \in \mathcal{F}_L^S(X \times X)} \Lambda(\mathcal{G}) \to \Sigma\Big(ev \times ev\big(\eta(\Psi \times (\mathcal{G}^{-1} \circ \mathcal{G}))\big)\Big) \\
\geq & \Lambda_C(\Phi) \wedge \Lambda_C(\Psi).
\end{aligned}$$

■

We have now shown that the proposed stratified L-uniform convergence structure on $UC(X, Y)$ did in fact satisfy the required properties. Now we move on to further results with the goal of showing cartesian closedness.

Proposition 8.2.6. *[27] Let $(X, \Lambda), (Y, \Sigma) \in |SL - UCS|$. Then the evaluation map*

$$ev : \big(UC(X,Y) \times X, \Lambda_C \times \Lambda) \longrightarrow (Y, \Sigma)$$

is uniformly continuous.

PROOF: Let $\mathcal{H} \in \mathcal{F}_L^S\big((UC \times X) \times (UC \times X)\big)$. Note that here the projections are defined as $P_1 : UC \times X \longrightarrow UC$ and $P_2 : UC \times X \longrightarrow X$. Then

$$\begin{aligned}
&(\Lambda_C \times \Lambda)(\mathcal{H}) \\
&= \Lambda_C\big(P_1 \times P_1(\mathcal{H})\big) \wedge \Lambda\big(P_2 \times P_2(\mathcal{H})\big) \\
&\leq \Bigg(\Lambda\big(P_2 \times P_2(\mathcal{H})\big) \to \Sigma\Big(ev \times ev\big(\eta((P_1 \times P_1)(\mathcal{H}) \times (P_2 \times P_2)(\mathcal{H}))\big)\Big)\Bigg) \\
&\qquad\qquad\qquad\qquad \wedge \Lambda\big(P_2 \times P_2(\mathcal{H})\big) \\
&\leq \Sigma\Bigg(ev \times ev\Big(\eta\big((P_1 \times P_1)(\mathcal{H}) \times (P_2 \times P_2)(\mathcal{H})\big)\Big)\Bigg).
\end{aligned}$$

For the second inequality we use the fact that for $\alpha, \beta \in L$, $\big((\alpha \to \beta) \wedge \alpha\big) \leq \beta$. Finally, we use the result of Lemma 8.1.4 to get

$$(\Lambda_C \times \Lambda)(\mathcal{H}) \leq \Sigma\big(ev \times ev(\mathcal{H})\big).$$

■

We now define two further mappings. Let $\varphi : X \times Y \longrightarrow Z$ and let $x \in X$. Define $\varphi_x : Y \longrightarrow Z$ by $\varphi_x(y) = \varphi(x, y)$. Now we define $\varphi^* : X \longrightarrow Z^Y, x \longmapsto \varphi_x$. The mapping:

$$E : \begin{cases} Z^{X \times Y} & \longrightarrow (Z^Y)^X \\ \varphi & \longmapsto \varphi^* \end{cases}$$

is known as the *exponential map* (see Definition 1.2.1).

Lemma 8.2.7. *[27] Let $\varphi : X \times Y \longrightarrow Z$ and let $x \in X$. Further, let $\mathcal{F} \in \mathcal{F}_L^S(Y \times Y)$. Then*

$$\varphi_x \times \varphi_x(\mathcal{F}) \geq (\varphi \times \varphi)\Big(\eta\big(([x] \times [x]) \times \mathcal{F}\big)\Big).$$

PROOF: Let $a \in L^{Z \times Z}$. Then

$$(\varphi \times \varphi)\Big(\eta\big(([x] \times [x]) \times \mathcal{F}\big)\Big)(a) = \bigvee_{\substack{a_1 \in L^{X \times X}, a_2 \in L^{Y \times Y}: \\ a_1 \times a_2 \leq \eta^{\leftarrow}\big((\varphi \times \varphi)^{\leftarrow}(a)\big)}} a_1(x, x) \wedge \mathcal{F}(a_2).$$

Since $\mathcal{F}$ is a stratified L-filter, we use **(LFS)** to show that the right hand side of the equality above

$$\leq \bigvee_{\substack{a_1 \in L^{X\times X}, a_2 \in L^{Y\times Y}: \\ a_1 \times a_2 \leq \eta^{\leftarrow}\left((\varphi\times\varphi)^{\leftarrow}(a)\right)}} \mathcal{F}\big(a_1(x,x) \wedge a_2\big).$$

We now need to show that for $a_1 \times a_2 \leq \eta^{\leftarrow}\big((\varphi \times \varphi)^{\leftarrow}(a)\big)$ it will follow that $a_1(x,x) \wedge a_2 \leq (\varphi_x \times \varphi_x)^{\leftarrow}(a)$. For $(y_1, y_2) \in Y \times Y$,

$$\begin{aligned} a_1(x,x) \wedge a_2(y_1,y_2) &= a_1 \times a_2\big((x,x),(y_1,y_2)\big) \\ &\leq \eta^{\leftarrow}\big((\varphi\times\varphi)^{\leftarrow}(a)\big)\big((x,x),(y_1,y_2)\big) \\ &= a\big(\varphi(x,y_1), \varphi(x,y_2)\big) \\ &= a\big(\varphi_x(y_1), \varphi_x(y_2)\big) \\ &= (\varphi_x \times \varphi_x)^{\leftarrow}(a)(y_1,y_2). \end{aligned}$$

Therefore we have that

$$(\varphi\times\varphi)\Big(\eta\big(([x]\times[x])\times\mathcal{F}\big)\Big)(a) \leq \mathcal{F}\big((\varphi_x\times\varphi_x)^{\leftarrow}(a)\big) = (\varphi_x\times\varphi_x)(\mathcal{F})(a).$$

■

Lemma 8.2.8. *[27] Let $\varphi : (X,\Lambda)\times(Y,\Sigma) \longrightarrow (Z,\Gamma)$ be uniformly continuous. Then for $x \in X$, the mapping $\varphi_x : (Y,\Sigma) \longrightarrow (Z,\Gamma)$ is also uniformly continuous.*

Proof: Let $\mathcal{F} \in \mathcal{F}_L^S(Y\times Y)$. Using both **(UC2)** and Lemma 8.2.7 we get

$$\Gamma\big(\varphi_x\times\varphi_x(\mathcal{F})\big) \geq \Gamma\bigg(\varphi\times\varphi\Big(\eta\big(([\varphi]\times[\varphi])\times\mathcal{F}\big)\Big)\bigg).$$

By the uniform continuity of φ we get that

$$\begin{aligned} &\Gamma\bigg(\varphi\times\varphi\Big(\eta\big(([\varphi]\times[\varphi])\times\mathcal{F}\big)\Big)\bigg) \\ &\geq \Lambda\times\Sigma\Big(\eta\big(([x]\times[x])\times\mathcal{F}\big)\Big) \\ &= \Lambda\bigg(P_1\times P_1\Big(\eta\big(([x]\times[x])\times\mathcal{F}\big)\Big)\bigg)\wedge\Sigma\bigg(P_2\times P_2\Big(\eta\big(([x]\times[x])\times\mathcal{F}\big)\Big)\bigg) \\ &\geq \Lambda([x]\times[x])\wedge\Sigma(\mathcal{F}) = \Sigma(\mathcal{F}). \end{aligned}$$

This last step uses Lemma 8.1.3.

■

We now present two technical results that are needed in proving our final lemma. The first of these can be shown in the more general setting of L a pseudo-bisymmetric enriched cl-premonoid.

Lemma 8.2.9. *Let X, Y and Z be sets, $\varphi : X \times Y \longrightarrow Z$ and let $(L, \leq, \otimes, *)$ be a pseudo-bisymmetric enriched cl-premonoid. Let $f \in L^{X\times X}, g \in L^{Y\times Y}$. Then*

$$ev \times ev\Big(\eta\Big(\big(E(\varphi) \times E(\varphi)\big)(f) \times g\Big)\Big) = \varphi \times \varphi\big(\eta(f \times g)\big).$$

Proof: Clearly both of the L-sets above are elements of $L^{Z\times Z}$. Let $(z_1, z_2) \in Z\times Z$. For the left hand side:

$$ev \times ev\Big(\eta\big(E(\varphi) \times E(\varphi)(f) \times g\big)\Big)(z_1, z_2)$$

$$= \bigvee_{\substack{\big((\phi_1,y_1),(\phi_2,y_2)\big) \\ \in (Z^Y\times Y) \\ \times (Z^Y\times Y)}} \Big\{\eta\big(E(\varphi) \times E(\varphi)(f) \times g\big)\big((\phi_1, y_1), (\phi_2, y_2)\big) \ : \ ev \times ev\big((\phi_1, y_1), (\phi_2, y_2)\big) = (z_1, z_2)\Big\}$$

$$= \bigvee_{\substack{\big((\phi_1,y_1),(\phi_2,y_2)\big) \\ \in (Z^Y\times Y) \\ \times (Z^Y\times Y)}} \Big\{\eta\big(E(\varphi) \times E(\varphi)(f) \times g\big)\big((\phi_1, y_1), (\phi_2, y_2)\big) \ : \ \phi_1(y_1) = z_1, \phi_2(y_2) = z_2\Big\}$$

$$= \bigvee_{\substack{\big((\phi_1,\phi_2),(y_1,y_2)\big) \\ \in (Z^Y\times Z^Y) \\ \times (Y\times Y)}} \Big\{\big(E(\varphi) \times E(\varphi)(f) \times g\big)\big((\phi_1, \phi_2), (y_1, y_2)\big) \ : \ \phi_1(y_1) = z_1, \phi_2(y_2) = z_2\Big\}$$

$$= \bigvee_{\substack{\big((\phi_1,\phi_2),(y_1,y_2)\big) \\ \in (Z^Y\times Z^Y) \\ \times (Y\times Y)}} \Big\{\Big\{\bigvee_{\substack{(x_1,x_2) \\ \in X\times X}} (f \times g)\big((x_1, x_2), (y_1, y_2)\big) \ : \ E(\varphi) \times E(\varphi)(x_1, x_2) = (\phi_1, \phi_2)\Big\} \ : \ \phi_1(y_1) = z_1, \phi_2(y_2) = z_2\Big\}$$

$$= \bigvee_{\substack{\big((\phi_1,\phi_2),(y_1,y_2)\big) \\ \in (Z^Y\times Z^Y) \\ \times (Y\times Y)}} \Big\{\Big\{\bigvee_{\substack{(x_1,x_2) \\ \in X\times X}} (f \times g)\big((x_1, x_2), (y_1, y_2)\big) \ : \ \big(\varphi^*(x_1), \varphi^*(x_2)\big) = (\phi_1, \phi_2)\Big\} \ : \ \phi_1(y_1) = z_1, \phi_2(y_2) = z_2\Big\}$$

$$= \bigvee_{\substack{\left((\phi_1,\phi_2),(y_1,y_2)\right)\\ \in (Z^Y\times Z^Y)\\ \times(Y\times Y)}} \left\{ \left\{ \bigvee_{\substack{(x_1,x_2)\\ \in X\times X}} (f \times g)\big((x_1,x_2),(y_1,y_2)\big) \ : \ (\varphi_{x_1},\varphi_{x_2}) = (\phi_1,\phi_2) \right\} \ : \right.$$

$$\left. \phi_1(y_1) = z_1, \phi_2(y_2) = z_2 \right\}$$

$$= \bigvee_{\substack{\left((x_1,y_1),(x_2,y_2)\right)\\ \in(X\times Y)\times(X\times Y)}} \left\{ (f \times g)\big((x_1,x_2),(y_1,y_2)\big) \ : \ \varphi_{x_1}(y_1) = z_1, \varphi_{x_2}(y_2) = z_2 \right\}$$

$$= \bigvee_{\substack{\left((x_1,y_1),(x_2,y_2)\right)\\ \in(X\times Y)\times(X\times Y)}} \left\{ (f \times g)\big((x_1,x_2),(y_1,y_2)\big) \ : \ \varphi(x_1,y_1) = z_1, \varphi(x_2,y_2) = z_2 \right\}$$

$$= \bigvee_{\substack{\left((x_1,y_1),(x_2,y_2)\right)\\ \in(X\times Y)\times(X\times Y)}} \left\{ (f \times g)\big((x_1,x_2),(y_1,y_2)\big) \ : \ \varphi \times \varphi\big((x_1,y_1),(x_2,y_2)\big) = (z_1,z_2) \right\}.$$

Now for the right hand side we get

$$(\varphi \times \varphi)\big(\eta(f \times g)\big)(z_1,z_2)$$

$$= \bigvee_{\substack{\left((x_1,y_1),(x_2,y_2)\right)\\ \in(X\times Y)\times(X\times Y)}} \left\{ \eta(f \times g)\big((x_1,y_1),(x_2,y_2)\big) \ : \ (\varphi \times \varphi)\big((x_1,y_1),(x_2,y_2)\big) = (z_1,z_2) \right\}$$

$$= \bigvee_{\substack{\left((x_1,y_1),(x_2,y_2)\right)\\ \in(X\times Y)\times(X\times Y)}} \left\{ (f \times g)\big((x_1,x_2),(y_1,y_2)\big) \ : \ (\varphi \times \varphi)\big((x_1,y_1),(x_2,y_2)\big) = (z_1,z_2) \right\}.$$

■

Lemma 8.2.10. *[27] Let L be a complete Heyting algebra and let $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$ and $\mathcal{G} \in \mathcal{F}_L^S(Y \times Y)$ then*

$$ev \times ev\bigg(\eta\Big(\big(E(\varphi) \times E(\varphi)\big)(\mathcal{F}) \times \mathcal{G}\Big)\bigg) = \varphi \times \varphi\big(\eta(\mathcal{F} \times \mathcal{G})\big).$$

PROOF: For $c \in L^{Z\times Z}$,

$$
\begin{aligned}
& ev \times ev\Big(\eta\big(E(\varphi) \times E(\varphi)(\mathcal{F}) \times \mathcal{G}\big)\Big)(c) \\
&= \bigvee_{\substack{\phi \in L^{UC\times UC}, g \in L^{Y\times Y}: \\ ev\times ev\left(\eta(\phi\times g)\right)\le c}} \mathcal{F}\Big(\big(E(\varphi) \times E(\varphi)\big)^{\leftarrow}(\phi)\Big) \wedge \mathcal{G}(g) \\
&= \bigvee_{\substack{\phi \in L^{UC\times UC}, g \in L^{Y\times Y}: \\ ev\times ev\left(\eta(\phi\times g)\right)\le c}} \bigvee_{\substack{f \in L^{X\times X}: \\ E(\varphi)\times E(\varphi)(f)\le \phi}} \mathcal{F}(f) \wedge \mathcal{G}(g) \\
&= \bigvee_{\substack{f \in L^{Xx\times X}, g \in L^{Y\times Y}: \\ ev\times ev\left(\eta\left((E(\varphi)\times E(\varphi)(f))\times g\right)\right)\le c}} \mathcal{F}(f) \wedge \mathcal{G}(g) \\
&= \bigvee_{\substack{f \in L^{Xx\times X}, g \in L^{Y\times Y}: \\ \varphi\times\varphi\left(\eta(f\times g)\right)\le c}} \mathcal{F}(f) \wedge \mathcal{G}(g) \\
&= \mathcal{F} \times \mathcal{G}\Big(\eta^{\leftarrow}\big((\varphi \times \varphi)^{\leftarrow}(c)\big)\Big) \\
&= \varphi \times \varphi\big(\eta(\mathcal{F} \times \mathcal{G})\big)(c).
\end{aligned}
$$

■

Lemma 8.2.11. *[27] Let L be a complete Heyting algebra and let $(X,\Lambda),(Y,\Sigma)$ and $(Z,\Gamma) \in |SL-UCS|$. If $\varphi : (X \times Y, \Lambda \times \Sigma) \longrightarrow (Z,\Gamma)$ is uniformly continuous, then $E(\varphi) : (X,\Lambda) \longrightarrow \big(UC(Y,Z),\Lambda_C\big)$ is also uniformly continuous.*

PROOF: If we let $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$ we will have as a consequence of the definition of Λ_C that

$$\Lambda_C\Big(\big(E(\varphi)\times E(\varphi)\big)(\mathcal{F})\Big) = \bigwedge_{\mathcal{G}\in\mathcal{F}_L^S(Y\times Y)} \Sigma(\mathcal{G}) \to \Gamma\Bigg(ev\times ev\bigg(\eta\Big(\big((E(\varphi)\times E(\varphi))(\mathcal{F})\big)\times\mathcal{G}\Big)\bigg)\Bigg).$$

From the result of Lemma 8.2.10 we can see that

$$\Lambda_C\Big(\big(E(\varphi) \times E(\varphi)\big)(\mathcal{F})\Big) = \bigwedge_{\mathcal{G}\in\mathcal{F}_L^S(Y\times Y)} \Sigma(\mathcal{G}) \to \Gamma\Big((\varphi \times \varphi)\big(\eta(\mathcal{F} \times \mathcal{G})\big)\Big).$$

Since φ is uniformly continuous

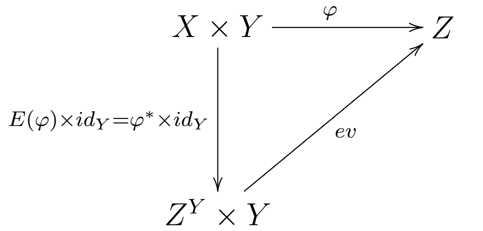
$$\begin{aligned}
&\bigwedge_{\mathcal{G}\in\mathcal{F}_L^S(Y\times Y)} \Sigma(\mathcal{G}) \to \Gamma\Big((\varphi\times\varphi)\big(\eta(\mathcal{F}\times\mathcal{G})\big)\Big) \\
&\geq \bigwedge_{\mathcal{G}\in\mathcal{F}_L^S(Y\times Y)} \Sigma(\mathcal{G}) \to \Lambda\times\Sigma\big(\eta(\mathcal{F}\times\mathcal{G})\big) \\
&= \bigwedge_{\mathcal{G}\in\mathcal{F}_L^S(Y\times Y)} \Sigma(\mathcal{G}) \to \Bigg(\Lambda\Big(P_1\times P_1\big(\eta(\mathcal{F}\times\mathcal{G})\big)\Big)\wedge\Sigma\Big(P_2\times P_2\big(\eta(\mathcal{F}\times\mathcal{G})\big)\Big)\Bigg) \\
&\geq \bigwedge_{\mathcal{G}\in\mathcal{F}_L^S(Y\times Y)} \Sigma(\mathcal{G}) \to \big(\Lambda(\mathcal{F})\wedge\Sigma(\mathcal{G})\big) \\
&\geq \Lambda(\mathcal{F}).
\end{aligned}$$

■

Proposition 8.2.12. *[27] Let L be a complete Heyting algebra. Then $SL-UCS$ is cartesian closed.*

PROOF: Let $(X,\Lambda),(Y,\Sigma)$ and $(Z,\Gamma)\in|SL-UCS|$. We have already shown in 7.2.2 that finite products exist and in 8.2.1 that we can define a stratified L-uniform convergence structure on the set of morphisms from one $SL-UCS$ object to another.

We claim that ev defined in 8.2 will satisfy the conditions of the evaluation map required in the definition of cartesian closedness (see 1.2.1). To show this, we let $\varphi\in UC(X\times Y,Z)$. We have just shown that $\varphi^*=E(\varphi)$ is uniformly continuous and therefore we can use it to be our unique mapping (called $\hat{f}$ in 1.2.1).



Clearly, for any $\varphi\in UC(X\times Y,Z),\varphi^*$ is unique. Further, we have shown in 8.2.8 that $\varphi_x\in UC(Y,Z)$. We now show that $ev\circ(\varphi^*\times id_Y)=\varphi$. For $(x,y)\in X\times Y$,

$$\begin{aligned}
ev\circ(\varphi^*\times id_Y)(x,y) &= ev\big(\varphi^*(x),y\big) \\
&= ev\big(\varphi_x,y\big) \\
&= \varphi_x(y) \\
&= \varphi(x,y).
\end{aligned}$$

That is, for L a complete Heyting algebra, the category of stratified L-uniform convergence spaces is cartesian closed.

■

Chapter 9

Conclusions

We have attempted to produce definitions and results for a category of stratified L-uniform convergence spaces, using an enriched cl-premonoid as our underlying lattice. This has largely proven successful, although some restrictions have had to be made to the lattice context through the work. Our work with the product L-filters required that we use a pseudo-bisymmetric enriched cl-premonoid so as to guarantee the existence of the product of two stratified L-filters.

We successfully generalised the definition of a stratified L-uniform convergence space to our new lattice context. The result that the category of stratified L-uniform spaces should form a reflective subcategory of our new category, did however require a further restriction as outlined in Lemma 7.3.7. Our new generalised definition of a stratified L-uniform convergence space was successfully used in defining the induced stratified L-limit space. Having done this we were able to show that the initial structures are preserved by the forgetful functor, and that a stratified L-limit structure can be induced via two different pathways to produce the same structure.

The investigations into possible function spaces of $SL-UCS$ were very difficult. For the case of a general pseudo-bisymmetric enriched cl-premonoid these investigations did not prove successful because of the inability to evaluate the product L-filters. We attempted the case of $* = \otimes$, but still could not make progress.

The function spaces aside, all of our work here now extends the theory of lattice-valued uniform convergence spaces to the case of $(L, \leq, \circledast, *)$ where $\circledast$ is the monoidal mean operator. An idea for possible further work would be to investigate the function spaces for the two specific cases of the product and Lukasiewicz t-norms on $[0, 1]$.

Although we did not succeed in producing a function space structure, we did not prove that it can not be done for the examples that we attempted. Thus our question remains unanswered and could be attempted sometime in the future, perhaps with an approach that does not rely so heavily on the evaluation of the product L-filters.

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