A STUDY OF MAXIMUM AND MINIMUM OPERATORS WITH APPLICATIONS TO PIECEWISE LINEAR OPTION PAYOFF FUNCTIONS

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EBRAHIM D. SEEDAT

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ABSTRACT

The payoff functions of contingent claims (options) of one variable are prominent in Financial Economics and thus assume a fundamental role in option pricing theory. Some of these payoff functions are continuous, piecewise-defined and linear or affine. Such option payoff functions can be analysed in a useful way when they are represented in additive, Boolean normal, graphical and linear form.

The issue of converting such payoff functions expressed in the additive, linear or graphical form into an equivalent Boolean normal form, has been considered by several authors for more than half-a-century to better-understand the role of such functions.

One aspect of our study is to unify the foregoing different forms of representation, by creating algorithms that convert a payoff function expressed in graphical form into Boolean normal form and then into the additive form and vice versa. Applications of these algorithms are considered in a general theoretical sense and also in the context of specific option contracts wherever relevant.

The use of these algorithms have yielded easy computation of the area enclosed by the graph of various functions using min and max operators in several ways, which, in our opinion, are important in option pricing. To summarise, this study effectively dealt with maximum and minimum operators from several perspectives.

Keywords: Piecewise linear functions, Piecewise affine functions, Lattices, Max and min operators, Riesz spaces, Boolean normal forms, Option pricing.

American Mathematical Society Subject Classification (2010): 03G05, 03G10, 06A06, 91G20.

Journal of Economic Literature Classification: C61, C63, G13.

DECLARATION

Except for the references that have been accurately cited and discussed herein, the content of this thesis represents my own efforts. The entire thesis has neither been nor is concurrently being submitted to any other academic institution for the purpose of obtaining a degree.

Ebrahim D. Seedat

Grahamstown, R.S.A.

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I am grateful to Professor V. Murali of the Department of Mathematics of Rhodes University who has, apart from supervising this study, graciously and constantly assisted me from an intellectual perspective during the last three years.

I dedicate this work to my parents.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF ABBREVIATIONS & NOTATION

max	Maximum
min	Minimum
sup	Supremum
inf	Infimum
m	slope
p	Boolean polynomial in P_n
comp	Component
ℓ	Lower bound
u	Upper bound
w	Weight
L_s	Set of all lower bounds of a subset S
U_s	Set of all upper bounds of a subset S
epi	Epigraph
sub	Subgraph
log	Common logarithm
cp	Call option's premium
pp	Put option's premium
C_{IV}	Intrinsic Value of a European-style call option
C_{EV}	Extrinsic Value of a European-style call option
CF	Characteristic Function
B	Bond price
$\overline{B}$	Boolean lattice
C[a,b]	Space of functions on [a,b]
D	USD/GBP exchange rate
E	Cone/Expectation operator
E_+	Positive cone
$\hat{E}$	Expectation operator in a risk-neutral world
CNF	Conjunctive Normal Form
DNF	Disjunctive Normal Form
ECH	European-style call option held
ECW	European-style call option written
EPH	European-style put option held
EPW	European-style put option written
F	Riesz subspace
F'	Set of all real single-valued functions
$\hat{F}$	Forward or futures price
H	Linear hyperplane
I	Class
J	Affine hyperplane
K	Forward/futures delivery price or Option exercise price
$\overline{K}$	Field (numbers)
L	Lattice
L'	Sublattice
$\hat{L}$	Number of units of GBP
P	Poset
P_n	Set of all Boolean polynomials
$\hat{P}$	Power set of all subsets of a set X
PA	Payoff function
PL	Profit/Loss function
PDF	Probability Density Function
S	Spot price of underlying (e.g. asset)/Subset
$\overline{S}$	Family of subsets
V	Vector space (real)

LIST OF ABBREVIATIONS & NOTATION (CONTINUED)

r	Risk-free rate of interest
v	Element of a real vector space V
X	Set
X_n	Set of n symbols (variables or indeterminates)
$f(x), g(x)$	Functions of a variable x
α	Fixed non-zero vector
$\delta(a)$	Dirac delta/unit impulse function
$H(a)$	Heaviside function
$I(a)$	Indicator function
$N(.)$	Cumulative distribution function for a standardised normal distribution
$\hat{P}(X)$	Powerset of set a X
$R(a)$	Ramp function
$\cap$	Set intersection
$\cup$	Set union
$>, \geq$	Greater than, greater than or equal to
$<, \leq$	Less than, less than or equal to
$\subseteq$	Proper subset
$()$	Open interval
$[]$	Closed interval
$(], [)$	Half-open intervals
$\{\}$	Set
N	Set of natural numbers
R	Set of real numbers
R_+	Set of positive real numbers
R^n	Set of real numbers in n dimensions
R_+^n	Set of positive real numbers in n dimensions
Z	Set of integers
Z_+	Set of positive integers
Z_-	Set of negative integers
$Z \setminus \{0\}$	Set of integers excluding zero
Q	Set of rational numbers
$\vee$	Boolean symbol for maximum (max)/join/disjunction
$\wedge$	Boolean symbol for minimum (min)/meet/conjunction
$\lvert . \rvert$	Modulus or determinant of a square real matrix
$\lVert . \rVert$	Lattice norm
$\equiv$	Is the same as
$\cong$	Isomorphism
$:=$	Equals
$\rightarrow$	Mapping
$\Rightarrow$	Implies that
$\in$	Is an element of
$\setminus$	Relative complement
$\forall$	For all
$\square$	End of proof
Δ	Discriminant
d	Derivative
’	First derivative/complement (of a set)
λ	Parameter with $0 \leq \lambda \leq 1$
ρ	Parameter
σ	Standard deviation of natural logarithm of a stock’s price
Γ	Gamma function

LIST OF ABBREVIATIONS & NOTATION (CONTINUED)

$\psi^{(n)}$	n^{th} order derivative of the trigamma function
$\int_x^y$	Integral with lower limit of integration of x and upper limit of y
$\sum$	Summation
$\prod$	Product
$\bullet$	Dot product
!	Factorial
.	Multiplication/Multiplication (Boolean lattice)
+	Addition/Addition (Boolean lattice)
ϕ	Void class
$\perp$	Orthogonality

Subscripts not related to notation appearing earlier in this list

T	Terminal/expiry date of an option
0	Date on which option transaction is initiated
t	A date between 0 and T ($0 < t < T$)

Superscripts not related to notation appearing earlier in this list

T	Transpose of a real vector
+	Positive part (e.g. a^+ is the positive part of a)
-	Negative part (e.g. a^- is the negative part of a)
$\pm$	Positive part or negative part

CHAPTER 1. INTRODUCTION

1.1 Brief Context

The maximum (max) and minimum (min) of a finite set of real numbers $\{x_1,...,x_n\}$, are its largest and smallest elements respectively, with the usual ordering. If a sequence is infinite, the min (max) does not exist if it is unbounded from below (above); thus the concept of supremum (sup) and infimum (inf) must replace max and min. The foregoing concepts are considered in Real Analysis.[1]

Algebras involving max and min operators which generally consist of a real-valued input a and b yielding a real-valued output a or b via the application of certain rules, are deduced from the sup and inf in the context of a lattice. Boolean algebras are related to a class of algebraic structures called lattices.

Such algebras were introduced by the British mathematician G. Boole (1854) as part of research which led to the establishment of mathematical logic, and comprise a special class of lattices. Subsequently, the elementary theory of lattices appeared in an article written by the American philosopher and mathematician C.S. Peirce (1880), who first considered the use of the inf and sup in partially ordered sets.[2] Later, the German mathematician E. Schroder developed the theory of lattices in monographs published between 1890 and 1905.[3] Almost contemporaneously, Schroder's compatriot, J.W.R. Dedekind devised the basic theory of lattices (*Dualgruppen*) which culminated in the publication of Dedekind (1897) and Dedekind (1900). Dedekind was the first to recognise the link between modern algebra, mathematical logic and lattice theory. In particular, Dedekind defined distributive and modular lattices in modern terminology.

[1] Bartle & Sherbert (2011) offer an introduction to this topic while more advanced treatments appear in Rudin (1976) and Folland (1999).

[2] Like C.S. Peirce, the British logician and economist W.S. Jevons studied the theory of Boolean algebra during the 1860's. Jevons (1863) is considered to represent the first use of mathematical methods in Economics and Jevons (1864) consists of a monograph which develops Boolean algebra.

[3] The three volumes of Schroder's work were re-printed in 1966 (see References section of this study). Schroder proved that Peirce (1880) erred in asserting that all lattices were distributive.

Between the publication of an article written by Huntington (1904) and Birkhoff's (1935) article on universal algebra (which is considered particularly important), the study of lattice theory fell into abeyance. Thereafter, considerable advances in lattice theory have occurred, with Birkhoff's (1940) book containing a comprehensive exposition of the topic.[4] In addition, monographs written by Rennie (1950), Rutherford (1965), Grätzer (1971, 1978) and Schmidt (1982) represent important contributions to knowledge of this field. The important max and min operators are explicitly defined by Donnellan (1968), Aliprantis & Burkinshaw (1985) and Trybulec & Bylinski (2004). Such operators can be represented using Boolean normal forms – the well-known and so-called disjunctive and conjunctive normal forms which are discussed by Lidl & Pilz (1984) and, more recently by Steinbach & Posthoff (2007) among others. These normal forms were considered as a means of checking whether a given Boolean expression could be written in different ways.

In one dimension, max and min operators represent a special case of a vector lattice or Riesz space. An address by F. Riesz in 1928 to participants attending the International Mathematical Congress, which was subsequently published as Riesz (1930), impelled the study of Riesz spaces, as vector lattices became known. Freudenthal (1936) and Kantorovich (1936a, 1936b) independently introduced axioms and established numerous properties for the lattice structure of Riesz spaces. Substantial developments have occurred in this field since then and have been considered by Luxemburg & Zaanen (1971) and Zaanen (1983, 1997), among many others.

In any dimension, max and min operators are useful in solving various optimisation problems in Mathematics, Statistics (including the cognate fields of Actuarial Science and Operations Research), Economics and Information Science. Such operators are important in Economics given that when economic agents make decisions affecting the level of their wealth based on numerous variables, a specific value that maximises gains or minimises losses must be attained where a sequence representing asset prices (in a maximization problem) is of finite length. Economists apply max and min operators to

[4] The most recent edition is Birkhoff (1995).

models of production technology such as the Leontief production function (Leontief, 1941) and production issues in general (Topkis, 1998), payoffs to economic agents yielded by adopting particular strategies in game theory (Chiang, 1974), market efficiency[5] (McCauley, 2004) and models of the circumstances in which firms merge (Gorton, Kahl & Rosen, 2005).

Arguably, C. Aliprantis was the most prolific author of journal articles and books pertaining to applications of max and min operators to problems in Economics. Aliprantis & Burkinshaw (2003) claim that Aliprantis & Brown (1983) first applied the theory of vector lattices or Riesz spaces to Economics. Aliprantis, Brown & Burkinshaw (1990) systematically applied such operators to various economic problems in a monograph on microeconomic theory. Mas-Colell & Richard (1991) offered an infinite-dimensional equilibrium existence theorem for exchange economies. This theorem incorporated all the hitherto known and important equilibrium results as corollaries and was proven by considering the lattice structure of Riesz spaces. Topkis (1998) applies the notion of supermodularity to, for example, the existence of fixed points in general equilibrium theory. The microeconomic applications of locally solid Riesz spaces are discussed by Aliprantis & Burkinshaw (2003) from a topological perspective. Aliprantis & Border (2006) consider Riesz spaces in an infinite-dimensional setting.

Given that the fields of “Financial Economics” and “Microeconomics” are often considered to be related, authors that have studied microeconomic issues have sometimes inadvertently studied issues in “Financial Economics” and vice versa[6] in the context of Riesz spaces and more generally through the ideas of lattice theory.

[5] For example, McCauley (2004) defines market efficiency (in the microeconomic sense) as $e(p) = \min(S/D, D/S)$ with e depending on both bid and ask prices to a greater extent, the wider the spread is. If $e = 1$ there is equilibrium and market-clearing is associated with 100% efficiency. Note p can be thought of as the ask price and/or bid price.

[6] Inverted commas are used because these concepts are defined loosely considering the Journal of Economic Literature classification system.

Economists since Samuelson[7] (1965) have applied max and min operators in writing payoff (date of maturity/terminal or otherwise) and corresponding profit/loss functions to holders and writers of contingent claims or options. Options confer holders the right without the obligation to buy or sell a financial or real asset on a given date (European-style) or during a given period (American-style). The value of such options depend on an entity called the underlying, with examples being:

1. the prices of currencies, securities, commodities[8], energy[9] and money (interest rates);
2. borrower default probabilities (credit risk);
3. weather patterns such as temperature and rainfall;
4. capital project outcomes[10]; and
5. the value of insurance claims.

Options are standard ("plain vanilla") or non-standard ("exotic"). Generally, standard options are traded on organised derivative exchanges whilst non-standard options are traded outside such exchanges or "Over-the-Counter" (Hull, 2003).

Expositions of all categories of standard and non-standard option transactions except the third are captured in books by Hull (2003), Shreve (2004a, 2004b) and Musiela & Rutkowski (2005).[11] Jewson, Brix & Ziehmann (2005) describes standard and non-standard options for the third category of underlying. Nelken (1996) gives a comprehensive treatment of non-standard options for the first and second category of underlying.

[7] P. Samuelson was one of the first economists to apply mathematical principles designed in solving problems in thermodynamics to those in Economics. The book by Samuelson (1947) contains evidence of such applications.

[8] Examples include all precious metals, some base metals, certain food crops and by-products, some cash crops and by-products as well as certain livestock by-products.

[9] Examples include non-agricultural commodities that can be used in the production of fuel directly such as crude oil and natural gas and indirectly, with coal being an example. Contracts on electricity produced are also traded.

[10] These relate to real options and are discussed by Hull (2003).

[11] In addition, Johnson & Tian (2000) and Gerstner & Holtz (2008) demonstrate the application of max and min operators to the firm manager performance-dependent option terminal payoffs with the latter authors considering options discussed by Stulz (2004). Chance, Hillebrand & Hilliard (2006) use such operators in assigning a price to options on motion picture (movie) revenues.

The notion of an option's value discussed above is nuanced. One way of considering an option's value is by examining its terminal payoff function. Nelken (1996:13) states that:

> "terminal payoff functions act as one of the fundamental boundary conditions in the contingent claim. The other two are when the underlying price goes to either zero or infinity, for times at and away from the inception time. Together, these three boundary conditions determine the unique payoff of the contingent claim for all other times. The terminal payoff of an option…reveals a lot of information about the behavior of the option at all other times".

Some option payoffs (whether terminal or otherwise) are functions of one underlying variable. Among that class, are piecewise-defined functions that are linear (and affine) and continuous. Examples include the European call and put option. For functions that are discontinuous, examples include the binary/digital call and put. Of those that are piecewise non-linear and continuous, examples include the Esser (2003) power call option for exponents greater than two and calendar spreads created with call or put options. Combinations of these categories are also possible.

Sakurai & Murofushi (2004) state that continuous piecewise linear (and affine) of one variable can be expressed in several ways.[12] Thus, option payoff functions of the same form can be represented in several ways. In the option pricing literature, the linear form, additive form, Boolean normal form and graphical form have been used interchangeably and in a disparate manner. Additive form is useful because options can be considered as portfolios containing assets. Graphical form admits ease of understanding to a greater extent than the use of explanations with and/or without equations. Linear form is well-known while Boolean normal forms are the most concise.

[12] Sakurai & Murofushi (2004) state that the max-min representation, including Boolean normal form of Ovchinnikov (2000, 2002), is one among several representations of piecewise linear functions. These representations include Chua & Deng's (1988) canonical form, Leenaerts & van Bokhoven's (1998) state variable representation and Murofushi & Narukawa's (2002) Choquet integral representation.

Samuelson (1965) first used max operators to construct the additive form for a warrant payoff.[13] Musiela & Rutkowski (1997) write certain option payoff functions in Boolean normal form and an equivalent additive form. The use of the graphical form may have started with Lefèvre (1873)[14] who used rather cumbersome notation and prolix explanations in constructing such payoff functions. One of the functions considered in Lefèvre's (1873) book consisted of eleven linear components. The graphical representation can be constructed by the combination of component graphs following Lefèvre (1873) or obtaining the slopes of component graphs in the "direction vector" technique of Cuthbertson & Nitzsche (2006). The more complex the terminal payoff becomes, the more the effective the latter method is in sketching terminal payoff diagrams.

Wilkinson (1963) recognized that continuous piecewise linear functions of several variables can be represented in Boolean normal form. More than three decades later, Gorokhovik & Zorko (1994) present several characterisations of piecewise linear functions defined on normed vector spaces, with one of them including the Boolean normal form. Tarela & Martinèz (1999) considered the same issue but suggest two domain partitions that enable the realisation of lattice models.

Ovchinnikov (2002) refines the ideas of Wilkinson (1963), Gorokhovik & Zorko (1994) and Tarela & Martinèz (1999) by presenting a more effective Boolean representation of the foregoing functions. Aliprantis, Harris & Tourky (2006) use slopes to discuss the ideas which are used by Ovchinnikov (2002) and demonstrate the link between linear form, additive form, Boolean normal form and graphical form in one and higher dimensions. Aliprantis & Tourky (2007) are the first to suggest the applicability of Ovchinnikov's (2002) approach to the representation of certain terminal option payoff functions in Boolean normal form.

[13] The only difference beteen a stock warrant and option is that the former entitles the holder to convert the warrant into a share of common stock whilst the option's holder obtains cash.

[14] Weber (2009) states that, since the 1860's, options having one security as the underlying asset (or the sum of such options) had payoff functions that were represented by continuous piecewise linear or affine functions.

The foregoing facts mean that one or multiple representations of continuous piecewise linear and affine functions of one variable (of the four mentioned above) can be used in obtaining another. The problem of how this can be done systematically arises.

Another problem arises from the absence of studies that unify the concepts of Boolean normal form, additive form, graphical and linear form with integral calculus, in the context of continuous, piecewise linear and affine option payoff functions. Among such payoff functions, is the standard European-style call option's payoff function. Black & Scholes (1973) combine some of the foregoing forms to represent the call option's payoff function with integral calculus to compute a corresponding call option price. The segmentation of literature concerning the link between the forms discussed above with integral calculus in an option pricing context, has been inimical to a proper understanding of option prices. Evidence of this assertion is provided in the form of Cheung & Chung (1996) and Carroll and Brask (1999) depicting the area between the graphs of piecewise linear non-standard option payoff functions as the Black & Scholes (1973) "building block" price of such options when this is not the case.

1.2 Aims Of This Study

The first aim of this study is to identify the max and min operators which are applicable to the pricing of standard and non-standard options associated with piecewise linear and affine payoff functions.

The second aim of this study is to convert the graph of a continuous piecewise linear or affine option payoff functions, where the underlying assets are stocks and currencies,[15] to an appropriate Boolean normal form. Writing terminal option payoffs in the appropriate Boolean normal form assists computer programmers in assigning non-standard option prices more easily as Jewson, Brix & Ziehmann (2005) suggest.[16] It also permits market participants to understand non-standard transactions to a greater extent. This is done because such normal forms most concisely represent terminal payoff and profit/loss functions, as mentioned earlier.

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[15] Hull (2003) discusses options with continuous but non-linear payoff functions such as calendar spreads which are not discussed further in this study.

[16] The author has corresponded with Steve Jewson on this topic.

The third aim of this study is to obtain the additive form from the Boolean normal form because the additive form can readily be interpreted as a portfolio constructed with certain assets.

The fourth aim of this study is to connect the concepts of the graphical, linear, additive and Boolean normal forms of continuous, piecewise and linear option payoff functions on the one hand, with integral calculus on the other, in order to measure the area between the graphs of such function and the horizontal axis (for a given value of a single underlying asset and over a given interval). These values can be used for several purposes. For example, the area of a given region can be compared with another when an investor assesses the extent of their gains/losses in various scenarios (represented by different terminal values of an asset's price). In addition, these computed values can be used in determining the extent to which one option payoff function dominates another. Finally, such values can be used in demonstrating the difference between option payoffs and prices.

1.3 Methods and Structure

Firstly, we collect several definitions and results relevant to our study from lattice theory as discussed by Birkhoff (1940) among others. Secondly, we identify max and min operators relevant to option pricing when one underlying asset is considered. In the interests of transparency, complete proofs of some lattice propositions deemed crucial to the analysis of continuous piecewise linear and affine functions are reproduced or reformulated in this study. The max and min operators discussed by Donnellan (1968) and Trybulec & Bylinski (2004) are also included.

Max and min operators comprising real numbers along with random variables are then associated with standard options and non-standard options' payoff functions (of one variable or underlying asset), whether terminal or otherwise. Non-standard options are designed with reference to the extant literature or deduction. Then, apt max and/or min operators will be used in constructing such payoff functions. The issue of duality between max and min operators, wherein max operators are substituted by min operators and vice versa, will be considered. Duality permits switching from one form of max/min expression to another and vice versa to represent "buyer" and "writer" positions in options[17] in certain cases. As part of this analysis, inadmissible transactions based on the properties of max and min operators are also described.

To convert the graphical form of each terminal payoff function into the Boolean normal form, we create the reverse decomposition algorithm. The foregoing algorithm will be constructed by following parts of the Ovchinnikov (2002) approach in particular. The steps used in constructing the algorithm of Ovchinnikov (2002) will be explained in greater detail and extra steps are added to ensure transparency. This algorithm can be applied to appropriate standard and non-standard European-style options. Given that the algorithm designed by Ovchinnikov (2002) relies to some extent on a hyperplane argument, the points at which hyperplanes intersect for the graphs of the option payoff functions considered will be computed in order to express such functions in Boolean normal form. To verify a part of the reverse decomposition algorithm developed herein, Cuthbertson & Nitzsche's (2006) direction vector technique will be used.

The decomposition algorithm is devised in order to convert the aforementioned option payoff functions in an appropriate Boolean normal form into an equivalent additive form by using the properties of inequalities.

To obtain a precise numeric value that measures the area between the horizontal axis (representing terminal asset prices) and an option's payoff function, for a given value of a single underlying asset, three methods belonging to two categories are used. The first category consists of the calculation of the area between the horizontal axis and graph to obtain value of gain or loss using a geometric approach. The results of the application of this approach are shown to be consistent with those obtained from the second category – the integral approach. It is shown that the methods comprising this approach can be used in obtaining the aforementioned area with one of these methods relying on the definition

[17] Max and min operators are less apt to the case of representing "long" and "short" positions in forward commitments traded on organised exchanges. However, the terminal payoff to forward options, which are a class of non-standard financial derivatives, can be represented by max and min operators.

of the max operator. Complete workings are offered in certain cases to ensure transparency, especially for the more complicated non-standard options. Standard calculations of area are provided and can be used for several purposes. Among them, the upper limit of integration can be changed in order to determine whether an investor gains or loses based on the difference in the area for the original function and its vertical translation.

Given that the study's emphasis will be on studying non-standard options' terminal payoff functions of one underlying asset, transactions contained in various books and journal articles for case will be discussed.

The organisation of the remainder of this study follows. Chapter 2 contains definitions of max and min operators. Chapter 3 comprises a discussion of the theory of Riesz spaces. The exposition of Chapter 2 and Chapter 3 is used in setting notation and the theoretical basis for the concepts used in later chapters. The central theme of this study is captured in Chapter 4 wherein two algorithms are devised and applied. Chapter 5 consists of a discussion of the link between max and min operators and standard option payoff functions. Applications of both algorithms to standard option payoff functions are also considered in Chapter 5. In Chapter 6, the algorithms are applied to the payoff functions of several non-standard options. The final chapter concludes the study with a discussion of future research prospects and topics.

For ease of reference, in each chapter (appendices excluded), the numbers of all theorems, definitions, lemmata, corollaries, notes, remarks, and examples on the one hand and some propositions on the other, are assigned serially. For example, 4.3.1 refers to the first entity of the third section of the study's fourth chapter. A different scheme is used in the assignment of numbers to axioms and the remaining propositions, although the practice of assigning numbers in serial order is retained. Thus, LA1 (for instance) will refer to the first axiom of a series, with the acronym L, which precedes the letter A, denoting the subject of lattices. To use another example, VLP3 will similarly refer to the third proposition of a series, with the acronym VL, which precedes P, denoting the subject of vector lattices (Riesz spaces).

Equations are assigned numbers only if these entities are referred to in the exposition, with the chapter number appearing before the number of the equation in a given series. Thus, 3.4 would (for example) refer to the fourth equation of the study's third chapter. Note that the proofs appearing in this study are, in accordance with convention, bereft of numbers assigned through the use of any scheme.

To the best of our knowledge, the results stated hereunder are our own:

Proposition 5.3.1, Proposition 5.3.3.

CHAPTER 2. RELATING LATTICE THEORY TO MAX AND MIN OPERATORS

In this chapter, the max and min operators relevant to the assignment of prices to options will be considered by the theory of lattices. Such operators correspond to the Boolean operations "AND" ($a \wedge b$, a.b, ab), conjunction, "OR" ($a \vee b$, a+b) and disjunction discussed by Steinbach & Posthoff (2007).

The max and min operators applicable to the set of real numbers as envisaged in this chapter are related to the theory of lattices. Henceforth, the Boolean symbols $\vee$ and $\wedge$ will be used exclusively except where it is deemed critical to juxtapose the notation sup, inf, max and min with these Boolean symbols as a prelude to the discussion of additive forms and Boolean normal forms. The latter concepts will be defined, explained and applied from Chapter 4 onwards.

In Section 2.1, we will discuss the basic lattice concepts from which max and min operators originate. Distributive lattices and the implications of distributivity on max and min operators are discussed in Section 2.2. Finally, Section 2.3 consists of a discussion of *Boolean lattices* which are complemented distributive lattices.

2.1 Basic Lattice Concepts

The book by Birkhoff (1940) contains a comprehensive discussion of lattices. A more recent and succinct exposition of this topic is given by Lidl & Pilz (1984) and Topkis (1998). Following Birkhoff (1940), we begin with the concept of a partially ordered set, which underlies the concept of lattices.

Definition. 2.1.1 A partially ordered set or *poset* is a non-empty set on which a binary relation denoted by $\leq$ is defined. Then (P, $\leq$) is called a poset if the following conditions are satisfied:

[PP1] Reflexivity: $a \leq a$ for all $a \in P$;

[PP2] Antisymmetry: $a \leq b$ and $b \leq a$ implies $a = b$ for $a, b \in P$;

[PP3] Transitivity: $a \leq b$ and $b \leq c$ implies $a \leq c$ for $a,b,c \in P$.

Note that $\leq$ is read as "less than or equal to", "included in" or "contained in" and vice versa for the symbol $\geq$, as the context dictates. The expression "a is less than b" or "b strictly dominates a" is denoted by $a < b$ and vice versa for the symbol $>$. An element ℓ (u) is called a least (greatest) element of P if $\ell \leq a$ ($a \leq u$) for all $a \in P$. If a and b are elements of P, $a < b$ and in the absence of c in P with $a < c < b$, we say that b covers a in P. Two elements a and b in P are ordered if either $a \leq b$ or $b \leq a$. If not, a and b are unordered or not comparable.

Numerous examples of mathematical relations satisfying [PP1]-[PP3] exist. A selection of these are considered below, as variations of the same will appear later in this study.

Examples. 2.1.2

1. Let N denote the set of natural numbers. Then N is a poset under the usual or natural ordering as in $a < b$. The same is also true for the set of integers (Z), rational numbers (Q) and real numbers (R). From an algebraic perspective, integers form a *ring* with respect to the usual operations of addition and multiplication whereas rationals and reals form a *field*.

2. Let R_+ denote the set of positive real numbers, which is a poset under the natural ordering.

3. The usual ordering $\leq$ among R^n by $(a_1, a_2, \ldots, a_n) \leq (b_1, b_2, \ldots, b_n)$ occurs if and only if $a_i \leq b_i$ for $i = 1, 2, \ldots, n$ is a partial ordering on both R and R^n rendering them as posets. Note that for R^n, the usual ordering is component-wise.

4. The usual ordering $\leq$ among R_+^n by $(a_1, a_2, \ldots, a_n) \leq (b_1, b_2, \ldots, b_n)$ occurs if and only if $a_i \leq b_i$ for $i = 1, 2, \ldots, n$ is a partial ordering on both R_+ and R_+^n rendering them as posets. Note that for R_+^n, the usual ordering is component-wise.

5. Let X be a non-empty set and $\hat{P}(X)$ be the power set of all subsets of X. Then $(\hat{P}(X), \leq)$ is a poset wherein $\leq$ means subset inclusion. To be specific, A $\leq$ B if and only if A is contained in B as sets.

6. The lexicographic ordering relation $\leq_{lex}$ on the set R^n consists of $x \leq_{lex} x'$ in R^n if either x = x′ or if there exists some i′ with 1 $\leq$ i′ $\leq$ n such that $x_i = x'_i$ for each $1 \leq i < i'$, and $x_{i'} < x'_{i'}$.

7. Let F′ comprise the set of all real single-valued functions defined on the closed unit interval 0 $\leq$ a $\leq$ 1. Then $f \geq g$ means that for two functions f and g in F′ f(a) $\geq$ g(a) for all a with 0 $\leq$ a $\leq$ 1.

Many posets satisfy the following relation:

[PP4] Given a and b, either a $\leq$ b or b $\leq$ a.

In that case we have:

Definition. 2.1.3 A poset P is a *chain* if any two elements of P are comparable under $\leq$ such as a $\leq$ b or b $\leq$ a for a, $b \in P$. A chain is also called *totally* or *simply ordered*.

In Examples 2.1.2, Part 7 is an example of a poset which is not a chain because pairs of incomparable elements are involved where neither x $\leq$ y nor y $\leq$ x.

The concept of a poset is used in defining a lattice once the bounds of a given set are defined.

Definition. 2.1.4 Suppose P is a poset and S is a non-empty subset of P. An element $\ell \in P$ ($u \in P$) is said to be a *lower bound* (*upper bound*) for $S \subseteq P$ if $\ell \leq$ s (s $\leq$ u) for every $s \in S$. The set of all lower (upper) bounds of S is denoted by L_s (U_s). The greatest (least) element L_s (U_s) is called the greatest lower (least upper) bound or *infimum* (*supremum*).

Definition. 2.1.5 An element z of a poset P is the supremum (infimum) of a pair of elements $a,b \in P$ if $a \leq z$ ($z \leq a$) and $b \leq z$ ($z \leq b$) together with the property that $a \leq u$ ($l \leq a$) and $b \leq u$ ($l \leq b$) implies that $z \leq u$ ($l \leq z$). The supremum (sup) and infimum (inf) respectively are denoted by:

$$\sup\{a, b\} = a \vee b \tag{2.1}$$

$$\inf\{a, b\} = a \wedge b\,. \tag{2.2}$$

Note. 2.1.6

1. Henceforth and in general, the Boolean notation $\vee$ and $\wedge$ will replace or appear concurrently with the alphabetic abbreviations "sup" and "inf", as was the case in Equation 2.1 and Equation 2.2. The Boolean notation most concisely represents max and min operators because fewer letters and symbols are needed than for the alphabetic abbreviations.
2. Nomenclature: We refer to sup as max and inf as min for finite sets. For infinite sets, max and sup may differ, with sup being a more subtle concept.

Following Birkhoff (1940), the concept of duality of posets is introduced because it is the prelude to the concept of duality of max and min operators which is, in turn, crucial to the study of continuous piecewise linear and payoff functions.

Theorem. 2.1.7 [Duality Principle] Let P be a poset endowed with a binary relation $\leq$. Then the dual binary relation of $\leq$ on P is $\geq$ wherein $b \geq a$ if and only if $a \leq b$.

We note that the changes of negative to positive signs in front of real numbers or vice versa, are related to the duality of the usual partial order relation $\leq$ on the real numbers. For example, given two real numbers a and b,

$$a \leq b \text{ if and only if} -a \geq -b$$

$$a \geq b \text{ if and only if } -a \leq -b\,. \tag{2.3}$$

This observation is of significance throughout the study and it will be exemplified at appropriate places.

Definition. 2.1.8 A *lattice* L is a poset with any two elements having a sup and an inf. A lattice is complete when a sup and an inf of each of its subsets exists.

R is a lattice under the natural partial order $a \leq b$ for $a,b \in R$. The natural partial order on R leads to lattices of min and max on R. The min and max on R are also the natural minimum and maximum as used universally. The partial order is connected to the max and min in the following way:

$$a \leq b \text{ if } a \wedge b = a\ ,$$

or analogously:

$$a \leq b \text{ if } a \vee b = b\ .$$

In general, the binary operations $(a,b) \to a \wedge b$ and $(a,b) \to a \vee b$ on a poset P are lattice operations. Thus, in its simplest form, a lattice (L) is a non-empty set closed under two binary operations called meet/conjunction $(\wedge)$ and join/disjunction $(\vee)$[18] which both satisfy the following equations for $a,b,c \in L$:

[LA1] commutative law

$$a \wedge b = b \wedge a$$
$$a \vee b = b \vee a.$$

[LA2] associative law

$$(a \wedge b) \wedge c = a \wedge (b \wedge c)$$
$$(a \vee b) \vee c = a \vee (b \vee c)$$

[LA3] idempotent law

$$a \vee a = a$$
$$a \wedge a = a.$$

[LA4] absorption law

$a \leq b$ if and only if:

$$a \wedge (a \vee b) = a$$
$$a \vee (a \wedge b) = a.$$

[18] Lidl & Pilz (1984) denote join by the symbol $\sqcup$ and meet by the symbol $\sqcap$.

Remarks. 2.1.9

1. [LA4] implies that:

$$a \le b \Rightarrow a \wedge (a \vee b) = a \wedge b = a$$
$$a \ge b \Rightarrow a \wedge (a \wedge b) = a \wedge a = a.$$

2. $a \le b$ is equivalent to the consistency condition:

$$a \wedge b = a \;\;\&\;\; a \vee b = b.$$

3. An element 0 (1) in a lattice L is called a least (greatest) element if $0 \le a$ $(a \le 1)$ for all $a \in L$. The least and greatest elements are unique.

The following series of lemmata are discussed by Birkhoff (1940).

Lemma. 2.1.10 If a poset P has a least element ℓ then:

$$\ell \wedge x = \ell \text{ and } \ell \vee x = x \text{ for all } x \in P.$$

Similarly, if P has a greatest element u then:

$$x \wedge u = x \text{ and } x \vee u = u \text{ for all } x \in P.$$

Proof. This is left to the reader.

Before stating the next lemma, some mappings on posets must be defined.

Definition 2.1.11 Let $(P_1, \le)$ and $(P_2, \le)$ be any two posets. A mapping $\theta : P_1 \to P_2$ is called an *order-preserving* or *isotone* mapping from $(P_1, \le)$ to $(P_2, \le)$ if $x_1 \le x_2 \Rightarrow \theta(x_1) \le \theta(x_2)$ for $x_1, x_2 \in P_1$.

Isotone and *montone* are often used interchangeably in the literature.

A *bijective* mapping $\theta : P_1 \to P_2$ is called an *isomorphism* if θ and its inverse (θ^{-1}) are isotone mappings. A bijection or one-to-one correspondence θ satisfying both $x_1 \le x_2 \Rightarrow \theta(x_1) \ge \theta(x_2)$ and $\theta(x_1) \le \theta(x_2) \Rightarrow x_1 \ge x_2$, is called a *dual isomorphism*.

Definition 2.1.12 The mapping $\theta : P_1 \to P_2$ is called *order-inverting* or *antitone* if $x_1 \le x_2 \Rightarrow \theta(x_2) \le \theta(x_1)$ for $x_1, x_2 \in P_1$.

Lemma. 2.1.13 The operations of join and meet are isotone in any lattice (L). If b $\le$ c then $a \wedge b \le a \wedge c$ and $a \vee b \le a \vee c$.

Proof. See Birkhoff (1940).

Lemma. 2.1.14 [Birkhoff] In any lattice, the following *distributive inequalities* are valid:

$$a \wedge (b \vee c) \ge (a \wedge b) \vee (a \wedge c) \tag{2.4}$$

$$a \vee (b \wedge c) \le (a \vee b) \wedge (a \vee c)\,. \tag{2.5}$$

Proof. See Birkhoff (1940).

Lemma. 2.1.15 The *modular inequality* is satisfied by the elements a,b and c of any lattice $a \le c \Rightarrow a \vee (b \wedge c) \le (a \vee b) \wedge c$.

Proof. [Birkhoff]

Let $a \le (a \vee b)$ and a $\le$ c. Thus $a \le (a \vee b) \wedge c$. In addition $b \wedge c \le b \le a \vee b$, and $b \wedge c \le c$. Thus, $b \wedge c \le (a \vee b) \wedge c$. Then $a \vee (b \wedge c) \le (a \vee b) \wedge c$. □

Any non-empty complete lattice has a least and greatest element. Given any lattice, its dual is a lattice but joins become meets and meets become joins.

Only some lattices are complete. An example of a lattice which is not complete is the set of rational numbers (Q) or real numbers (R).

Note that the concepts of “max” and “min” mentioned in these examples will be considered more comprehensively in the sequel. Some examples of lattices which are of importance to and used in this study are described below.

Examples. 2.1.16

1. The set of positive real numbers R_+ is a lattice with $a \wedge a' = \min\{a, a'\}$ and $a \vee a' = \max\{a, a'\}$ for a, $a' \in R_+$.

2. Any chain is a lattice wherein $a \wedge b$ is the least and $a \vee b$ is the greater of a and b.

3. For any positive integer $n \in Z_+$, R^n is a lattice with:

$$a \vee b = (a_1 \vee b_1, ..., a_n \vee b_n)$$
$$\max(a, b) = [\max(a_1, b_1), ..., \max(a_n, b_n)]$$

and:

$$a \wedge b = (a_1 \wedge b_1, ..., a_n \wedge b_n)$$
$$\min(a, b) = [\min(a_1, b_1), ..., \min(a_n, b_n)]$$

for $a, b \in R^n$.

Definition. 2.1.17 If L' is a subset of a lattice L and L' contains the meet and join in relation to L, of each pair of elements of L' then L' is a *sublattice* of L. A sublattice is still a lattice endowed with the same join and meet operations.

Example. 2.1.18

If L is a lattice and is (itself) the set of real numbers R and L = R, then any subset of L is called a sublattice of L.

In a lattice L, if a $\leq$ b the closed interval of all elements satisfying a $\leq$ x $\leq$ b is a sublattice. Given a poset P, its convex subset is a subset containing [a,b] when are and b are contained in this subset and a $\leq$ b. A subset L' of a lattice L is a convex sublattice whenever $a, b \in L'$ imply that for the interval $[a \wedge b, a \vee b] \subset L'$. A subset of a lattice can be a lattice under the identical relative order without being a sublattice (L'). Birkhoff (1940) discusses this matter in greater detail.

2.1.1 Max and Min Operators

Given that the concepts of upper and lower bounds and sup and inf have been defined, we can now define the max and min. It will be apparent that these definitions follow naturally.

Definition. 2.1.19 A real number is call the minimum (maximum) of a given set of real numbers if and only if it is an element of the given set and is less (greater) than every other element belonging to the set.[19]

In the ensuing discussions, we tacitly assume that a,b,c and d are real numbers. Some sets, expressed as intervals, have both a maximum, whilst some sets possess one without the other and other sets have neither a maximum nor a minimum, with bounded intervals being an example.

Denote open intervals by (,), closed intervals by [,] and half-open/closed sets by (,] or [,). A maximum and minimum exists in cases wherein:

$$a = \min[a,b] = \min[a,b)$$

$$b = \max[a,b] = \max(a,b].$$

In cases where one exists without the other:

i. (a,b] lacks a minimum and [a,b) lacks a maximum.

ii. (a,b) has neither a maximum nor a minimum.

[19] A related concept is the running or cumulative maximum over the set of real numbers wherein each component is compared pair-wise (starting from a sequence's first term) such that maximal number prevails until a number greater than the first maximal number is identified and the latter replaces the original component and so forth. The running maximum of a sequence of values $\{a_k\}_{k=1}^{n}$ is given by $\{\max(a_1,\ldots,a_k)\}_{k=1}^{n}$. Given a sequence (11,12,13,20,19,18,17,18,23,21) its running maxima are (11,12,13,20,20,20,20,20,23,23). The running minimum is obtained analogously. For example, the use of another sequence yields a running minimum of:
{10,19,17,20,11,29,5,25,25,1}={10,10,10,10,10,10,5,5,5,1}.
In higher dimensions, the running maximum of a sequence of vectors with the component-wise ordering is not relevant but for the lexicographic ordering, the running or cumulative maximum can be computed. Poulsen (2006) constructs a zero-rebate barrier option's terminal payoff function with the running maximum. In this study, a discussion of such options will be excluded.

We now cease the practice of calling any entity adjoining or enclosed by (.) and [.] as denoting open and closed intervals respectively, as was done above.

The formulae for max and min, using | . | to denote the modulus, are:

$$\max(a,b) = \left(\frac{a+b+|a-b|}{2}\right) = \begin{cases} b & \text{if} \quad a \le b \\ a & \text{if} \quad \text{otherwise} \end{cases} \tag{2.6}$$

$$\min(a,b) = \left(\frac{a+b-|a-b|}{2}\right) = \begin{cases} a & \text{if} \quad a \le b \\ b & \text{if} \quad \text{otherwise.} \end{cases} \tag{2.7}$$

The foregoing formulae are obtained from Trybulec & Bylinski (2004) and Boukezzoula & Galichet (2007).

The *positive part* of a is:

$$a^{+} = \max(a,0) = \left(\frac{a+|a|}{2}\right). \tag{2.8}$$

Note that Equation 2.8 is also called a *positive form* and that sometimes the notation a_{+} is used instead. Similarly, the *negative part* of a is:

$$a^{-} = \max(-a,0). \tag{2.9}$$

Equation 2.9 is also called *a negative form*. Note that the notation a_{-} is sometimes used. Later, the concepts of a^{+} and a^{-} will be discussed in greater detail.[20]

Remark. 2.1.20

1. The components (a and b) of Equation 2.6 and Equation 2.7 are:

[20] Folland (1999:46) decomposes functions with max operators. Given $f : X \to R$ the positive and negative parts of f are $f^{+}(x) = \max[f(x),0]$ and $f^{-}(x) = \max[-f(x),0]$ which means that $f = f^{+} - f^{-}$. For a given function f, the positive part function f^{+} is defined by $f(x)^{+} = \max[f(x),0]$. Similarly, the negative part function is $f(x)^{-} = \max[-f(x),0]$.

i. both deterministic constants which are linear, polynomial functions such as a^2+a-1, either or both max and min operators which are added such as:

$$\max[(\max(a,0)+\max(b,0)), \max(c,0)] = [(a \vee 0)+(b \vee 0)] \vee (c,0);$$

ii. both stochastic; and

iii. stochastic and deterministic.

2. The deterministic components themselves can be determined by stochastic processes such as the expected value of a max operator with at least one of the components being stochastic.

3. The min or max of a component is the component itself:

$$\min(a) = \wedge a = a \ \ \& \ \ \max(a) = \vee a = a.$$

4. Gunawardena (1994) states that when one variable is stochastic another is deterministic, certain lattice operations are forbidden and others are permitted. The forbidden operations are that one side of the maximum or minimum symbol contains a stochastic variable only or a deterministic variable only as in $a \wedge 2$. Permitted operations contain a stochastic and a deterministic variable on either side of the maximum and minimum symbol with examples being $a-1 \wedge b+1$ and $a+b \wedge c+2$. Gunawardena (1994) also states that the positive sign has a "higher binding" than the Boolean symbols above.

Remarks. 2.1.21

1. For a finite subset S={a_1, a_2, … a_n} of L written as $S \subseteq L$ and natural number $n \in N$:

$$\bigwedge S = \wedge S = a_1 \wedge a_2 \wedge \ldots \wedge a_n$$

$$\bigvee S = \vee S = a_1 \vee a_2 \vee \ldots \vee a_n.$$

If S={a,b} is a set comprising two elements then $\vee S = \vee\{a,b\} = a \vee b$ and $\wedge S = \wedge\{a,b\} = a \wedge b$.

2. In general:

$$\underbrace{\ell}_{\substack{\text{lower}\\\text{bound}}} \leq \underbrace{\bigwedge S}_{\text{infimum}} \leq \underbrace{a_i}_{\text{component}} \leq \underbrace{\bigvee S}_{\text{supremum}} \leq \underbrace{u}_{\substack{\text{upper}\\\text{bound}}} \equiv \underbrace{-\ell}_{\substack{\text{lower}\\\text{bound}}} \geq \underbrace{\bigwedge -S}_{\text{infimum}} \geq \underbrace{-a_i}_{\text{component}} \geq \underbrace{\bigvee -S}_{\text{supremum}} \geq \underbrace{-u}_{\substack{\text{upper}\\\text{bound}}}.$$

The expression "u majorises the supremum" or the "supremum majorises u" will henceforth be used.

2.1.2 Additive Properties Of Max and Min Operators

The additive properties of max and operators can be achieved by the use of the following device. A constant can be eliminated by subtracting the constant from itself and adding the constant to all terms contained in a max operator. If a positive real constant r is added to a and b we obtain:

$$(a + r) \vee (b + r) = (a \vee b) + r$$

$$(r + a) \vee (r + b) = r + (a \vee b)\,.$$

Subtraction of r contained in the expression appearing to the right of the equality sign from r and each component (a and b), retrieves the expression appearing on the left.

The aforementioned concept applies to a real number added to a max (min) term but if one of the max (min) terms components (either a or b) consists of two or more numbers joined by addition or subtraction and the elimination of one or more of these numbers is sought, add (subtract) the number or numbers from a and b if the number or numbers are negative (positive) and subtract (add) the number or numbers to any real number appearing outside of the max (min) operator.

In the following example, we will write in different equivalent forms some expressions which will be useful later.

Examples. 2.1.22 For real numbers a,b,c and d, the use of the foregoing device yields:

1. Four equivalent forms, with c being shifted to the left hand side and then placed inside the max operator:

$$\begin{aligned} \max(a, b) &= c \\ \max(a, b) - c &= 0 \\ \max(a - c, b - c) - c + c &= 0 \\ \max(a - c, b - c) &= 0. \end{aligned}$$

2. Three equivalent forms[21], with dc being shifted to the left hand side and then placed inside the max operator:

$$\max(a,b) = dc$$
$$\max(a,b) - dc = 0$$
$$\max(a - dc, b - dc) = 0.$$

2.1.3 Duality Of Reals

The dual of the max operator is a min operator together with the interchange of positive and negative signs and vice versa. The application of the following rules to a max or min operator (with the suitable interchange of signs) permits its dual to be obtained:

If a negative sign preceding a max operator is converted into a positive sign preceding a min operator with components of the opposite sign to and reversed relative to the original max operator then repeated use of Equation 2.3 yields $-(a \vee b) = -a \wedge -b$. It follows that $-(a \vee (-b)) = +((-a) \wedge b)$.

Similarly, a positive sign preceding a max operator is converted into a negative sign preceding a min operator with components of the opposite sign to and reversed relative to the original max operator. To be specific, $(a \vee (-b)) = -(-a \wedge (b))$.

A negative sign preceding a min operator is converted into a positive sign preceding a max operator with components of the opposite sign to and reversed relative to the original max operator. That is, $-(a \wedge (-b)) = +((-a) \vee b)$.

Similarly, a positive sign preceding a min operator is converted into a negative sign preceding a max operator with components of the opposite sign to and reversed relative to the original max operator. Specifically, $(a \wedge (-b)) = -(-a \vee b)$.

One or all components of max and min operators and the dual of each can be raised to the power of any real number (which is a distinct concept from raising the entire max operator to such powers). The same applies to max and min operators and the dual of such operators comprising more than two components.

[21] We have omitted max(a-dc,b-dc)-dc+dc=0 for illustrative purposes (although the corresponding expression was included in Part 1 of Examples 2.1.22).

2.2 Distributive Lattices

Numerous lattices require the distributive law $a(b+c) = ab + ac$ in order to admit the analogy between the arithmetic operations + and . and the lattice operations $\vee$ and $\wedge$. In such circumstances, Equation 2.3 and Equation 2.4 become identities involving =, which hold in certain lattices only.

Theorem. 2.2.1 [Birkhoff] For all a,b and c in a lattice L, the identities:
[DLA1] $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$; and
[DLA1′] $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,
are equivalent.

[DLA1] can be written alternatively as $\min[a, \max(b.c)] = \max[\min(a,b), \min(a,c)]$.

Definition. 2.2.2 A lattice is said to be *distributive* if and only if either [DLA1] or its equivalent [DLA1′] hold.

The set of real numbers forms a distributive lattice under the usual ordering. This occurs because of the following:

Lemma. 2.2.3 Any chain is a distributive lattice.

To demonstrate this concept, if a is less than b or c and $a \wedge (b \vee c)$, the result will be a. The dual of a distributive lattice is distributive. Similarly, the sublattice of a distributive lattice is distributive. In addition, the direct product of a distributive lattice is distributive.

Clearly, Part 5 of Examples 2.1.2 is an example of a distributive lattice. That is, $\hat{P}(X)$ consisting of all subsets of any class X under the ordering $x \leq y$ where x is a subset of y under inclusion, is a distributive lattice.

More generally:

Definition. 2.2.4 A family $\bar{S}$ of subsets of a set X, which, along with any two sets S and T in $\bar{S}$ contains their union $S \cup T$ and intersection $S \cap T$, is called a *ring* of sets. A ring of sets containing with any S its set complement S′, is called a field of sets.

Any ring of sets under the natural ordering $x \subseteq y$ is a distributive lattice.

The following property of distributive lattices is of particular importance.

Theorem. 2.2.5 In a distributive lattice L if $\rho \wedge a = \rho \wedge b$ and $\rho \vee a = \rho \vee b$ then a = b for $a, b, \rho \in L$.

Proof. See Birkhoff (1940).

Remark. 2.2.6 The above property is called cancellation and is valid in any distributive lattice.

Any distributive lattice satisfies the following:

[DLA2] If a $\leq$ c then $a \vee (b \wedge c) = (a \vee b) \wedge c$ assuming that we set a < c in [DLA1] or [DLA1′] thereby ensuring that $c = a \vee c$.

This is called the modular identity. We note that although every distributive lattice is modular, there are lattices which are modular but not distributive.

Operations on join and meet yield a series of propositions which are of particular importance in respect of writing max and min operators comprising three real numbers a,b and c.

[DLP1] Maximisation is distributive with respect to minimization in R:

$$\begin{aligned} a \vee (b \wedge c) &= (a \vee b) \wedge (a \vee c) \\ &= [(a \vee b) \wedge a] \vee [(a \vee b) \wedge c]. \end{aligned}$$

This can also be expressed as:

$$\max[a,\min(b,c)] = \min[\max(a,b),\max(a,c)]$$
$$= \max[\min[\max(a,b),a],\min[\max(a,b),c]].$$

Proof. Prove that left hand side of the following is less than or equal to the right hand side:

$$\underbrace{a \vee (b \wedge c)}_{L} \leq \underbrace{(a \vee b) \wedge (a \vee c)}_{\min}.$$

Subtract the left hand side expression above to get:

$$0 \leq (a \vee b) \wedge (a \vee c) - [a \vee (b \wedge c)].$$

Take the dual of the right hand side expression to obtain:

$$0 \leq (a \vee b) \wedge (a \vee c) + [-a \wedge (-b \vee -c)]$$
$$= (a \vee b) + \underbrace{[-a \wedge (-b \vee -c)]}_{=u} \wedge [(a \vee c) + [-a \wedge (-b \vee -c)]].$$

Applying the majorising argument yields:

$$= \underbrace{[0 \wedge ((a-b) \vee (a-c))]}_{\substack{\leq 0 \\ \text{using } a \wedge b \leq a}} \vee \underbrace{[(b-a) \wedge ((0 \vee (b-c))]}_{\leq (b-c)^+} \wedge \underbrace{[0 \wedge ((a-b) \vee (a-c))]}_{\substack{=0 \\ \text{using } a \wedge b \leq a}}$$
$$\vee \underbrace{[(c-b) \wedge (((c-b) \vee 0)]}_{\leq (c-b)^+}$$
$$\leq (b-c)^+ \wedge (c-b)^+ = 0$$

which follows from $a^+ \wedge (-a)^+ = 0$. Now, since $0 \leq a^+$ and $0 \leq (-a)^+$, by taking the min we get:

$$0 \leq a^+ \wedge (-a)^+ = (a \vee 0) \wedge (-a \vee 0) = (-a \wedge 0) \wedge (a \wedge 0) = a \wedge -a \wedge 0 \wedge 0 \leq 0.$$

Thus, $a^+ \wedge (-a)^+ = 0$. □

Remark. 2.2.7

1. The foregoing equality (see [DLP1]) can be verified to be an identity by considering all of the following six possible cases:

i. $b \geq c \geq a$

ii. $b \geq a \geq c$

iii. $c \geq b \geq a$

iv. $c \geq a \geq b$

v. $a \ge b \ge c$

vi. $a \ge c \ge b$

2. The reversal of the order in which components of the expression appearing [DLP1] yields:

$$(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c),$$

which can also be expressed as:

$$\begin{aligned} \max[\min(a,b),c] &= \min[\max(a,c),\max(b,c)] \\ &= \min[\min(a,c),\max(a,b)] \\ &= \begin{cases} a \text{ if } a-c>0 \text{ and } a-b \le 0 \\ b \text{ if } a-b>0 \text{ and } b-c>0 \\ \quad c \quad \text{elsewhere.} \end{cases} \end{aligned}$$

[DLP2] Minimisation is distributive with respect to maximisation

$$\begin{aligned} a \wedge (b \vee c) &= (a \wedge b) \vee (a \wedge c) \\ &= [a \vee (a \wedge b)] \wedge [c \vee (a \wedge b)] \end{aligned}$$

which can also be expressed as:

$$\begin{aligned} \min[a,\max(b,c)] &= \max[\min(a,b),\min(a,c)] \\ &= \min[\max[a,\min(a,b)],\max[c,\min(a,b)]]. \end{aligned}$$

Proof. Given $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ its dual is $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and then consider the right hand side of the latter expression to obtain:

$$\begin{aligned} (a \wedge b) \vee (a \wedge c) &= [(a \wedge b) \vee a] \wedge [(a \wedge b) \vee c] \\ &= a \wedge [(a \vee c) \wedge (b \vee c)] \\ &= \underbrace{[a \wedge (a \vee c)]}_{a} \wedge (b \vee c) \\ &= a \wedge (b \vee c) \end{aligned}$$

□

[DLP3]

$$(a \wedge c) \vee b = (a \vee b) \wedge (b \vee c)$$
$$\max(\min(a,c),b) = \min[\max(a,b),\max(b,c)].$$

Remark. 2.2.8 The max or min for three components is found by applying the foregoing rules to a pair of operators at a time.

Maximum and minimum functions contained within maximum and minimum functions can be subtracted as demonstrated below:

$$(a \vee b) \wedge c = \min[\max(a,b), c].$$

This expression can be re-written as:

$$\begin{aligned}\min[\max(a,b), c] &= \max(a,b) + \min[\max(a,b) - \max(a,b), c - \max(a,b)] \\ &= \max(a,b) + \min[0, c - \max(a,b)].\end{aligned}$$

Equivalently, we can write:

$$(a \vee b) \wedge c = (a \vee b) + [0 \wedge (c - (a \vee b))].$$

The next series of propositions are of particular importance in respect of writing max and min operators comprising four components. Consider the real numbers a,b,c and d:

[DLP4] Minimisation is distributive with respect to maximization:

$$\begin{aligned}(a \vee b) \wedge (c \vee d) &= (a \wedge c) \vee (a \wedge d) \vee (b \wedge c) \vee (b \wedge d) \\ &= (a \wedge c) \vee (b \wedge d) \\ \min(\max(a,b), \max(c,d)) &= \max[\min(a,c), \min(a,d), \min(b,c), \min(b,d)] \\ &= \max[\min(a,c), \min(b,d)].\end{aligned}$$

[DLP5] Maximisation is distributive with respect to minimization:

$$\begin{aligned}(a \wedge b) \vee (c \wedge d) &= (a \vee c) \wedge (a \vee d) \wedge (b \vee c) \wedge (b \vee d) \\ \max(\min(a,b), \min(c,d)) &= \min[\max(a,c), \max(a,d), \max(b,c), \max(b,d)].\end{aligned}$$

Examples. 2.2.9

1. A fuzzy set is characterized by a set X and a mapping from X to the unit interval. The value of the function is the grade of membership of the input to the fuzzy set (Zadeh (1965)). The set of all fuzzy sets of a non-empty set X is a distributive lattice with a 0 and 1 with point-wise ordering.[22]

[22] Boukezzoula & Galichet (2007:1135) state that Equation 2.6 and 2.7 must be discarded when studying fuzzy intervals given that fuzzy numbers are not reliant on the total order relation. Zadeh's (1965) "extension principle" is used in order to extend the lattice operations of min and max on real numbers to the

2. A *Boolean algebra* is a bounded and complemented distributive lattice. This concept is discussed in Section 2.3.

3. A lattice-ordered vector space, which is also called a vector lattice or Riesz space, is a distributive lattice. The concept of Riesz spaces is discussed in Chapter 3.

Remark. 2.2.10 When a max operator is added to a min operator, the validity of the two foregoing operations of addition cease except if the duality result is applied in which case the min operator is converted to a max operator and then added to the first max operator.

[DLP7] The maximum is flat and order-less:

$$\max(a, b, c) = \max(\max(c, b), a)$$
$$a \vee b \vee c = (c \vee b) \vee a.$$

Dually, the minimum is flat and order-less:

$$\min(a, b, c) = \min(\min(c, b), a)$$
$$a \wedge b \wedge c = (c \wedge b) \wedge a.$$

[DLP8] The maximum is flat and order-less:

$$\max(a, b, c, d) = \max[\max(a, b), \max(c, d)]$$
$$a \vee b \vee c \vee d = [(a \vee b) \vee (c \vee d)].$$

Dually, the minimum is flat and order-less:

$$\min(a, b, c, d) = \min[\min(a, b), \min(c, d)]$$
$$a \wedge b \wedge c \wedge d = [(a \wedge b) \wedge (c \wedge d)].$$

Note. 2.2.11

1. [DLP7] can also be written as:

$$\max(a, b, c) = \max(\max(a, b), c)$$
$$a \vee b \vee c = (a \vee b) \vee c.$$

2. Given that one may be tempted to assume that Equation 2.6 has a three component analog, it is demonstrated that if three real numbers a,b and c are considered, then in general:

---

operations of min and max on fuzzy intervals which are stated in Equation 2 of Boukezzoula & Galichet's (2007) article.

$$a \vee b \vee c = \max(a,b,c) \neq \left(\frac{a+b+c+|a-b-c|}{2} \right).$$

and similarly for the corresponding min operator. Moreover, and in general:

$$a \vee b \vee c = \max(a,b,c) \neq \left(\frac{a+b+c+|a-b-c|}{3} \right)$$

$$a \wedge b \wedge c = \min(a,b,c) \neq \left(\frac{a+b+c-|a-b-c|}{3} \right).$$

3. Three-component max operators, with one of the components being equal to zero, cannot be expressed in positive form.

4. The associative and commutative laws and application of the Principle of Mathematical Induction imply the following result (additive form). For any natural number n and given $\max(a_1, a_2, a_i, \ldots, a_n)$, if a_i is removed from the max operator for any i=1, 2, …, n:

$$\begin{aligned} \max(a_1, a_2, a_i, \ldots, a_n) &= a_i + \max(a_1 - a_i, a_2 - a_i, \ldots, a_i - a_i, \ldots, a_n - a_i) \\ &= a_i + \max(a_1 - a_i, a_2 - a_i, \ldots, 0, \ldots, a_n - a_i). \end{aligned}$$

We could remove any one of the components from the max operator instead of a_i. The same device can be applied to any min operator.

If the formula for the max and min operator (for two real numbers) is applied to max(a,b,c) and min(a,b,c) respectively a pair at a time, the correct results can be obtained. Consider max(6,4,2) which yields the pairs created in Figure 2.1.

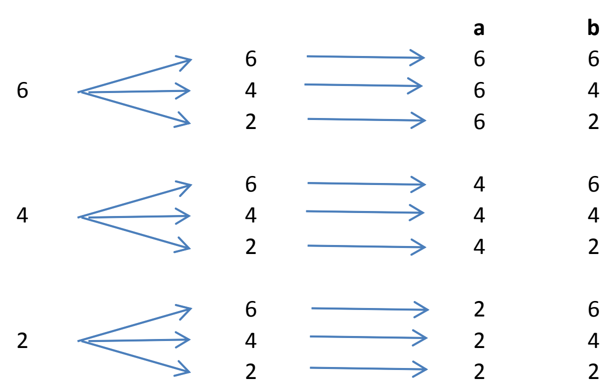

Figure 2.1: Pair Creation Method

The pairs (2,2), (4,4) and (6,6) are dropped with the remaining pairs being available for substitution into the formula.

Remark. 2.2.12 The foregoing approach holds for four and more components.

The last distributive lattice proposition has a third component of 0:

[DLP9]

$$\max(a,b,0) = \max[\max(a,0),\max(b,0)]$$
$$a \vee b \vee 0 = (a \vee 0) \vee (b \vee 0).$$

2.3 Boolean Lattices

Definition 2.3.1 A bounded lattice L with 0 and 1 is said to be *complemented*, if for each $a \in L$ there is one element b such that $a \wedge b = 0$ and $a \vee b = 1$.

We call b a *complement* of a.

Note 2.3.2 If L is a distributive lattice then each $a \in L$, denoted by a′, has at most one complement.

Definition 2.3.3 (Birkhoff) A bounded complemented distributive lattice is called a *Boolean lattice* $\overline{B}$.

In other words, $\overline{B}$ is a distributive lattice with 0 and 1 in which every element has a complement and $0 \neq 1$.

Definition 2.3.4 A *Boolean algebra* is a Boolean lattice in which complementation is considered to be an operation. The elements of 0 and 1 are regarded as constant operations.

Henceforth, the terms Boolean lattice and Boolean algebra will be considered synonymous and will be denoted by $\overline{\mathrm{B}}$. Moreover, $\overline{\mathrm{B}}$ denotes a set with binary operations $\wedge$ and $\vee$, the unary operation of complement along with 0 and 1.

Example. 2.3.5 A typical example of a Boolean algebra is the power set $\hat{\mathrm{P}}(\mathrm{X})$ of a non-empty set X consisting of all subsets of X under the inclusion as a partial order.

Theorem. 2.3.6 In every Boolean algebra, for all $a, b \in \overline{\mathrm{B}}$:

$$(a \vee b)' = a' \wedge b' \ \& \ (a \wedge b)' = a' \vee b'.$$

These equations are called DeMorgan's Laws.

Proof. Due to the elementary nature of these laws, the proof is left to the reader.

It follows from Theorem. 2.3.6 that in a Boolean algebra $\overline{\mathrm{B}}$ for all $a, b \in \overline{\mathrm{B}}$, $a \leq b$ if and only if $a' \geq b'$. In the following, we use the word ring as defined in Algebra.

Definition. 2.3.7 A ring with 1 is called a *Boolean ring* if all elements of the Boolean ring are idempotent.

Every Boolean ring is commutative.

Theorem. 2.3.8 (Lidl & Pilz) If the operations + and . on a Boolean algebra $\overline{\mathrm{B}}$ are defined by $a + b := (a \wedge b') \vee (a' \wedge b)$ and $ab := a \wedge b$, then a Boolean ring with identity is obtained.

If the operation $\wedge$ and $\vee$ on a Boolean ring with 1 is defined by $a \vee b := a + b + ab$ and $a \wedge b := ab$ and the complement of some element x' of x by $x' := x+1$, then a Boolean algebra is obtained.

Proof. See Lidl & Pilz, (1984).

There is a symbolic way of dealing with max and min expressions in a Boolean lattice with multiplication representing the minimum and addition representing the maximum. Algebra on these expressions exist if the maximum is called join and the minimum is called meet in the sense of the axioms of set theory. If there is no cause for confusion, we can write max and min with the symbolic notation of addition and multiplication respectively.

For real-valued components a,b,c and d the following Boolean lattice propositions hold:

[BLP1]

$$(a+b)c = ac + bc$$
$$[(a \vee b) \wedge c] = (a \wedge c) \vee (b \wedge c).$$

[BLP2]

$$\begin{aligned} a + bc &= (a+b)(a+c) \\ &= \underbrace{(a+b)a}_{=a} + (a+b)c \\ &= \underbrace{a + ac}_{=a} + bc \\ &= a \vee (b \wedge c) \\ &= (a \vee b) \wedge (a \vee c) \\ &= [(a \vee b) \wedge a] \vee [(a \vee b) \wedge c] \end{aligned}$$

$$\begin{aligned} a \vee (b \wedge c) &= (a \vee b) \wedge (a \vee c) \\ &= [(a \vee b) \wedge a] \vee [(a \vee b) \wedge c]. \end{aligned}$$

[BLP3]

$$ab + c = (a+c)(b+c)$$

$$(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c).$$

[BLP4]

$$\begin{aligned} a(b+c) &= ab+ac \\ &= \underbrace{(ab+a)}_{=a}(ab+c) \\ &= a(ab+c) \\ &= \underbrace{a(a+c)}_{=a}(b+c). \end{aligned}$$

$$\begin{aligned} a \wedge (b \vee c) &= (a \wedge b) \vee (a \wedge c) \\ &= [(a \wedge b) \vee a] \wedge [(a \wedge b) \vee c] \\ &= [a \vee (a \wedge b)] \wedge [c \vee (a \wedge b)]. \end{aligned}$$

[BLP5]

$$(a+b)c = ac+bc$$

$$[(a \vee b) \wedge c] = (a \wedge c) \vee (b \wedge c).$$

[BLP6]

$$ac+b = (a+b)(b+c)$$

$$(a \wedge c) \vee b = (a \vee b) \wedge (b \vee c).$$

[BLP7]

$$(a+b)(c+d) = (ac)+(bd)$$

$$\begin{aligned} (a \vee b) \wedge (c \vee d) &= (a \wedge c) \vee (a \wedge d) \vee (b \wedge c) \vee (b \wedge d) \\ &= (a \wedge c) \vee (b \wedge d). \end{aligned}$$

Example. 2.3.9 For real-valued components x, y and z:

$$\min[\max(x,y), \max(x,z), \max(y,z)] = \max[\min(x,y), \min(x,z), \min(y,z)]$$
$$(x \vee y) \wedge (x \vee z) \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$$
$$(x+y)(x+z)(y+z) = (xy)+(xz)+(yz).$$

CHAPTER 3. MAX AND MIN OPERATORS IN RIESZ SPACES

This chapter comprises the refinement of certain concepts discussed in Chapter 2. It is demonstrated that the basic theory of max and min operators discussed in the previous chapter is situated within the context of Riesz spaces. We begin by stating the properties of a real vector space as stated in the classic monograph by Gantmacher (1959). A more recent treatment of real vector spaces is contained in the book by Anton & Rorres (2000), among others. In Section 3.1, we define basic concepts. Section 3.2 contains Riesz space propositions, with proofs of important propositions being given.

3.1 Basic Riesz Space Concepts

Definition. 3.1.1 A *real vector space* is a pair (V, R, +, .) where V is non-empty set of elements called vectors and R is the field of real numbers called scalars such that (V, +) is a commutative group under addition, and V is closed under a scalar multiplication "." written as $\rho.v \in V$ where ρ is a scalar and v is a vector satisfying the usual distributive and associative laws.

Example 3.1.2 Suppose $V = R^2$ consisting of all ordered pairs of real numbers (r_1, r_2) with vector addition defined by component-wise addition and scalar multiplication is again defined by component-wise multiplication. Then V is a real vector space of dimension 2.

Similarly, R^3 is vector space of dimension 3 with vectors represented by a row of a triplet of real numbers (r_1, r_2, r_3). In a similar way R^n, denoted by a row of n real numbers with $(r_1, r_2, \ldots, r_n)$ with component-wise vector addition and scalar multiplication, is a real vector space of dimension n. These vectors are called row vectors for obvious reasons because they are represented by rows of real numbers. It is useful to consider the n real numbers written in a column and called a column vector.

For example, we illustrate in R^2 the example of row and column vectors. Let a = (1,1) and b = (3,2) be two vectors in V. Then component-wise addition, subtraction and scalar multiplication of a and b are represented by a + b = (4,3), a – b = (-2,-1) and -2a = (-2,-2).

Note. 3.1.3 A row vector will be represented by $(a_1, b_1) = [a_1 \quad b_1]$ while the transpose of the row vector is a column vector represented by $(a_1, b_1)^T = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$.

Definition. 3.1.4 A real vector space E, with a partial order $\leq$ such that (E, $\leq$) is a lattice consistent with vector addition and scalar multiplication, is called a *vector lattice*. That is, if any two elements $a, b \in E$ have a sup and inf denoted by $a \vee b$ and $a \wedge b$ respectively satisfying:

[VLA1] $a \leq b$ implies $a + c \leq b + c$ for all $a, b, c \in E$.

[VLA2] $0 \leq a$ implies $0 \leq \rho a$ for all $a \in E$ and $\rho \in R_+$.

A vector lattice is also called a *Riesz space*. The modulus | . | of an element a in a Riesz space is defined as $|a| = -a \vee a$.

[VLA1] implies that for $a, b, c, d \in E$ if $a \leq b$ and $c \leq d$, the addition of c and b to both sides of $a \leq b$ and $c \leq d$ respectively yields $a + c \leq b + d$.

Examples. 3.1.5

1. The Euclidean space R^n under the component-wise ordering $a = (a_1, \ldots, a_n) \geq b = (b_1, \ldots, b_n)$ whenever $a_i \geq b_i$ for each i = 1, …, n. The infimum of two vectors is:

$$a \wedge b = [\min\{a_1, b_1\}, \ldots, \min\{a_n, b_n\}],$$

while the supremum is:

$$a \vee b = [\max\{a_1, b_1\}, \ldots, \max\{a_n, b_n\}].$$

2. A Banach lattice is a Riesz space which is also a Banach space with R^n being the best example since it has lattice and vector properties and is complete in the lattice norm (Aliprantis & Border, 2007). A norm ‖ . ‖ on a vector lattice is called a lattice norm if $|a| \leq |b|$ implies $\| a \| \leq \| b \|$ for $a, b \in E$.

3. The space of piecewise linear functions on an interval of R_+ with pointwise ordering representing the payoff and profit/loss functions associated with the purchase or sale of financial options (discussed in Chapter 5).

Remark. 3.1.6 Another useful vector lattice uses the lexicographic or dictionary ordering wherein:

$$(x_1, x_2) \le (y_1, y_2) .$$

Such an ordering means that either:

(i) $x_1 < y_1$; or

(ii) $x_1 = y_1$ and $x_2 \le y_2$.

An example of (i) is x = (3,4) and y = (8,1) since $x_1 < y_1$ (3 < 8) while an example of (ii) is x = (3,4) and y = (3,9) since $x_1 = y_1$ (3=3) and $x_2 \le y_2$ (4 < 9).

The *positive cone* of E is denoted by $E_+ = \{a \in E : 0 \le a\}$.

For $a \in E$, the positive part, negative part and modulus are denoted by $a^+ = a \vee 0$, $a^- = -a \vee 0$ and $|a| = -a \vee a$.

3.2 Fundamental Results On Riesz Spaces

Numerous studies contain various vector lattice propositions. Recent examples include Rhandi (2002), Aliprantis & Border (2006) and Aliprantis & Tourky (2007). In the absence of a study which unifies the aforementioned propositions, the content of this section contains the collection of all known vector lattice propositions. In addition, a few newer propositions which may be useful to mathematicians, mathematical economists and financial economists but are absent from the extant literature, are also included.

Every lattice identity which is true for $a \in R$ and $b \in R$ is also true in every Riesz space E and, more generally, in any E. If the maximum and minimum of both a and b exist, the following theorem holds.

Theorem. 3.2.1 For elements a,b,c and d of a Riesz space E, the following propositions are true:

[VLP1] $a \wedge b = -[(-a \vee -b)]$ & $a \vee b = -[(-a) \wedge (-b)]$,

[VLP2] For a constant $\rho \geq 0$, $(\rho a) \wedge (\rho b) = \rho(a \wedge b)$ & $(\rho a) \vee (\rho b) = \rho(a \vee b)$**,**

[VLP3] $a + b = (a \vee b) + (a \wedge b)$ & $a - b = (a \wedge -b) + (a \vee -b)$**,**

[VLP4] $a^{+} = a \vee 0$ & $a^{-} = -a \vee 0$. Moreover, for

$\rho \geq 0$, $\rho(a^{+}) = (\rho a^{+})$ & $\rho(a^{-}) = (\rho a^{-})$,

[VLP5] $(a^{+})^{+} = a$ if $a \leq 0$ & $(a^{-})^{-} = 0$ for all $a \in R^{n}$,

[VLP6] For $|a| = a^{+} + a^{-} = a \vee -a$**,**

[VLP7] $a = a^{+} - a^{-}$**,**

[VLP8]

$$a \vee b = \left[\frac{a + b + |a - b|}{2}\right] = \begin{cases} a & \text{if } a \geq b \\ b & \text{if } a < b \end{cases}$$

$$\max(a, b) = -(-a \wedge -b),$$

and:

$$a \wedge b = \min(a, b) = \left[\frac{a + b - |a - b|}{2}\right] = \begin{cases} a & \text{if } a \leq b \\ b & \text{if } a > b \end{cases} = a \wedge b = -(-a \vee -b).$$

Having stated the basic properties of Riesz spaces, the modulus properties of Riesz spaces comprise the next group of propositions:

[VLP9] $|a| = a^{+} + a^{-} = a \vee -a$ & $-|a| = -(a^{+} + a^{-}) = -a \wedge a$ thus $|a| = 0$ if and only if $a = 0$,

[VLP10] $|a| = a^{+} \vee a^{-}$**,**

[VLP11] $a^{+} \wedge a^{-} = 0$ & $|a^{+}| \wedge |a^{-}| = 0$,

[VLP12] $|\rho a| = |\rho|\,|a|$ for any $\rho \in R$**,**

[VLP13] $|a - b| = (a \vee b) - (a \wedge b)$**,**

[VLP14] $|a|\vee|b| = \dfrac{\big|\,|a+b|+|a-b|\,\big|}{2}$ & $|a|\wedge|b| = \dfrac{\big|\,|a+b|-|a-b|\,\big|}{2}$,

[VLP15] $|a+b|\vee|a-b| = |a|+|b|$,

[VLP16] $a \perp b$ implies $|a|\vee|b| = |a|+|b|$ thus $|a+b| = |a|+|b|$,

[VLP17] $a^{+} \perp a^{-}$,

[VLP18] $|a-b| = |(a\vee c)-(b\vee c)|+|(a\wedge c)-(b\wedge c)|$,

[VLP19] $\big|\ a-b\ \big| = 2(a\vee b)-(a+b)$.

The following group of propositions relates to distributivity:

[VLP20] $a+b\vee c = (a+b)\vee(a+c)$ & $a+b\wedge c = (a+b)\wedge(a+c)$,

[VLP21] $(a\vee b)\wedge c = (a\wedge c)\vee(b\wedge c)$ & $(a\wedge b)\vee c = (a\vee c)\wedge(b\vee c)$,

[VLP22] $c+(a\vee b) = (c+a)\vee(c+b)$ & $c-(a\wedge b) = (c-a)\wedge(c-b)$,[23]

[VLP23] $(a\vee b)+(c\vee d) = (a+c)\vee(b+d)$ & $(a\wedge b)+(c\wedge d) = (a+c)\wedge(b+d)$,

[VLP24] $(a\vee b).(c\vee d) = \rho_1$ for $\rho_1 \in R$ & $(a\wedge b).(c\wedge d) = \rho_2$ for $\rho_2 \in R$.

The following identity stated by Birkhoff (1940:296) serves as the prelude to a part of the next group of propositions.

[VLP25] $|a\vee c-b\vee c|+|a\wedge c-b\wedge c| = a\vee b-a\wedge b = |a-b|$.

The following group of propositions relates to vector lattice inequalities:

[VLP26] $a \le b$ implies that $a^{+} \le b^{+}$ and $b^{-} \le a^{-}$,

[VLP27] $|a+b| \le |a|+|b|$ & $\big|\ |a|-|b|\ \big| \le |a-b|$,

[VLP28] $|a\vee c-b\vee c| \le |a-c|$ & $|a\wedge c-b\wedge c| \le |a-c|$,

[VLP29] $|a^{+}-b^{+}| \le |a-b|$ & $|a^{-}-b^{-}| \le |a-b|$,

[VLP30] $a\wedge(b+c) \le (a\wedge b)+(a\wedge c)$ for $a,b,c \ge 0$,

[VLP31] $(a+c)\wedge(b+c) \le (a\wedge b)+c$.

[23] Addition distributes over the maximum.

We now include the proofs for some of the foregoing vector lattice propositions. Those that are omitted are considered to be elementary.

Proofs.

VLP1. Since Part 1 and Part 2 are similar, we only prove Part 2.The max majorizes its components:

$$\begin{matrix} a \le a \vee b \\ b \le a \vee b \end{matrix} \Rightarrow \begin{matrix} -a \le a \wedge -b \\ -b \le a \wedge -b. \end{matrix}$$

□

VLP2. For any $\rho > 0$, if $a \le b$ then $\rho a \le \rho b$, $\rho(a \vee b) = \rho a \vee \rho b$.

For the a component we have:

$$a \le a \vee b$$
$$\rho a \le \rho(a \vee b).$$

Similarly, for the b component:

$$b \le a \vee b$$
$$\rho b \le \rho(a \vee b).$$

Taking the max yields $(\rho a \vee \rho b) \le \rho(a \vee b)$ but conversely $\rho a \le (\rho a \vee \rho b)$. Thus, $a \le \frac{1}{\rho}(\rho a \vee \rho b)$. In a similar way, $ba \le (\rho a \vee \rho b)$ implies that $b \le \frac{1}{\rho}(\rho a \vee \rho b)$. It follows that:

$$(a \vee b) \le \frac{1}{\rho}(\rho a \vee \rho b)$$

$$\rho(a \vee b) \le (\rho a \vee \rho b).$$

Thus, $\rho(a \vee b) = (\rho a \vee \rho b)$. □

VLP3. For a+b, add the identities contained in [VLP8] to obtain:

$$\begin{aligned} (a \vee b) + (a \wedge b) &= \left[\frac{a + b + |a - b|}{2}\right] + \left[\frac{a + b - |a - b|}{2}\right] \\ &= \frac{1}{2}\left[a + b + |a - b| + a + b - |a - b|\right] \\ &= a + b. \end{aligned}$$

and similarly for a–b, subtract the same identities to obtain the desired result. □

VLP4. If $\rho > 0$:

$$\begin{aligned}\rho a^- &= \rho((-a \vee 0) \\ &= (-(\rho a) \vee 0) \\ &= (\rho a)^-.\end{aligned}$$

Given that $a^- = (-a \vee 0)$ the max majorizes the –a component and 0 component which are demonstrated in turn. For –a:

$$\begin{aligned}-a &\le a^- \\ \rho(-a) &\le \rho a^- \\ -(\rho a) &\le \rho a^-.\end{aligned}$$

For 0 we have:

$$\begin{aligned}0 &\le a^- \\ \rho(0) &\le \rho a^- \\ 0 &\le \rho a^- \\ -\rho(a \vee 0) &\le \rho a^- \\ (\rho a)^- &\le \rho a^-.\end{aligned}$$

For $(\rho a)^- = \rho(-a \vee 0) = (-\rho a \vee 0)$ the max majorises the $-\rho a$ component and 0 component:

$$\begin{aligned}-\rho a &\le (\rho a)^- \\ 0 &\le (\rho a)^-\end{aligned}$$

$$\begin{aligned}\rho(-a) &\le (\rho a)^- \\ -a &\le \frac{1}{\rho}(\rho a)^-\end{aligned}$$

Thus:

$$\begin{aligned}\rho 0 &\le (\rho a)^- \\ 0 &\le \frac{1}{\rho}(\rho a)^-\end{aligned}$$

$$\begin{aligned}(-a \vee 0) &\le \frac{1}{\rho}(\rho a)^- \\ \rho(a^-) &\le (\rho a)^-.\end{aligned}$$

□

VLP7.

$$\begin{aligned} a &= a^{+} - a^{-} \\ &= (a \vee 0) - (-(a \wedge 0)) \\ &= (a \vee 0) - (-a \vee 0) \end{aligned}$$

or:

$$\begin{aligned} a &= a^{+} - a^{-} \\ &= (a \vee 0) - (-a \vee 0) \\ &= (a \vee 0) + (-(-a) \wedge -0) \\ &= (a \vee 0) + (a \wedge 0) \\ &= a + 0 \end{aligned}$$

Substitute b = 0 into the first part of [VLP3] to get:

$$\begin{aligned} a + 0 &= (a \vee 0) + (a \wedge 0) \\ a &= a^{+} - a^{-}. \end{aligned}$$

□

VLP8. For the proof of $(a \vee b) = -(-a \wedge -b)$ consider the left hand side then:

$$\begin{aligned} a &\le (a \vee b) \\ b &\le (a \vee b). \end{aligned}$$

Rearrangement yields:

$$\begin{aligned} -a &\ge -(a \vee b) \\ -b &\ge -(a \vee b) \end{aligned}$$

Reverse the inequality signs to get:

$$\begin{aligned} -(a \vee b) &\le -a \\ -(a \vee b) &\le -b. \end{aligned}$$

These conditions imply $-(a \vee b) \le (-a) \wedge (-b)$. Adjusting the direction of the inequality yields $(a \vee b) \ge -[(-a) \wedge (-b)]$. To prove that $a \vee b = -[(-a) \wedge (-b)]$ we begin with:

$$\begin{aligned} (-a) \wedge (-b) &\le -a \\ (-a) \wedge (-b) &\le -b. \end{aligned}$$

Adjusting the direction of the inequality we obtain:

$$\begin{aligned} -[(-a) \wedge (-b)] &\ge a \\ -[(-a) \wedge (-b)] &\ge b. \end{aligned}$$

Taking the maximum yields $-[(-a)\wedge(-b)]\geq(a\vee b)$. Therefore, we have proved that $a\vee b=-[(-a)\wedge(-b)]$. The proof of the min operator is similar to that constructed for the max operator, except that inequality signs are reversed in the left and right hand side.

Another approach to proving the first identity arises from [VLP8], which leads to the following:

$$\begin{aligned}\frac{1}{2}\left[a+b+|a-b|\right]&=\frac{1}{2}\left[a+b+(a-b)\vee(b-a)\right]\\&=\frac{1}{2}\left[(2a)\vee(2b)\right]\\&=a\vee b.\end{aligned}$$

□

VLP9. Use:

$$\begin{aligned}|a|&=a^{+}+a^{-}\\&=a+a^{-}+a^{-}\\&=a+2a^{-}\\&=a+(2a)^{-}\\&=a+(-2a\vee 0)\\&=a-2a\vee a\\&=-a\vee a.\end{aligned}$$

It follows that $|a|=a^{+}+a^{-}\geq 0$. The proof can also be devised thus:

$$\begin{aligned}|a|=a^{+}+a^{-}&=(a\vee 0)+(-a\vee 0)\\&=((a\vee 0)-a)\vee((a\vee 0)+0).\end{aligned}$$

Given that:

$$\begin{aligned}a^{+}&=\underbrace{|a|}_{=\underbrace{a^{+}}_{=a+a^{-}}+a^{-}}-a^{-}\\&=\underbrace{a+a^{-}+a^{-}}_{=|a|}-a^{-}.\end{aligned}$$

Thus:

$$
\begin{aligned}
\underbrace{|a|}_{=a^{+}+a^{-}} &= a + a^{-} + a^{-} \\
&= a + 2a^{-} \\
&= a + (2a)^{-} \\
&= a + ((-2a) \vee a + 0) \\
&= -a \vee a.
\end{aligned}
$$

□

VLP10. Consider:

$$
\begin{aligned}
a^{+} &\le a^{+} \wedge a^{-} \\
a^{-} &\le a^{+} \wedge a^{-}.
\end{aligned}
$$

Rearrangement yields:

$$
\begin{aligned}
a &\le a^{+} \wedge a^{-} \\
0 &\le a^{+} \wedge a^{-}.
\end{aligned}
$$

Then:

$$
\begin{aligned}
-a &\le a^{+} \wedge a^{-} \\
0 &\le a^{+} \wedge a^{-}.
\end{aligned}
$$

Thus:

$$
\begin{aligned}
\underbrace{a^{+}}_{=a\vee 0} &\le \underbrace{a^{+} \wedge a^{-}}_{=U} \\
\underbrace{a^{-}}_{=-a\vee 0} &\le \underbrace{a^{+} \wedge a^{-}}_{=U}.
\end{aligned}
$$

Finally we obtain:

$$
\begin{aligned}
-(a^{+} \wedge a^{-}) &\le 0 \\
-(a^{+}) \wedge [-(a)^{-}] &\le 0.
\end{aligned}
$$

□

VLP12. If $0 \le \rho$, $|\rho a| = (-\rho a) \vee (\rho a) = \rho((-a) \vee a) = \rho\,|a| = |\rho|\,|a|$. If $\rho < 0$, then for any scalar $|\rho a| = (-\rho a) \vee (\rho a) = (-\rho)(a \vee (-a)) = -\rho\,|a| = |\rho|\,|a|$. Alternatively, $|\rho a|$ can be expressed as follows. For any scalar ρ:

$$\begin{aligned}|\rho a| &= |\rho|\ |a|\\ &= |\rho|\ [a^{+} + a^{-}]\\ &= |\rho|\ [\max(a,0) + [-\min(a,0)]]\\ &= |\rho|\ [\max(a,0) + [\max(-a,0)]]\\ &= |\rho|\ [(a \vee 0) + (-a \vee 0)]].\end{aligned}$$

□

VLP15.

$$\begin{aligned}|(a+b)| \vee |(-a-b)| &= \left[(a+b) \vee (-a-b)\right] \vee \left[(a-b) \vee (b-a)\right]\\ &= \left[(a+b) \vee (a-b)\right] \vee \left[(-a-b) \vee (b-a)\right]\\ &= \left[a + b \vee (-b)\right] \vee \left[-a + (-b) \vee b\right]\\ &= \left[a + |b|\right] \vee \left[-a + |b|\right]\\ &= \left[a \vee (-b)\right] + |b|\\ &= |a| + |b|.\end{aligned}$$

□

VLP16. For two elements $a,b \in E$, $a \perp b$ are orthogonal or lattice disjoint if $|a| \wedge |b| = 0$. Generally, using [VLP7] $|a| + |b| = |a| \vee |b| + |a| \wedge |b|$.

□

VLP19. From the right hand side we get:

$$\begin{aligned}2(a \vee b) - (a+b) &= 2(a \vee b) - (a \vee b + a \wedge b)\\ &= a \vee b - a \wedge b.\end{aligned}$$

By [VLP13], the last expression is | a - b |. □

VLP20. When the a component appears before the max or min operator (to which it is added), we have $a + (b \vee c) = (a+b) \vee (a+c)$ & $a + (b \wedge c) = (a+b) \wedge (a+c)$. Now:

$$\begin{aligned}b &\le (b \vee c)\\ c &\le (b \vee c).\end{aligned}$$

Add a to both sides to obtain:

$$\begin{aligned}a + b &\le a + (b \vee c)\\ a + c &\le a + (b \vee c).\end{aligned}$$

Take the max of the left-most expression to get $(a+b)\vee(a+c)\le a+(b\vee c)$. For the reverse inequality, clearly:

$$a+b\le(a+b)\vee(a+c)$$
$$a+c\le(a+b)\vee(a+c).$$

Then:

$$b\le(a+b)\vee(a+c)-a$$
$$c\le(a+b)\vee(a+c)-a.$$

Thus, $b\vee c\le(a+b)\vee(a+c)-a$ which implies that $a+(b\vee c)\le(a+b)\vee(a+c)$.

Therefore, as required $a+(b\vee c)=(a+b)\vee(a+c)$. □

VLP22. Replacing addition by subtraction yields the identities:

$$c-(a\vee b)=(c-a)\wedge(c-b)\ \&\ c-(a\wedge b)=(c-a)\vee(c-b).$$

The proof of the right-most expression appearing immediately above is constructed by using [VLP8]:

$$a\wedge b\le a$$
$$a\wedge b\le b.$$

Since the negative of the minimum majorises the negative of its components:

$$-(a\wedge b)\ge -a$$
$$-(a\wedge b)\ge -b.$$

Add c to both sides:

$$c-(a\wedge b)\ge c-a$$
$$c-(a\wedge b)\ge c-b.$$

Take the max of the right-most terms yields $c-(a\wedge b)\ge(c-a)\vee(c-b)$. Now consider the right hand side expression:

$$c-a\le(c-a)\vee(c-b)$$
$$c-b\le(c-a)\vee(c-b).$$

Add –c to both sides:

$$-c+c-a\le -c+[(c-a)\vee(c-b)]$$
$$-c+c-b\le -c+[(c-a)\vee(c-b)].$$

Thus:

$$-a \leq -c + [(c-a) \vee (c-b)]$$
$$-b \leq -c + [(c-a) \vee (c-b)].$$

Take the max of the left-most expression to get $(-a \vee -b) \leq -c + [(c-a) \vee (c-b)]$, with the dual of this expression being $-(a \wedge b) \leq -c + [(c-a) \vee (c-b)]$. Rearrangement yields $c - (a \wedge b) \leq [(c-a) \vee (c-b)]$ and therefore $c - (a \wedge b) = (c-a) \vee (c-b)$. □

VLP23. Two max expressions can be added. Let $z = (a \vee b)$ to obtain:

$$\begin{aligned}(a \vee b) + (c \vee d) &= z + (c \vee d)\\ &= (z + c) \vee (z + d).\end{aligned}$$

Substitute for z = (.) which yields:

$$\begin{aligned}(a \vee b) + (c \vee d) &= [(a \vee b) + c] \vee [(a \vee b) + d]\\ &= (a + c) \vee (a + d) \vee (b + c) \vee (b + d).\end{aligned}$$

Similarly, two min expressions can be added:

$$\begin{aligned}(a \wedge b) + (c \wedge d) &= (a + c) \wedge (a + d) \wedge (b + c) \wedge (b + d)\\ &= (a + c) \wedge (b + d).\end{aligned}$$

□

VLP25. We reproduce the proof from Aliprantis & Burkinshaw (2003). The use of [VLP13] and [VLP3] together with the distributive laws yields:

$$\begin{aligned}&| a \vee c - b \vee c | + | a \wedge c - b \wedge c |\\ &= [(a \vee c) \vee (b \vee c) - (a \vee c) \wedge (b \vee c)] + ... + [(a \wedge c) \vee (b \wedge c) - (a \wedge c) \wedge (b \wedge c)]\\ &= [c \vee (a \vee b) - c \vee (a \wedge b)] + [c \wedge (a \vee b) - c \wedge (a \wedge b)]\\ &= [c \vee (a \vee b) + c \vee (a \vee b)] - [c \vee (a \wedge b) + c \wedge (a \wedge b)]\\ &= [c + a \vee b] - [c + a \wedge b]\\ &= a \vee b - a \wedge b\\ &= | a - b |.\end{aligned}$$

□

VLP26. $a \leq b$ implies $a \vee 0 \leq b \vee 0$ therefore $a^+ \leq b^+$. Similarly for the other part. □

VLP27. Consider the right hand side of the first part of [VLP27] and use $|a| = a \vee (-a)$ and $|b| = b \vee (-b)$ then:

$$\begin{aligned}|a|+|b| &= [a\vee(-a)]+[b\vee(-b)]\\ &= [(a\vee(-a))+b]\vee[(a\vee(-a))-b)]\\ &= \underbrace{(a+b)}_{1}\vee(-a+b))\vee(a-b)\underbrace{\vee(-a-b)}_{2}.\end{aligned}$$

The use of 1 and 2 yields $|a+b| \;\geq\; |a+b|$. It follows from the triangle inequality that:

$$\big|\,|a|-|b|\,\big| \;\leq |a-b|.$$

□

VLP31. Using the min expression from [VLP8]

$$a\wedge b\leq a$$
$$a\wedge b\leq b.$$

Add c to both sides:

$$(a\wedge b)+c\leq a+c$$
$$(a\wedge b)+c\leq b+c$$

Expand left hand side expression:

$$(a+c)\wedge(b+c)\leq a+c$$
$$(a+c)\wedge(b+c)\leq b+c.$$

Add –c to both sides:

$$-c+[(a+c)\wedge(b+c)]\leq a+c-c$$
$$-c+[(a+c)\wedge(b+c)]\leq b+c-c.$$

Thus:

$$-c+[(a+c)\wedge(b+c)]\leq a$$
$$-c+[(a+c)\wedge(b+c)]\leq b.$$

Take the min of the right hand side expression to get $-c+[(a+c)\wedge(b+c)]\leq a\wedge b$.

Then add c to both sides to obtain $[(a+c)\wedge(b+c)]\leq(a\wedge b)+c$. □

Remarks. 3.2.2 The functions that arise from [VLP4] and [VLP9] in the form of E to E_+ are given by, for $a\in E$, $f_1: a\to a^+$, $f_2: a\to a^-$ and $f_3: a\to |a|$ which have the following properties:

1. $a^+=(-a)^-$, $a^-=(-a)^-$ & $|a|=a^+\vee a^-$.

2. If $0 \le a$ then $a^{+} = a$, $a^{-} = 0$ and |a| = a. In addition, $a < 0$ implies that $a^{+} = 0$, $a^{-} = -a$ and $|a| = -a$.

3. |a| = 0 if and only if a = 0. The "only if" is obvious unlike the "if" portion, with the latter being constructed by arguing that if |a| = 0 then $-a \vee a = 0$, which clearly implies that a = 0.

4. [VLP15] yields the triangle inequality:

$$|a+b| \le |a|+|b| \text{ since } |a+b| \le |a+b| \vee |a-b| = |a|+|b|.$$

We now discuss notes which are specifically related to some of the foregoing vector lattice propositions. Others are considered elementary, therefore notes pertaining to them will be omitted.

Note. 3.2.3

VLP1.

i. $a \vee b = -[(-a) \wedge (-b)] = (a-b)^{+} + b = (b-a)^{+} + a$.

ii. $(a \vee b) - (-a \wedge -b) = (a \vee b) + (a \vee b) = 2(a \vee b) = \begin{cases} 2a \text{ if } a \ge b \\ 2b \text{ if } a < b. \end{cases}$

VLP2.

i. [VLP2] may not be valid for negative real numbers in the sense for $\rho \geq 0$,

$$-\rho(a \vee b) \ne \text{-}\rho a \vee \text{-}\rho b \text{ and similarly } -\rho(a \wedge b) \ne \text{-}\rho a \wedge \text{-}\rho b,$$

but for $\rho \geq 0$ the corresponding [VLP2] for negative real numbers are:

$$-\rho(a \wedge b) = \text{-}\rho a \vee \text{-}\rho b \text{ and similarly } -\rho(a \vee b) = \text{-}\rho a \wedge \text{-}\rho b.$$

ii. Max and min expressions are preceded by either a positive and negative sign which is attached to ρ $(\rho \in R : \rho > 0)$ in [LP2][24] and can be re-written in the following ways:

$$\rho \max(a,b) = \max(\rho a, \rho b)$$

$$\rho\left[\frac{a+b+|a-b|}{2}\right] = \rho\left[\frac{a+b+|a-b|}{2}\right]$$

$$\rho(a \vee b) = (\rho a \vee \rho b).$$

[24] These propositions become useful later when we consider multiple contingent claims or forward commitments are bought or sold as part of a given transaction such as the purchase of 100 European-style call options.

Dually, for the min operator:

$$\rho\min(a,b)=\min(\rho a,\rho b)$$
$$\rho\left[\frac{a+b-|a-b|}{2}\right]=\rho\left[\frac{a+b-|a-b|}{2}\right]$$
$$\rho(a\wedge b)=(\rho a\wedge\rho b).$$

However, max and min expressions can be also be succeeded by ρ which has a positive or negative sign because of the commutative property under multiplication. That is, $ab=ba$ admits a positive[25] real constant $(\rho\in R:\rho>0)$ to succeed the max operator, to which it is multiplied:

$$\max(a,b)\rho=\max(a\rho,b\rho)$$
$$\left[\frac{a+b+|a-b|}{2}\right]\rho=\left[\frac{a\rho+b\rho+|a-b|\rho}{2}\right]$$
$$(a\vee b)\rho=(a\rho\vee b\rho).$$

If a positive real constant $(\rho\in R:\rho>0)$ succeeds min, to which it is multiplied then:

$$\min(a,b)\rho=\min(a\rho,b\rho)$$
$$\left[\frac{a+b-|a-b|}{2}\right]\rho=\left[\frac{a\rho+b\rho-|a-b|\rho}{2}\right]$$
$$(a\wedge b)\rho=(a\rho\wedge b\rho).$$

If E is the Riesz space of real numbers R over itself, then the following hold. For $a,b\in R$ the components of a max or min operator can be multiplied by a fraction (1/b) where b > 0:

$$\max(a,b)=b\left[\max\left(\frac{a}{b},1\right)\right]$$
$$=b\left[\frac{\frac{a}{b}+1+\left|\frac{a}{b}-1\right|}{2}\right]$$
$$a\vee b=b\left[\frac{a}{b}\vee 1\right].$$

Dually we have:

[25] Negative numbers multiplying max or min operators yield a different result. Consider on the real line
max (6,5) .-10 = 6.-10 = -60 but max(6.-10,5.-10) = max(-60,-50) = -50
max(0,2) . -1 = 2.-1 = -2 but max(0.-1,2.-1) = max(0,-2) = 0

$$\min(a,b) = b\left[\min\left(\frac{a}{b},1\right)\right]$$

$$= b\left[\frac{\frac{a}{b}+1-\left|\frac{a}{b}-1\right|}{2}\right]$$

$$a \wedge b = b\left[\frac{a}{b} \wedge 1\right].$$

Generally, when E is a Riesz space and a,b belong to E, ρ is a real non-zero scalar we have:

$$\max(a,\rho b) = \rho \max\left(\frac{a}{\rho}, b\right)$$

$$\left[\frac{a+\rho b+|a-\rho b|}{2}\right] = \rho\left[\frac{\frac{a}{\rho}+b+\left|\frac{a}{\rho}-b\right|}{2}\right]$$

$$a \vee \rho b = \rho\left(\frac{a}{\rho} \vee b\right).$$

An example on the real line comprises a = 6, b = 4 and ρ = 2 then:

$$(6 \vee 2.4) = 2\left(\frac{6}{2} \vee 4\right) = 8\,.$$

A two dimensional example consists of the real vectors a = (1,3), b = (2,4) and real-valued scalar ρ = 2 then:

$$(1,3) \vee 2(2,4) = 2\left[\frac{1}{2}(1,3) \vee (2,4)\right] = (4,8)\,.$$

Similarly,

$$\max(\rho a, b) = \rho \max\left(a, \frac{b}{\rho}\right)$$

$$\left[\frac{\rho a + b + |\rho a - b|}{2}\right] = \rho\left[\frac{a + \frac{b}{\rho} + \left|a - \frac{b}{\rho}\right|}{2}\right]$$

$$\rho a \vee b = \rho\left(a \vee \frac{b}{\rho}\right).$$

For real-valued a = 6, b = 4 and ρ = 2:

$$2(6) \vee 4 = 2\left[6 \vee \frac{4}{2}\right] = 12.$$

A two-dimensional example consists of the real vectors a = (1,3), b = (2,4) and real-valued scalar ρ = 2 then:

$$2(1{,}3) \vee (2{,}4) = 2\left[(1{,}3) \vee \frac{1}{2}(2{,}4)\right] = (2{,}6).$$

In both cases, the same equalities are valid when the max operator is replaced by the min operator. When $\rho < 0$ precedes a max and min expression containing at least three components, De Schutter & Van den Boom (2004:183) take the negative of the absolute value of ρ and multiply each component.

VLP3.

i. Given a real number a and real number b = 0:

$$a^+ = \max(a,0) = a \vee 0 = \frac{1}{2}\left(a + 0 \ + |\, a - 0\,|\right) = \frac{1}{2}\left(a + |\, a\,|\right) = \begin{cases} a \text{ if } a \geq 0 \\ 0 \text{ if } a < 0. \end{cases}$$

Notice that the additive form can be represented most concisely in a symbolic fashion (i.e. in the form a^+) compared to the other representations in the equation above, as it contains the fewest letters and symbols. Thus, the foregoing form is considered superlative (although, as we will observe later, such a statement may be impugned in certain circumstances).[26] In the context of financial economics, Baaquie (2004), Ribeiro

[26] Later, we will demonstrate this concept with some examples.

& Webber (2006), Lamberton & Mikou (2008) and Bardou, Bouthemy & Pages (2009) among others denote max(a,0) by a_+.

ii. For a real number x, the application of the first part of [VLP8] to the following expression yields:

$$\max(0, \log x) = \log^+ x = 0 \vee \log x = \frac{0 + \log x + |0 - \log x|}{2} = \frac{\log x + |\log x|}{2}.$$

Barndorff-Nielsen, Mikosch & Resnick (2001) consider the modulus of a real number x:

$$\max(0, \log|x|) = \log^+ |x| = 0 \vee \log|x| = \frac{0 + \log|x| + |0 - \log|x||}{2}.$$

where log is the common logarithm, although the natural logarithm can also be used.[27] To demonstrate the use of this notation, for x = –2, max[0, log |–2|] = max[0, log |2|] = 0.30 and for x = 1, max[0, log |1|] = max[0, log |1|] = 0.00.

iii. For a real number x:

$$\log[\max(x,0)] = \log(x)^+ = \log(0 \vee x) = \log\left[\frac{0 + x + |0 - x|}{2}\right] = \log\left[\frac{x + |-x|}{2}\right].$$

Again, note the application of the first part of [VLP8] above. Carr & Lee (2009) express the logarithm of the positive part of (a/b) where x = (a/b) by:

$$\log\left[\max\left(\frac{a}{b}, 0\right)\right] = \log\left(\frac{a}{b} \vee 0\right) = \log\left(\frac{a}{b}\right)^+.$$

iv. Given a real number a and b = 0 and using Equation 2.8:

$$a^- = -\min(a,0) = -\left(\frac{1}{2}(a - |a|)\right) = -(a \wedge 0) = (-a) \vee 0 = \begin{cases} -a \text{ if } a \le 0 \\ 0 \text{ if } a > 0 \end{cases}.$$

with the dual being $-(a \vee 0) = -a \wedge 0$. Bardou, Bouthemy & Pages (2009) among others use the notation a_- to denote -min(a,0).

v. Karatzas & Shreve (1998:22) express the first part of [VLP4] as:

[27] If S is an asset's price, its common or natural logarithm may be computed for scaling purposes. Scaling applied by financial economists differs to the notion of scaling in mathematics in the sense that economists scale a time series to adjust frequency.

$$a^{\pm} = \max(\pm a, 0) = \pm a \vee 0 .$$

VLP5. If a > 0 then $(a^{+})^{+} = a$.

VLP6. This identity is consistent with:

$$a^{+} = \frac{a + 0 + |a - 0|}{2} = \frac{a + |a|}{2} .$$

Similarly,

$$a^{-} = (-a)^{+} = \frac{-a + |-a|}{2} = \frac{-a + |a|}{2} .$$

which is consistent with:

$$a^{-} = \frac{-a + 0 + |-a - 0|}{2} = \frac{-a + |-a|}{2} = \frac{-a + |a|}{2} .$$

An alternative approach to obtaining a^{+} is to use :

$$|a| + a = 2a^{+} .$$

VLP7. $a^{+} - a^{-} = (a \vee 0) - (-a) \vee 0 = (a \vee 0) + (a \wedge 0) = a + 0 = a$.

VLP8. For real components of the max and min operator, the modulus of one or both components can be taken. An example for a real number x is:

$$\max(1, |x|^2) = 1 \vee |x^2| = 1 \vee |x|^2 = \frac{1 + |x|^2 + \left|1 - |x|^2\right|}{2} .$$

VLP9. The max and min operators, which are connected to the modulus, can be written more concisely in the latter form using [VLP8]:

$$\max(a, -a) = \left(\frac{a + (-a) + |a - (-a)|}{2}\right) = \left(\frac{0 + |2a|}{2}\right) = \left(\frac{|2|\,|a|}{2}\right) = |a| ,$$

$$\min(-a, a) = \left(\frac{(-a) + a - |(-a) - a|}{2}\right) = \left(\frac{0 - |-2a|}{2}\right) = \left(\frac{-\left(|-2|\,|a|\right)}{2}\right) = -|a| .$$

In addition, given $a^{+} = \left(\frac{a + |a|}{2}\right)$, $|a| + a = 2a^{+}$.

VLP13. An equivalent form is:

$$|a-b| = \max(a-b, b-a).$$

Substitution into the first part of [VLP8] yields:

$$\begin{aligned}\max(a-b,b-a) &= \left(\frac{a-b+(b-a)+|a-b-(b-a)|}{2}\right)\\ &= \left(\frac{0+|2a-2b|}{2}\right)\\ &= \left(\frac{|2|\,|a-b|}{2}\right)\\ &= |a-b|.\end{aligned}$$

Similarly (and it can be easily verified that using the same approach):

$$-|a-b| = -[\max(a-b,b-a)] = \min(b-a,a-b).$$

VLP17. That is, the positive part and negative part of a are orthogonal or lattice disjoint because $|a^+| \wedge |a^-| = 0$. This allows the decomposition of a into the difference of two orthogonal positive elements and is unique.

VLP20. A similar expression replaces the addition sign with the subtraction sign:

$$a-b\vee c = (a-b)\wedge(a-c) \ \ \& \ \ a-b\wedge c = (a-b)\vee(a-c).$$

VLP22.

i. We demonstrate with the example of $\underbrace{6+\underbrace{(4\vee 8)}_{8}}_{14} = \underbrace{\underbrace{(6+4)}_{10}\vee\underbrace{(6+8)}_{14}}_{14}$.

Moreover, addition distributes over the minimum. Let c be some real constant then:

$$c+(a\wedge b) = (c+a)\wedge(c+b).$$

An example is $\underbrace{6+\underbrace{(4\wedge 8)}_{4}}_{10} = \underbrace{\underbrace{(6+4)}_{10}\wedge\underbrace{(6+8)}_{14}}_{10}$.

ii. Given $a+(b\vee 0)$ the distributive property applies but does not when $(a+b)\vee 0$.

VLP23. Addition is distributive with respect to maximization:

$$\max(a,b)+\max(c,d)=\max(a+c,a+d,b+c,b+d)$$
$$(a\vee b)+(c\vee d)=(a+c)\vee(a+d)\vee(b+c)\vee(b+d).$$

Dually, addition is distributive with respect to minimization:

$$\min(a,b)+\min(c,d)=\min(a+c,a+d,b+c,b+d)$$
$$(a\wedge b)+(c\wedge d)=(a+c)\wedge(a+d)\wedge(b+c)\wedge(b+d).$$

VLP24. It is possible to perform the product of a max with a max and a min with a min operator based on the vector space dot product. In dimension 1, this turns out to be the ordinary product.

Division of max or min operators is not possible. Other admissible forms of [VLP25] include for $\rho_3,\rho_4\in R$:

$$\min(a,b).\max(c,d) = \rho_3$$
$$\max(a,b).\min(c,d) = \rho_4.$$

We demonstrate this concept with an example. Given the vectors a = (1,4), b = (8,9), c = (3,6), d = (4,11) we obtain max(a,b) = (8,9) and max(c,d) = (4,11). The dot product is therefore $(8,9)\bullet(4,11)=131$.

VLP28. Both parts are the direct consequence of [VLP25].

We now consider Riesz subspaces. In order to do so, the concept of a vector subspace must be considered.

Definition. 3.2.4 A non-empty subset W of a vector space V is called a *vector subspace* of V if W is itself a vector space under the vector addition and scalar multiplication defined on V.

The following are simple examples of a vector subspaces.

Examples. 3.2.5

1. Subspaces of R^2 are $\{0\}$, lines through the origin and R^2 itself.

2. Subspaces of R^3 are $\{0\}$, lines and planes through the origin and R^3 itself.

Theorem. 3.2.6 If W is a non-empty set of one or more vectors in a vector space V, W is a vector subspace of V if and only if the following hold:

a. If a and b are vectors in W, then a + b is in W.

b. If a is any vector in W and ρ is any scalar the ρa is in W.

Definition. 3.2.7 A vector subspace F of Riesz space E is called a *vector sublattice* or *Riesz subspace* if for each $a, b \in F$, $a \vee b$ and $a \wedge b$ in F.

Note. 3.2.8 A vector subspace F of Riesz space E is called an *ideal* if $|b| \le |a|$ and $a \in F$ implies that $b \in F$.

We desist from elaborating on the concept of an ideal because it is a peripheral issue in the approach to the analysis of option transactions considered in the next chapter.

The following definitions will be useful in the following chapters.

Definition. 3.2.9 A *linear hyperplane* is an (n-1) dimensional subspace H of a real vector space V wherein α is a fixed non-zero vector in V and $\alpha \bullet v$ is the dot product. That is:

$$H = \{v \in V : \alpha \bullet v = 0\}.$$

Definition. 3.2.10 A translation J of a linear hyperplane is called an *affine hyperplane* wherein α is a fixed non-zero vector in V and a is an element of the field of real numbers. That is:

$$J = \{v \in V : \alpha \bullet v = a\}.$$

Definition. 3.2.11 An *isomorphism* between two vector spaces V_1 and V_2 is a mapping $f : V_1 \to V_2$ such that:

(i) f is one-to-one and onto; and

(ii) structure is preserved: if $v_1, v_2 \in V_1$ then

$$f(v_1 + v_2) = f(v_1) + f(v_2)$$

and if $v \in V_1$ and $\rho \in R$ then:

$$f(\rho v) = \rho f(v) .$$

When we want to say that V_1 is isomorphic to V_2, we will write $V_1 \cong V_2$. In finite dimensions, if a vector space V over a field $\overline{K}$ is of finite dimension, then $V \cong \overline{K}^n$.

Definition. 3.2.12 A finite set of affine hyperplanes in some vector space $V \cong \overline{K}^n$ where $\overline{K}$ is a field, is called a *finite hyperplane arrangement.*

Henceforth, we will use the field of real numbers ($\overline{K}$ = R). Although infinite hyperplane arrangements are considered in the literature, reference will be made in subsequent chapters to a *hyperplane arrangement* which is taken to be synonymous with the term *finite hyperplane arrangement*.

CHAPTER 4. MAX AND MIN OPERATORS, BOOLEAN NORMAL FORMS AND PIECEWISE LINEAR FUNCTIONS

This chapter consists of an investigation of the link between lattices, max and min operators along with *lattice polynomials* on the one hand and continuous piecewise linear (and related affine) functions on the other. It will be demonstrated that such functions can be written in a specific Boolean normal form suggested by Ovchinnikov (2002)[28]. The aforementioned functions can also be expressed in the equivalent form $y = mx + b$ with $b = 0$ for a linear function and $b \neq 0$ for an affine function and the mx portion being represented by one or several max expressions. We then proceed to devise two algorithms. The first algorithm can be used in converting the function expressed in the form $y = mx + b$, to the equivalent Boolean normal form. The second algorithm is applied when a function expressed in Boolean normal form must converted to the form $y = mx + b$.

We now describe the plan of this chapter. In Section 4.1, we discuss lattice polynomials and the related concept of *Boolean polynomials*. Informally, lattice polynomials are expressions that comprise a finite number of variables $x_1, \ldots, x_n$, two symbols $\vee$ and $\wedge$ (along with parentheses on certain occasions) in a constructive manner. The concept of Boolean normal forms, which arise from lattice polynomials, is then discussed.

Section 4.2 contains the study of the properties of continuous piecewise linear and affine functions, with the concepts of translation and differentiability and integrability of functions involving max and min operators raised to the power one being discussed.

In Section 4.3, we discuss Ovchinnikov's (2002) algorithm for obtaining a function expressed in Boolean normal form from the graph of a continuous piecewise linear function.

In Section 4.4 we construct the two algorithms. The steps of the first algorithm (mentioned in the first paragraph of this chapter) will subsume that used by Ovchinnikov (2002) for obtaining a function expressed in Boolean normal form from the graph of a continuous piecewise linear function. Of those steps that are described by Ovchinnikov

[28] Ovchinnikov (2000) represents a refinement of the work of Wilkinson (1963) and Tarela & Martinèz (1999) in particular.

(2002) and used henceforth, such steps are explained in much greater detail to ensure transparency. Note the hyperplane arrangement approach used by Ovchinnikov (2002) is omitted because it best suits functions of more than one variable.[29]

Finally, Section 4.5 contains a few applications of the algorithm devised in this study. Most of the examples considered represent a new perspective on the study of extant functions that appear in related literature.

4.1 Lattice Polynomials, Boolean Polynomials and Normal Forms

We begin with a brief discussion on lattice polynomials such as definitions and some examples along with their simple properties.

Definition 4.1.1 Let $X_n = \{x_1,...,x_n\}$ be a set of n symbols (called variables or indeterminates) which are polynomials of *weight* one. The lattice polynomials over X_n are obtained from a finite number of successive applications of min $\vee$ and max $\wedge$ starting with:

(i) $x_1,...,x_n$ are lattice polynomials;

(ii) if x_1 and x_2 are lattice polynomials, the same applies to $x_1 \vee x_2$ and $x_1 \wedge x_2$.

If there are only relatively few variables used, then we write x_1 as a x_2 as b and so forth.

Examples 4.1.2

1. Lattice polynomials over $\{a, b\}$ are a, b, $a \vee b$ and $a \wedge b$.
2. Some lattice polynomials over are $a, b, c, a \wedge c, a \wedge b$, $a \wedge c \vee a \wedge b$.
3. $a \vee b \wedge c \vee d$ is another example of a lattice polynomial in four variables.

In (ii) of Definition 4.1.1, let a have weight w and b have weight w′. Then, (recursively) $a \wedge b$ and $a \vee b$ are called lattice polynomials of weight w + w′. The forms $a \wedge a$ (of weight two) and $a \vee (b \wedge a)$ (of weight three) are treated as different polynomials.

[29] The hyperplane arrangement could be used for functions of a single variable but such an approach is more tedious from a computational perspective than the approach that we discuss.

However, these forms are equivalent because each form represents the same function $p : L \to L$ in any lattice.

The different *lattice polynomial functions* of a and b are enumerated given:

Lemma 4.1.3 [Birkhoff] In any lattice, the sublattice L' generated by a pair of elements a and b comprises a, b, $a \vee b = u$ and $a \wedge b = v$.

Proof. See Birkhoff (1940:30).

Lemma 4.1.4 [Birkhoff] In any distributive lattice every polynomial in r symbols is equivalent to a join of meets and dually:

$$p(x_1, \ldots x_r) = \bigvee_{\alpha \in A} \left\{ \bigwedge_{S_\alpha} x_i \right\} = \bigwedge_{\delta \in D} \left\{ \bigvee_{T_\delta} x_j \right\},$$

where are S_α and T_δ are non-empty sets of indices of the non-empty sets A and D (with elements α and δ respectively).

Proof. See Birkhoff (1940:30).

Note. 4.1.5 The sets A and D can satisfy a certain condition, depending on the application. We will discuss a particular condition later in this chapter, because this fairly abstract concept (like other abstract concepts) is best demonstrated by application.

The concept of Boolean polynomial can now be defined in a Boolean lattice. Whereas lattice polynomials are constructed with the operations $\vee$ and $\wedge$, Boolean polynomials (which are lattice polynomials) are constructed with the operations $\vee$, $\wedge$, $'$, 0 and 1. Specifically, the definition of a Boolean polynomial is the same as Definition 4.1.1 except that we add the following clauses:

(i) 0 and 1 are Boolean polynomials;

(ii) If p is a Boolean polynomial then p' is also a Boolean polynomial.

The set of all polynomials over X_n is denoted by P_n.

If a polynomial p must be expressed more systematically or in a more concise form, so-called normal forms are considered.

A pair of Boolean polynomials are equal if they are endowed with identical sequences of symbols. In a Boolean lattice, a polynomial is said to be in *Conjunctive Normal Form* (CNF) if the polynomial is a conjunction of clauses wherein each clause is a disjunction of the variables in X_n. Dually, one can consider *Disjunctive Normal Form* (DNF) where the same polynomial is a disjunction of clauses wherein each clause is a conjunction of the variables in X_n.

Some examples of CNF are $(a \vee c) \wedge (b \vee c)$, $a \wedge (b \vee c)$, $a' \wedge (b \vee c)$, $(a \vee b) \wedge (b' \vee c \vee c')$ and $a \wedge b$. Examples of polynomials that are not in CNF are $(b \vee c)'$, $(a \wedge b) \vee c$ and $a \wedge (b \vee (d \wedge e))$. Respectively, the foregoing three formulae are equivalent to the next three formulae in CNF which are $b' \vee c'$, $(a \vee c) \wedge (b \vee c)$ and $a \wedge (b \vee d) \wedge (b \vee e)$.

Similarly, examples of the DNF are $(a \wedge c) \vee (b \wedge c)$, $a \vee (b \wedge c)$ and $(a \wedge b' \wedge c') \vee (d' \wedge e \wedge f)$. Examples of polynomials that are not in DNF are $a \vee (b \wedge (c \vee d))$ and $(a \vee b)'$.

The most general formulae of DNF and CNF of normal forms in P_n are respectively:

(i) $N_{DNF} = \left\{ \bigvee_{(i_1,\dots,i_n) \in \{1,-1\}^n} d_{i_1,\dots,i_n} \wedge x_1^{i_1} \wedge x_2^{i_1} \wedge \dots \wedge x_n^{i_n} \right\}$ where $d_{i_1,\dots,i_n} \in \{0,1\}$;

(ii) $N_{CNF} = \left\{ \bigwedge_{(i_1,\dots,i_n) \in \{1,-1\}^n} c_{i_1,\dots,i_n} \vee x_1^{i_1} \vee x_2^{i_1} \vee \dots \vee x_n^{i_n} \right\}$ where $c_{i_1,\dots,i_n} \in \{0,1\}$;

where $x_i^1 := x_i$, $x_i^{-1} := x_i^{'}$, $0^1 := 0$, $0^{-1} := 1$, $1^1 := 1$, $1^{-1} := 0$.

The proof of the above statement is found in Lidl & Pilz (1984:35-36).

Informally expressed, (i) represents the most general DNF in "join of meet form" and (ii) represents the most general CNF in "meet of join" form. For N_{DNF} and N_{CNF}, no order is imposed upon the joins and meets respectively, thus precluding the unique determination required for normal forms. To be specific, the join of meets is unique only up to rearrangement of joins and dually meet of joins is unique only up to rearrangement of meets. The foregoing formulae (i) and (ii) have been written with the Boolean symbols $\vee$ and $\wedge$, which can be replaced by max and min respectively wherever relevant.[30]

4.2 Properties Of Piecewise Linear and Affine Functions

We begin with the definitions of a *line segment*, *convex set*, *convex function* and analogous concept of *concave function*. These concepts are required to define a *convex domain* and ascertain whether a function is concave or convex. We will be mainly concerned with convex functions arising out of min and max operators.

Definition. 4.2.1 Let u and v be points belonging to R^n. The *line segment* joining u and v is the set:

$$L(u, v) = \{x \in R^n : x = \lambda u + (1-\lambda)v,\ 0 \le \lambda \le 1\},$$

where x is a point situated on the line segment for a given λ.

A convex set can now be defined.

Definition. 4.2.2 X is a *convex set* if whenever u,v belong to X, the line segment joining u and v is contained in X. That is:

$$u, v \in X \Rightarrow \{\lambda u + (1-\lambda)v \in X, \forall\ 0 \le \lambda \le 1\}.$$

Definition. 4.2.3 Given a function $f : R^n \supseteq X \to R$ with X being a convex set, its *subgraph*[31] is the set of points lying on or below the graph of f:

$$\text{sub } f = \{(y, x) \in R^{n+1}, x \in X \ \ \& \ \ y \le f(x)\}.$$

Conversely, the *epigraph* lies on or below the same graph with:

[30] Thus, the DNF and CNF are said to have a symbolic (arising from Boolean symbols) and alphabetic (arising from the abbreviations max and min) representation.
[31] An alternative name for subgraph which often appears in the literature is *hypograph*.

$$\text{epi } f = \{(y,x) \in R^{n+1}, x \in X \ \ \& \ \ y \geq f(x)\}.$$

Theorem. 4.2.4 The function $f: R^n \supseteq X \to R$ is *concave (convex)* if an only if its subgraph (epigraph) is a convex set.

Note. 4.2.5 Kamien & Schwartz (1991:298) state that y = f(x) is a concave function if for $f: R^n \supseteq X \to R$:

$$\lambda f(u) + (1-\lambda) f(v) \leq f[\lambda u + (1-\lambda)v],$$

and will be convex if :

$$\lambda f(u) + (1-\lambda) f(v) \geq f[\lambda u + (1-\lambda)v].$$

The weighted arithmetic mean of the values of the function at any two points is no greater (less) than the same weighted arithmetic mean of the same arguments (which may be thought of as a straight line superimposed on the function) for concavity (convexity).

If the foregoing inequalities are strict, then f(x) is called strictly convex or strictly concave.

Definition. 4.2.6 A domain in R^n is called a *convex domain* if it is a convex set.

An operation that preserves convexity consists in obtaining the pointwise maximum. If f_1, …, f_m are convex then $f(x) = \max [f_1(x), \ldots, f_m(x)]$ is convex.

We now define some elementary functions.

Definition. 4.2.7 A real-valued function f of a real variable x is a *linear function* if f(x) = ax + b for real numbers a and b.

A linear function is often called a linear form. We will use the term linear form henceforth.

Definition. 4.2.8 The function defined by f(x) = c where c is a constant (which may be a specific real number), is called a *constant function*.

Remark. 4.2.9 For all real numbers x, let f(x) = c. The graph of f(x) = c is a horizontal line comprising all the points (x, f(x)) having f(x) value equal to c for any real x. The constant function is continuous. For piecewise constant functions taking different constant values on different intervals, there are discontinuities.

The piecewise affine or piecewise linear function represents a special case of a piecewise-defined function. Aliprantis & Tourky (2007:221) state that "piecewise affine functions on R^n are real-valued continuous functions that agree with a finite number of affine functions". The space of such functions can be shown to coincide with the Riesz subspace which affine functions on R^n generate.
The ensuing discussion is based on the exposition of Aliprantis, Harris & Tourky (2006).

Definition. 4.2.10 [Aliprantis, Harris & Tourky] A function $f : R \to R$ is called a *piecewise linear function* if there exist real numbers $-\infty < a_0 < a_1 < ... < a_k < \infty$ and pairs of real numbers (m_i,b_i) where m_i is the slope of the i^{th} function and b_i is the vertical intercept of the i^{th} function with i = 0, 1,…, k, k+1 such that:

$$f(x) = \begin{cases} m_i x + b_i & \text{if} \quad a_{i-1} \le x \le a_i \text{ for some } 1 \le i \le k \\ m_0 x + b_0 & \text{if} \quad x \le a_0 \\ m_{k+1} x + b_{k+1} & \text{if} \quad x \ge a_k \end{cases}$$

The functions $f_i(x) = m_i x + b_i$ are referred to as *components* of the representation.[32] A representation of f consists of the parameters {a_0, a_1, …, a_k} representing the bounds of an interval and the pairs (m_i,b_i) for i = 0, 1, …, k, k+1.

We can consider $f : R \to R$ as consisting of three components with f restricted to (-∞,a_0] and [a_k,∞) with a_0 = a and a_k = b, and a function $f : [a, b] \to R$. Such a piecewise linear

[32] Note that Aliprantis, Harris & Tourky (2006) inadvertently write $x \ge a_{k+1}$ rather than $x \ge a_k$ on the third line of the set of equations appearing immediately above and the same occurs in Aliprantis & Tourky (2007).

function is also referred to as *piecewise affine* if there exists a partition $a = a_0 < a_1 < ... < a_k = b$ of the closed interval [a,b] and pairs of real numbers (m_i,b_i), i = 1, 2, …, k, as above in Definition 4.2.10, such that:

$$f(x) = m_i x + b_i \text{ for all } a_{i-1} < x < a_i \qquad (4.1)$$

These definitions imply automatically the continuity of piecewise linear (affine) functions.

Note. 4.2.11

1. The absence of a point of intersection between components implies the presence of a linear function.

2. The length of the graph of each component of a piecewise linear and affine function can be computed from:

$$\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x\,.$$

3. An arbitrary value of x can be set which affects the last component's length and thus the entire function's length. The addition of the length of several pieces yields:

$$\sum_{i=1}^{n}\left(\sqrt{1+\left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i\right).$$

Taking the limit of this expression and assuming that $\max_i(\Delta x_i) \to 0$ the sum becomes the following with suitable limits of integration:

$$\int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx\,,$$

which is the usual arc length.

4. The discriminant associated with the difference between two parameters comprising a representation of a piecewise linear function is denoted by Δ. If $\Delta = 0$ two parameters are equal, Δ > 0 one parameter is greater than another and Δ < 0 means that one parameter is

less than another. As will be demonstrated later, the concept of Δ is useful in revealing how a given piecewise linear or affine function associated with a certain number of components for a given value of Δ can be associated with more or fewer components for another value of Δ (if different).

5. Functions consisting of more than two components alternate from convex to concave (and vice versa) over certain intervals.

The link between a piecewise linear function and piecewise affine function will become clearer with the use of an example, once the two concepts of translation (vertical and horizontal) are considered. Vertical reflection will be discussed immediately after both concepts of translation because it will be needed later.[33]

Definition. 4.2.12 A real-valued function of a real variable is said to have been expressed in the *additive form* if it is a finite sum consisting of constants and/or positive forms of input.

Examples. 4.2.13

1. The function $f(x) = x^+$ is expressed in additive form.
2. For a real-valued constant a, the function $f(x) = (x-a)^+$ is expressed in additive form.
3. For real-valued constants a, b and c the function $f(x) = b + (x-a)^+ + (x-c)^+$ is expressed in additive form.

Note. 4.2.14

1. Max operators and positive forms (Equation 2.8) are used interchangeably.
2. Positive forms are used in the Mathematical context whilst max operators are used in the Economics context.

[33] It is natural for an *isometry* like vertical reflection to be studied immediately after the isometry of vertical translation and horizontal translation.

Vertical Translation

Given a function containing a max operator:

$$f(x) = (x - a)^{+} \qquad (4.2)$$

If a constant c is added (subtracted) from Equation 4. 2 is translated vertically along the f(x) axis upwards (downwards).

Horizontal Translation

If a constant c is added (subtracted) from the input a of Equation 4.2, the graph is translated horizontally along the x axis such that it shifts rightwards (leftwards) respectively by c.

Remarks. 4.2.15

1. Note that if $f(x) = \max(a,x) = a + (x–a)^{+}$, the presence of the constant (a) outside of the max operator implies the translation of max(0,x–a) by a units.
2. If $f(x) = \max(a,x) = a + (x–a)^{+} – c$, the presence of (a–c) outside of the max operator implies the translation of a + max(0, x–a) by c units.
3. Vertical Translation and Horizontal Translation of a piecewise linear function can be done concurrently.

Vertical Reflection

A reflection in the x axis of the function f(x) is given by the mapping $(x, f(x)) \rightarrow (x, -f(x))$. The vertical reflection of Equation 4.2 is:

$$f(x) = – [(x-a)^{+}].$$

Notice that the concept of duality discussed earlier implies that $f(x) = –[(x–a)^{+}] = \min(a–x,0)$.

We are now in a position to proceed with the example. Consider again the piecewise linear function of Equation 4.2. Then, the addition or subtraction of a constant c to and from the max operator with $c \neq 0$ (i.e. vertical translation) yields, respectively, $f(x) = (x - a)^{+} + c$ and $f(x) = (x - a)^{+} - c$ which are piecewise affine functions.

When a piecewise affine function is "unbundled" it consists of a piecewise linear function represented by a max operator and a constant.

Lemma. 4.2.16 [Aliprantis, Harris & Tourky] If $f : R \to R$ is piecewise linear and is restricted to any closed interval of R, then its restriction is also piecewise linear. In addition if [a,b] is a closed sub-interval of R, then the piecewise linear function's components $f : [a, b] \to R$ belong to the components of $f : R \to R$.

It is tedious but simple to check that the collection of all piecewise linear functions form a vector space.

Lemma. 4.2.17 [Aliprantis, Harris & Tourky] The vector space of all piecewise affine functions which are defined on a closed interval [a,b] is a Riesz subspace which is uniformly dense in C[a,b].

Proof. See Aliprantis, Harris & Tourky, 2006.

A piecewise affine function can now be constructed with a max operator from which the relation between piecewise affine functions and Boolean normal forms become apparent.

Aliprantis, Harris & Tourky (2006) give an example of a piecewise affine function of the following form (when $a_0 = 0$):

$$f(x) = b_1 + x^+ - 2(x - a_1)^+ + 2(x - a_2)^+ \qquad (4.3)$$

Figure 4.1 depicts the graph associated with Equation 4.3 by a solid line assuming that $a_0 < a_1 < a_2$ and each value of a is equi-spaced along the x axis.[34]

[34] The depiction of Figure 4.1 in this study differs to the approach taken by Aliprantis, Harris & Tourky (2006) in the sense that they draw the similar graph (Figure 1) of $(t-a_2)^+$, $t^+-(t-a_1)^+$ and $(t-a_2)^+-(t-a_1)^+$ and then deduce the graph of f(t) which corresponds to our f(x) as defined in Equation 4.3. The same comments apply to the relevant part of Aliprantis & Tourky (2007).

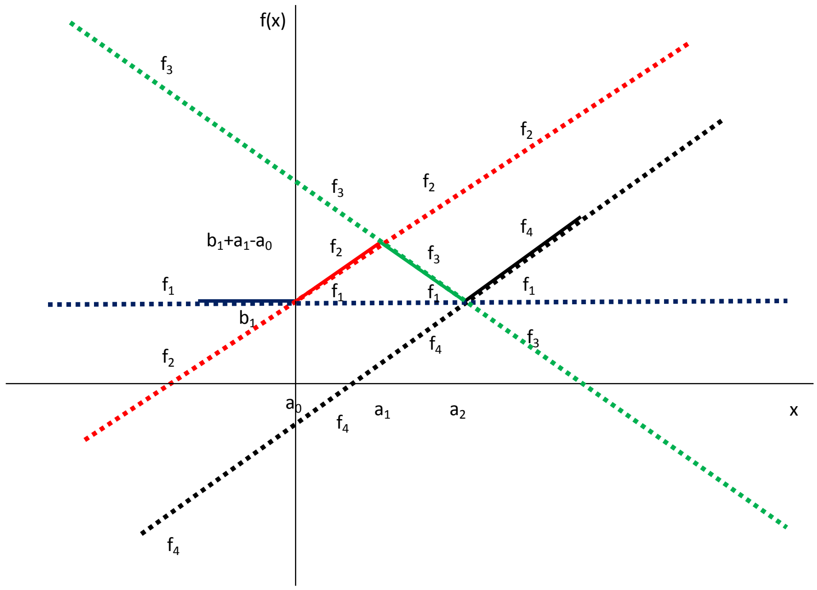

Figure 4.1: Graph Of Aliprantis, Harris & Tourky (2006) Function

Source: Adapted from Aliprantis, Harris & Tourky (2006:82) assuming that a_0, a_1 and a_2 are situated an equal distance apart.

Lemma. 4.2.18 [Aliprantis, Harris & Tourky] Let an arbitrary piecewise affine function be $f:[a,b] \to R$. If a representation of f consists of the parameters $\{a_0, a_1, \ldots, a_k\}$ representing the bounds of an interval and the pairs (m_i, b_i) for i = 0, 1, …, k then for each $x \in [a,b]$[35]:

$$\begin{aligned} f(x) &= b_1 + m_1 x + \sum_{i=1}^{k-1} (m_{i+1} - m_i)(x - a_i)^+ \\ &= b_1 + m_1 x + \sum_{i=1}^{k-1} (m_{i+1} - m_i)[\max(x - a_i, 0)] \\ &= b_1 + m_1 x + \sum_{i=1}^{k-1} (m_{i+1} - m_i)\left[\frac{x - a_i + |x - a_i|}{2}\right] \\ &= b_1 + m_1 x + \sum_{i=1}^{k-1} (m_{i+1} - m_i)(x - a_i \vee 0). \end{aligned}$$

Note. 4.2.19

1. We have written the expression above in four different but equivalent ways to demonstrate that the additive form can be represented symbolically (i.e. a^+ from Equation

[35] Aliprantis & Tourky (2007) inadvertently write $x \in R$ where R denotes the set of real numbers which represents the transfer of the error contained in Aliprantis, Harris & Tourky (2006).

2.8) and alphabetically (i.e. as max(a,0)). Moreover, the additive form can also be expressed in modulus form (using Equation 2.6) as well as Boolean normal form. This practice of writing the same functions in different ways will continue for some of the following examples.

2. The $(x-a)^{+}$ term of the foregoing equation represents a special case of the following function which is defined by Esser (2003) for $x, a \in R$ and $\alpha \in R$:

$$f(x) = [(x-a)^{+}]^{\alpha} = (x-a)^{\alpha} I(x>a) = \sum_{n=0}^{\alpha} \binom{\alpha}{n} x^{\alpha-n} (-a)^{n} I(x>a)\,,$$

where $\binom{\alpha}{n} = \dfrac{\alpha!}{\alpha!(\alpha-n)!}$, the Indicator Function is defined as $I_{x>a} = \begin{cases} 0 & \text{if} \quad x \le a \\ 1 & \text{if} \quad x > a \end{cases}$

and $[(x-a)^{+}]^{\alpha} = (x-a)^{\alpha}$ using the fact that | x–a | = x–a and $|x-a|^{\alpha} = (x-a)^{\alpha}$ (if x > a).

When $\alpha = 1$, we have a piecewise linear or affine function, otherwise the function is non-linear. We define the indicator function and related functions as well as state how such functions relate to max operators in Appendix C.

Corollary. 4.2.20 [**Aliprantis, Harris & Tourky]** A function $f:[a,b] \to R$ is piecewise affine if and only if a partition $a = a_0 < a_1 < ... < a_k = b$ of [a,b] exists together with constants $c_0, c_1, \ldots, c_k$ such that for each $x \in [a,b]$:

$$\begin{aligned} f(x) &= c + \sum_{i=0}^{k} c_i (x-a_i)^{+} \\ &= c + \sum_{i=0}^{k} c_i [\max(x-a_i, 0)] \\ &= c + \sum_{i=0}^{k} c_i \left[\frac{x-a_i + |x-a_i|}{2} \right] \\ &= c + \sum_{i=0}^{k} c_i [x-a_i \vee 0]. \end{aligned}$$

Proof. [Aliprantis, Harris & Tourky]

Let $a \le x \le b$. If $a_0 \le x \le a_1$, then:

$$b_1+m_1x+\sum_{i=1}^{k-1}(m_{i+1}-m_i)(x-a_i)^+ = b_1+m_1x = f(x)\,\mathbf{,}$$

since $(x–a_i)^+$ is 0 for all i = 1, 2, …, k–1. Thus, for some $1\le j\le k$ the assumption that $a_{j-1}\le x\le a_j$ is valid and for each $1<i\le k-1$ $m_i\,a_i+b_i=m_{i+1}\,a_i+b_{i+1}$ or $(m_{i+1}-m_i)a_i=-(b_{i+1}-b_i)$ implying that:

$$\begin{aligned}
f(x) &= b_1+m_1x+\sum_{i=1}^{j-1}(m_{i+1}-m_i)(x-a_i)^+ \\
&= b_1+m_1x+\sum_{i=1}^{j-1}(m_{i+1}-m_i)(x-a_i) \\
&= b_1+m_1x+\sum_{i=1}^{j-1}m_{i+1}x-m_{i+1}a_i-m_ix+m_ia_i \\
&= b_1+m_1x+\left[\sum_{i=1}^{j-1}(m_{i+1}-m_i)\right]x+\sum_{i=1}^{j-1}(m_{i+1}-m_i)\,a_i \\
&= b_1+m_1x+\left[\sum_{i=1}^{j-1}(m_{i+1}-m_i)\right]x+\sum_{i=1}^{j-1}(b_{i+1}-b_i).
\end{aligned}$$

□

A consequence of this lemma follows.

Corollary. 4.2.21 [Brown, Huijsmans & De Pagter] The vector subspace generated by $\{\mathbf{1},x\}\cup\{(\alpha-x)^+:\alpha\in R\}$ and the Riesz subspace of all piecewise affine functions coincide on [0,1].

In the applications considered later, we will assume that $x\in R$ and will consider domains other than [0,1].

Corollary. 4.2.22 [Aliprantis, Harris & Tourky] Consider a piecewise affine function $g:R\to R$. If an arbitrary representation of g consists of $\{a_0, a_1, \ldots, a_k\}$ and (m_i,b_i) for i = 0, 1,…, k, k+1 then for each $x\in R$:

$$g(x)=b_0+m_0x+\sum_{i=0}^{k}(m_{i+1}-m_i)(x-a_i)^+ \,.$$

A function $f : R \to R$ is piecewise affine if and only if real constants m_0, b_0, a_0, a_1, …, a_k and c_0, c_1, …, c_k such that for every $x \in R$:

$$f(x) = b_0 + m_0 x + \sum_{i=0}^{k} c_i (x - a_i)^+ .$$

Proof. [Aliprantis, Harris & Tourky]

Consider a piecewise affine function $h : R \to R$

$$h(x) = b_1 + m_1 x + \sum_{i=1}^{k-1} (m_{i+1} - m_i)(x - a_i)^+ .$$

For all $a_0 \le x \le a_k$, h(x) = f(x) and:

$$h(x) = m_k x + b_k \text{ for all } x \ge a_k$$

$$h(x) = m_1 x + b_1 \text{ for all } x \le a_0 .$$

Given all of the following:

$$m_1 x + b_1 + m_1 (a_0 - x)^+ - m_0 (a_0 - x)^+ = b_0 + m_0 x \text{ for all } x \le a_0$$

$$m_1 x + b_1 + m_1 (a_0 - x)^+ - m_0 (a_0 - x)^+ = m_1 x + b_1 \text{ for all } x \ge a_0$$

$$m_k x + b_k + (m_{k+1} - m_k)(x - a_k)^+ = m_k x + b_k \text{ for all } x \le a_k$$

$$m_k x + b_k + (m_{k+1} - m_k)(x - a_k)^+ = m_{k+1} x + b_{k+1} \text{ for all } x \ge a_k$$

it follows that:

$$\begin{aligned} g(x) &= m_1 (a_0 - x)^+ - m_0 (a_0 - x)^+ h(x) + (m_{k+1} - m_k)(x - a_k)^+ \\ &= m_1 (a_0 - x)^+ - m_0 (a_0 - x)^+ + b_1 + m_1 x \\ &\quad + \ldots + \sum_{i=1}^{k-1} (m_{i+1} - m_i)(x - a_i)^+ + (m_{k+1} - m_k)(x - a_k)^+ \\ &= b_0 + m_0 x + (m_1 - m_0)(x - a_0)^+ + \sum_{i=1}^{k} (m_{i+1} - m_i)(x - a_i)^+ \\ &= b_0 + m_0 x + \sum_{i=0}^{k} (m_{i+1} - m_i)(x - a_i)^+ . \end{aligned}$$

□

4.2.1 Differentiability and Integrability Of Power One Max and Min Operators

In this section we develop the calculus of max and min operators as functions of one variable. In particular, this section contains a discussion of the differentiability and then integrability of max and min functions. Throughout this section, functions of max and min operators are assumed to be raised to the power one. Although these concepts are considered to be elementary, we discuss them in this section to ensure a proper understanding given the ease with which errors can enter into computations when many components are contained in a given piecewise linear or affine function.

4.2.1.1 Differentiability

As for other linear functions, differentiation of a function containing max or min operators which are raised to the power one (therefore the function is piecewise linear) yields the following. Over a given interval, a function is increasing if $f'(x) > 0$ and decreasing if $f'(x) < 0$. If $f'(x) = 0$, the function neither increases nor decreases (a turning point is reached). It is easiest to differentiate such functions when they are expressed in linear form compared to the additive form and DNF. In particular, when the additive form and DNF are represented symbolically, the advantage of using the linear form is most conspicuous.

Example. 4.2.23

Consider $f(x) = (x\text{-}a)^{+}$. When this function is converted to linear form, differentiation of with respect to x yields:

$$\frac{df(x)}{dx} = \begin{cases} 0 \text{ if } x \leq a \\ 1 \text{ if } x > a \end{cases}$$

Next consider $f(x) = \min(x–a,0)$. Differentiation of $f(x)$ with respect to x yields:

$$\frac{df(x)}{dx} = \begin{cases} 1 \text{ if } x \leq a \\ 0 \text{ if } x > a \end{cases}$$

We have obtained the above results through the use of either of the aforementioned approaches to differentiation.

4.2.1.2 Integrability

Given a function $f(x)$ which contains max or min expressions, in computing a precise numeric value of area between the horizontal axis and the graph of $f(x)$ over a given interval, several methods can be used. Note that the acronyms appearing next to the appellation or description of each approach will be used henceforth:

a. plane figures (PF): area is obtained by the computation of plane figures such as squares, rectangles and triangles as well as combinations thereof.

b. integration via the following methods:

i) whole or parts of an interval wherein the original integrand and therefore its upper and lower limits of integration are kept intact with:

- direct evaluation of the max or min operator noting that when parts of the whole interval are considered signs must never be adjusted when splitting expressions (IA1); and
- evaluation by applying Equation 2.6 (assign a negative (positive) sign to the integrand associated with the interval lying to the left (right) of a given threshold) (IA2).

ii) evaluation by considering values of the function (expressed in linear form) in each interval (IB).

Remarks. 4.2.24

1. The IB approach is the easiest to apply is as it requires the least computation given the simplicity of functions expressed in linear form.

2. Should an area associated with one interval among several only be required, IA1 and IA2 proves useful (note the absence of summation as done before).

3. Approach IA1 is useful when there are two or more intervals with non-zero areas whereas IA2 is useful when only one interval has a non-zero area.

4. IA1 and IA2 can be extended to the integral of the minimum of two numbers. Note that any integral of a function comprising max(x,0) (where one or both of the limits are plus/minus infinity) is without a solution since the integral does not converge.

5. Ensure that the co-ordinates for each interval are obtained by substitution into f(x) some of which can be also be obtained by determination of the point(s) at which component functions intersect.

6. IA1 and IA2 are similar in the sense that they arise from the additive form.

7. If the upper limit of integration is $+\infty$ then integration can proceed by the insertion of a Probability Density Function (PDF) or statistical Characteristic Function (CF) in the integrand.

8. If f and g are integrable and f dominates g then the integral of f dominates the integral of g yielding the *Domination Inequality*[36] for Definite Integrals where if f(x) $\geq$ g(x) on [a,b]:

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx.$$

Example. 4.2.25

Suppose that $f(x) = x \vee 0$ with domain [0,1] and range [0,1] is depicted by:

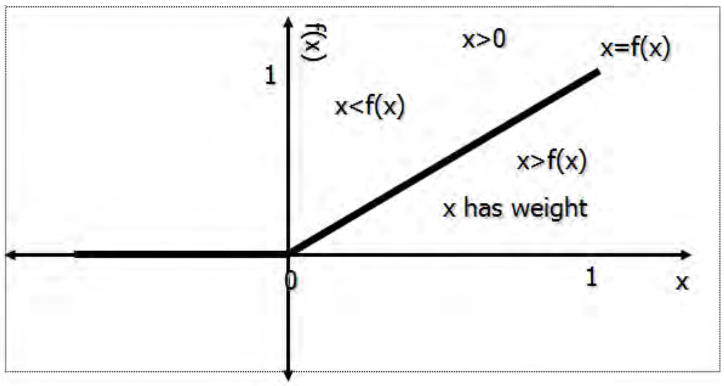

Figure 4.2: Identifying Regions Of Integration

In Figure 4.2, consider the first quadrant. Below the line where x = f(x), x is said to have weight because x > f(x). The area of the triangle demarcated in this region is calculated with the following methods.

PF. The area of the triangle is ½(base)(perpendicular height) = 1/2(1)(1) = 1/2.

To obtain the same result using the methods of integration we write the integrand both in DNF and additive form (symbolically):

$$\int_0^1 f(x)\,dx = \int_0^1 (x \vee 0)\,dx = \int_0^1 x^+\,dx.$$

This expression will be used in the exposition that follows.

[36] This concept is defined by Finney & Thomas (1994). Note that if f(x) is negative over all or a portion of [a,b], the required area will not be obtained from $\int_a^b f(x)\,dx$ and must be computed using $\int_a^b |f(x)|\,dx$.

IA1. Given that x prevails on [0,1]:

$$\int_0^1 f(x)\,dx = \int_0^1 (x \vee 0)\,dx$$
$$= \int_0^1 x^+\,dx$$
$$= \int_0^1 x\,dx$$
$$= \frac{1}{2}.$$

IA2. Applying Equation 2.8 we obtain:

$$\int_0^1 f(x)\,dx = \int_0^1 (x \vee 0)\,dx$$
$$= \int_0^1 x^+\,dx$$
$$= \int_0^1 \frac{x + 0 + |x - 0|}{2}\,dx$$
$$= \int_0^1 \frac{x}{2}\,dx + \int_0^1 \frac{|x|}{2}\,dx$$
$$= \frac{1}{4} + \frac{1}{4}$$
$$= \frac{1}{2}$$

noting that |x| only takes the value x over the interval [0,1].

IB. Given that the values of the function in each interval are:

$$f(x) = x \vee 0 = x^+ = \begin{cases} 0 & \text{if} \quad x \le 0 \\ x & \text{if} \quad x > 0 \end{cases}$$

Clearly, over [0,1], f(x) = x therefore:

$$\int_0^1 f(x)\,dx = \int_0^1 x\;dx = \frac{1}{2}.$$

Note that in three dimensions and higher, max and min functions are also integrable although manual computation is tedious.

4.3 Ovchinnikov's (2002) Algorithm and Related Concepts

The Ovchinnikov (2002) representation theorem, which also appears in Ovchinnikov (2000), will be studied in this section. The representation theorem is based on a fundamental result stated as follows:

Theorem 4.3.1 Any continuous piecewise linear function f on a convex domain in R^n is a lattice polynomial consisting of variables which are linear components of f.

Proof. See Ovchinnikov, 2002.

The following lemma is from Ovchinnikov's (2002) central result.

Lemma. 4.3.2 [Ovchinnikov][37] Let f be a piecewise linear function on $[a, b] \subset R$ and denote the set of its components by $\{f_1, ..., f_n\}$. For given points $[a, b] \in R$, there is i such that $f_i(a) \leq f(a)$ and $f_i(b) \geq f(b)$.

Proof. See Ovchinnikov, 2002.

Figure 4.3 depicts a line segment that connects the points [(a,f(a)), (b,f(b))] with slope $m = \frac{f(b) - f(a)}{b - a}$. Lemma 4.3.2 means that there exists a line segment which has a slope greater than m which intersects the line segment connecting [(a,f(a)), (b,f(b))] thereby satisfying $f_i(a) \leq f(a)$ and $f_i(b) \geq f(b)$ for i = 1, 2, …, n.

[37] Aliprantis, Harris & Tourky (2006) re-cast Lemma 4.3.2 and give a proof as well.

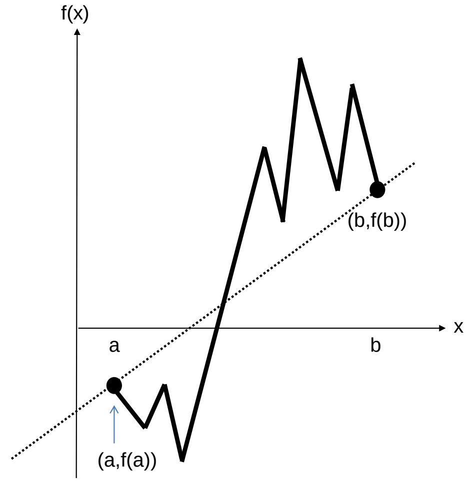

Figure 4.3: Constructing The Graph Of f(x) With Line Segments

The graph of a piecewise linear or affine function can be used in order to obtain the DNF of a max and min polynomial comprising the variables or components f_i which is apparent from the following theorem.

Theorem. 4.3.3 [Ovchinnikov] Let f be a piecewise linear function on a convex domain Γ and $\{f_1, \ldots, f_n\}$ be the set of its distinct components. There exists a family $\{S_j\}_{j\in J}$ of incomparable (with respect to $\subseteq$) subsets of the index set $\{1, \ldots, n\}$ such that:

$$f(x) = \bigvee_{j\in J} \bigwedge_{i\in S_j} f_i(x) \qquad \forall x \in [a,b] \tag{4.4}$$

Proof. [Ovchinnikov]

For any $a \in \Gamma$, $S_a = \{i \in \{1,\ldots n\} : f_i(a) \geq f(a)\}$ and $F_a(x) = \bigwedge_{i\in S_a} f_i(x)$. Therefore $F_a(a) = f(a)$. Using Lemma 4.3.4, for any $b \in [a,b]$, $F_b(a) = \bigwedge_{i\in S_b} f_i(a) \leq f(a)$. Thus:

$$f(x) = \bigvee_{a\in \Gamma} \bigwedge_{i\in S_a} f_i(x) \qquad \forall x \in [a,b] \tag{4.5}$$

Equation 4.5 implies Equation 4.4 when we let $\{S_j\}_{j\in J}$ be the family of distinct minimal elements with respect to $\subseteq$ in the family $\{S_a\}_{a\in\Gamma}$.

□

Equation 4.4 is considered the most transparent Boolean representation and most easily computed when compared with an alternative "hyperplane arrangement approach".[38]

For the purpose of applications, we will use Equation 4.5 later. However, in its current form, Equation 4.5 is rather abstract and thus difficult to interpret without reference to an example, which we now create without giving a specific functional form to each $f_i(x)$.

Example. 4.3.4

Suppose that there are three components of f(x) (i = 3) and over the intervals $[a_1, a_2]$, $[a_2, a_3]$ and $[a_3, a_4]$ with $a_1 < a_2 < a_3 < a_4$, $S_a = \{1,3\}$, $S_a = \{2,3\}$ and $S_a = \{3,2\}$ respectively for each interval. Then, convert the indices into component functions for each S_a therefore $S_a=\{1,3\}$, $S_a=\{2,3\}$ and $S_a=\{3,2\}$ will become $f_1(x) \wedge f_3(x)$, $f_2(x) \wedge f_3(x)$ and $f_3(x) \wedge f_2(x)$ respectively. Next, take the maximum of $f_1(x) \wedge f_3(x)$, $f_2(x) \wedge f_3(x)$ and $f_3(x) \wedge f_2(x)$ which yields:

$$f(x) = [f_1(x) \wedge f_3(x)] \vee [f_2(x) \wedge f_3(x)] \vee [f_3(x) \wedge f_2(x)].$$

In this particular example, besides recognising that f(x) is in DNF, the min of $f_2(x)$ and $f_3(x)$ are repeated. Applying idempotency [LA3], we can obtain the more succinct equivalent DNF expression:

$$f(x) = [f_1(x) \wedge f_3(x)] \vee [f_2(x) \wedge f_3(x)] \qquad (4.6)$$

If the equations of the component functions (f_i) are given, these must be substituted into Equation 4.6 to obtain f(x).

4.4 Construction Of Algorithms

We will construct two algorithms in this section and refer to the first as the *reverse decomposition algorithm* whilst we will refer to the second as the *decomposition algorithm*. We describe the reverse decomposition algorithm first because it differs to the established terminology in the literature, wherein reference is made to decomposition. In addition, the use of the word "reverse" is in fact appropriate since the method used in

[38] Refer to Ovchninnikov (2002) for details.

reverse decomposition is the converse of its decomposition counterpart. The steps comprising each algorithm will be stated in detail to ensure transparency.

4.4.1 Reverse Decomposition Algorithm

To construct a function and its associated graph in DNF, we introduce the reverse decomposition algorithm. In order to implement this algorithm, it is essential to identify at least one piecewise linear or affine function to which a linear function or another piecewise linear or affine function is to be added or subtracted. The piecewise linear functions/affine functions must be of the form represented by Equation 4.1. The domain is x and co-domain is f(x). In addition, it is also necessary to identify whether any of these functions will be multiplied by a positive integer ($\rho \in Z^{+}$) or negative integer ($\rho \in Z^{-}$) noting that $-\rho \max(a,b) \neq \max(-\rho a, -\rho b)$. Either addition or subtraction of the foregoing functions will yield another piecewise linear or affine function. To the latter function, the process of addition or subtraction of the linear function and piecewise linear or affine functions can continue. The process mentioned can continue *ad infinitum*.

The graph of a function can then be drawn in stages. First, divide the domain into sub-intervals. At least two pieces and therefore two intervals must comprise a piecewise linear function otherwise a function is linear in which case application of the algorithm must cease. Denote the function in the first interval (left-most interval) f_1 and denote f_2 the interval to its immediate right and so forth. Find the value in each interval of the component function by substituting values for x into the f(x). Then identify the points at which each component function agrees (the intersection of hyperplanes) taking pairs of components at a time. The point at which the pieces intersect is obtained by solving a set of linear equations using Cramer's Rule. If points of intersection appear in any sub-interval, that sub-interval must be split thereby creating two intervals. Finally, determine the payoff function's slope by differentiating it with respect to x. The result will be a real number with a positive (negative) sign meaning a positive (negative) slope. The graph of the function can then be drawn.

The DNF of the function can then be obtained by following the proceeding steps:

Step 1. Choose point a, which must lie within the domain of the first interval identified.

Step 2. Determine the value of the function corresponding to this point i.e. f(a).

Step 3. Determine whether the component function $f_1 \geq f(a)$. If $f_1 < f(a)$, exclude the index of f_1 i.e. exclude 1 from a set containing indices of all component function but include it if $f_1 \geq f(a)$.

Step 4. Attach the min symbol to indices contained in the set. If one index is contained in the set we have $\wedge 1$ for example and if the set contains (for instance) two indices, we have $1 \wedge 2$.

Step 5. Choose a point called a, which lies within the domain of the second interval identified and repeat Step 2 to Step 4.

Step 6. If all intervals have been considered in accordance with the aforementioned steps, take the max of all sets created and express the result as a function of x. The DNF of f(x) has been created.

Step 7. Use the distributive laws to condense the expression created in Step 6. In other words, each component of the expression will appear exactly once.[39]

Step 8. Replace indices by the values of each function to which they correspond.

A useful feature of the reverse decomposition algorithm is that if we are only given the graph of f(x) with co-ordinates identified we can, besides deducing the components of f(x) in each interval using the point-slope equation; obtain f(x) in DNF. As will be demonstrated in the sequel, the reverse decomposition algorithm can be applied to f(x) in DNF to obtain the equivalent form consistent with Equation 4.1.

4.4.2 Decomposition Algorithm

It has been demonstrated that the reverse decomposition algorithm can be used in creating a function in DNF. Sometimes, a function expressed in the form of Equation 4.1 is needed because it is more familiar and tractable than the equivalent DNF form. In addition, since f(x) expressed in the form of Equation 4.1 is in additive form, it may often be more concise than the DNF of f(x).

[39] Ovchinninkov (2002) states that the distributive law cannot be used in compressing a less concise Boolean normal form of a function in the higher-dimensional case.

In general, the DNF form can be converted into the form of Equation 4.1 using a basic decomposition algorithm. The aim is to ensure that as many max operators (containing two components) as possible have one component equal to 0 so that the expression obtained can be written in the form of Equation 2.8. In general, all of the max operators comprising a given expression must satisfy the following conditions. That is, either all components match the point at which all functions comprising f(x) intersect or one component must match the point of intersection of two functions with repetition. If not, then when the function in addition form is evaluated, some or all of the component functions of f(x) will be of the wrong form. For example, Equation 4.3 is written in a form that is amenable to the identification of the correct functions prevailing in the intervals shown in Figure 4.1. The importance of identifying points at which component functions intersect is thus apparent from the aforementioned comments.

From this observation, we create the required steps of the algorithm which are:

Step 1. Replace the $\wedge$ and $\vee$ symbols by min and max respectively if required.

Step 2. If a max expression is contained within a min expression (or vice versa) that max expression must be removed by addition and subtraction of components thus yielding (for example) the sum of a max expression and a min expression.

Step 3. The remaining components of the max operator which contain a min operator (or vice versa) which are not equal to zero should be eliminated by evaluation wherein the value of x is compared to the values $\{a_0, a_1, \ldots, a_k\}$, (considering the relation between each value of a) to convert a max or min expression to one or several functions.

Step 4. Convert the remaining expressions from min to max if necessary, ensuring that all terms amenable to being written in the additive form (Equation 2.8) must be written in that way.

The result of Step 3 may yield two additive forms of f(x) which yield the correct form of each component function for all intervals, in the sense that points of intersection of component functions are incorporated.

It should be emphasised that as many terms as possible must be expressed in the form of Equation 2.8 because, as mentioned earlier this notation is generally more concise than the DNF (comprising $\wedge$ and $\vee$ notation). This can only occur if max or min operators containing more than two components are compressed by addition and subtraction along with evaluation into two component max and min operators.

The equivalence of DNF and additive form can be checked in two ways. First, by substituting values of x into the function expressed in either form and confirming that values of f(x) in either case are the same, for any value of x in the domain where the functions considered above are defined. The decomposition algorithm and the reverse decomposition algorithm are considered to have been applied properly if the DNF form and the equivalent expression in the form of Equation 4.1 both yield identical values of f(x), for any value of x.

Note. 4.4.1

1. In the two component case only, it is possible to substitute the max operator's components into Equation 2.6 considering the bounds of each interval arising from $\{a_0, a_1, \ldots, a_k\}$ in succession. Thus, use of the decomposition algorithm in the two-component case may not be worthwhile from a computational perspective.
2. The precise area between the graph of a given function and the horizontal axis (over a given interval) can be computed more easily when the function is in the form of Equation 4.1 than when it is in DNF.

4.5 General Applications Of Reverse Decomposition and Decomposition Algorithms

While the reverse decomposition algorithm and the decomposition algorithm devised in the previous section can also be applied to two component functions we will only consider three and four component functions in applying our algorithms in this section. However, in Chapter 5 we will apply our algorithms to two-component functions. We have included functions in Section 4.5.1-Section 4.5.4 on the basis of some or all of the following criteria:

i. relevance to the work of Ovchinnikov (2002);

ii. comprise three or four components; and

iii. complicated functional representation.

Note that we will defer our study of the computation of the area between the graph of a given function and the horizontal axis to Chapter 5 and Chapter 6 where we attach an economic meaning to area in the study of option payoff functions.

In addition, note that in Section 4.5.1-Section 4.5.5 the steps of the reverse decomposition algorithm and decomposition algorithm will be followed implicitly to ensure brevity of exposition.

4.5.1 Ovchinnikov (2002) & Ovchinnikov (2010) Functions

The following function is defined by its components:

$$f(x) = \begin{cases} 2x+1 & \text{if} \quad x \le 1 \\ 5-2x & \text{if} \quad 1 \le x \le 2 \\ 0.5x & \text{if} \quad x \ge 2 \end{cases}$$

and differentiation with respect to x of each component function yields:

$$\frac{df(x)}{dx} = f'(x) = \begin{cases} 2 & \text{if} \quad x \le 1 \\ -2 & \text{if} \quad 1 \le x \le 2 \\ 0.5 & \text{if} \quad x \ge 2 \end{cases}$$

We will adhere to the convention in assigning names to functions that f_1 will denote the function corresponding to the left-most interval on the real line, f_2 will denote the function corresponding to the interval adjacent and to the right of the interval to which f_1 corresponds and so forth. This practice will continue henceforth wherever similar examples are studied.

The points of intersection of each function are:

f_1 and f_2: (1,3)

f_1 and f_3: (-0.67, -0.33)

f_2 and f_3: (2,1)

The information required for obtaining the DNF appears in Table 4.1. A comment on the content of the format of Table 4.1 is required, because the format used herein will apply to other piecewise linear functions which are studied later in this chapter as well as the next. *Column a* represents the intervals $[a_0,a_1]$ which in this example is [-0.67,1], $[a_1,a_2]$ corresponds to [1,2] and $[a_2,\infty)$ corresponds to $[2,\infty)$. Then the value of a must be chosen from which f(a) is obtained. *Column b* represents the component functions $f_i(x)$ that prevail in a given interval in the sense that $f_i(x) = f(a)$. To be specific, over [-0.67,1], f(a) $= f_1(x) = 2x+1$ and similarly for the remaining intervals. *Column c* represents the i^{th} (i = 1,2,3) function over a given interval which is greater than or equal to f(a). Again, to be specific and using Figure 4.4, over [-0.67,1] $f_2(x)$ and $f_1(x)$ are greater than or equal to f(a) where we defined $f(a) = f_1(x) = 2x+1$ and similarly for the other intervals. For each interval, the corresponding S_a appears in *Column d* and $F_a(x)$ appears in *Column e*. Clearly, the progression from Column d to Column e of Table 4.1 involves the conversion of an index to the number of a function. Once this process is complete, the min of all functions over a given interval can be taken.

Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c) $S_a =$	Minimum of functions corresponding to indices in S_a $F_a(x) =$
a	**b**	**c**	**d**	**e**
[-0.67,1]	$f_1(x) = 2x+1$	$f_2(x),f_1(x)$	{2,1}	$f_2(x) \wedge f_1(x)$
[1,2]	$f_2(x) = 5-2x$	$f_1(x),f_2(x)$	{1,2}	$f_1(x) \wedge f_2(x)$
$[2,\infty)$	$f_3(x) = 0.5x$	$f_1(x),f_3(x)$	{1,3}	$f_1(x) \wedge f_3(x)$

Table 4.1: Algorithm Inputs – Ovchinnikov (2002) Function

The max of the expressions contained in all of the rows in Column e of Table 4.1 is then taken to obtain:

$$f(x) = [f_1(x) \wedge f_2(x)] \vee [f_1(x) \wedge f_3(x)].$$

Next, using the distributive law and [LA3] we obtain:

$$f(x) = f_1(x) \wedge [f_2(x) \vee f_3(x)].$$

This result is consistent with Ovchinnikov (2002). The values of each function could be substituted into the equation above if required. We convert f(x) expressed in DNF symbolically into its equivalent alphabetic representation so that f(x) can be written in the form of Equation 2.8. Thus:

$$
\begin{aligned}
f(x) &= \min(f_1(x), \max[f_2(x), f_3(x)] \\
&= \min(2x+1, \max[(5-2x), (0.5x)]) \\
&= \max[(5-2x), (0.5x)] \\
&\quad + \min(2x+1-\max[(5-2x), (0.5x)], \max[(5-2x), (0.5x)] - \max[(5-2x), (0.5x)]) \\
&= \max[(5-2x), (0.5x)] + \min(2x+1-\max[(5-2x), (0.5x)], 0) \\
&= 0.5x + \max[(5-2x) - 0.5x, 0.5x - 0.5x] \\
&\quad + \min(2x+1-\{0.5x + \max[(5-2x)-0.5x, 0.5x-0.5x]\}, 0) \\
&= 0.5x + \max[(5-2x) - 0.5x, 0.5x - 0.5x] \\
&\quad + \min(2x+1-\{0.5x + \max[(5-2x)-0.5x, 0]\}, 0)
\end{aligned}
$$

If $x \geq 2$

$$
\begin{aligned}
f(x) &= 0.5x + 2.5\max(2-x, 0) - \max[-1.5x-1, 0] \\
&= 0.5x + 2.5(2-x)^{+} - [(-1.5x-1)^{+}].
\end{aligned}
$$

If $x < 2$ then:

$$
f(x) = 0.5x + 2.5\max[2-x, 0] + 2.5\min[x-1, 0].
$$

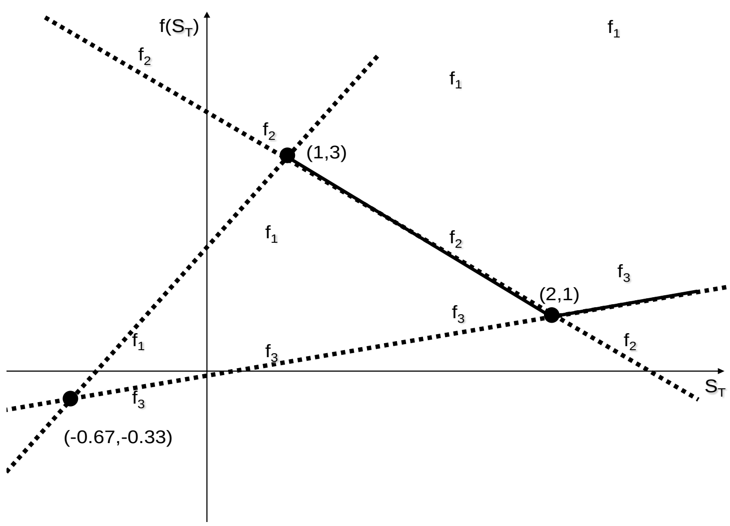

Figure 4.4: Graph Of Ovchinnikov (2002) Function

Adapted from Ovchinnikov, (2002).

Ovchinnikov (2010) defines the following function by its components:

$$
f(x) = \begin{cases} x+2 & \text{if} \quad x \leq -1 \\ -x & \text{if} \quad -1 \leq x \leq 1 \\ 0.5x - 1.5 & \text{if} \quad x \geq 1 \end{cases}
$$

and:

$$f'(x) = \begin{cases} 1 & \text{if} \quad x \le -1 \\ -1 & \text{if} \quad -1 \le x \le 1 \\ 0.5 & \text{if} \quad x \ge 1 \end{cases}$$

Intersections of the following functions include:

f_1 and f_2: (-1,1)

f_1 and f_3: (-7,-5)

f_2 and f_3: (1,-1)

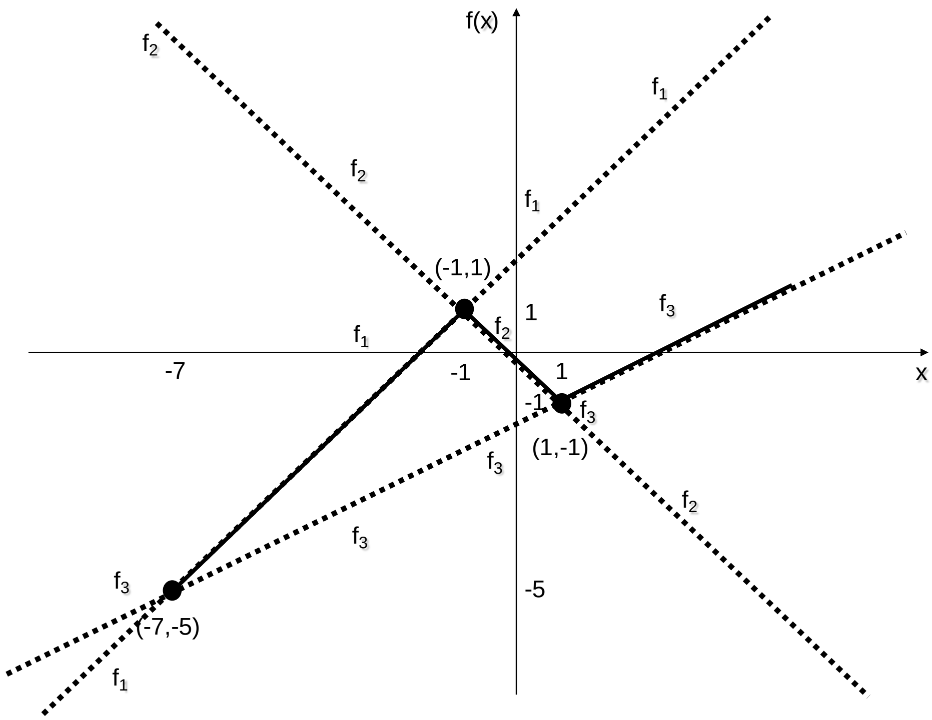

Figure 4.5: Graph Of Ovchinnikov (2010) Function

Adapted from Ovchinnikov, (2010).

The information required for obtaining DNF appears in Table 4.2 with the use of Figure 4.5 to obtain Column c and Column d in particular of the same table:

Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c) $S_a =$	Minimum of functions corresponding to indices in S_a $F_a(x) =$
a	**b**	**c**	**d**	**e**
[-7,-1]	$f_1(x) = x+2$	$f_2(x), f_1(x)$	{2,1}	$f_2(x) \wedge f_1(x)$
[-1,1]	$f_2(x) = -x$	$f_1(x), f_2(x)$	{1,2}	$f_1(x) \wedge f_2(x)$
$[1,\infty)$	$f_3(x) = 0.5x-1.5$	$f_1(x), f_3(x)$	{1,3}	$f_1(x) \wedge f_3(x)$

Table 4.2: Algorithm Inputs – Ovchinnikov (2010) Function

Taking the max and applying [LA3] yields:

$$\begin{aligned} f(x) &= [f_1(x) \wedge f_2(x)] \vee [f_1(x) \wedge f_3(x)] \\ &= f_1(x) \wedge [f_2(x) \vee f_3(x)] \\ &= \min\left[f_1(x), \max[f_2(x), f_3(x)]\right] \\ &= \min[(x+2), \max(-x, 0.5x-1.5)] \\ &= \max(-x, 0.5x-1.5) + \min\left[x+2-\max(-x, 0.5x-1.5), 0\right] \\ &= -x + \max(0, 0.5x-1.5+x) + \min\left[x+2-(-x+\max(0, 0.5x-1.5+x)), 0\right] \\ &= -x + \max(0, 1.5x-1.5) + \min\left[2x+2-\max(0, 1.5x-1.5), 0\right] \\ &= -x + 1.5\max(0, x-1) + \min\left[2x+2-1.5\max(0, x-1), 0\right]. \end{aligned}$$

If $x \geq 1$ then:

$$f(x) = -x + 1.5(x-1)^+ - 0.5(-x-7)^+ .$$

If $x < 1$ then:

$$f(x) = -x + 1.5(x-1)^+ - 2(-1-x)^+ .$$

It can be verified that these expressions are both equivalent to each other as well as the original equations. However, we prefer the additive form which is obtained by assuming that $x < 1$ because the point of intersection at $x = 1$ is considered.

4.5.2 Aliprantis, Harris & Tourky (2006) Function

Recall from Equation 4.3 which is in additive form that if $a_0 = 0$ the linear form will be:[40]

[40] From the perspective of option payoffs, Equation 4.3 can be interpreted as combination consisting of a bond with a terminal payoff of b_1 (b_0 is the bond's price on the date that the contract is created), one call option with strike a_0 held (with price t on the date the contract matures), two call options with a strike price of a_1 on a stock written and two call options with strike a_2 being held.

$$f(x)=\begin{cases} b_1 & \text{if} \quad x<a_0 \\ b_1+x-a_0 & \text{if } a_0 \le x < a_1 \\ b_1-x-a_0+2a_1 & \text{if } a_1 \le x \le a_2 \\ b_1+x-a_0+2a_1-2a_2 & \text{if} \quad x > a_2 \end{cases}$$

and:

$$f'(x)=\begin{cases} 0 & \text{if} \quad x<a_0 \\ 1 & \text{if} \quad a_0 \le x < a_1 \\ -1 & \text{if} \quad a_1 \le x \le a_2 \\ 1 & \text{if} \quad x > a_2 \end{cases}$$

The points of intersection are:

f_1 and f_2: (a_0, b_1)

f_2 and f_3: $(a_1, b_1+a_1-a_0)$

f_3 and f_4: $(a_2, b_1-a_0+2a_1-a_2)$

The point of intersection is $(a_2, b_1-a_0+2a_1-a_2)$ in Figure 4.1 but an inspection of the diagram appearing in Aliprantis, Harris & Tourky (2006) implies that $f(a_2)=b_1$. This is a special case of a_0, a_1 and a_2 being an equal distance apart.

Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c) $S_a =$	Minimum of functions corresponding to indices in S_a $F_a(x) =$
a	**b**	**c**	**d**	**e**
$(-\infty,a_0]$	$f_1(x) = b_1$	$f_3(x),f_1(x)$	{3,1}	$[f_3(x) \wedge f_1(x)]$
$[a_0,a_1]$	$f_2(x) = b_1+x-a_0$	$f_3(x),f_2(x)$	{3,2}	$[f_3(x) \wedge f_2(x)]$
$[a_1,a_2]$	$f_3(x) = b_1-x-a_0+2a_1$	$f_2(x),f_3(x)$	{2,3}	$[f_2(x) \wedge f_3(x)]$
$[a_2,\infty)$	$f_4(x) = b_1+x-a_0+2a_1-2a_2$	$f_2(x),f_4(x)$	{2,4}	$[f_2(x) \wedge f_4(x)]$

Table 4.3: Algorithm Inputs – Aliprantis, Harris & Tourky (2006) Function

Using Table 4.3, when the maximum is taken we obtain the function in DNF:

$$\begin{aligned}
f(x) &= [f_3(x)\wedge f_1(x)]\vee[f_3(x)\wedge f_2(x)]\vee[f_2(x)\wedge f_3(x)]\vee[f_2(x)\wedge f_4(x)]\\
&= [f_3(x)\wedge f_1(x)]\vee[f_2(x)\wedge f_3(x)]\vee[f_2(x)\wedge f_4(x)]\\
&= \left[b_1 - x - a_0 + 2a_1 \wedge b_1\right]\vee\left[b_1 - x - a_0 + 2a_1 \wedge b_1 + x - a_0\right]\\
&\ \vee\left[b_1 + x - a_0 \wedge b_1 + x - a_0 + 2a_1 - 2a_2\right]\\
&= \max(\min\left[b_1 - x - a_0 + 2a_1, b_1\right], \min\left[b_1 - x - a_0 + 2a_1, b_1 + x - a_0\right]\\
&\ ,\left[\min(b_1 + x - a_0),(b_1 + x - a_0 + 2a_1 - 2a_2)\right]).
\end{aligned}$$

It can be easily verified that this result is equivalent to Equation 4.3.

4.5.3 Max-In-Modulus Function

Consider the interval S = I = [0,1] with $n \in R$ and $s \in S$ and define $f_n : S \to I^I$ as:

$$f_n(s)(x) = (1 - n\,|s - x|)^+ = \max(1 - n\,|s - x|, 0) \text{ …………..(4.7)}$$

Observe that the line of symmetry occurs at s = x and this will be associated with the turning point (s,1) because f(s) = 1, which is one of the three points of intersection. The other two can be obtained by solving systems of equations. Since s = x, intervals can be de-limited as:

$$f_n(s)(x) = \frac{1 - n\,|s - x| + 0 + \left|\ 1 - n\,|s - x| - 0\ \right|}{2} = \frac{1 - n\,|s - x| + \left|\ 1 - n\,|s - x|\ \right|}{2}.$$

First compare s < x and s > x thus:

$$f(x) = \begin{cases} \dfrac{1 - n(x - s) + \left|\ 1 - n(x - s)\ \right|}{2} & \text{if} \quad x > s \\ \dfrac{1 - n(s - x) + \left|\ 1 - n(s - x)\ \right|}{2} & \text{if} \quad x \le s \end{cases}$$

Next compare for x > s, 1 < n(x–s) and 1 > n(x–s) then for x $\le$ s compare 1 < n(s–x) and 1 > n(s–x), which implies the creation of four pieces of f(x):

$$f(x) = \begin{cases} \dfrac{1 - n(x - s) + n(x - s) - 1}{2} = 0 & \text{if} \quad x > s\ \&\ 1 < n(x - s) \\ \dfrac{1 - n(x - s) + 1 - n(x - s)}{2} = 1 + n(s - x) & \text{if} \quad x > s\ \&\ 1 > n(x - s) \\ \dfrac{1 - n(s - x) + n(s - x) - 1}{2} = 0 & \text{if} \quad x < s\ \&\ 1 < n(s - x) \\ \dfrac{1 - n(s - x) - n(s - x) + 1}{2} = 1 + n(x - s) & \text{if} \quad x < s\ \&\ 1 > n(s - x) \end{cases}$$

The functions defined on each interval must, unlike in other cases, be considered with particular caution. Set:

$$f_1(x) = 0 \ \{x > s \ \ \& \ \ 1 < n(x - s)\}$$

$$f_2(x) = 1 + n(x - s) \ \{x < s \ \ \& \ \ 1 > n(s - x)\}$$

$$f_3(x) = 1 + n(s - x) \ \{x > s \ \ \& \ \ 1 > n(x - s)\}$$

$$f_4(x) = 0 \ \{x < s \ \ \& \ \ 1 < n(s - x)\}$$

Differentiation yields:

$$f_1'(x) = 0 \ \{x > s \ \ \& \ \ 1 < n(x - s)\}$$

$$f_2'(x) = n \ \{x < s \ \ \& \ \ 1 > n(s - x)\}$$

$$f_3'(x) = -n \ \{x < s \ \ \& \ \ 1 > n(s - x)\}$$

$$f_4'(x) = 0 \ \{x < s \ \ \& \ \ 1 < n(s - x)\}$$

The point at which each pair of pieces intersect can now be determined by solving systems of equations. The x axis co-ordinates can be identified in this case by identifying the point at which each component function intersects. The point at which:

$f_1(x)$ intersects $f_2(x)$ is $\left(s - \frac{1}{n}, 0\right)$

$f_2(x)$ intersects $f_3(x)$ is (s,1)

$f_3(x)$ intersects $f_4(x)$ is $\left(s + \frac{1}{n}, 0\right)$

The graph of the function (Figure 4.6) contains four “pieces”. It is apparent that:

i. s is the arithmetic mean of s – (1/n) and s + (1/n).

ii. when 1 > n(.) a component has a positive or negative slope and if 1 < n, a component has neither.

The remaining steps of the reverse decomposition algorithm can be applied with the use of Figure 4.6.

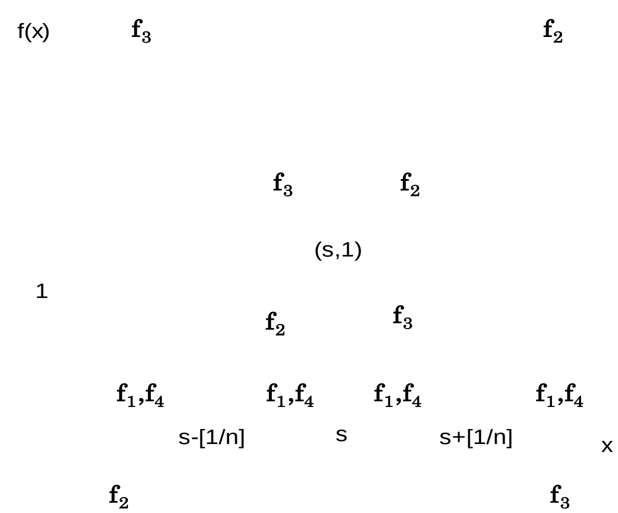

Figure 4.6: Graph Of Max-In-Modulus Function

Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c) $S_a =$	Minimum of functions corresponding to indices in S_a $F_a(x) =$
a	**b**	**c**	**d**	**e**
[0,(s-1/n)]	$f_1(x) = 0$	$f_3(x),f_1(x),f_4(x)$	{3,1,4}	$f_3(x) \wedge f_1(x) \wedge f_4(x)$
[(s-1/n),s]	$f_2(x) = 1+n(x-s)$	$f_3(x),f_2(x)$	{3,2}	$f_3(x) \wedge f_2(x)$
[s,(s+1/n)]	$f_3(x) = 1+n(s-x)$	$f_2(x),f_3(x)$	{2,3}	$f_2(x) \wedge f_3(x)$
[(s+1/n),∞)	$f_4(x) = 0$	$f_2(x),f_1(x),f_4(x)$	{2,1,4}	$f_2(x) \wedge f_1(x) \wedge f_4(x)$

Table 4.4: Algorithm Inputs – Modulus Function

Taking the max of each row of Column e entries (Table 4.4) yields the DNF of the modulus function:

$$
\begin{aligned}
f(x) &= [f_3(x) \wedge f_1(x) \wedge f_4(x)] \vee [f_2(x) \wedge f_3(x)] \vee [f_2(x) \wedge f_1(x) \wedge f_4(x)] \\
&= [(1+n(s-x)) \wedge 0 \wedge 0] \vee [(1+n(x-s)) \wedge (1+n(s-x))] \vee [(1+n(x-s)) \wedge 0 \wedge 0] \\
&= [(1+n(s-x))] \vee [(1+n(x-s)) \wedge (1+n(s-x))] \vee [(1+n(x-s))] \\
&= \max[[(1+n(s-x))], [\min(1+n(x-s)), (1+n(s-x))], [(1+n(x-s))]]
\end{aligned}
$$

It can be checked that this expression is equivalent to Equation 4.7.

4.5.4 Four-Component Function

The following function is defined by its components:

$$
f(x) = \begin{cases} -x-3 & \text{if} \quad x \le -3 \\ x+3 & \text{if} \quad -3 < x < 0 \\ -2x+3 & \text{if} \quad 0 \le x < 3 \\ x-6 & \text{if} \quad x \ge 3 \end{cases}
$$

The points of intersection are:

f_1 and f_2: (-3,0)

f_2 and f_3: (0,3)

f_3 and f_4: (3,-3)

f_1 and f_3: (6,-9)

f_1 and f_4: (1.5,-4.5)

Figure 4.7 depicts the graph of the four-component function and the information required for obtaining f(x) in DNF appears in Table 4.5.

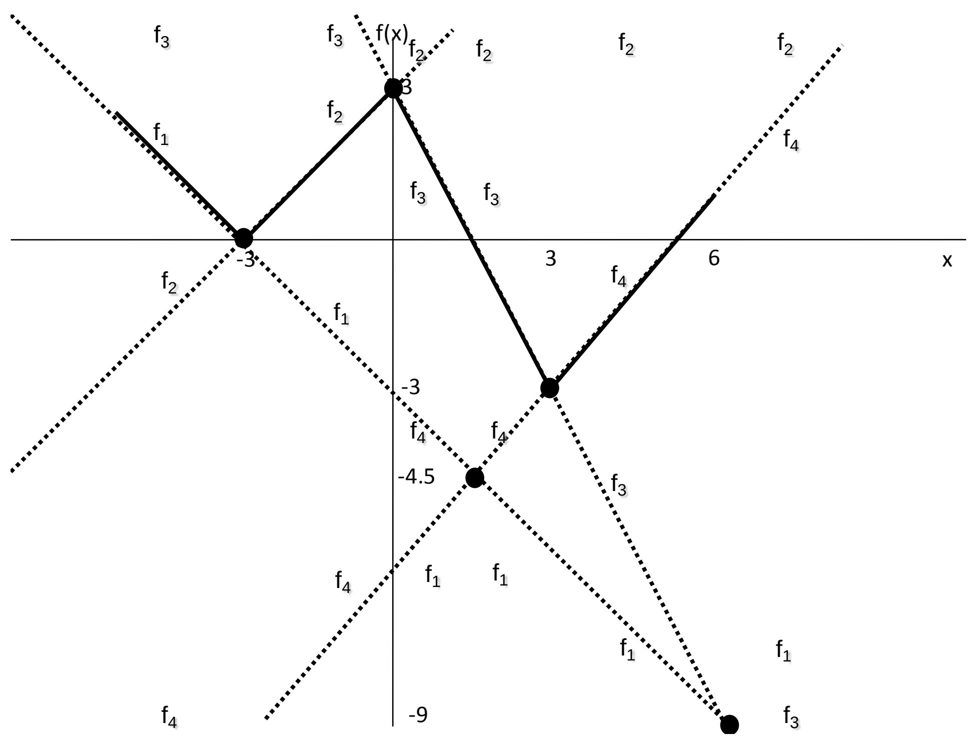

Figure 4.7: Graph Of Four-Component Function

Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c)	Minimum of functions corresponding to indices in S_a
			$S_a =$	$F_a(x) =$
a	**b**	**c**	**d**	**e**
[-4,-3]	$f_1(x) = -x-3$	$f_3(x), f_1(x)$	{3,1}	$[f_3(x) \wedge f_1(x)]$
[-3,0]	$f_2(x) = x+3$	$f_3(x), f_2(x)$	{3,2}	$[f_3(x) \wedge f_2(x)]$
[0,1.5]	$f_3(x) = -2x+3$	$f_2(x), f_3(x)$	{2,3}	$[f_2(x) \wedge f_3(x)]$
[1.5,3]	$f_3(x) = -2x+3$	$f_2(x), f_3(x)$	{2,3}	$[f_2(x) \wedge f_3(x)]$
[3,6]	$f_4(x) = x-6$	$f_2(x), f_4(x)$	{2,4}	$[f_2(x) \wedge f_4(x)]$
[6,∞)	$f_4(x) = x-6$	$f_2(x), f_4(x)$	{2,4}	$[f_2(x) \wedge f_4(x)]$

Table 4.5: Algorithm Inputs – Four-Component Function

Taking the max and applying [LA3] yields the DNF:

$$\begin{aligned} f(x) &= [f_3(x) \wedge f_1(x)] \vee [f_3(x) \wedge f_2(x)] \vee [f_2(x) \wedge f_4(x)] \\ &= \max[\min(f_3(x), f_1(x)), \min(f_3(x) \wedge f_2(x)), \min(f_2(x) \wedge f_4(x))] \\ &= \max[\min(-2x+3, -x-3), \min(-2x+3 \wedge x+3), \min(-2x+3) \wedge x-6))] \end{aligned}$$

CHAPTER 5. THE ROLE OF MAX AND MIN OPERATORS IN OPTION PRICING: THEORY AND APPLICATIONS TO STANDARD OPTIONS

The reverse decomposition algorithm and the decomposition algorithm which were introduced in Section 4.4, can be applied to options that have piecewise linear or affine payoff functions (of one variable), whether terminal or otherwise. This statement is true irrespective of whether an option's payoff function consists of two components or more than ten components[41] in the case of some Over-the-Counter derivatives and options. All of the functions considered in Section 4.5, with the exception of the modulus function discussed in 4.5.3, are inapplicable to option pricing for the simple reason that the domain on which these functions are defined is R rather than R_+, as is the case for the standard option payoff functions considered in this chapter.

In Section 5.1 we discuss the literature pertaining to the use of max and min operators in option pricing, using the literature applications on Riesz spaces in financial economics as a starting point. The relation between such operators and piecewise linear functions of one variable is considered in the foregoing discussion. Section 5.2 contains an explanation of how the reverse decomposition and decomposition algorithms (discussed in 4.4.1 and 4.4.2 respectively) are adapted to suit option pricing theory. In Section 5.3 we apply both algorithms to four standard European-style option payoff functions. Finally, in Section 5.4, we demonstrate the properties of lattice theory that are useful in standard option pricing theory.

Throughout Section 5.3 and Section 5.4, several methods of computing a given value of the area will be discussed (where these exist), to verify results and ensure transparency. Most of the option transactions considered henceforth will consist of parameters that are assigned numbers within a given range which permits greater flexibility in studying option transactions, although these numbers can be adjusted within limits to accommodate analytical needs.

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[41] An example of the latter is the Lefèvre (1873) "complex strategy" which is represented by a continuous piecewise linear function containing as many as eleven pieces.

5.1 Max and Min Operators In Option Pricing: A Literature Review

The American economist and mathematician C.D. Aliprantis was pre-eminent among authors who studied the application of max and min operators or Riesz spaces to issues in financial economics. However, the first authors to propound the application of Riesz spaces to such matters were Brown & Ross (1991). These authors assert that models of the commodity space or the set of marketed asset portfolios for options markets should be constructed with a Riesz space. Early attempts at arbitrage pricing consisted of the assumption that the commodity space was a linear vector space (see Ross (1976)). Ross (1975) and Ross (1978) state that arbitrage occurred because economic agents prefer more to less in making decisions related to portfolio holdings. Such decisions never affected the commodity space.

Kreps (1981) first constructs a model of the commodity space wherein the latter is considered to be a partially ordered linear vector space. Such vector spaces admit positivity. Consideration of the positive cone E_+ represents the idea that in some states of the world an asset a has greater returns than an asset b does and that in all states of the world, the returns to asset a are at least as great as those of asset b. If the space of assets have a partial ordering denoted by $\geq$, it can thus be stated that a $\geq$ b or $a - b \in E_+$. Consideration of the partial order enables the positive price system to be maintained which are "linear functionals on the linear space of portfolios such that positive vectors are mapped into positive numbers" (Brown & Ross 1991:4).

Brown & Ross (1991) construct a model of option markets' commodity space as a Riesz space E, which is a partially ordered linear vector space endowed with the extra feature that every pair of elements {a,b} in the space has a supremum and infimum. If $a \in E$ then the positive part is $a^+ = a \vee 0$, the negative part is $a^- = (-a) \vee 0$ and $|a| = a \vee (-a)$. Brown & Ross (1991) represent the payoff to options such as call options and put options as non-linear functions of a stock (S), bond (B) and strike price (K) using linear operations for constructing portfolios with the lattice operations of sup (or max) and inf (or min).

Aliprantis, Brown, Polyrakis & Werner (1998) mention that the commodity space and portfolio space are different in the sense of order structure. The commodity space has the usual component-wise order but the portfolio space has an order obtained from its payoff rather than the component-wise order. This is because the authors state that by comparing the payoffs to portfolios, portfolios can be ordered using the natural ordering in any dimension. A portfolio dominates another portfolio if its payoff is greater than that of any other portfolio. Such portfolio dominance is a lattice order if, for any two portfolios, the sup and inf are well-defined. Aliprantis, Brown, Polyrakis & Werner (1998) consider security markets comprising an infinite number of assets and portfolio holdings which may be arbitrary in order to demonstrate that optimal portfolio allocations and equilibria in such markets exist if the portfolio dominance order is a lattice order with the asset span having a Yudin basis[42].

Aliprantis, Brown & Werner (2000) use the theory of vector lattice subspaces to represent incomplete derivative security markets wherein minimum-cost portfolio insurance does not depend on security prices which are arbitrage free. Minimum-cost portfolio insurance is defined by Aliprantis, Brown & Werner (2000:1703) as "an investment strategy that enables an investor to avoid losses while still capturing the gains of a payoff of a portfolio at minimum cost". Aliprantis, Polyrakis & Tourky (2002) utilise, where appropriate, the order properties of Riesz spaces in devising the portfolio dominance approach to the solution of the minimum-premium insurance problem. Aliprantis, Polyrakis & Tourky (2002:270) define the minimum-insurance portfolio as "a portfolio whose payoff dominates the desired insured payoff and which has the lowest insurance-premium".

Issues in financial economics, specifically the assignment of prices to financial options were also studied beginning with a discussion paper by Aliprantis, Brown & Werner (1996). Later, an article by Aliprantis, Harris & Tourky (2006) and monograph by Aliprantis & Tourky (2007) contained explicit references to the link between piecewise linear functions, Riesz spaces and option payoffs constructed in DNF.

[42] The Yudin basis is described in the Appendix of Aliprantis, Brown & Werner (2000).

Samuelson (1965) first introduced the additive form (and therefore the simplest form of DNF) expressed alphabetically, to represent a warrant's payoff function wherein the first component of the max operator comprises a stochastic variable such as the value of an observed underlying asset. The second component comprises a deterministic variable which is a pre-determined strike price.[43] Samuelson (1965) also used the graphical form to represent a warrant's payoff function.[44] The content of Samuelson's (1965) article implies that the link between lattice theory and certain continuous piecewise linear functions was identified during the first half of the twentieth century.[45]

Chen (1970) is considered to be the first to use the additive form (expressed alphabetically) in expressing warrant payoff function of a single variable in the *Journal of Finance* (which is arguably the world's leading peer-reviewed finance journal by ISI impact factor).[46] Articles about option pricing appearing until 2011 in the *Journal of Finance* contain various representations but there is a lack of unification of the four forms.[47]

Mathematical Finance, which is considered to be a leading specialised peer-reviewed journal also contains the aforementioned forms in a disparate manner. In particular, the additive form represented symbolically appears in numerous articles on option pricing since this publication's inception in 1991.[48] The same variation in the use of forms

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[43] Fischer (1978) demonstrates that the strike price in a contract can be deterministic.

[44] Like Sprenkle (1961), Samuelson (1965) assumes that the underlying asset (share price) follows a lognormal distribution which is more reasonable than the assumption of Bachelier (1900) and Bronzin (1908) who assume that it follows the normal distribution which implies negative asset prices.

[45] Bartle & Sherbert (2011) mention that mathematicians of the nineteenth century thought that functions implied definite formulae in such a way that each real number x was associated with another number f(x) which prevented functions from being defined piecewise.

[46] This assertion is based on a search of the EBSCO Business Source Premier database where "Options" was used as the keyword and the search was restricted to the *Journal of Finance*, yielding 263 records during the 1957-2011 period.

[47] For a very recent example, Collin-Dufresne, Goldstein & Yang (2011) use additive form and DNF (without Boolean notation) to represent a continuous piecewise linear loss function. This article will be published in a forthcoming issue of the *Journal of Finance*.

[48] In rather arcane journals like *Probability In The Engineering And Informational Sciences*, articles such as Ross & Shanthikumar (2000) represent continuous piecewise option payoff functions of a single variable in the additive form symbolically.

applies to articles contained in the lower-ranked but older (published since 1994) specialised peer-reviewed journal *Applied Mathematical Finance*.

Generally in the option pricing literature, DNF is represented alphabetically rather than symbolically. Stulz (1982) writes option payoff functions of two variables in both DNF and additive form (and a combination of the two forms) alphabetically. Windcliff, Forsyth & Vetzal (2006) use DNF without Boolean symbols in writing the payoff function for a Cliquet option which can be shown to be a piecewise linear function of one variable.

The predecessor of option payoff functions were profit charts. Weber (2009:456) states that:

> "[t]he invention of profit charts…contributed much to the understanding of derivative contracts. Profit charts clarified the nature of forward contracts and options and…made it possible to combine derivatives in novel ways, achieving payoffs that had hitherto been impossible. The invention of profit charts was a decisive step in the evolution of derivative markets. They made it possible to explain a derivative contract with a single graph instead of long-winded explanations, numerical examples and tables".

In modern parlance, a continuous piecewise linear function constructed with one or several equations using one or several max operators, sometimes together with other linear constants represented a profit chart. This is possible because it is well-known that the sum f+g of two functions of x is a function of x, which is defined for any point x lying in both domains. The sum f+g represents the purchase of a single asset (+f(x)) and (+g(x)) where both functions may contain max expressions, a linear function may be added to a max expression or a constant can be added to a max expression. In addition, continuous piecewise linear functions are obtained for the differences of such g–f and f–g. In the first case a single unit (+g(x)) of an asset is held and one unit (–f(x)) is sold.

Lefèvre (1873) may have been the first to add option payoff functions which are all continuous piecewise linear functions to obtain the payoff to a "complex strategy". This is because Jovanovic (2006:169) asserts that Lefèvre in 1870 "constructed the first graphical representations...for analysing the outcome of complex combinations of stock market operations, most notably combined options" although Jovanovic cites three articles published by Lefèvre in 1870, without identifying the specific one in which such graphic representations first appear. In fact, Jovanovic uses examples of this approach from Lefèvre (1873). In reviewing the treatise of Moser (1875) wherein profit charts were used in studying the relations between various derivative contracts, Schmidt (2009) states that Moser (1875) legitimately claims to be the first author to represent derivative payoff and profit and loss equations with graphs. However, Weber (2009) states that it is nebulous as to whether Lefèvre (1873) or Moser (1875) introduced this concept because Moser began writing his treatise approximately three years before Lefèvre (1873) was published.

Given such controversy[49], Weber (2009) states that derivative dealers and bankers operating in Paris may have invented profit charts during the 1860s and Lefèvre's interaction with these individuals may have led to his use of profit charts. Moser claims to be the first author to represent derivative payoffs, profits and losses with equations but Schmidt (2009) desists from either confirming or contradicting this claim[50]. Bachelier (1900) and Bronzin (1908) may have emulated either or both Lefèvre (1873) and Moser (1875) in applying the "graphic" method as an aid in assigning prices to options. Schmidt (2009) states that this practice prevails until the present day.

Bachelier's (1900)[51] attempt to assign prices to warrants traded on the Paris exchange included drawing the Profit/Loss function of warrants by combining graphs as Lefèvre (1873) did. Examples of the graphs of payoff functions constructed by Lefèvre (1873) and Bachelier (1900) appear in Figure 5.1, with the use of modern terminology. In Chapter 6, we will demonstrate how to construct similar graphs analogous to Bachelier's (1900) example, which appears in the right panel of Figure 5.1.

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[49] The matter of identifying the first person to discover profit charts requires resolution.
[50] Moser's claim still requires investigation.
[51] The English translation appears in the References section of this study as Bachelier (1964).

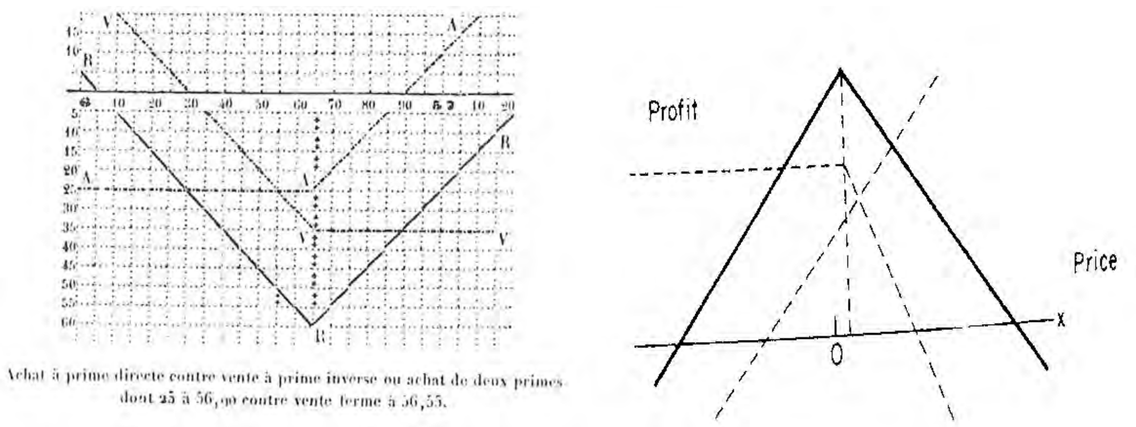

Figure 5.1: Combining Payoff/Profit Graphs

Lefèvre (1873) Call option held (achat à prime or AA) and Call option written (vente à prime or VV) yields position RR (left panel), Bachelier's (1900) Figure 8, Long position in futures and two Call options written yields the chevron-shaped position represented by the bold line (right panel).
Sources: Lefèvre (1873) and Bachelier (1900), p.58.

During the century-long period after the publication of Lefèvre (1873), with the exception of Bachelier (1900), Moser (1875), Bronzin (1908), Giguère (1958), Sprenkle (1961), Samuelson (1965), Black & Scholes (1973) and Merton (1973) present the graphs of payoff functions without combining other functions. However, prominent textbooks such as Hull (2003) contain examples of Lefèvre's (1873) and Bachelier's (1900) practice in constructing graphs.

Note that during the same period, Lefèvre (1873) and Moser (1875) lacked any explicit reference to max and min operators which would be introduced more than two decades later by Schroder and Dedekind. Bachelier (1900) and Bronzin (1908)[52] also used profit charts without referring to max operators in assigning a price to warrants and premium contracts (which are known today as options) respectively because lattice theory was still inchoate and esoteric at the time. In addition, note that Giguère (1958), Sprenkle (1961) and Samuelson (1965)[53] follow the same approach as Bachelier (1900) in assigning prices to warrants. Finally, we note the omission of several studies during this period because we are only interested in the *level* of an underlying asset's price. Studies which

[52] The English translation appears in the References section of this study as Bronzin (2009).
[53] A re-print of this article appears in the References section of this study as Sprenkle (1964).

involve the *first difference* of underlying asset's prices include Kruizenga (1956) who uses piecewise linear functions to represent call option and put option payoffs (and certain combinations thereof) without referring to max operators[54]. In addition, Kruizenga (1964a) studies piecewise linear functions which are associated with vectors of portfolio holdings (see Appendix A) – also without reference to max operators.

During the century-long period considered above, only Samuelson (1965) and Merton (1973) use max operators to represent payoff functions that they study. After this period, the use of max and min operators became common in articles, notes and books pertaining to option pricing. Wilmott, Howison & Dewynne (1995), Campbell, Lo & MacKinlay (1997), Hull (2003), Musiela & Rutkowski (2005) and Cuthbertson & Nitzsche (2006) also use max and min operators for such a purpose. Cuthbertson & Nitzsche (2006) propound a "direction vector" approach to drawing terminal payoff graphs in their book, which consists of differentiating the component functions of a given payoff function and then adding the values of the slope. Thus, combining payoff graphs to create another payoff graph is not required.

Contemporaneous to Cuthbertson & Nitzsche (2006) is the unpublished note of Anderson, Barber & Keys (2006). The latter authors study piecewise linear terminal payoff functions without using max or min operators. The information contained in Table 5.1 is used in constructing Figure 5.2, by considering terminal payoffs in a given interval. Anderson, Barber & Keys (2006) create a Butterfly Spread[55] (Figure 5.2) which is depicted by a piecewise linear functions through the use of Table 5.1. The following example is analogous to Bachelier's (1900) position (right panel of Table 5.1) and is depicted by the combination of graphs (Figure 5.2).

[54] Articles related to Kruizenga (1956) are Kruizenga (1964a) and Kruizenga (1964b). Kruizenga (1964a) is discussed in Appendix A of this study whereas Kruizenga (1964b) comprises an empirical study which excludes these concepts.

[55] Anderson, Barber & Keys (2006) create a Big W spread which is considered in Chapter 6.

	Price Range			
Position	P ≤ 35	35 < P ≤ 40	40 < P ≤ 45	P > 45
Buy 1 Jan 35 call at 8	0	P – 35	P - 35	P - 35
Sell 2 Jan 40 calls at 4 □	0	0	-2(P - 40)	-2(P - 40)
Buy 1 Jan 45 call at 1	0	0	0	P - 45
Portfolio	0	P – 35	-P + 45	0

Table 5.1: Anderson Barber & Keys' (2006) Payoff Table For Butterfly Spread
Source: Anderson, Barber & Keys (2006:9)

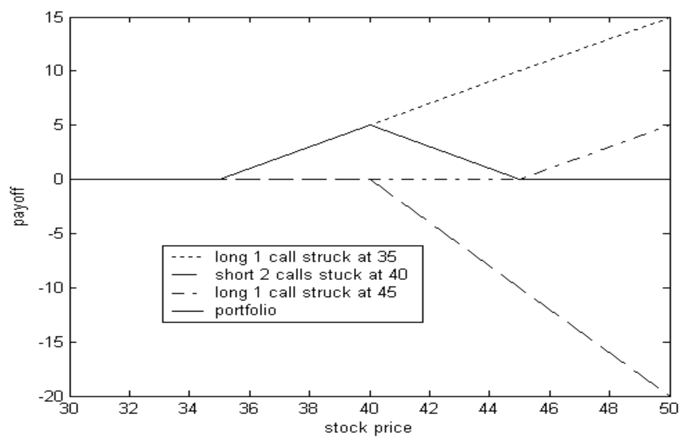

Figure 5.2: Graph Of Anderson Barber & Keys' (2006) Butterfly Spread
Source: Anderson, Barber & Keys (2006:6)

It should be noticed that Anderson, Barber & Keys (2006) as well as Hull (2003) use a table representing the component *functions* in each interval rather than a table containing the *slope of each function* in each interval which is required in the application of Cuthbertson & Nitzsche's (2006) direction vector technique.

The main deficiency of Hull (2003), Cuthbertson & Nitzsche (2006) and Anderson, Barber & Keys (2006), like their pre-1973 counterparts, is that all three-component terminal payoff functions (of those considered) are not in DNF.

Although Ovchinnikov (2002)[56] established an algorithm for graphing piecewise linear functions based on DNF in any dimension, DNF was applied to linear option payoff functions of one variable during the last decade of the twentieth century. In books by Musiela & Rutkowski (2005) and Jewson, Brix & Ziehmann (2005) and notes by Molchanov (2007), the linear payoff functions are in DNF but the Boolean symbols for max and min are not used. Musiela & Rutkowski (2005) demonstrate that several payoff functions in DNF can be expressed equivalently in additive form (as the sum of max operators and a constant). Jewson, Brix & Ziehmann (2005) represent some weather option payoff functions in the computer analog of DNF and graphical form. A useful feature of Jewson, Brix & Ziehmann's (2005) analysis is that the duality of payoff functions expressed with max operators is considered. None of these representations of such DNF reveal how the DNF was obtained in the first place, which means that trial-and-error has been used in obtaining such forms.

Aliprantis & Tourky (2007:221) state that a specific additive form of one-dimensional piecewise affine functions is "closely related to call and put options in finance". These authors refer to the piecewise linear terminal payoff functions of call and put options which can be expressed in DNF.

The combination of the theory of Riesz spaces[57], statistics and probability yielded a significant contribution to option pricing theory. Black & Scholes (1973) and Merton (1973) compute an explicit formula for assigning prices[58] to a standard European call

[56] This extends the work of Gorokhovik & Zorko (1994). In turn, Ovchinnikov (2010) extends the Ovchinnikov (2002) algorithm.

[57] Lattices in the context of the theory of financial derivatives also refers to diagram representing the period-by-period evolution in an underlying asset's price, with examples being the binomial lattice of Cox, Ross & Rubinstein (1979) and trinomial lattice of Kamrad & Ritchken (1991).

[58] In general, contingent claims can be assigned a value in the following ways:
1) Explicit formulae
2) Simulation with Fourier Transforms
3) Simulation with Monte Carlo experiments
4) Recursion in the form of finite differences
5) Stochastic calculus of variations (Malliavin Calculus) with Nualart (1995) and Okensdal (1997) discussing the topic in considerable detail.

None of the foregoing approaches are precise but some are better than others in the sense that estimated contingent claim prices approximate actual contingent claim prices fairly accurately.

option. In doing so, they connected the same option's terminal payoff function of with its price. Their method is discussed in the Appendix B.

When terminal payoff functions were drawn and the size of a gain or loss was sought when different terminal asset prices were considered, integral calculus was used by Bachelier (1900), Bronzin (1908), Samuelson (1965), Black & Scholes (1973), Merton (1973) and many others thereafter to compute such an area. Loosely speaking, these computations relied on the idea that the more a call option/warrant is worth to its holder, the larger the area between the payoff function and the horizontal axis would be using the strike price K as the lower limit of integration.

Black & Scholes (1973) assigned a price to a single European-style call option but the problem of how to assign prices to non-standard options, for example has remained. Carroll & Brask (1999:383) state that, for a class of non-standard options which can be referred to as compound financial instruments:

> "the U.S. Financial Accounting Standards Board (1991) designed an approach for estimating the values of compound financial instruments by breaking them down into fundamental contracts. These contracts are considered to be the building blocks of the entire instrument, whose value is estimated by the sum of these entire building blocks". (Carroll & Brask 1999:383)

Cheung & Chung (1996) and Carroll & Brask (1999) devised basic algorithms ("building blocks") for obtaining certain option prices with the Black & Scholes (1973) formula. In both cases, the terminal payoffs to the options are represented in graphical and linear form.

Cheung & Chung (1996) apply the "building block approach" to determine the price or extrinsic value of truncated standard European call options and put options which have continuous piecewise linear terminal payoff functions, truncated binary call options which have a discontinuous "step" payoff function and hybrid currency put option which has a discontinuity in its payoff function. This algorithm involves splitting the area between an option's terminal payoff function and the horizontal axis into several "blocks", assigning a value to each block and adding each of these values to obtain a given option's price. Carroll & Brask (1999) apply the building block approach to ascertain the price of a Bull Spread and several versions of a Butterfly Spread by decomposing such piecewise linear payoff diagrams into their linear components and then assigning a price to each block. The addition of each block yields the price of a given instrument.

Cheung & Chung (1996) exclude a discussion of how the area beneath each curve can be measured if the terminal payoff is sought. Cheung & Chung (1996) restrict their discussion to the calculation of prices rather than terminal payoffs which may be confusing because the two concepts are distinct. To be specific, while it would seem that Cheung & Chung (1996) and Carroll & Brask (1999) compute the area between a continuous piecewise linear payoff function and the horizontal axis (associated with positive payoffs) in reality, these authors compute the area of a non-linear region. The latter region is obtained from the curve representing an option's price which is non-linear, unlike the linear payoff function from which it is derived. In respect of the options considered by Carroll & Brask (1999), a simple calculation of the area (using any of the methods discussed in Chapter 4) of each "building block" yields a value which is of the same sign but different magnitude compared to the values provided by those authors. The discrepancy can be attributed to the use of the Black & Scholes (1973) formula by Cheung & Chung (1996) and Carroll & Brask (1999). Thus, the prices computed in these studies are correct but the graphical depictions are not.

It should be noted that American-style option valuation as discussed by Chiarella & Xiogas (2005) is impossible using the approach of Cheung & Chung (1996) and Carroll & Brask (1999)[59].

[59] For a share, an American call option's price without cash dividends equals a European call option's price without dividends since it is never optimal to exercise the American call without dividends before expiry. However, for American call options on shares with cash dividends, early exercise may be optimal but it is well-known that for an American call on a share yielding discrete cash dividends, the option is exercised when the dividend is disbursed. If the American call is on a share yielding continuous dividends the time of exercise is arbitrary. In the latter case, only approximations exist for assigning a value to the call. In the case of an American put option, both when cash dividends are disbursed discretely and continuously and

In conclusion, we note the following. In the literature, max operators are preferred to min operators in representing option payoff functions because the former representation accommodates economic agents' need for maximal gains from adopting an investment strategy. However, there are cases in which min operators appear, in which case the duality between max and min operators is seldom represented explicitly, despite representing the important difference between, for instance, "long" and "short" positions.

5.2 Applying The Algorithms To Option Payoff Functions

5.2.1 Basic Derivative and Option Concepts

A financial asset's price is denoted by S and expressed in currency units. The financial assets considered henceforth will be securities (bonds with deterministic S and shares of stock with stochastic S).[60] Thus, S can denote either a stock or bond's price and to avoid confusing matters, it will be mentioned which asset's price S refers to, wherever relevant.

The security's price S is considered to be an element of a Riesz space E in the sense that S may assume values in n states of the world which can be represented by the column vector $(S_1, S_2, \ldots, S_n)'$. Specifically, if the asset is a share of stock, $S \in E_+$ implies that the stock's issuer has limited liability or S is never negative. In other words, the foregoing vector's components belong to the set of positive real numbers.
The security S can either be bought or short sold in a financial market together with simple financial claims such as derivative contracts and options (described below) which have payoffs depending on S.

Definition. 5.2.1 The *payoff* of a simple claim is the amount gained or lost by an economic agent taking a given position in (buying or selling) a claim.

The payoff is measured in currency units such as American Dollars (USD), British Pounds (GBP) or Japanese Yen (JPY).[61]

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without such disbursements, early exercise may be optimal and only complex approximations exist for assigning values to such options.
[60] Later exchange rates will be considered.
[61] We mention these currencies as these are currently among the most heavily traded.

Definition 5.2.2 A *payoff function*, denoted by f(S), represents the payoff to a claim as some function of one stochastic variable which is an underlying asset's price S. The domain of S is R_+ and f is a one-to-one (injective) function.

The above definition is a general payoff function whereas we will be mostly concerned with such a function being continuous and piecewise linear (or affine) and related to the date of a claim's expiry S_T.

As mentioned earlier, an example of a simple claim or contract is a forward contract which is defined below. It is first discussed because it is related to and can be used in the construction of options having continuous piecewise linear and affine payoff functions that will be discussed later in this chapter. The ensuing discussion is based on Hull's (2003) exposition.

Definition. 5.2.3 A *forward contract* is an agreement to buy or sell a certain number of units of an asset worth S_T on a specific future date T, called the delivery/terminal date at a specific price K, called the delivery price where $K \in R_+$.

The terminal payoff to the long forward contract is $f(S_T) = S_T – K$ whereas its short counterpart has a terminal payoff of $f(S_T) = K – S_T$. The buyer (seller) of such a contract who takes a long (short) position in S, gains a certain amount when $S_T > K$.

Theoretically, the stochastic variable $S_T \in R_+$ but S_T is closer to the constant K than it is to $+\infty$ for all securities, despite the magnitude of the difference between S_T and K being distinct for each security.

Note. 5.2.4

1. A futures contract is similar to a forward contract. The main difference between the two is that a futures contract is always traded on organised exchanges with the standardisation of contract amounts and delivery dates.
2. A forward or futures price will be denoted by $\hat{F}$ where $\hat{F} \in R_+$.

With information about the terminal payoff function, a terminal payoff diagram can be drawn. The graph of the terminal payoff function of a forward/futures contract is obtained by differentiating the function with respect to the underlying asset and then determining whether the slope of the graph is positive or negative, as was done in Chapter 2. The differentiation procedure for forward/futures contracts appears in Table 5.2.

The differentiation procedure for share positions and bond positions also appear in the same table. It will be noticed that for the bond positions considered, that a bond's price is set equal to the underlying asset's price, thereby maintaining a constant value for the bond at time T.

The salient features of the claims contained in Table 5.2 are used in depicting the graphs of the payoff functions corresponding to such claims in Figure 5.3.

Simple claim	Panel of corresponding diagram	$f(S_T) =$	Domain	Co-domain	$f'(S_T) = \frac{df(S_T)}{S_T} =$	Slope of graph
Share held	a	S_T	R_+	R_+	1	Positive
Forward/futures (short position)	b	$-(S_T - K)$	R_+	R	-1	Negative
Share short sold	c	$-S_T$	R_+	R_-	-1	Negative
Forward/futures (long position)	d	$(S_T - K)$	R_+	R	1	Positive
Bond held	e	$S_T = B$	R_+	R	0	Flat
Bond sold	f	$S_T = -B$	R_+	R_-	0	Flat

Table 5.2: Procedure for Obtaining Terminal Payoff Graphs of Simple Claims

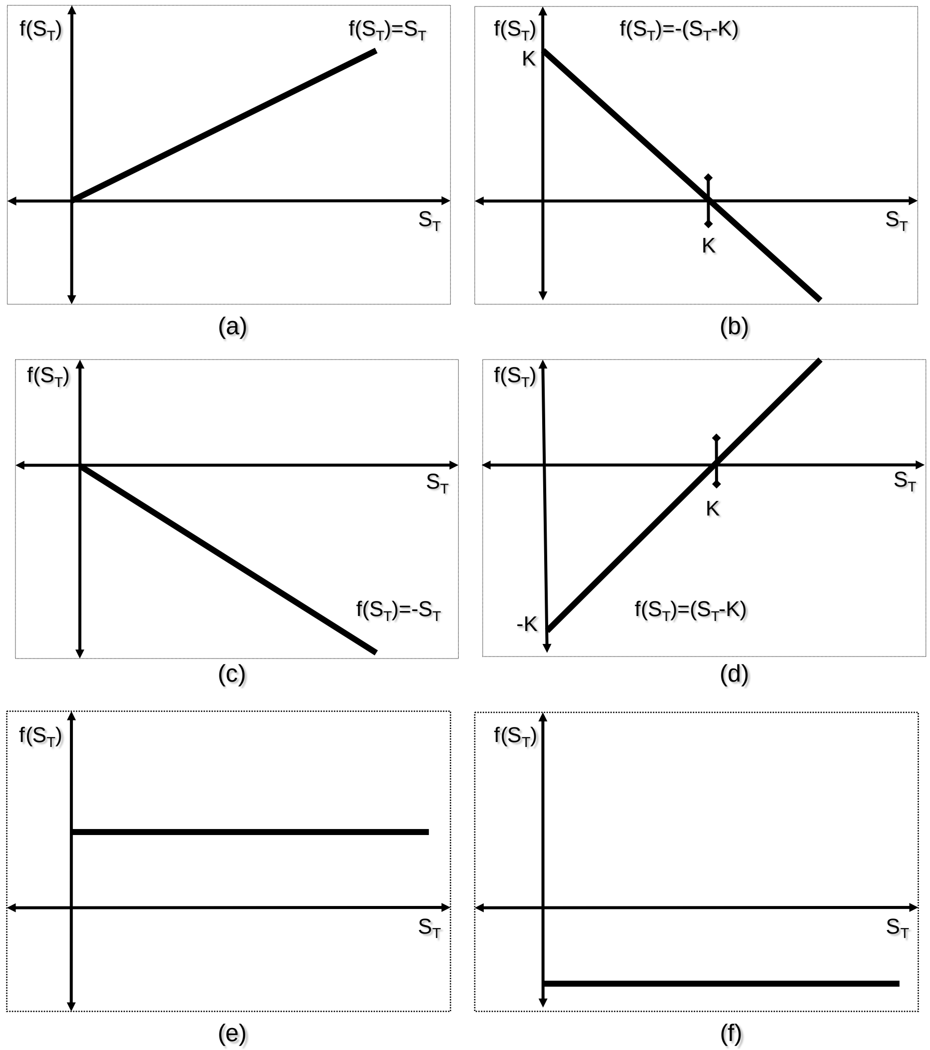

Figure 5.3: Graphs Of General Terminal Payoff Functions

Panel (a) represents a share of stock held (b) short forward/futures contract (c) short selling of a share of stock (d) long forward (e) bond held and (f) bond sold.

An inspection of Figure 5.3 reveals that the stock and forward/futures contract has a linear terminal payoff function whereas a bond has a constant terminal payoff function.[62] Clearly, each graph satisfies the Vertical Line Test of a function.

[62] It is possible that stocks which are seldom traded may have only a single expected terminal price based on the recent history of prices which implies that the terminal payoff function in this special case resembles a bond's terminal payoff function.

Having considered the terminal payoff functions associated with simple claims, we now introduce standard options' terminal payoff functions which are central to the algorithms devised later in this chapter.

Definition. 5.2.5 A *standard European-style call (put) option* on an underlying asset S confers its holder the right without the obligation to buy (sell) S on a specific date T at a specific price K called the exercise price where $K \in R_+$.

Note that we have used K interchangeably to refer to the forward/future delivery price and an option's exercise price.

Instead of holding such an option an economic agent can write (sell) it in what is referred to as an uncovered option trade. However, institutional rather retail traders do this given that the latter traders cannot maintain margin requirements.

The functions of Definition 5.2.5 are represented by continuous piecewise linear functions of one underlying asset. A summary of standard European-style options' terminal payoff functions appear in Table 5.3. Max and min operators comprise the terminal payoff functions of options if the outcome of an economic agent's decision is such that nothing is received or a non-negative amount is received.[63]

$f(S_T)=$	Domain	Co-domain	Feature	Additive Form	
				Symbolic	Alphabetic
$PAECH_T$	R_+	$R_+ \cup \{0\}$	Convex	$(S_T\text{-}K)^+$	$\max(S_T\text{-}K,0)$
$PAEPH_T$	R_+	$R_+ \cup \{0\}$	Convex	$(K\text{-}S_T)^+$	$\max(K\text{-}S_T,0)$
$PAECW_T$	R_+	$R_- \cup \{0\}$	Concave	$-\{(S_T\text{-}K)^+\}$	$-\{\max(S_T\text{-}K,0)\}$
$PAEPW_T$	R_+	$R_- \cup \{0\}$	Concave	$-\{(K\text{-}S_T)^+\}$	$-\{\max(K\text{-}S_T,0)\}$

Table 5.3: Characteristics of Standard Option Payoff Functions

The commonest representation of a standard option terminal payoff functions,[64] is the additive form wherein max operators are written alphabetically. If terminal payoffs are

[63] The terminal payoff of forward and futures contracts which has a co-domain comprising the set of real numbers, cannot be represented by max operators.
[64] This is also referred to as a "contract function" by Janson & Tysk (2002) and Björk (2005:90).

written as min operators, then these should be re-written as the max of terminal payoffs using the notion of duality mentioned earlier. This is necessary given that economic agents maximise gains from taking various positions in options or dually minimise losses as the case may be.

The functions considered in Table 5.3 wherein max operators contain two components, can also be represented expressed in DNF with the symbols $\vee$ and $\wedge$, although this is seldom done in the literature. Even when multiple max/min operators and/or constants are added thereby justifying the use of DNF symbolically (written with $\vee$ and $\wedge$) to maintain conciseness, DNF is generally written alphabetically (with the abbreviations max and min) in the literature.

The main advantage of writing the expression in DNF, besides compactness, is that the distinguishing points on the piecewise linear option payoff function can easily be identified by substitution. This in turn permits the delimitation of areas situated beneath or above such a function for the purpose of calculating areas. In all cases, the domain of the terminal payoff functions considered are the set of positive real numbers (R_+), which matches the domain of S_T.

In order to proceed, a few comments on notation are required. Denote the standard European-style call option by EC and the standard European-style put option by EP. Then attach H and W to EC or EL to indicate option positions bought (held) and sold (written). For example, ECL means a position in a European call such that it is held. Associated with each of these positions are payoffs (denoted by PA). The difference between terminal payoff and profit/loss is explained in the sequel. Using the aforementioned example, PAECH means the payoff earned by the European-style call option's holder. The terminal payoff function represents what a European-style option will be worth if it is exercised at time T.

The standard European-style call option's terminal payoff function ($PAECH_T$) is obtained by the substitution of $a = 1$ into Esser's (2003) Powered Option terminal payoff function which is given by:

$$f(S_T) = [\max(S_T - K, 0)]^a = (S_T - K)^a I(S_T > K) = \sum_{j=0}^{a} \binom{a}{j} S_T^{a-j} (-K)^j I(S_T > K) \qquad (5.1)$$

where $\binom{a}{j} = \frac{a!}{a!(a-j)!}$, $a \in N$.[65]

Salient features of $PAECH_T$ along with those of other standard options appear in Table 5.3. Variants of the Powered Option terminal payoff function, which yield continuous piecewise but non-linear graphs, appear in Appendix D.

Note. 5.2.6

1. The terminal payoff functions to the other standard options are obtained analogously from Equation 5.1.
2. The European call option's terminal payoff function can also be obtained from the digital call option's payoff function (see Appendix E).
3. The Indictor Function term of Equation 5.1 can be replaced by, for example, the Heaviside Function (see Appendix C).
4. Shreve (2004a:111) represents a call option's[66] payoff function in the form of the positive part function (which we discussed in Footnote 20 of Chapter 2) as:

$$f(S) = f^+(S) = (S\text{-}K)^+$$

because f^+ has non-negative values and satisfies $f^+(0) = 0$ whereas Shreve (2004a) initially assumed that f can take negative values. This representation is useful in that f^+ is shown to be convex.

[65] When a = 1:

$$f(S_T) = \left[\binom{1}{0} S_T^{1-0}(-K)^0 + \binom{1}{1} S_T^{1-1}(-K)^1\right] I(S_T > K) = (S_T - K) I(S_T > K).$$

[66] Strictly speaking, American-style but the concept can be applied to the standard European-style option's payoff function.

The payoff functions of standard option positions are classified as concave or convex in Table 5.1 because Bergman, Grundy & Wiener (1996) state that in specific circumstances convex (concave) option payoff functions lead to option prices that are convex (concave) functions of S.

We have deferred drawing the graph of standard option positions' terminal payoff functions to the sequel because such graphs are depicted for the algorithm which is now discussed.

5.2.2 Adapting The Algorithms To Option Payoff Functions

Once slight adjustments are made to the reverse decomposition algorithm and the decomposition algorithm devised in the previous chapter, each can be applied to certain option payoff functions. For the functions considered henceforth, the left end points of the domain will be the origin because we will assume that the terminal asset prices (S_T) considered are non-negative. In applying the foregoing algorithms, it should be noted that the use of S_T can easily be replaced by asset prices on any date preceding T (i.e. S). Once the algorithms have been applied, we will then compute the region over which investors gain or lose (if neither a gain is enjoyed nor a loss incurred).

5.2.2.1 Reverse Decomposition Algorithm Applied To Option Payoff Functions

In adjusting some steps of the algorithm discussed in 4.4.1 we use the exposition of Section 5.2.1. Given that option payoffs are now considered, first identify one or more assets such as stocks, bonds, forwards/futures (terminal payoff functions of these assets appear in Table 5.2) and European-style options (terminal payoff functions of such options appear in Table 5.3) to be held/short sold or a combination of such assets. Then identify the number of assets to be:

i. bought/held (with $\rho \in Z^+$);

ii. short sold/borrowed (with $\rho \in Z^-$) noting that $-\rho \max(a,b) \neq \max(-\rho a, -\rho b)$

Either add or subtract each of the terminal payoff functions associated with each asset yielding a terminal payoff function to a given strategy.[67] In doing so, divide the domain $[0,\infty)$ into intervals and to find the value in each interval of the component function, we substitute values for S_T into the $f(S_T)$. The payoff function's slope is obtained by differentiating it with respect to S_T. In implementing the algorithm, Step 1 – Step 5 are the same as stated in Section 5.2.1. Step 6 differs in the sense that the max of all sets created are expressed as a function of S_T. Step 7 and Step 8 are also the same.

Note that the slope of each function can be compared with the tabular approach of Anderson, Barber & Keys (2006) from two perspectives. First, in what we will call Tabular Approach 1, derivatives related to component functions appearing in the original equation are considered. Second, for Tabular Approach 2, derivatives of the component functions that were created by evaluation of the original equation are considered. Both of these approaches are equivalent for piecewise linear functions containing two pieces, otherwise not.

5.2.2.2 Decomposition Algorithm Applied To Option Payoff Functions

The steps of the decomposition algorithm applied to option payoff functions are identical to those discussed in 4.5.2. Note that in evaluating functions of S_T in DNF, we compare S_T to one or multiple strike prices (as the case may be) in the domain of S_T, to eliminate max in min, min in max, min in min or max in max expressions.

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[67] Nelken (1996) considers a Shout Option which has a continuous and piecewise linear terminal payoff function of one stochastic variable S_T. However, on a single date before T, which we denote by t, the option's holder can specify a price S_{tShout} (the act of "shouting"). The terminal payoff function is of the form:

$$f(S_T) = \max[S_T - K, S_{tShout} - K, 0].$$

Note that S_{tShout} is deterministic. Although the max operator comprises three components, it can be demonstrated that the function is continuous and piecewise linear by re-writing it as:

$$\begin{aligned} f(S_T) &= \max[S_T - K - (S_{tShout} - K), S_{tShout} - K - (S_{tShout} - K), 0 - (S_{tShout} - K)] + [S_{tShout} - K] \\ &= \max[S_T - S_{tShout}, 0, -(S_{tShout} - K)] + S_{tShout} - K. \end{aligned}$$

Since it is assumed that $S_{tShout} > K$, the negative component above can be eliminated which yields:

$$f(S_T) = \max[S_T - S_{tShout}, 0] + S_{tShout} - K.$$

Given the above facts, our algorithm can readily be applied. Thus, the algorithm's applicability when three components are contained in a max (or min operator for that matter) is affected insofar as such functions must be re-written in more tractable two-component max operator form.

Remarks. 5.2.7

1. Arbitrary values of S_T can be substituted into a given option's payoff function derived with this algorithm.

2. To depict the graphs of two component functions more easily, identify the line splitting the entire terminal payoff function in half which is obtained by inspection of the modulus expression. The line of symmetry can be determined by setting the modulus expression contained in the max operator equal to zero and solving for S_T.

3. Discard the decomposition algorithm when an option's payoff function is not continuous, piecewise and linear (or affine) functions of one variable. Cases of functions that do not satisfy these requirements are considered in the Appendices. For example, piecewise non-linear or affine functions of one variable are either discontinuous (Appendix F) or continuous (Appendix G). Another example, contained in Appendix H, features a piecewise continuous linear or affine function of several variables.

Using the graph of piecewise linear terminal option payoff functions, we can compute the extent of gains enjoyed or losses incurred by an economic agent. For the interval of gain (loss) the area above (beneath) the horizontal axis will be computed using the PF, IA1, IA2 and IB methods discussed earlier. If the result obtained by the use of any of these methods is positive (negative) the option's holder can gain (lose) from the transaction.
The integration methods can be adjusted if the area associated with a certain proportion of all interval's areas is required (see protective put example of Section 6.1.1).

5.3 Applications Of The Algorithms To Standard Option Payoff Functions

Both algorithms are applied to two-component standard European-style terminal payoff functions. In applying our algorithms, complete workings appear in many instances to ensure transparency.

5.3.1 Standard European-Style Call Option Held

Use Equation 2.8 and let a = (S_T − K) the call's terminal payoff function in additive (symbolic and alphabetic) and linear form is:

$$f(S_T) = PAECH_T = (S_T - K)^+ = \max(S_T - K, 0) = \begin{cases} 0 \text{ if } S_T \leq K \\ S_T - K \text{ if } S_T > K \end{cases} \quad \ldots(5.2)$$

In Column b of Table 5.5, the function corresponding to left-most interval (in this case [0,K]) will be denoted by $f_1(S_T) = 0$ and similarly for $[K,\infty]$ which corresponds to $f_2(S_T) = S_T - K$. This convention will be adhered to henceforth wherever similar tables appear.

Proposition. 5.3.1 The terminal payoff to a European-style call option from a holder's perspective in the most favourable state of the world is obtained directly from Equation 2.8.

Proof. The terminal payoff to a European-style call option is $(S_T - K)^+$. The most favourable state of the world occurs when $S_T > K$. We recall from Equation 2.8 with $a = S_T - K$ that:[68].

$$(S_T - K)^+ = \frac{S_T - K + |S_T - K|}{2}.$$

When $S_T > K$, $(S_T - K)^+ = S_T - K$ therefore the call option's terminal payoff can also be expressed as:

$$\frac{S_T - K + |S_T - K|}{2}.$$

□

Observe that in the foregoing proof, for the unfavourable state of the world, from the call option holder's perspective $S_T - K < 0 \Rightarrow -(S_T - K)$ therefore:

$$(S_T - K)^+ = \frac{S_T - K - S_T + K}{2} = 0.$$

In turn, $S_T - K$ can be re-written in terms of the second part of [VLP3]:

[68] The same approach can be applied to equivalent expressions of the Payoff function as well as in obtaining the associated Profit/Loss function.

$$
\begin{aligned}
S_T - K &= [(S_T - K) \vee 0] + [(S_T - K) \wedge 0] \\
&= [(S_T - K) \vee 0] + [-[-(S_T - K) \vee -0]] \\
&= \frac{S_T - K + 0 + |S_T - K - 0|}{2} + \frac{-K + S_T - |K - S_T|}{2} \\
&= \frac{S_T - K + |S_T - K| + S_T - K - |K - S_T|}{2}.
\end{aligned}
$$

The terminal payoff function is depicted in Panel a of Figure 5.4 by ascertaining whether each component increases or decreases in a certain interval using slopes. For

$$
\frac{d\,PAECH_T}{d\,S_T} = \begin{cases} 0 & \text{if } S_T \le K \\ 1 & \text{if } S_T > K. \end{cases}
$$

The graph of this function has a positive slope on $[K,\infty)$ and neither increases nor decreases on $[0,K]$. The same result is obtained by the use of the tabular approaches (Table 5.4).

Slope of		$S_T \le K$	$S_T > K$
Approach 1	Approach 2		
0	0	0	0
$+(S_T - K)^+$	$+(S_T - K)$	0	1
Result		**0**	**1**

Table 5.4: Tabular Approaches – European Call Held

The value for $PAECH_T$ when $S_T = 0$ is 0, yielding the first co-ordinate of (0,0) for the interval $[0,K]$ whereas the second co-ordinate is $(K,0)$ when $S_T = K$ by substitution into Equation 5.2. The point at which $f_1(S_T)$ and $f_2(S_T)$ intersect can be determined by solving a system of two equations in two unknowns, with the latter being S_T and $f(S_T)$ by suppressing the subscripts for each component function.

Using Equation 5.2, a system of linear equations can be arranged in the form $Ax = b$ where A is an $n \times n$ matrix and x and b are vectors of dimension $n \times 1$:

$$f_1(S_T) = 0$$

$$f_2(S_T) = S_T - K$$

In dot product form this system can be written as:

$$
\begin{aligned}
(0,1) \bullet (S_T, f_1(S_T)) &= 0 \\
(-1,1) \bullet (S_T, f_2(S_T)) &= -K
\end{aligned}
$$

but sometimes it may be more convenient to write:

$$\underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}}_{=A}\underbrace{\begin{bmatrix} S_T \\ f(S_T) \end{bmatrix}}_{=x} = \underbrace{\begin{bmatrix} 0 \\ -K \end{bmatrix}}_{=b}$$

The determinant of A is denoted by $|A| = a_{11}a_{22} - a_{12}a_{21}$ with $a \in A$. This is |A| = 1. Using Cramer's Rule we compute:

$$S_T = \frac{|A_1|}{|A|} = \frac{K}{1} = K$$

$$f(S_T) = \frac{|A_2|}{|A|} = \frac{0}{1} = 0$$

where:

$$|A_1| = \begin{vmatrix} 0 & 1 \\ -K & 1 \end{vmatrix} = K$$

$$|A_2| = \begin{vmatrix} 0 & 0 \\ -1 & -K \end{vmatrix} = 0.$$

In Panel a of Figure 5.4, (K,0) represents the point at which the two components intersect which is consistent with the line of symmetry at $S_T = K$.

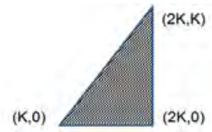
Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c) S_a =	Minimum of functions corresponding to indices in S_a $F_a(S_T)$ =
a	**b**	**c**	**d**	**e**
[0,K]	$f_1(S_T) = 0$	$f_1(S_T)$	{1}	$\wedge f_1(S_T)$
[K,∞)	$f_2(S_T) = S_T – K$	$f_2(S_T)$	{2}	$\wedge f_2(S_T)$

Table 5.5: Algorithm Inputs – European Call Held

Using column e of Table 5.5 (for each interval), take the max of $f_1(S_T)$ and $f_2(S_T)$ as the min symbol is discarded when there is only one component to get the following DNF:

$$f(S_T) = PAECH_T = (S_T - K) \vee 0 .$$

The extent of the interval over which gains can be received by the holder of this option can be calculated in several ways assuming that an upper limit is set on S_T. It will be assumed that an arbitrary truncation point of S_T = 2K exists as in Figure 5.4, so that the extent can be computed in several ways. Thus, the following triangle is obtained:



Its area is calculated in the following ways:

PF. The triangle's co-ordinates are obtained by substitution of 0, K and 2K into f(S_T), which admits the calculation of its area with:

$$
\begin{aligned}
\text{Area of Triangle} &= \frac{1}{2}\text{base.perpendicular height} \\
&= \frac{1}{2}(2K - K).(K - 0) \\
&= \frac{1}{2}K^2.
\end{aligned}
$$

IA1.

$$\int_0^{2K} f(S_T)dS_T = \int_0^{2K} PAECH_T dS_T = \int_0^{2K} (S_T - K)^+ dS_T = \int_0^{2K} \max(S_T - K, 0)dS_T \text{ .}$$

When $S_T > K$, the function S_T –K will apply over the interval [K,2K] and $S_T < K$ implies that the function 0 occurs over [0,K] thus:

$$
\begin{aligned}
\int_0^{2K} f(S_T)dS_T &= \int_0^{K} 0dS_T + \int_K^{2K} (S_T - K)dS_T = \underbrace{1\Big|_0^K}_{=0} + \left(\frac{S_T^2}{2} - KS_T\right)\Bigg|_K^{2K} \\
&= \frac{(2K)^2}{2} - K(2K) - \left(\frac{K^2}{2} - K(K)\right) \\
&= \frac{1}{2}K^2.
\end{aligned}
$$

IA2.

$$\int_0^{2K} f(S_T)dS_T = \int_0^{2K} \frac{S_T - K + 0 + |S_T - K - 0|}{2}dS_T = \int_0^{2K} \frac{S_T - K}{2}dS_T + \int_0^{2K} \frac{|S_T - K|}{2}dS_T \text{ .}$$

The integral expression associated with the modulus can be split yielding:

$$\int_0^{2K} f(S_T)dS_T = \int_0^{2K}\frac{S_T-K}{2}dS_T + \int_0^{K}\frac{-(S_T-K)}{2}dS_T + \int_K^{2K}\frac{S_T-K}{2}dS_T$$
$$=\left[\frac{1}{2}\left(\frac{S_T^2}{2}-KS_T\right)\right]\Bigg|_0^{2K} - \left[\frac{1}{2}\left(\frac{S_T^2}{2}-KS_T\right)\right]\Bigg|_0^{K} + \left[\frac{1}{2}\left(\frac{S_T^2}{2}-KS_T\right)\right]\Bigg|_K^{2K}$$
$$=\frac{1}{2}K^2.$$

If $S_T = 2K$ the region of gain from the holder's perspective is $\frac{1}{2}K^2$ units.

IB. Using the linear form of the function and assuming that the area over [0,2K] is sought:

$$\int_0^{2K} f(S_T)dS_T = \int_0^{2K} PAECH_T dS_T = \int_0^{K} 0dS_T \int_K^{2K}(S_T-K)dS_T = \frac{1}{2}K^2.$$

This method is preferred to the other three because the fewest computations are required.

If we applied Cheung & Chung's (1996) building block approach, the call option's price would be obtained and this price will not be equivalent to the area of the triangle computed above. The price is analogous to the area between the graph of the payoff function and the horizontal axis in the sense that the triangular region is situated in the first quadrant, implying a positive payoff and therefore price. The price measures the area between the graph of the expected value of the piecewise linear payoff function and the horizontal axis over [0,2K] and is analogous but different to the area between the graph of the call option's piecewise linear payoff function and the horizontal axis over the same interval.

Remarks. 5.3.2

1. For a standard European-style call option, the maximal profit approaches $+\infty$ since the co-domain of the profit/loss function is R. However, Chance (2007:296) states that the maximal profit is $+\infty$, thus the co-domain must be adjusted to the set of extended real numbers. The use of the latter approach is impractical given that S_T has a finite number of values during a given period and since profit/loss depends on S_T, profit/loss is thus finite as well. The maximal loss may be c_p and break-even thus occurs where $S_T = K + c_p$.

2. The convexity of $PAECH_T$ demonstrates a holder's gain (loss) beyond (at or before) K.

3. For the terminal payoff function, if the upper limit of integration was positive infinity, it would not be possible to integrate via the methods used above. Black & Scholes (1973) and Merton (1973) resorted to taking the expected value of $PAECH_T$ in computing the European-style call option's extrinsic value (C_{EV}) with a discussion of this approach by Hull (2003:262).

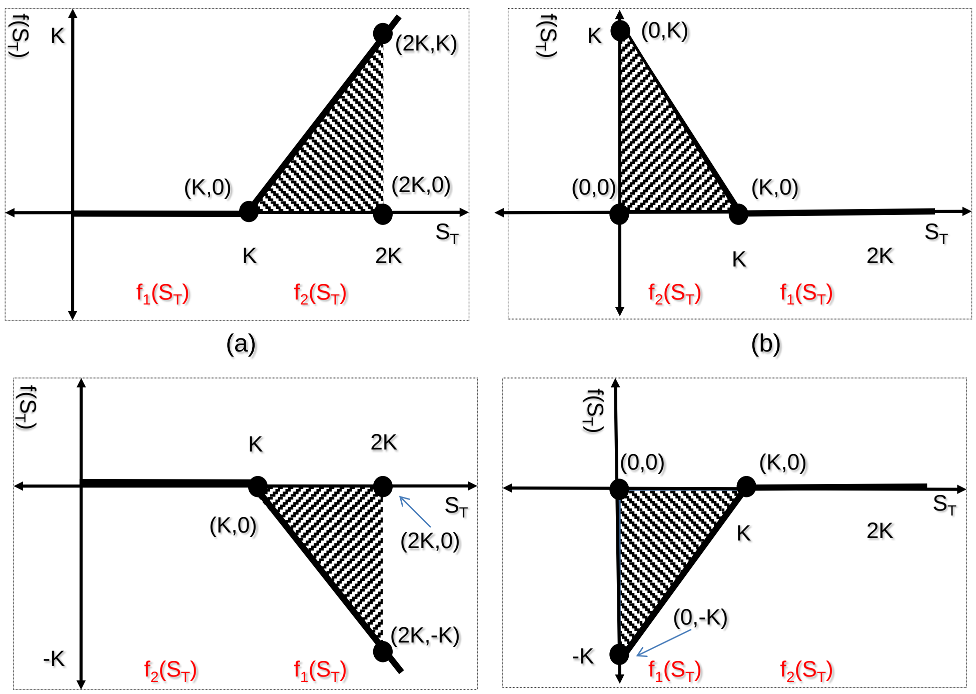


Figure 5.4: Graphs Of Standard European Options' Payoff Functions

Panel (a) contains the terminal payoff diagram for a call held (b) put held (c) call written and (d) put written. The dominant component function in a given interval is identified by a red label which is situated at the bottom of each panel.

5.3.2 Standard European-Style Call Option Written

The terminal payoff function in additive form (alphabetic and symbolic) and linear form is:

$$f(S_T) = PAECW_T = -[(S_T - K)^+] = -\max(S_T - K, 0) = \begin{cases} 0 \text{ if } S_T \le K \\ -(S_T - K) \text{ if } S_T > K. \end{cases}$$

Thus:

$$\frac{d\,PAECW_T}{d\,S_T} = \begin{cases} 0 \text{ if } S_T \le K \\ -1 \text{ if } S_T > K \end{cases}$$

which is consistent with Table 5.6.

Slope of		$S_T \le K$	$S_T > K$
Approach 1	Approach 2		
0	0	0	0
$-(S_T - K)^+$	$-(S_T - K)$	0	-1
Result		**0**	**-1**

Table 5.6: Tabular Approaches – European Call Written

Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c) $S_a =$	Minimum of functions corresponding to indices in S_a $F_a(S_T) =$
a	**b**	**c**	**d**	**e**
[0,K]	$f_1(S_T) = 0$	$f_2(S_T), f_1(S_T)$	{2,1}	$f_2(S_T) \wedge f_1(S_T)$
[K,∞)	$f_2(S_T) = -(S_T - K)$	$f_1(S_T)$, $f_2(S_T)$	{1,2}	$f_1(S_T) \wedge f_2(S_T)$

Table 5.7: Algorithm Inputs – European Call Written

Using Panel c of Figure 5.4 to construct Table 5.7 and taking the max for each interval yields the following DNF:

$$f(S_T) = PAECW_T = [f_2(S_T) \wedge f_1(S_T)] \vee [f_1(S_T) \wedge f_2(S_T)] = -[(S_T - K) \vee 0]$$

Dually, we have:

$$f(S_T) = PAECW_T = [(K - S_T) \wedge 0]$$

The additive form can be obtained by mere conversion of notation.

5.3.3 Standard European-Style Put Option Held

Use Equation 2.8 then let a = $(K - S_T)$ thus the terminal payoff function in additive and linear forms comprise:

$$f(S_T) = PAEPH_T = (K - S_T)^+ = \begin{cases} K - S_T \text{ if } S_T < K \\ 0 \text{ if } S_T \ge K \end{cases}.$$

The terminal payoff function is drawn by finding the derivative:

$$\frac{d\,PAEPH_T}{dS_T} = \begin{cases} -1 \text{ if } S_T \le K \\ 0 \text{ if } S_T > K \end{cases}$$

which yields Panel b of Figure 5.4. These results are consistent with the tabular approach (Table 5.8):

Slope of		$S_T \le K$	$S_T > K$
Approach 1	Approach 2		
0	0	0	0
$(K-S_T)^+$	$(K-S_T)$	-1	0
Result		**-1**	**0**

Table 5.8: Tabular Approaches – European Put Held

Using Table 5.9, we take the max to obtain the following DNF:

$$f(S_T) = PAEPH_T = (K - S_T) \vee 0.$$

Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c) $S_a =$	Minimum of functions corresponding to indices in S_a $F_a(S_T) =$
a	**b**	**c**	**d**	**e**
[0,K]	$f_1(S_T) = K–S_T$	$f_1(S_T)$	{1}	$\wedge f_1(S_T)$
[K,∞)	$f_2(S_T) = 0$	$f_2(S_T)$	{2}	$\wedge f_2(S_T)$

Table 5.9: Algorithm Inputs – European Put Held

Proposition. 5.3.3 The terminal payoff to a European-style put option from a holder's point of view in the most favourable state of the world is obtained directly from Equation 2.8.

Proof. The terminal payoff to a European-style put option is $(K - S_T)^+$. The most favourable state of the world occurs when $K > S_T$. We recall from Equation 2.8 with $a = K - S_T$ that:

$$(K - S_T)^+ = \frac{K - S_T + |K - S_T|}{2}.$$

When $K > S_T$, $(K - S_T)^+ = K - S_T$. Thus, the put option's terminal payoff can also be expressed as:

$$\frac{K - S_T + |K - S_T|}{2}.$$

□

The point at which $f_1(S_T)$ and $f_2(S_T)$ intersect is (K,0), which is consistent with the line of symmetry at $S_T = K$. The extent of the interval over which gains can be received by the holder of this option can be calculated in several ways, assuming that an upper limit is set on S_T.

5.3.4 Standard European-Style Put Option Written

The terminal payoff function in additive form considering duality and linear form is:

$$f(S_T) = PAEPW_T = \min(S_T - K, 0) - \max[K - S_T, 0] = -[(K - S_T)]^+ = \begin{cases} -(K - S_T) & \text{if } S_T < K \\ 0 & \text{if } S_T \geq K \end{cases}$$

By duality of the min operator, the max operator is retrieved. The derivative is:

$$\frac{d\,PAEPW_T}{d\,S_T} = \begin{cases} 1 & \text{if } S_T \leq K \\ 0 & \text{if } S_T > K \end{cases}$$

which yields Panel d of Figure 5.4. The tabular approach yields the same result (Table 5.10).

Slope of		$S_T \leq K$	$S_T > K$
Approach 1	Approach 2		
0	0	0	0
$-(K-S_T)^+$	$-(K-S_T)$	1	0
Result		**1**	**0**

Table 5.10: Tabular Approaches – European Put Written

Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c) $S_a =$	Minimum of functions corresponding to indices in S_a $F_a(S_T) =$
a	**b**	**c**	**d**	**e**
[0,K]	$f_1(S_T) = (S_T - K)$	$f_2(S_T), f_1(S_T)$	{2,1}	$f_2(S_T) \wedge f_1(S_T)$
[K,∞)	$f_2(S_T) = 0$	$f_1(S_T), f_2(S_T)$	{1,2}	$f_1(S_T) \wedge f_2(S_T)$

Table 5.11: Algorithm Inputs – European Put Written

Taking the max of each row of e in Table 5.11, we the following DNF:

$$f(S_T) = PAEPW_T = [-(K - S_T) \vee 0] = [(S_T - K) \wedge 0].$$

5.4 Lattice Concepts Relevant To Standard Options

Several mathematical concepts, especially those relevant to lattice theory, which were discussed in the previous three chapters, can be applied to the theory of option pricing, thereby appropriating economic significance.

Vertical Translation

We now consider the effect of a vertical translation of standard European-style call and put option terminal payoff functions. The terminal payoff function to a European-style call or put option which is either held or written is a piecewise linear function. Application of this principle to the European-style call held terminal payoff function yields:

$$\max(S_T - K, 0) = \max(S_T - K + K, 0 + K) - K = \max(S_T, K) - K .$$

It will be noticed that these expressions are equal, thus for this concept to be useful, quoted option prices are introduced later.[69]

Horizontal Translation and Change of Strike Price

The effect of a horizontal translation of any European-style option's terminal payoff is that a new strike price will be obtained.

Distinguishing terminal payoff from profit/loss is essential in understanding the technique(s) applied in assigning prices and values to options, especially when the specification of the option becomes more complex.

Definition. 5.4.1 The *Profit/Loss function* or *Net Terminal Payoff function* associated with a given standard European-style call option's terminal payoff function is its vertical translation.

Vertical translation of such a payoff function requires the subtraction of the option's premium from a terminal payoff function for a given position yielding the profit/loss

[69] Vertical translation is often used in the option pricing literature. For example, Kuhn & Kyprianou (2007) create a continuous piecewise affine gain function which is a vertical translation of a piecewise linear lower gain function based on the payoff of an American put option.

function for the corresponding position. The graph of the terminal payoff function is translated vertically by the premium's value. A profit/loss function differs from the terminal payoff function because option premia must, for instance, be paid by holders of call options and put options. Such option premia are generally paid when an option contract is struck (t = 0).[70] The call option's quoted price or premium (c_p) where $c_p \in R_+$ is defined as:

$$c_p = C_{IV} + C_{EV},$$

where C_{IV} is call option's Intrinsic Value or the payoff that would be received if the underlying is at its current level (S_0) when the option expires and C_{EV} is the Extrinsic Value or the premium that a rational investor would pay in excess of C_{IV} based on the probability it will increase in value before expiry on the date the option contract is struck (t = 0). If $C_{IV} = 0$ then $C_{EV} = 0$ at $t = 0$. C_{EV} can be obtained using the risk-neutral approach[71] or closed-form solutions such as the Black-Scholes formula discussed in Appendix B.

A put option's quoted price or premium (p_p) where $p_p \in R_+$ is defined by:

$$p_p = P_{IV} + P_{EV},$$

where P_{IV} is and P_{EV} is the put option's intrinsic and extrinsic value respectively.[72]

The four standard European-style option payoff functions are translated and represented in the additive form (symbolic and alphabetic) and linear form. In addition, DNF is also used to represent the functions below. For the two cases of options being written, the max and min operators will appear together to demonstrate the concept of duality. The European-style call held profit/loss function considering the call option's premium (c_p) is:

[70] Davis (2001) states that Bachelier (1900) discussed warrants traded on the Paris Bourse wherein premia were paid when contracts expired – a practice that would endure for the next three decades on organised exchanges.

[71] See Hull (2003).

[72] Using the put-call parity condition P_{EV} can be obtained from C_{EV}.

$$
\begin{aligned}
\text{PLECH}_T &= \max(S_T - K, 0) - c_p \\
&= (S_T - K)^+ - c_p \\
&= (S_T - K \vee 0) - c_p \\
&= \begin{cases} -c_p \text{ if } S_T \le K \\ S_T - K - c_p \text{ if } S_T > K \end{cases}
\end{aligned}
$$

The corresponding position of the call's writer considering duality is:

$$
\begin{aligned}
\text{PLECW}_T &= -[\max(S_T - K, 0) - c_p] \\
&= -[(S_T - K)^+ - c_p] \\
&= -[[(S_T - K) \vee 0] - c_p] \\
&= \min(K - S_T, 0) + c_p \\
&= [(K - S_T) \wedge 0] + c_p \\
&= \begin{cases} (K - S_T) + c_p \text{ if } S_T > K \\ c_p \text{ if } S_T \le K \end{cases}
\end{aligned}
$$

The European-style put held profit/loss function is:

$$
\begin{aligned}
\text{PLEPH}_T &= [\max(K - S_T, 0) - p_p] \\
&= (K - S_T)^+ - p_p \\
&= [(K - S_T) \vee 0] - p_p \\
&= \begin{cases} K - S_T - p_p \text{ if } S_T \le K \\ -p_p \text{ if } S_T > K \end{cases}
\end{aligned}
$$

Considering duality, the put writer has a corresponding position of:

$$
\begin{aligned}
\text{PLEPW}_T &= -[\max\{K - S_T, 0) - p_p] \\
&= -[(K - S_T)]^+ - p_p] \\
&= -[[(K - S_T) \vee 0] - p_p] \\
&= -[-\min\{-K + S_T, 0) - p_p] \\
&= -[-\{-K + S_T \wedge 0\} - p_p] \\
&= \begin{cases} -[(K - S_T) - p_p] \text{ if } S_T < K \\ p_p \text{ if } S_T \ge K \end{cases}
\end{aligned}
$$

A summary of the characteristics of the profit and loss functions described above appear in Table 5.12.

Option	Domain	Co-domain	Feature	Profit/Loss Function =
$PLECH_T$	R_+	R	Convex	$(S_T-K)^+ - c_p$
$PLEPH_T$	R_+	R	Convex	$(K-S_T)^+ - p_p$
$PLECW_T$	R_+	R	Concave	$-[(S_T-K)^+] + c_p$
$PLEPW_T$	R_+	R	Concave	$- [(K-S_T)^+] + p_p$

Table 5.12: Characteristics of Standard Option Profit/Loss Functions

The profit/loss functions of standard European-style call and put options are piecewise affine functions if $c_p \neq 0$ and $p_p \neq 0$ respectively. For example, consider the profit/loss function for a European-style call held (with the piecewise linear part indicated):

$$\max(S_T - K,0) - c_p = \max(S_T - K + K,0 + K) - K - c_p = \underbrace{\max(S_T, K) - K}_{\text{Linear}} - c_p.$$

Similarly, the remaining option positions can be shown to be piecewise affine. It is apparent from Table 5.12 that the terminal payoff and profit/loss equations have the same domain but different co-domains, given the consideration of option premia for the latter equations.

Profit/loss functions, like payoff functions are generally represented in additive form, DNF, linear and graphical form. The terminal payoff function and associated profit/loss function appear in a common diagram in Figure 5.5. The graph of profit/loss function is also drawn by finding whether each component increases or decreases in a certain interval.

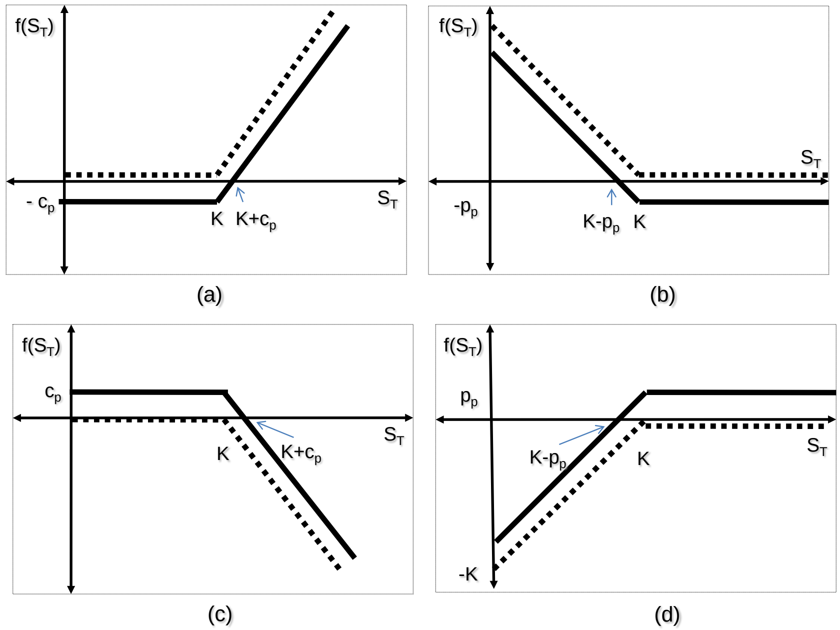


Figure 5.5: Graphs Of Combined Payoff and Profit/Loss Functions – Standard European-Style Options

The combined terminal payoff (dashed lines) and profit/loss (solid lines) diagram - Panel (a) call held (b) put held (c) call written and (d) put written.

From a geometric perspective, a terminal payoff function's position beneath or above the profit/loss function, when the two are drawn in a common diagram (i.e. co-domain is expressed as currency units and domain is expressed as S_T), represents the concept of dominance of a function discussed in Part 8 of Remarks 4.2.24. If we replace x by S_T, the function $f(S_T)$ is said to dominate a function $g(S_T)$ on [a,b] if $f(S_T) > g(S_T)$ for all S_T in [a,b]. This concept serves as the basis for the following proposition.

Proposition. 5.4.2 The terminal payoff functions of standard European-style options dominate corresponding profit/loss functions when positions are held but are dominated by corresponding profit/loss functions when positions are written.

Proof. For the European-style call option held, recall from Chapter Four that the pointwise maximum of two functions f(x) and g(x) yields h(x) = max[f(x),g(x)]. Thus, if x is called S_T, then:

$$h(S_T) = \max[\ f(S_T), g(S_T)\ ] = \max[\ (S_T - K)^+, (S_T - K)^+ - c_p\ ] = (S_T - K)^+.$$

The result for the put option held position is obtained in a similar way. For the dominance of a profit/loss function over the terminal payoff function, consider the European call option written wherein $h(S_T) = \max[\ f(S_T), g(S_T)\ ] = \max[-(S_T-K)^+, -[(S_T-K)^+ - c_p]] = -[(S_T-K)^+ - c_p]$. The result for put option written is obtained similarly. □

Consider Panel a of Figure 5.5 wherein the presence of the vertical intercept ($-c_p$) causes the standard European-style option's terminal payoff function (represented by the dotted line) to dominate the profit/loss function (represented by the solid line).

The extent of the dominance of a given function over another will depend on option premia values. Using the Domination Inequality for the European-style call option held, the dominance of its payoff function over its profit and loss function over (0,K] (for example) can be represented by:

$$\int_0^K (S_T - K)^+ dS_T - \int_0^K [(S_T - K)^+ - c_p] dS_T \geq 0$$

where:

$$\int_0^K [(S_T - K)^+] dS_T = 0$$

$$\int_0^K [(S_T - K)^+ - c_p] dS_T = -c_p K .$$

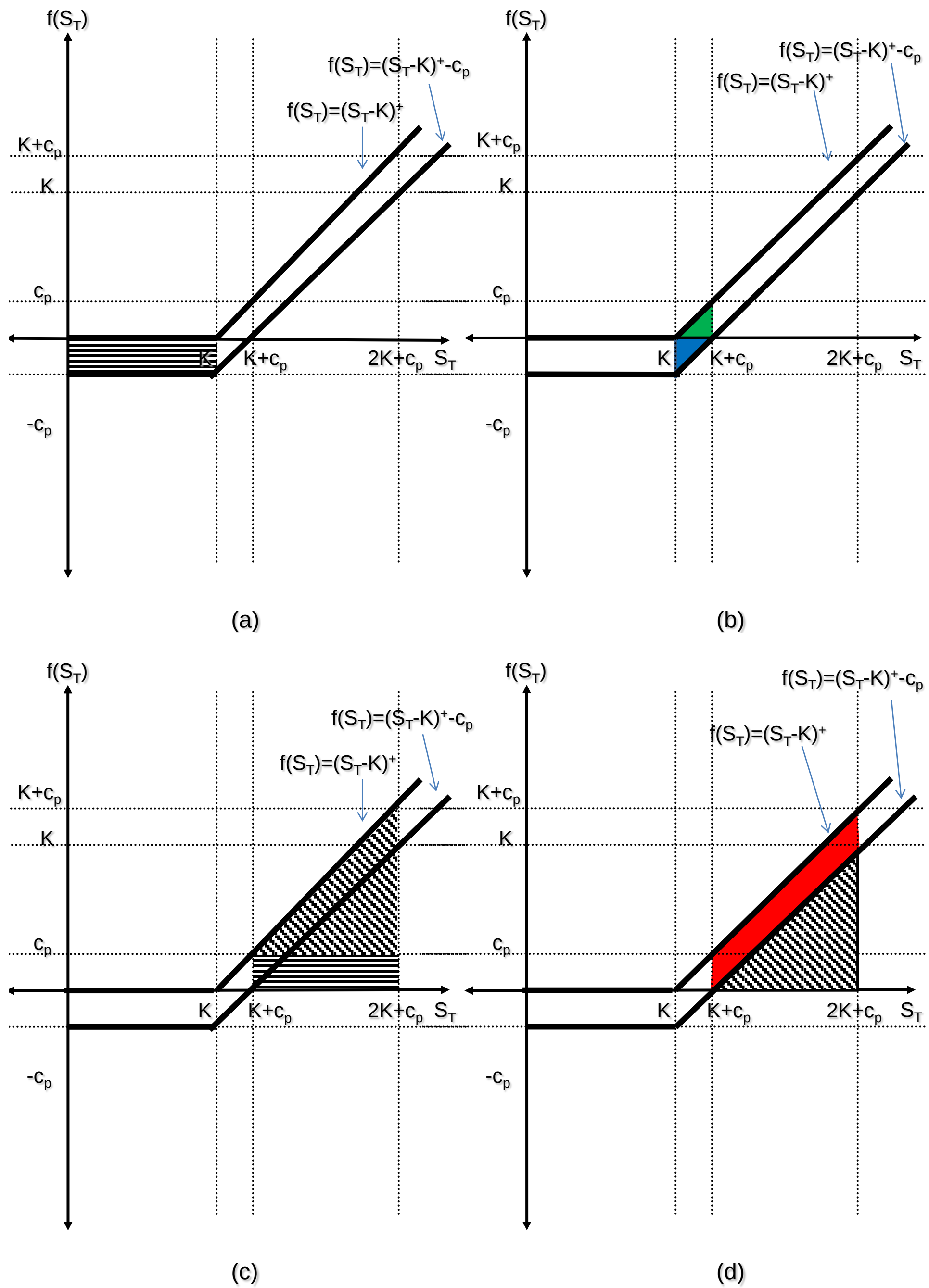


Figure 5.6: Graphs Of European-Style Call Option Payoff and Profit/Loss Functions

The call option will not be exercised and only a loss will be incurred by the option's holder (equal to the call option premium that c_p must be paid) when S_T takes on values in (0,K]. The area of the region shaded in grey in Panel a of Figure 5.6 is equal to $c_p K$ if only magnitude is considered. However, the attachment of a negative sign to $c_p K$ is necessary to demonstrate the payment of c_p over (0,K]. This payment, along with the fact that $S_T \leq K$, can also be demonstrated by the following expression:

$$\int_0^K (S_T - K)^+ dS_T + \int_0^K [(S_T - K)^+ - c_p] dS_T = 0 + (-c_p K) = -c_p K \,.$$

Now consider Panel b of Figure 5.6. It can be verified that the triangular region shaded in green has an area of $\frac{1}{2}c_p^2$ which is exactly offset by the area of the triangular region shaded in red, thus for any value of S_T in (K,K+c_p], the payment of c_p detracts from the amount gained when $S_T > K$. The fact that K+c_p is a break-even point for the call option's holder can be demonstrated by:

$$\int_K^{K+c_p} (S_T - K)^+ dS_T + \int_K^{K+c_p} [(S_T - K)^+ - c_p] dS_T = \frac{1}{2}c_p^2 + \left(-\frac{1}{2}c_p^2\right) = 0$$

The value of 0 means that the option's holder neither gains nor loses over the interval considered. Finally, inspect Panel c of Figure 5.6. It can be checked that:

$$\int_{K+c_p}^{2K+c_p} (S_T - K)^+ dS_T = \frac{1}{2}K^2 + c_p K \,.$$

From Panel d of Figure 5.6 we observe that:

$$\int_{K+c_p}^{2K+c_p} [(S_T - K)^+ - c_p] dS_T = \frac{1}{2}K^2 \,.$$

Despite the payment of c_p, the excess of S_T over K over (K+c_p,2K+c_p] leads to the holder gaining, as indicated by a positive value of the difference between the call option's payoff and profit and loss function:

$$\int_{K+c_p}^{2K+c_p} (S_T - K)^+ dS_T - \int_{K+c_p}^{2K+c_p} [(S_T - K)^+ - c_p] dS_T = c_p K$$

which is the area of the region shaded in red in Panel d of Figure 5.6.

Duality and Vertical Reflection

Recall that duality in the lattice-theoretic sense means that, via the procedure described in Chapter 2, a max operator is converted into a min operator and vice versa. In addition, recall the relation between duality and a vertical reflection in the horizontal axis of a given function. These concepts of duality and vertical reflection can be applied to the terminal payoff functions of the four positions considered. A vertical reflection along the S_T axis of $f(S_T)$ is given by the mapping $(S_T, f(S_T)) \rightarrow (S_T, -f(S_T))$. Thus, the vertical translation of the following functions means that $f(S_T)$ = $PAECH_T$ becomes $PAECW_T$; and $f(S_T)$ = $PAEPH_T$ becomes $PAEPW_T$.

The vertical reflection of corresponding profit and loss functions have the same effect. It is less obvious that the consequence of the vertical reflection of a standard European-style option's concave (convex) terminal payoff or profit and loss function will be the creation of a convex (concave) terminal payoff or profit and loss function.

For example, the terminal payoff and profit/loss (expressed with max operators) for ECH is convex. When the negative of such terminal payoff and profit/loss functions (negative of a max operator which yields a positive min operator with components having the opposite sign to the max operator) are written to obtain the dual which then serves as the terminal payoff or profit/loss function for ECW, the graph of PAECW and its corresponding profit/loss function are both concave. The same comments apply to both put option positions. We now discuss a concept which is related to vertical translation.

Scaling

Hitherto, single units of an option were held or written. However, multiple options are generally held or written by economic agents. Since max operators can be multiplied by a constant and a terminal payoff and profit/loss function can be written with max operators, the same functions can also be multiplied by a constant when multiple options are struck. Thus, if x claims are bought (sold) then a payoff function consisting of a max operator must be multiplied by ρ where if Z is the set of integers $\rho \in Z \setminus \{0\}$ which precludes fractions of an option and no option from being held.

The concept of scaling by ρ = -1 is equivalent to vertical reflection because the terminal payoff and profit/loss function for an option being held yields the terminal payoff and profit/loss function for an option being written and vice versa.

	(1 claim held)	(1 claim written)	(2 claims held)	(2 claims written)
$PAECH_T$	$(S_T\text{-}K)^+$	$-(S_T\text{-}K)^+$	$2(S_T\text{-}K)^+$	$-2(S_T\text{-}K)^+$
$PAEPH_T$	$(K\text{-}S_T)^+$	$-(K\text{-}S_T)^+$	$(2)(K\text{-}S_T)^+$	$(-2)(K\text{-}S_T)^+$
$PAECW_T$	$[-\{(S_T\text{-}K)^+\}]$	$-[-\{(S_T\text{-}K)^+\}]$	$(2)[-\{(S_T\text{-}K)^+\}]$	$(-2)[-\{(S_T\text{-}K)^+\}]$
$PAEPW_T$	$[-\{(K\text{-}S_T)^+\}]$	$-[-\{(K\text{-}S_T)^+\}]$	$(2)[-\{(K\text{-}S_T)^+\}]$	$(-2)[-\{(K\text{-}S_T)^+\}]$

Table 5.13: Summary of Standard European-Style Options' Payoff Functions

	(1 claim held)	(1 claim written)	(2 claims held)	(2 claims written)
$PLECH_T$	$(S_T\text{-}K)^+ - c_p$	$-\{(S_T\text{-}K)^+ - c_p\}$	$2\{(S_T\text{-}K)^+ - c_p\}$	$-2\{(S_T\text{-}K)^+ - c_p\}$
$PLEPH_T$	$(K\text{-}S_T)^+ - p_p$	$-\{(K\text{-}S_T)^+ - p_p\}$	$2\{(K\text{-}S_T)^+ - p_p\}$	$-2\{(K\text{-}S_T)^+ - p_p\}$
$PLECW_T$	$-\{(S_T\text{-}K)^+\} + c_p$	$-\{[-\{(S_T\text{-}K)^+\}] + c_p\}$	$2\{-\{(S_T\text{-}K)^+\} + c_p\}$	$-2\{-\{(S_T\text{-}K)^+\} + c_p\}$
$PLEPW_T$	$- \{(K\text{-}S_T)^+\} + p_p$	$-\{[-\{(K\text{-}S_T)^+\}] + p_p\}$	$2\{- \{(K\text{-}S_T)^+\} + p_p\}$	$-2\{- \{(K\text{-}S_T)^+\} + p_p\}$

Table 5.14: Summary of Standard European-Style Options' Profit/Loss Functions

The graphical implication of scaling is that the payoff function's graph for a single option pivots only in intervals where the terminal payoff is non-zero (in the same quadrant) if ρ is positive which is depicted in Figure 5.7 (assuming values of K and S_T) by the purchase of one claim as the bold line and the purchase of two claims yielding the thin line. For $\rho = 2$, this concept is demonstrated below for terminal payoff and profit/loss.[73]

For multiple units of an option being written, multiplication of a terminal payoff or profit/loss function by a negative integer is necessary (see Table 5.13 and Table 5.14). Consider a call option which can be held as well as written (sold). If 2 units of such an option are written, $PAECH_T$ would be multiplied by -2 which would yield the same terminal payoff as for 2 units of the European style call option being written ($PAECW_T$), with the geometric implications being obvious.

Note that the superimposition of the line representing the purchase of multiple units of S_T would still dominate terminal payoff function corresponding to the case when multiple options are held but would be dominated by the same payoff function when multiple options are written.

[73] Lewis (2010, p.5) considers the sale of two standard European-style call options.

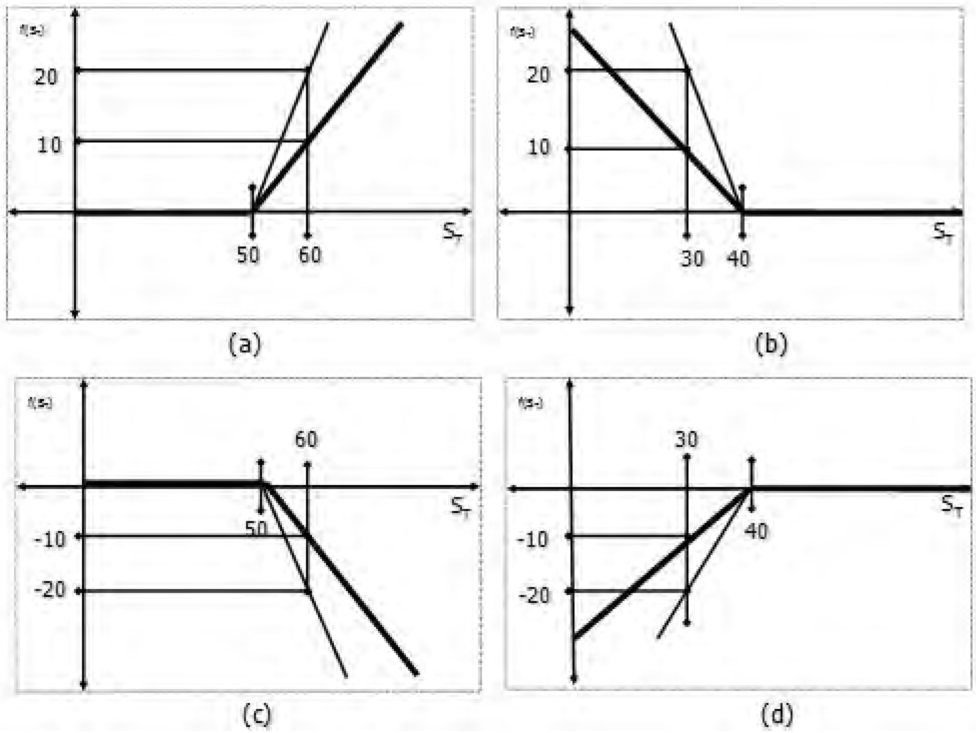


Figure 5.7: Graphs Of European-Style Options' Payoff Functions (2 Held)

Panel (a) contains the terminal payoff diagram for a call held (b) put held (c) call written and (d) put written when 2 claims are held with the bold line indicating 1 unit of the option while the thin line indicates -2 units of the option.

The geometric implication of the graphs contained in Figure 5.7 is that in non-zero intervals, the area between the graph and the horizontal axis in (a) and (b) and above the line (c) and (d) doubles, meaning that an economic agent's gain/loss also doubles, using the formula for the area of a triangle or integration.

Horizontal Translation

If a constant x is added (subtracted) from the K component of a terminal payoff function, the graph of the function is translated horizontally along the S_T axis such that it shifts leftwards (rightwards) respectively by x, which represents either call or put option premia.

When single or multiple strike prices are deterministic, horizontal translation procedures must be discarded. However, a stochastic strike price admits horizontal translation for the terminal payoff and profit/loss functions and the associated graphs.

The horizontal translation of the European call option's (held) terminal payoff by $-c_p$ units can be represented by:

$$\begin{aligned}[S_T-(K-c_p)]^+ &= \max[S_T-(K-c_p),0] \\ &= \max[S_T-(K-c_p)+(K-c_p),0+(K-c_p)]-(K-c_p) \\ &= \max[S_T,(K-c_p)]-(K-c_p).\end{aligned}$$

Notice that if:

$$\begin{aligned}&S_T>(K-c_p)\Rightarrow S_T-(K-c_p) \\ &S_T<(K-c_p)\Rightarrow (K-c_p)-(K-c_p)=0.\end{aligned}$$

The same result is obtained by using Equation 2.6 which yields:

$$\max[S_T,(K-c_p)]-(K-c_p)=\frac{S_T+(K-c_p)+|S_T-(K-c_p)|}{2}-(K-c_p).$$

Evaluation of the modulus expression yields:

$$\begin{aligned}&S_T>(K-c_p)\Rightarrow S_T-(K-c_p)>0\Rightarrow S_T-(K-c_p)\Rightarrow \\ &\Rightarrow \frac{S_T+(K-c_p)+S_T-(K-c_p)}{2}-(K-c_p)=\frac{2S_T}{2}-(K-c_p)=S_T-(K-c_p).\end{aligned}$$

In addition:

$$\begin{aligned}&S_T<(K-c_p)\Rightarrow S_T-(K-c_p)<0\Rightarrow -S_T+(K-c_p)\Rightarrow \\ &\Rightarrow \frac{S_T+(K-c_p)+[-S_T+(K-c_p)]}{2}-(K-c_p)=\frac{2(K-c_p)}{2}-(K-c_p)=0.\end{aligned}$$

Note that the line of symmetry of $S_T = K–c_p$ is obtained from:

$$|S_T-(K-c_p)|=0\rightarrow S_T-(K-c_p)=0 \text{ thus } \underbrace{K-c_p}_{=S_T}-(K-c_p)=0.$$

Similarly, the addition of $+c_p$ to a component of $PAECH_T$ yields:

$$\begin{aligned}[S_T-(K+c_p)]^+ &= \max[S_T-(K+c_p),0] \\ &= \max[S_T-(K+c_p)+(K+c_p),0+(K+c_p)]-(K+c_p) \\ &= \max[S_T,(K+c_p)]-(K+c_p).\end{aligned}$$

If:

$$\begin{aligned}&S_T>(K+c_p)\Rightarrow S_T-(K+c_p) \\ &S_T<(K+c_p)\Rightarrow (K+c_p)-(K+c_p)=0.\end{aligned}$$

The use of Equation 2.6 yields the same result:

$$\max[S_T,(K+c_p)]-(K+c_p)=\frac{S_T+(K+c_p)+|S_T-(K+c_p)|}{2}-(K+c_p)\,.$$

Again, evaluation of the modulus expression yields:

$$S_T>(K+c_p)\Rightarrow S_T-(K+c_p)>0\Rightarrow$$
$$S_T-(K+c_p)\Rightarrow\frac{S_T+(K+c_p)+S_T-(K+c_p)}{2}-(K+c_p)=\frac{2S_T}{2}-(K+c_p)=S_T-(K+c_p).$$

along with:

$$S_T<(K+c_p)\Rightarrow S_T-(K+c_p)<0\Rightarrow -S_T+(K+c_p)\Rightarrow$$
$$\frac{S_T+(K+c_p)+[-S_T+(K+c_p)]}{2}-(K+c_p)=\frac{2(K+c_p)}{2}-(K+c_p)=0.$$

Notice that the line of symmetry of S_T = K + c_p is obtained from $|S_T-(K+c_p)|=0\rightarrow S_T-(K+c_p)=0$ thus:

$$\underbrace{K+c_p}_{=S_T}-(K+c_p)=0$$

This result appears in Panel a and b of Figure 5.8.

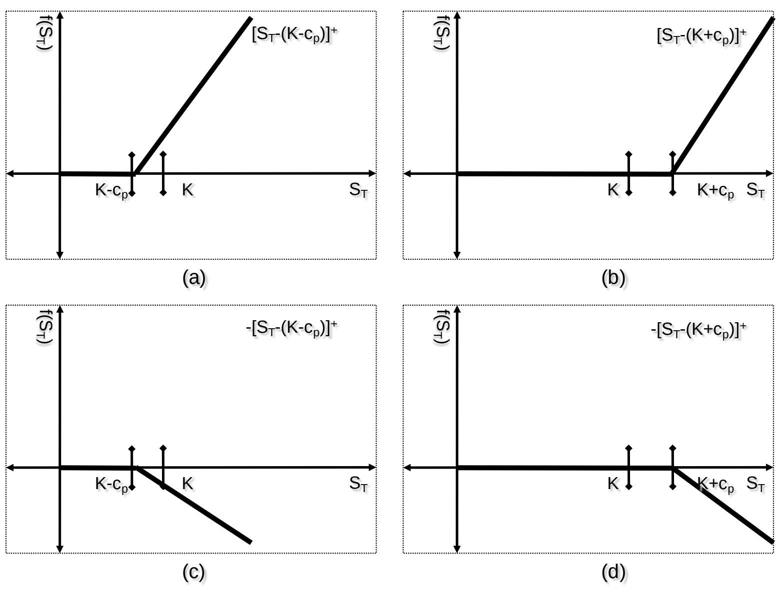


Figure 5.8: Graph Of $PAEC_T$ - Horizontal Translation By c_p Units

Panel (a) contains the terminal payoff diagram for a $PAECH_T$ translated by –c_p units (b) $PAECH_T$ translated by +c_p units (c) $PAECW_T$ translated by –c_p units (d) $PAECW_T$ translated by +c_p units.

Similar reasoning can be applied to horizontal translation of the European put option, with the results appearing in Figure 5.9.

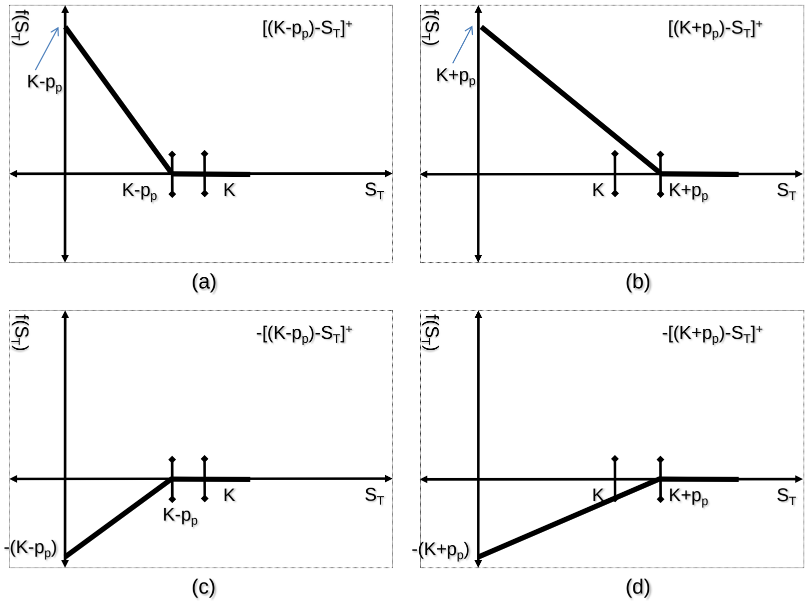


Figure 5.9: Graphs Of PAEP Functions - Horizontal Translation By p_p Units

Panel (a) contains $PAEPH_T$ translated by – p_p units (b) $PAEPH_T$ translated by $+p_p$ units (c) $PAEPW_T$ translated by $–p_p$ units (d) $PAEPW_T$ translated by $+p_p$ units.

Effects of Simultaneous Vertical and Horizontal Translation

Vertical and horizontal translation of the terminal payoff functions to the options considered is also possible but only for the purpose of demonstrating that premia terms must be carefully placed in payoff functions consisting of max operators.

Assume that $c_p = x$ for the expression contained within and outside of the max operator in Figure 5.10. The graphs of every function contained in each panel of Figure 5.10 cannot be construed as profit and loss functions because c_p is added to or subtracted from one component of the max operator. In addition, when c_p is added to the max operator, as in Panel c and Panel d of Figure 5.10, the wrong graph is also obtained. The foregoing exercise demonstrates that circumspection is needed when option premia are added to or subtracted from standard option payoff functions.

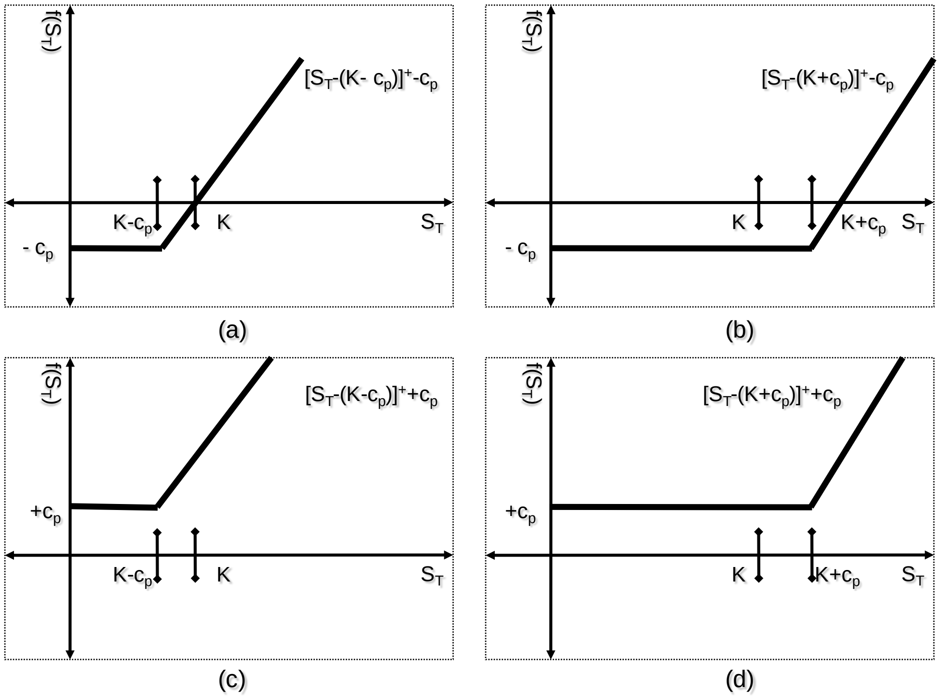


Figure 5.10: Graphs Of ECH Functions – Horizontal and Vertical Translation

Panel (a) contains ECH translated by –c_p units horizontally and vertically (b) ECH translated by +c_p units horizontally and –c_p units vertically (c) ECH translated by –c_p units horizontally and +c_p units vertically (d) ECH translated by +c_p units horizontally and vertically.

Adding Payoff Functions

Karatzas & Shreve (1998) state that the terminal payoff functions to strategies are continuous, piecewise and linear functions and can be constructed with the terminal payoff to

i. options and forward asset prices;

ii. options and spot asset prices with examples including the covered call and its reverse, protective put and its reverse;

iii. two or more options of the same kind which are referred to as spreads with examples being the bull bear, butterfly and Big W option;[74]

iv. call and put options on same asset, which are referred to as combinations with examples being the long and short straddle, strip, strap and strangle;

[74] Carroll & Brask (1999) discuss this transaction.

v. options, cash and the underlying asset which are referred to as packages with examples being break forwards, range forwards and collars; and

vi. combinations of any of the above strategies.

Given that the sum of a piecewise linear function and linear function yields another piecewise linear function, the four standard European-style option positions considered earlier can be constructed or replicated from combinations of forward contracts and other standard European-style options.

In the sequel, we let $PALF_T$ denote the terminal payoff to a long forward contract and $PASF_T$ denote the terminal payoff to a short forward contract. Thus, by applying max operators a new perspective is brought to the following well-known identities used in option pricing:

$$PAECH_T = PAEPH_T + PALF_T = PAEPH_T - PASF_T.$$

Therefore:

$$\begin{aligned}
(S_T - K)^+ &= (K - S_T)^+ - [-(S_T - K)] \\
&= \frac{K - S_T + 0 + |K - S_T - 0|}{2} - \{-[S_T - K]\} \\
&= \frac{K - S_T + |K - S_T|}{2} + S_T - K \\
&= \begin{cases} \dfrac{K - S_T + K - S_T}{2} + S_T - K = 0 & \text{if} \quad S_T \le K \\ \dfrac{K - S_T + S_T - K}{2} + S_T - K = S_T - K & \text{if} \quad S_T > K. \end{cases}
\end{aligned}$$

Similarly,

$$PAECW_T = PAEPW_T + PASF_T = PAEPW_T - PALF_T.$$

Thus:

$$\begin{aligned}
-(S_T-K)^+ &= -(K-S_T)^+ - (S_T-K)\\
&= -\left\{\frac{K-S_T+0+|K-S_T|-0}{2}\right\}-(S_T-K)\\
&= \frac{-K+S_T-|K-S_T|}{2}-S_T+K\\
&= \begin{cases} \dfrac{-K+S_T-(K-S_T)}{2}-S_T+K=0 & \text{if} \quad S_T \le K\\ \dfrac{-K+S_T-(S_T-K)}{2}-S_T+K=-(S_T-K) & \text{if} \quad S_T > K. \end{cases}
\end{aligned}$$

Moreover,

$$PAEPH_T = PASF_T + PAECH_T.$$

Thus:

$$\begin{aligned}
(K-S_T)^+ &= -(S_T-K)+(S_T-K)^+\\
&= K-S_T+\frac{S_T-K+0+|S_T-K-0|}{2}\\
&= \begin{cases} K-S_T+\dfrac{S_T-K+(K-S_T)}{2}=K-S_T & \text{if} \quad S_T < K\\ K-S_T+\dfrac{S_T-K+(S_T-K)}{2}=0 & \text{if} \quad S_T \ge K. \end{cases}
\end{aligned}$$

Finally,

$$PAEPW_T = PALF_T + PAECW_T.$$

Therefore:

$$\begin{aligned}
-(K-S_T)^+ &= (S_T-K)+[-(S_T-K)^+]\\
&= S_T-K+-\left\{\frac{S_T-K+0+|S_T-K-0|}{2}\right\}\\
&= S_T-K+\frac{-S_T+K-|S_T-K|}{2}\\
&= \begin{cases} S_T-K+\dfrac{-S_T+K-(K-S_T)}{2}=S_T-K & \text{if} \quad S_T < K\\ S_T-K+\dfrac{-S_T+K-(S_T-K)}{2}=0 & \text{if} \quad S_T \ge K. \end{cases}
\end{aligned}$$

An implication of the foregoing exposition is that long and short forward contracts can be constructed with standard European-style options, which can be easily verified by rearrangement of above set of equations.[75]

Translation and "scaling" via multiplication by a positive constant represent respectively a schedule's shift rightwards, leftwards, upwards and downwards on the one hand and the "pivot" of a graph around a given point, without any other shift. Calculation of area in non-zero intervals in both cases again yields the graphical analog (area) of an economic agent's gain/loss, with greater gains/losses being associated with larger areas.

The concepts discussed above apply to the non-standard option payoff functions that are discussed in the next chapter.

[75] Analogous to the option strategies considered for standard European-style terminal payoffs are prices of the share (S) and bond (B) in different states of the world being used in the creation of a very important strategy called the put-call parity relationship. Brown & Ross (1991) demonstrate that if $S \in E$, $B \in E$ and $K \in E$ then a call option on the stock with strike K is $(S\text{-}KB)^{+}$ and a put on the stock with same strike value is $(KB\text{-}S)^{+}$. Thus, the put-call parity relationship is:

$$\text{Stock} + \text{Put} = \text{Bond} + \text{Call} = S + (KB - S)^{+} = S + (KB - S) + (KB - S)^{-} = KB + (KB - S)^{-}.$$

CHAPTER 6. APPLICATIONS OF ALGORITHMS TO NON-STANDARD OPTION PAYOFF FUNCTIONS

Recall from Chapter 5 (specifically Section 5.4) that the piecewise linear and affine functions representing the four standard option positions' payoffs can be added to and subtracted from each other to create non-standard option payoff functions. In addition, linear functions representing the payoffs to holding or selling any security such as an equity can be added to or subtracted from any one of the four standard option payoff functions. In doing so, other continuous piecewise linear or affine non-standard option payoff functions arise and are created.

The remainder of this chapter is organised as follows. In Section 6.1 we study two-component non-standard option payoff functions constructed with standard options and shares. Section 6.2 contains the study of mainly three-component package options. The exposition includes a demonstration of the link between a payoff function in DNF and distributive (and related Boolean) lattice propositions stated in Chapter 2. In Section 6.3, spread options which consist of a varying number of components are considered. Finally, in Section 6.4, option combinations will be studied. A common feature of some of the options discussed in Section 6.3 and Section 6.4 is that a given option payoff function can implicitly "nest" another one.

We will compute values of the area between the horizontal axis and a given payoff function using several methods to demonstrate the usefulness of writing functions containing relatively many components in linear form to apply the IB method[76]. "Base" values for the area of some functions will be computed for several purposes. For example, the the extent to which the one function dominates or is dominated by its vertical translation can be computed if the "base" value is known. This computation enables the extent of an option holder's gain or loss to be compared for a given value of the underlying asset S. We will also demonstrate in Section 6.3.4 the extent to which the area between an option's payoff function differs from the same option's price. Note that the decomposition algorithm will be applied selectively throughout this chapter.

[76] In doing so, we will demonstrate that the IB method is superlative compared to the plane figure method and the IA1 and IA2 methods which are related to the additive form of a function.

6.1 Options and Shares

Hull (2003) discusses four option strategies comprising a stock and a single standard European-style option on the stock. Subtle, yet important differences justify the inclusion of complete workings for each of these positions.

6.1.1 Protective Put

The protective put comprises one share of stock being held and one standard European-style put option held and its terminal payoff function is defined in additive and linear form as:

$$f(S_T) = S_T + (K - S_T)^+ = \begin{cases} S_T + K - S_T \\ S_T + 0 \end{cases} = \begin{cases} K \text{ if } S_T \leq K \\ S_T \text{ if } S_T > K. \end{cases}$$

For future use, denote $f_1 = K$ and $f_2 = S_T$. These functions intersect at (K,K). Since a two-component function is still being considered, the line of symmetry is $K=S_T$. Differentiation yields:

$$f'(S_T) = \begin{cases} 0 \text{ if } S_T \leq K \\ 1 \text{ if } S_T > K. \end{cases}$$

The tabular approaches yields the same result (Table 6.1).

Slope of: Approach 1	Slope of: Approach 2	$S_T \leq K$	$S_T > K$
S_T	S_T	1	1
$+(K - S_T)^+$	$+(K - S_T)$	-1	0
Result		**0**	**1**

Table 6.1: Tabular Approaches – Protective Put

Note that the co-ordinates of the protective put's terminal payoff function appear in Table 6.2 along with the option and share positions to be discussed below. The graph of this function appears in Panel a of Figure 6.1, from which we obtain the information contained in Table 6.3.

$f(S_T)$	Protective Put	Reverse Covered Call	Reverse Protective Put	Covered Call
f(0) =	K	0	-K	0
f(K) =	K	-K	-K	K
f(2K) =	2K	-K	-2K	-K

Table 6.2: Co-ordinates of Option and Share Position Graphs

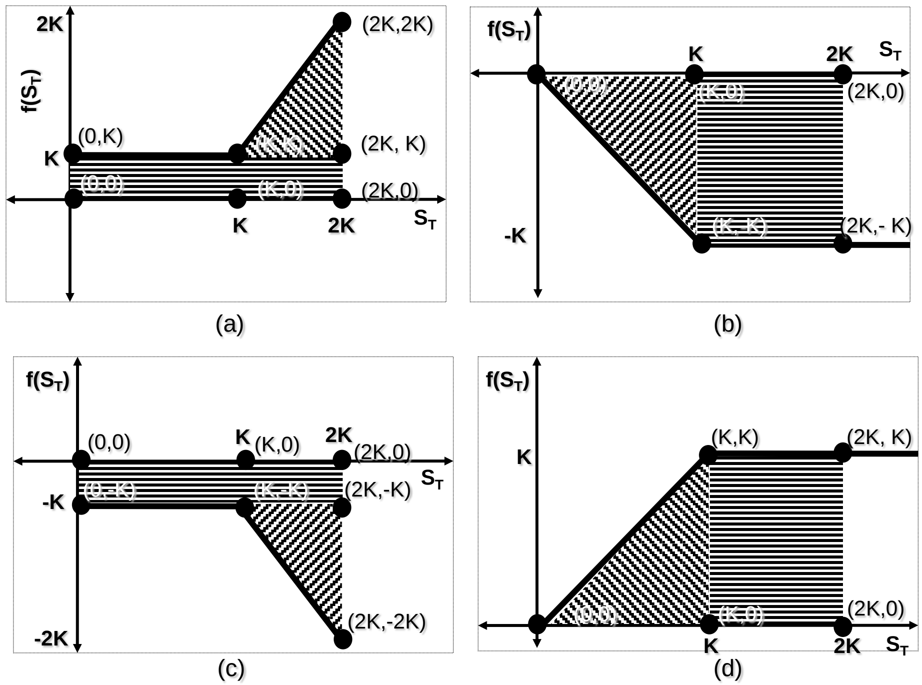


Figure 6.1: Graphs Of Option and Stock Functions

Panel (a) contains the Profit/Loss diagram for (a) a protective put (b) reverse covered call (c) reverse protective put and (d) covered call.

Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c) S_a =	Minimum of functions corresponding to indices in S_a $F_a(S_T)$ =
a	**b**	**c**	**d**	**e**
[0,K]	$f_1(S_T) = K$	$f_1(S_T)$	{1}	$\wedge f_1(S_T)$
[K,2K]	$f_2(S_T) = S_T$	$f_2(S_T)$	{2}	$\wedge f_2(S_T)$

Table 6.3: Algorithm Inputs – Protective Put

Taking the max of f_1 and f_2 in Table 6.3, $f(S_T) = K \vee S_T = S_T + \max(K - S_T, 0) = S_T + (K - S_T)^+$ which is consistent with the original terminal payoff function.

6.1.2 Reverse Protective Put

The reverse protective put comprises the short sale of a share and the writing of a European-style put option with a terminal payoff in additive and linear form of:

$$f(S_T) = -S_T + \left[-(K - S_T)\right]^+ = \begin{cases} -K \text{ if } S_T < K \\ -S_T \text{ if } S_T > K. \end{cases}$$

Differentiation yields:

$$f'(S_T) = \begin{cases} 0 \text{ if } S_T < K \\ -1 \text{ if } S_T > K. \end{cases}$$

This result is equivalent to that obtained in Table 6.4.

Slope of		$S_T \le K$	$S_T > K$
Approach 1	Approach 2		
$-S_T$	$-S_T$	-1	-1
$-(K-S_T)^+$	$-(K-S_T)$	1	0
Result		**0**	**-1**

Table 6.4: Tabular Approaches – Reverse Protective Put

Panel c of Figure 6.1 contains the graph of the reverse protective put's terminal payoff function and the contents of Table 6.5 are based on this diagram.

Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c) $S_a =$	Minimum of functions corresponding to indices in S_a $F_a(S_T) =$
a	**b**	**c**	**d**	**e**
[0,K]	$f_1(S_T) = -K$	$f_2(S_T)$,$f_1(S_T)$	{2,1}	$f_2(S_T) \wedge f_1(S_T)$
[K,2K]	$f_2(S_T) = -S_T$	$f_1(S_T)$, $f_2(S_T)$	{1,2}	$f_1(S_T) \wedge f_2(S_T)$

Table 6.5: Algorithm Inputs – Reverse Protective Put

Taking the max of f_1 and f_2 yields:

$$f(S_T) = -K \wedge -S_T = \min(-K, -S_T) = -S_T + \left[-(K - S_T)^+\right].$$

The equivalent preferred expression using the max operator is obtained by taking the dual:

$$f(S_T) = -(K \vee S_T) = -[\max(K, S_T)] = -S_T + \left[-(K - S_T)^+\right],$$

which is consistent with the original terminal payoff function.

6.1.3 Covered Call

The covered call consists of the purchase of a share and the writing of a European-style call option with a terminal payoff in additive and linear form of:

$$f(S_T) = S_T + [-(S_T - K)^+] = \begin{cases} S_T \text{ if } S_T \le K \\ K \text{ if } S_T > K. \end{cases}$$

Differentiation yields:

$$f'(S_T) = \begin{cases} 1 \text{ if } S_T \le K \\ 0 \text{ if } S_T > K. \end{cases}$$

This result and the result of Table 6.6 are equivalent.

Slope of: Approach 1	Slope of: Approach 2	$S_T \le K$	$S_T > K$
S_T	S_T	1	1
$-(S_T - K)^+$	$-(S_T - K)$	0	-1
Result		**1**	**0**

Table 6.6: Tabular Approaches – Covered Call

Table 6.7 is obtained from the use of Panel d of Figure 6.1.

Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c) S_a =	Minimum of functions corresponding to indices in S_a $F_a(S_T)$ =
a	**b**	**c**	**d**	**e**
[0,K]	$f_1(S_T) = S_T$	$f_2(S_T), f_1(S_T)$	{2,1}	$f_2(S_T) \wedge f_1(S_T)$
[K,2K]	$f_2(S_T) = K$	$f_1(S_T)$, $f_2(S_T)$	{1,2}	$f_1(S_T) \wedge f_2(S_T)$

Table 6.7: Algorithm Inputs – Covered Call

Taking the max yields:

$$f(S_T) = (S_T \wedge K) = \min(S_T, K) = S_T + [-(S_T - K)^+].$$

The equivalent preferred expression using the max operator is obtained by taking the dual:

$$f(S_T) = -(-S_T \vee -K) = -[\max(-S_T, -K)] = S_T + [-(S_T - K)^+],$$

which is consistent with the original terminal payoff function.

6.1.4 Reverse Covered Call

The reverse covered call consists of the short sale of a share and the holding of a European-style call option with a terminal payoff in additive and linear form of:

$$f(S_T) = -S_T + (S_T - K)^+ = \begin{cases} -S_T \text{ if } S_T < K \\ -K \text{ if } S_T > K. \end{cases}$$

The point of intersection is (K,-K) and the remaining steps are followed by using the information contained in Table 6.8 which in turn has been obtain from Panel b of Figure 6.1.

Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c) $S_a =$	Minimum of functions corresponding to indices in S_a $F_a(S_T) =$
a	**b**	**c**	**d**	**e**
[0,K]	$f_1(S_T) = -S_T$	$f_2(S_T)$	{2}	$\wedge f_2(S_T)$
[K,2K]	$f_2(S_T) = -K$	$f_1(S_T)$	{1}	$\wedge f_1(S_T)$

Table 6.8: Algorithm Inputs – Reverse Covered Call

Taking the max yields $f(S_T) = -S_T \vee -K = -S_T + (S_T - K)^+$, which is consistent with the original terminal payoff function.

6.2 Package Options

Musiela & Rutkowski (2005:194) defines a Package option as "[a]n arbitrary financial contract whose terminal payoff is a piecewise linear function of the terminal price of the underlying asset...a combination of standard options, cash and the underlying asset".

The three categories of Package options considered are Break Forwards, Collars and Range Forwards (Flexible Forwards), with the exposition of the last two options appearing later given that these comprise three pieces. Although the Break Forward belongs to the previous section because its terminal payoff graph consists of two pieces and more than two pieces are considered in this section, it is treated in this section for two reasons. First, to preserve continuity with the other Package options and second because its payoff function contains the price of an asset which has specific graphical implications and has been omitted in the cases studied in Chapter 5 and Section 6.1.

6.2.1 Break Forward

Consider the forward price ($\hat{F}$) of an asset (or commodity) along with the usual terminal asset price (S_T). It is well-known that $\hat{F} = S_0 e^{rT}$ where S_0 is the asset's price on the date on which the break forward contract is stuck, r is a risk-free rate of interest and T is the date of the break forward's expiry. If the strike price $K > \hat{F}$ or $0 < \hat{F} < K$ then the terminal payoff in additive form is:

$$f(S_T) = (S_T - \hat{F})^+ + \hat{F} - K.$$

Since the ($\hat{F} - K$) of the function above omits a stochastic component, the same function is preferred because it consists of a max operator with one component being 0 and the other containing a stochastic component (comprising S_T and $\hat{F}$). The above function is split into two components, denoted by $f_1 = \hat{F} - K$ and $f_2 = S_T - K$ whose domains are non-intersecting intervals. The linear form of this function and its corresponding interval domains are obtained in the usual way:

$$f(S_T) = \begin{cases} 0 + \hat{F} - K \\ S_T - \hat{F} + \hat{F} - K \end{cases} = \begin{cases} \hat{F} - K \text{ if } S_T < \hat{F} \\ S_T - K \text{ if } S_T \geq \hat{F}. \end{cases}$$

and:

$$f'(S_T) = \begin{cases} 0 \text{ if } S_T < \hat{F} \\ 1 \text{ if } S_T \geq \hat{F}. \end{cases}$$

Despite the presence of $\hat{F}$, the tabular approach still yields the same result (Table 6.9).

Slope of		$S_T \le \hat{F}$	$S_T > \hat{F}$
Approach 1	Approach 2		
$(S_T - \hat{F})^+$	$(S_T - \hat{F})$	0	1
$+(\hat{F} - K)$	$+(\hat{F} - K)$	0	0
Result		**0**	**1**

Table 6.9: Tabular Approaches – Break Forward

The graph of the Break Forward option's terminal payoff function appears in Figure 6.2. For ease of interpretation of these graphs, we have depicted the actual functions given the presence of the $\hat{F}$ term. Figure 6.2 leads to the construction of Table 6.10.

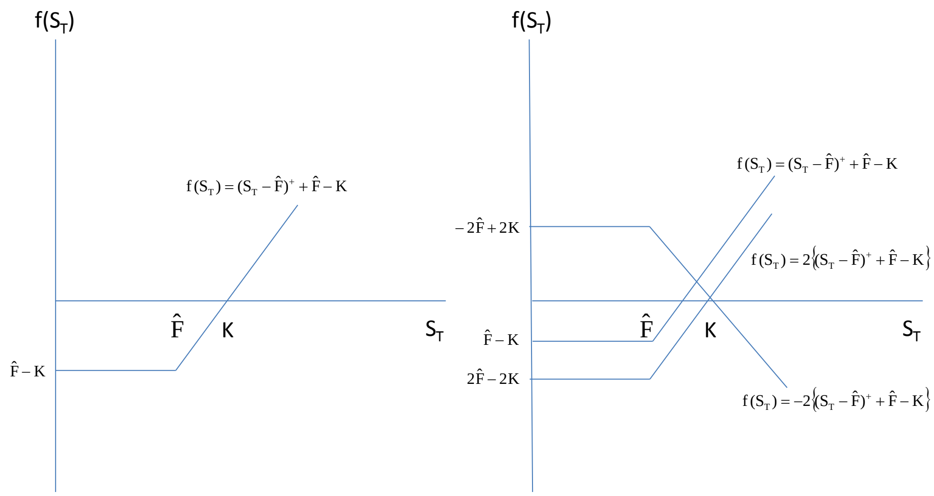


Figure 6.2: Graph Of Break Forward Function

Panel (a) represents 1 position in a Break Forward and Panel (b) multiple positions in Break Forwards

Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c) S_a =	Minimum of functions corresponding to indices in S_a $F_a(S_T)$ =
a	**b**	**c**	**d**	**e**
$[0, \hat{F}]$	$f_1(S_T) = \hat{F} - K$	$f_1(S_T)$	{1}	$\wedge f_1(S_T)$
$[\hat{F}, \infty)$	$f_2(S_T) = S_T - K$	$f_2(S_T)$	{2}	$\wedge f_2(S_T)$

Table 6.10: Algorithm Inputs – Break Forward

Taking the max of f_1 and f_2 the DNF of the function (together with its additive form) is:

$$f(S_T) = (\hat{F} - K) \vee (S_T - K) = (S_T - \hat{F})^+ + \hat{F} - K\,,$$

which is consistent with the original terminal payoff function.

Note that over $[0, \hat{F}]$, $\hat{F} < K$ or $\hat{F} - K < 0$. The Break Forward's terminal payoff can be negative ($\hat{F} - K$) where $S_T > \hat{F}$. In addition, note that $\hat{F}$ can be measured with more advanced models which depend on the characteristics of the observed underlying asset, as discussed by Musiela & Rutkowski (2005) among others.

If an investor holds two Break Forward options the additive and linear form is:

$$f(S_T) = 2\left[(S_T - \hat{F})^+ + \hat{F} - K\right] = \begin{cases} 2[0 + \hat{F} - K] \\ 2[S_T - \hat{F} + \hat{F} - K] \end{cases} = \begin{cases} 2[\hat{F} - K] \text{ if } S_T < \hat{F} \\ 2[S_T - K] \text{ if } S_T \geq \hat{F}. \end{cases}$$

Notice how, after scaling by +2, the new piecewise linear function appears to the south (rather than the north when the protective put terminal payoff was scaled) given the restriction that $\hat{F} < K$. Note that in spite of the presence of $\hat{F}$, if two units are written:

$$f(S_T) = -2\left[(S_T - \hat{F})^+ + \hat{F} - K\right] = \begin{cases} -2[0 + \hat{F} - K] \\ -2[S_T - \hat{F} + \hat{F} - K] \end{cases} = \begin{cases} -2[\hat{F} - K] \text{ if } S_T < \hat{F} \\ -2[S_T - K] \text{ if } S_T \geq \hat{F} \end{cases}$$

which reflects the graph representing two Break Forward options being held.

6.2.2 Collar

A Collar's (or "risk reversal", "cylinder, "option fence") terminal payoff function consists of the sum of K_1 units of Cash, 1 unit of European-style call option being held with strike K_1 (matching the Cash amount) and 1 unit of European-style call option with strike K_2 being written. In additive and linear form:

$$f(S_T) = K_1 + (S_T - K_1)^+ - (S_T - K_2)^+ = \begin{cases} K_1 & \text{if} & S_T \leq K_1 \\ S_T & \text{if} & K_1 < S_T < K_2 \\ K_2 & \text{if} & S_T \geq K_2 \end{cases}$$

Notice the presence of two deterministic exercise prices K_1 and K_2 with $0 < K_1 < K_2$ (although it could also have been the case that $0 < K_2 < K_1$). These interact with a stochastic underlying asset price S_T with $S_T > 0$. Differentiation yields:

$$f'(S_T)=\begin{cases}0 & \text{if} \quad S_T \le K_1\\ 1 & \text{if} \quad K_1 < S_T < K_2\\ 0 & \text{if} \quad S_T \ge K_2\end{cases}$$

When compared to the tabular approach (Table 6.11 and Table 6.12), the results are identical.

Slope of	$S_T \le K_1$	$K_1 < S_T \le K_2$	$S_T \ge K_2$
K_1	0	0	0
$+(S_T - K_1)^+$	0	1	1
$-(S_T - K_2)^+$	0	0	-1
Result	**0**	**1**	**0**

Table 6.11: Tabular Approach 1 – Collar

Slope of	$S_T \le K_1$	$K_1 < S_T \le K_2$	$S_T \ge K_2$
K_1	0	0	0
S_T	0	1	1
K_2	0	0	-1
Result	**0**	**1**	**0**

Table 6.12: Tabular Approach 2 – Collar

The points of intersection of the graphs depicted in Figure 6.3 corresponding to pairs of functions are:

f_1 and f_2: (K_1,K_1),

f_2 and f_3: (K_2,K_2).

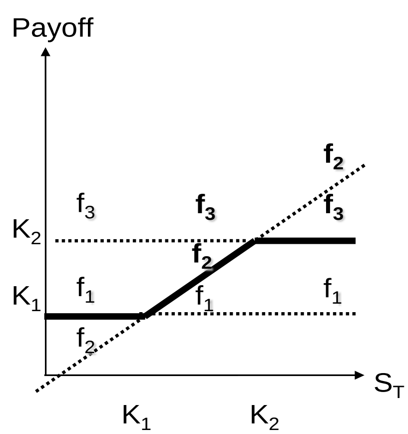


Figure 6.3: Graph Of Collar's Payoff Function

Note that f_1 and f_3 never intersect in Figure 6.3. In addition, note that the Collar's terminal payoff function alternates between convex and concave with the pointwise maximum preserving convexity of the interval before K_2.

Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c) S_a =	Minimum of functions corresponding to indices in S_a $F_a(S_T)$ =
a	**b**	**c**	**d**	**e**
$[0,K_1]$	$f_1(S_T) = K_1$	$f_3(S_T),f_1(S_T)$	{3,1}	$f_3(S_T) \wedge f_1(S_T)$
$[K_1,K_2]$	$f_2(S_T) = S_T$	$f_3(S_T),f_2(S_T)$	{3,2}	$f_3(S_T) \wedge f_2(S_T)$
$[K_2,\infty)$	$f_3(S_T) = K_2$	$f_2(S_T),f_3(S_T)$	{2,3}	$f_2(S_T) \wedge f_3(S_T)$

Table 6.13: Algorithm Inputs – Collar

Taking the maximum of the first and second row and then second and third row of $F_a(S_T)$ appearing in column e of Table 6.13 and using the idempotent law [LA3] yields $f(S_T)$ in DNF:

$$\begin{aligned} f(S_T) &= [f_1(S_T) \wedge f_3(S_T)] \vee [f_2(S_T) \wedge f_3(S_T)] \\ &= (K_1 \wedge K_2) \vee (S_T \wedge K_2) \\ &= (K_1 \vee S_T) \wedge K_2 \\ &= (S_T \vee K_1) \wedge K_2. \end{aligned}$$

The same result is obtained by using the relevant Boolean lattice property [BLP1]:

$$f(S_T) = (S_T \vee K_1) \wedge K_2 = (S_T + K_1)K_2 = S_T K_2 + K_1 K_2 = (S_T \wedge K_2) \vee (K_1 \wedge K_2).$$

To retrieve the additive form of the Collar's terminal payoff function, it is easiest to convert it to the alphabetic representation:

$$\begin{aligned} f(S_T) &= (S_T \vee K_1) \wedge K_2 \\ &= \min[\max(S_T, K_1),\ \ K_2] \\ &= \min[\max(S_T - K_1, K_1 - K_1) + K_1,\ \ K_2] \\ &= \min[\max(S_T - K_1, 0) + K_1,\ \ K_2]. \end{aligned}$$

Consider the min expression. Subtract from each of its components $\max(S_T - K_1, 0) + K_1$ which yields:

$$f(S_T) = \underbrace{K_1 + \max(S_T - K_1, 0)}_{A} + \underbrace{\min[0, K_2 - K_1 - \max(S_T - K_1, 0)]}_{B}.$$

To decompose part B use of the formula for max and min is futile. It only yields many modulus expressions (some of which may be the modulus of a modulus) and omits the restriction $0 < K_1 < K_2$ which must be considered for a valid solution to be obtained.

The original function containing three pieces is converted into the function appearing immediately above split into a part A, which is part of the correct solution and which will be ignored henceforth and a part B. The latter must be evaluated by considering either or both cases of the underlying asset's price (S_T) being less than the least exercise price (K_1) and greater than or equal to the greatest exercise price (K_2). A series of tedious yet basic calculations then yield rather cumbersome equations which comprise the solution.

Consider $S_T \leq K_1$

$$\begin{aligned} f(S_T) &= K_1 + \max(S_T - K_1, 0) + \min(0, K_2 - K_1 - 0) \\ &= K_1 + \max(S_T - K_1, 0) + \min(0, K_2 - K_1). \end{aligned}$$

Since $S_T \leq K_1$ then $S_T < K_2$. Then consider the min expression above and let $K_1 = S_T$ to obtain:

$$\begin{aligned} f(S_T) &= K_1 + \max(S_T - K_1, 0) + \min(0, K_2 - S_T) \\ &= K_1 + \max(S_T - K_1, 0) - \max(0, S_T - K_2). \end{aligned}$$

Consider $S_T \geq K_1$:

$$\begin{aligned} f(S_T) &= K_1 + \max(S_T - K_1, 0) + \min[0, K_2 - K_1 - (S_T - K_1)] \\ &= K_1 + \max(S_T - K_1, 0) + \min(0, K_2 - S_T) \\ &= K_1 + \max(S_T - K_1, 0) - \max(0, S_T - K_2). \end{aligned}$$

Thus:

$$\begin{aligned} f(S_T) &= K_1 + \max(S_T - K_1, 0) - \max(S_T - K_2, 0) \\ &= K_1 + [(S_T - K_1) \vee 0\} - \{(S_T - K_2) \vee 0] \\ &= K_1 + (S_T - K_1)^+ - (S_T - K_2)^+. \end{aligned}$$

Functions in DNF are differentiable but intervals must be given and apt lattice properties must be applied in splitting the nested expression into a sum of expressions which can then be differentiated with respect to the underlying asset over each interval.

Decomposition retrieves the original function assuming that $S_T > K_2$ which means that $S_T > K_1$:

$$
\begin{aligned}
f(x) &= \min[\max(S_T, K_1), K_2] \\
&= \max(S_T, K_1) + \min[\max(S_T, K_1) - \max(S_T, K_1), K_2 - \max(S_T, K_1)] \\
&= \max(S_T, K_1) + \min[0, K_2 - \max(S_T, K_1)].
\end{aligned}
$$

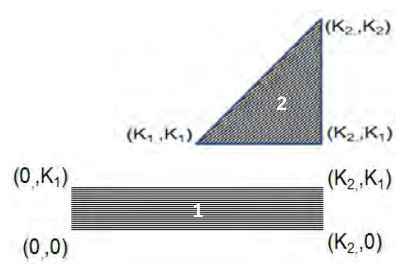
Thus:

$$f(S_T) = K_1 + (S_T - K_1)^+ - (S_T - K_2)^+ .$$

The case of $S_T < K_2$ is precluded because if $S_T < K_2$, $S_T > K_1$ or $S_T < K_1$.

If we apply Cheung & Chung's (1996) and Carroll & Brask's (1999) method was applied, the collar's price will measure the area between the graph of the expected value of the piecewise linear payoff function and the horizontal axis over $[0,K_2]$. This is different to the area between the graph of the piecewise linear payoff function and the horizontal axis over $[0,K_2]$ which is computed with the following methods:

PF. We assign a number to each of the following plane figures corresponding to the region $[0,K_2]$ of Figure 6.3:



Therefore:

$$\text{Area of } 1 + \text{Area of } 2 = K_2K_1 + \frac{1}{2}(K_2 - K_1)^2 = \frac{1}{2}(K_1^2 + K_2^2)$$

where:

$$\text{Area of } 1 = (K_2 - 0)(K_1 - 0) = K_2K_1$$

$$\text{Area of } 2 = \frac{1}{2}(K_2 - K_1).(K_2 - K_1) = \frac{1}{2}(K_2 - K_1)^2 .$$

IA1.

$$\int_0^{K_2} f(S_T)dS_T = \int_0^{K_2} K_1 dS_T + \int_0^{K_2} \max(S_T - K_1, 0)dS_T - \int_0^{K_2} \max(S_T - K_2, 0)dS_T = 0 .$$

Over $[0,K_2]$, $S_T < K_1$, $S_T > K_1$ and $S_T < K_2$ thus:

$$\int_0^{K_2} f(S_T)dS_T = \int_0^{K_2} K_1 dS_T + \int_0^{K_1} 0 dS_T + \int_{K_1}^{K_2} (S_T - K_1)dS_T + \int_0^{K_2} 0 dS_T = \frac{1}{2}K_1^2 + \frac{1}{2}K_2^2.$$

It was assumed that for $[0,K_2]$ $S_T < K_1$ implies that $\int_0^{K_1} 0\ dS_T$ and when $S_T > K_1$

$\int_{K_1}^{K_2} S_T - K_1 dS_T$ whereas $S_T < K_2$ implies $\int_0^{K_2} 0\ dS_T$.

IA2.

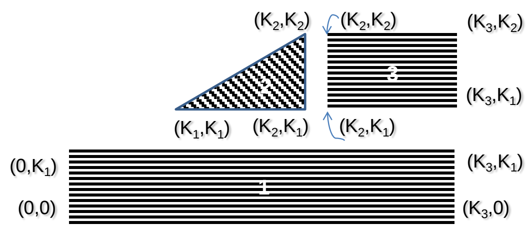
$$\int_0^{K_2} f(S_T)dS_T = \int_0^{K_2} K_1 dS_T$$
$$+ \int_0^{K_2} \frac{S_T - K_1 + 0 + |S_T - K_1 - 0|}{2} dS_T - \int_0^{K_2} \frac{S_T - K_2 + 0 + |S_T - K_2 - 0|}{2} dS_T$$
$$= \int_0^{K_2} K_1 dS_T + \int_0^{K_2} \frac{S_T - K_1}{2} dS_T + \int_0^{K_2} \frac{|S_T - K_1|}{2} dS_T$$
$$- \left\{ \int_0^{K_2} \frac{S_T - K_2}{2} dS_T + \int_0^{K_2} \frac{|S_T - K_2|}{2} dS_T \right\}$$
$$= \int_0^{K_2} K_1 dS_T + \int_0^{K_2} \frac{S_T - K_1}{2} dS_T + \int_0^{K_1} \frac{K_1 - S_T}{2} dS_T$$
$$+ \int_{K_1}^{K_2} \frac{S_T - K_1}{2} dS_T - \left\{ \int_0^{K_2} \frac{S_T - K_2}{2} dS_T + \int_0^{K_2} \frac{K_2 - S_T}{2} dS_T \right\}$$
$$= K_1K_2 + \frac{1}{4}K_2^2 - \frac{1}{2}K_1K_2 + \frac{1}{2}K_1^2 - \frac{1}{4}K_1^2 + \frac{1}{4}K_2^2$$
$$- \frac{1}{2}K_1K_2 - \frac{1}{4}K_1^2 + \frac{1}{2}K_1^2 - \frac{1}{4}K_2^2 + \frac{1}{2}K_2^2 - \frac{1}{2}K_2^2 + \frac{1}{4}K_2^2$$
$$= \frac{1}{2}K_1^2 + \frac{1}{2}K_2^2.$$

$$= K_1K_2 + \frac{1}{4}K_2^2 - \frac{1}{2}K_1K_2 + \frac{1}{2}K_1^2 - \frac{1}{4}K_1^2 + \frac{1}{4}K_2^2$$
$$- \frac{1}{2}K_1K_2 - \frac{1}{4}K_1^2 + \frac{1}{2}K_1^2 - \frac{1}{4}K_2^2 + \frac{1}{2}K_2^2 - \frac{1}{2}K_2^2 + \frac{1}{4}K_2^2$$
$$= \frac{1}{2}K_1^2 + \frac{1}{2}K_2^2.$$

It was assumed that for $[0,K_2]$ $S_T < K_1$ implying $|S_T–K_1|$ must be split therefore we get

$\int_0^{K_1} \frac{K_1 - S_T}{2} dS_T$ and $S_T > K_1$ for $\int_{K_1}^{K_2} \frac{S_T - K_1}{2} dS_T$. When $S_T < K_2$, $| S_T – K_2 |$ must be converted to $K_2 – S_T$ thus $\int_0^{K_2} \frac{K_2 - S_T}{2} dS_T$.

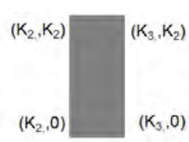
Suppose that the call option's price over $[0,K_3]$ is sought, then the area of a new region is identified using Figure 6.3 as:



with its area being most easily calculated using the IB. method. Using the linear form and considering $[0,K_3]$:

$$\int_0^{K_3} f(S_T) dS_T = \int_0^{K_1} K_1 dS_T + \int_{K_1}^{K_2} S_T dS_T + \int_{K_2}^{K_3} K_2 dS_T = K_2 K_3 + \frac{1}{2} K_1^2 - \frac{1}{2} K_2^2$$

If the Cheung & Chung (1996) and Carroll & Brask (1999) method was applied, two call option prices would be subtracted from each other and the difference in prices would also not match the area of the following:



which is equal to the difference between the area computed for $[0,K_3]$ and $[0,K_2]$:

$$\int_0^{K_3} f(S_T) dS_T - \int_0^{K_2} f(S_T) dS_T = K_2 K_3 + \frac{1}{2} K_1^2 - \frac{1}{2} K_2^2 - \left\{ \frac{1}{2} K_1^2 + \frac{1}{2} K_2^2 \right\} = K_2 K_3 - K_2^2 .$$

Scaling of the function comprising three pieces is obtained in the same way as for the two-component function. Consider 2 Collar options being held then the terminal payoff function is:

$$
\begin{aligned}
2f(S_T) &= 2[K_1 + (S_T - K_1)^+ - (S_T - K_2)^+] \\
&= 2\left[K_1 + \max(S_T - K_1, 0) - \max(S_T - K_2, 0)\right].
\end{aligned}
$$

The concept of the conversion of a collar's terminal payoff function in DNF to the same function expressed in additive form, is demonstrated below with a few examples.

Examples. 6.2.1

1. Standard Oil's Bond

Hull (2003:14) poses a problem wherein a payoff function must be assigned to a bond issued by Standard Oil in 1986. Holders of this bond never received interest and when the bond matured, Standard Oil would pay them USD 1000 plus an extra amount worth 170 times the difference (if it existed) between the USD price per barrel of crude oil at the time and USD 25. The maximum extra amount paid was set at USD 2550, if the oil price was USD 40 per barrel. Thus, the greater the crude oil price was, the more easily the company could disburse the extra amount to bond holders.

The bond's terminal payoff function can be constructed with call options and put options discussed earlier. Let S_T be the price of a barrel of crude oil at time T. Using European-style call options, this bond's terminal payoff function in DNF is:

$$
\begin{aligned}
f(S_T) &= \min\left[\max(1000, 170S_T - 3250), 3550\right] \\
&= \max(1000, 170S_T - 3250) \\
&\qquad + \min\left\{\begin{matrix} \max(1000, 170S_T - 3250) \\ -\max(1000, 170S_T - 3250), 3550 - \max(1000, 170S_T - 3250) \end{matrix}\right\} \\
&= \max(1000, 170S_T - 3250) + \min\left[0, 3550 - \max(1000, 170S_T - 3250)\right] \\
&= 1000 + \max(1000 - 1000, 170S_T - 3250 - 1000) \\
&\qquad + \min\left[0, 3550 - \max(1000, 170S_T - 3250)\right] \\
&= 1000 + \max(0, 170S_T - 4250) \\
&\qquad + \min\left[0,\ 3550 - \max(1000, 170S_T - 3250)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= 1000 + \max 170(0, S_T - 25) \\
&\qquad + \min\left[0,\ 3550 - \max(1000, 170S_T - 3250)\right] \\
&= 1000 + \max 170(0, S_T - 25) \\
&\qquad + \min\left[0,\ 3550 - [170S_T - 3250 + \max(1000 - 170S_T + 3250,\ 0)\right] \\
&= 1000 + \max 170(0, S_T - 25) \\
&\qquad + \min\left[0,\ 3550 - 170S_T + 3250 - \max(1000 - 170S_T + 3250,\ 0)\right] \\
&= 1000 + \max 170(0, S_T - 25) + \min\left[0,\ 6800 - 170S_T - \max(4250 - 170S_T,\ 0)\right] \\
&= 1000 + \max 170(0, S_T - 25) + \min\left[0,\ 170(40 - S_T) - \max[170(25 - S_T,\ 0)]\right].
\end{aligned}
$$

Consider only the component of the min operator now and assume that $S_T \geq 25$. If $S_T \geq$ 25:

i) $S_T > 0$.

ii) Either $S_T \leq 40$ or $S_T > 40$.

Therefore:

$$
\begin{aligned}
f(S_T) &= 1000 + 170\max(0, S_T - 25) + \min\left[0, 170(40 - S_T) - 0)\right] \\
&= 1000 + 170\max(0, S_T - 25) + 170\min\left[0, (40 - S_T)\right] \\
&= 1000 + 170\max(0, S_T - 25) - 170\max\left[0, -(40 - S_T)\right] \\
&= 1000 + 170\max(0, S_T - 25) - 170\max\left[0, (S_T - 40)\right] \\
&= 1000 + 170(S_T - 25)^+ - 170(S_T - 40)^+.
\end{aligned}
$$

In other words, the transaction consists of holding a bond worth USD 1000 (at T), buying 170 calls at USD 25 and writing 170 calls at USD 40. The same results should be obtained if $S_T \leq 25$ is applied to the min expression where:

i) $S_T > 0$.

ii) $S_T \leq 40$.

From the foregoing equation, the functions corresponding to each interval are:

$$
f(S_T) = \begin{cases} 1000 + 0 - 0 = 1000 & \text{if} \quad S_T < 25 \\ 1000 + 170(S_T - 25) - 0 = 170S_T - 3250 & \text{if} \quad 25 \leq S_T \leq 40 \\ 3550 & \text{if} \quad S_T > 40. \end{cases}
$$

Thus:

$$
f'(S_T) = \begin{cases} 0 & \text{if} \quad S_T < 25 \\ 170 & \text{if} \quad 25 \leq S_T \leq 40 \\ 0 & \text{if} \quad S_T > 40. \end{cases}
$$

An alternative means of constructing the terminal payoff function to Standard Oil's option consists in the use of European-style put options (from an option holder's perspective):

$$\begin{aligned} f(S_T) &= 3550 - 170\max(40 - S_T, 0) + 170\max(25 - S_T, 0) \\ &= 3550 - \max[170(40 - S_T, 0)] + \max[170(25 - S_T), 0]. \end{aligned}$$

The functions in each interval are:

$$f(S_T) = \begin{cases} 3550 - 170(40 - S_T) + 170(25 - S_T) = 1000 & \text{if} \quad S_T < 25 \\ 3550 - 170(40 - S_T) = 170S_T - 3250 & \text{if} \quad 25 \leq S_T \leq 40 \\ 3550 - 0 + 0 = 3550 & \text{if} \quad S_T > 40. \end{cases}$$

Clearly, the result is the same as for the call option construction except that it is obtained in a different way. The slope of functions in each interval when put options are used is the same as for the call option construction. Given these facts, the call and put terminal payoffs are equivalent numerically and Standard Oil's option terminal payoff diagram, irrespective of whether it was assembled with call options or put options is the same (Figure 6.4).

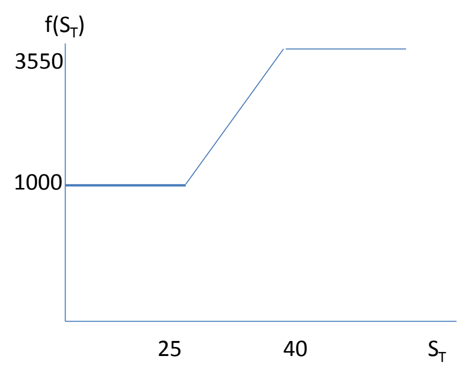


Figure 6.4: Graph Of Standard Oil Bond's Payoff Function

It can be easily checked that a common diagram represents the terminal payoff function of the bond when it is assembled with call options is the same as when it was assembled with put options.

2. Molchanov's Index Contract

Molchanov (2007:36) considers the difference between two standard European call options on a share price index held plus some cash yields a five-year contract that gives

90% times the ratio of a share price index's terminal and initial levels ($S = S_T/S_0$). Note that $S > 0$. Otherwise, the contract yields 130% if less than S or 180% if greater than S. In other words, a cash amount worth 1.8 units and the product of 0.9 and the sum of a call option held with strike price of 1.44 and call written with strike of 2.00 is considered. The contract's payoff function is expressed as:

$$
\begin{aligned}
f(S) &= \min\left[\max(1.30, 0.90S), 1.80\right] \\
&= \min\left[\max(0.90S - 1.30, 1.30 - 1.30) + 1.30, 1.80\right] \\
&= \min\left[\max(0.90S - 1.30, 0) + 1.30, 1.80\right] \\
&= 1.30 + \max(0.90S - 1.30, 0) + \min[\max(0.90S - 1.30, 0) \\
&\qquad + 1.30 - \max(0.90S - 1.30, 0) - 1.30, 1.80 - \max(0.90S - 1.30, 0) + 1.30] \\
&= 1.30 + \max(0.90S - 1.30, 0) + \min[0, 1.80 - \max(1.30, 0.90S)] \\
&= 1.30 + \max(0.90S - 1.30, 0) + 0.90\min[0, 2.00 - \max(1.44, S)].
\end{aligned}
$$

To decompose the payoff, recall that $0 < K_1 < K_2$ and $S \geq K_1 = 1.44$. Then apply these conditions to the max expression which is contained in the min expression above to get:

$$
\begin{aligned}
f(S) &= 1.30 + \max(0.90S - 1.30, 0) + 0.90\min\left[0, 2.00 - \max(S, S)\right] \\
&= 1.30 + \max(0.90S - 1.30, 0) + 0.90\min\left[0, 2.00 - S\right] \\
&= 1.30 + \max(0.90S - 1.30, 0) - 0.90\max\left[0, S - 2.00\right].
\end{aligned}
$$

Thus the additive and linear form is:

$$
f(S) = 1.30 + 0.90\left[(S - 1.44)^+ - (S - 2.00)^+\right] = \begin{cases} 1.30 & \text{if} \quad S \leq 1.44 \\ 0.9S & \text{if} \quad 1.44 < S < 2.00. \\ 2.00 & \text{if} \quad S \geq 2.00. \end{cases}
$$

The payoff diagram appears in Figure 6.5.

Figure 6.5: Graph Of Molchanov's Index Contract Function

6.2.3 Range Forwards

The Range Forward is used by major investment banks and corporate executives for hedging a short position in an underlying asset (generally a currency) at negligible cost. In this section, we will describe how different Range Forward contracts are created and decompose the associated payoff functions in DNF. The reason for doing so is that the inclusion of the steps followed to reach DNF from the additive form would be superfluous. This is so because the graphs of the Collar Options considered above are similar to those of Range Forwards.

Case 1. Hull's Range Forward

Hull (2003:435) mentions that if an American company on 16/08/2001 will need GBP three months hence, it could enter into a three month forward contract wherein GBP are purchased for 1.4407, which is obtained from USD/GBP = D = 1.4407 (refer to Table 6.14).[77] On the other hand, the same company could enter into a Range Forward contract instead of a forward contract. The Range Forward contract would consist of setting an exchange rate band surrounding USD/GBP = 1.4407, with the lower bound on this band being set at USD/GBP = 1.4200 whereas the upper bound is set at USD/GBP = 1.4600. If the spot USD/GBP rate three months hence is less than 1.4200, then 1.4200 will be paid by the American company, whereas if the same rate falls in the interval [1.4200, 1.4600], the spot USD/GBP rate is paid by the company. On the other hand, 1.4600 will be paid by the same company if this spot USD/GBP rate is greater than 1.4600. On 16/08/2001, the exchange rate quotations were:

	Bid	*Offer*
Spot	1.4452	1.4456
1-month forward	1.4435	1.4440
3-month forward	1.4402	1.4407
6-month forward	1.4353	1.4359
1-year forward	1.4262	1.4268

Table 6.14: Exchange Rates For Hull's (2003) Range Forward
Source: Hull (2003:435)

[77] Note that $D \in R_+$.

Eliminating two decimal places from the relevant thresholds, the terminal payoff function in DNF equals the USD cost of obtaining $\hat{L}$ units of GBP[78]:

$$f(S_T) = \min[1.46\hat{L}, \max(\hat{L}S_T, 1.42\hat{L})]$$

which, after decomposition, is expressed in additive and linear form as:

$$f(S_T) = 1.42\hat{L} + \max[\hat{L}(S_T - 1.42), 0] - \max[\hat{L}(S_T - 1.46), 0]$$

$$= \begin{cases} 1.42\hat{L} & \text{if} \quad S_T \le 1.42 \\ \hat{L}S_T & \text{if} \quad 1.42 < S_T < 1.46\,. \\ 1.46\hat{L} & \text{if} \quad S_T \ge 1.46. \end{cases}$$

Note that the foregoing expression can also be re-written using $\rho\max(a,b) = \max(\rho a, \rho b)$ with $\rho > 1$ (in this case $\hat{L} > 1$):

$$f(S_T) = \hat{L}[1.42 + \max[(S_T - K_1), 0] - \max[(S_T - K_2), 0]].$$

The graph of the terminal payoff function appears in Figure 6.6.

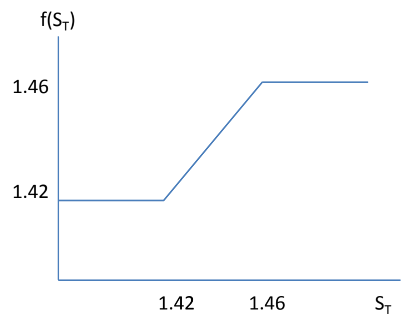


Figure 6.6: Graph Of Hull's Range Forward Function

Case 2. Musiela & Rutkowski's Range Forward

The DNF of the Range Forward's terminal payoff function as described by Musiela & Rutkowski (2005) is:

$$f(S_T) = \max[\min(S_T, K_2), K_1] - \hat{F}\,.$$

with $0 < K_1 < \hat{F} < K_2$. Placing $\hat{F}$ in the max operator yields:

$$f(S_T) = \max[\min(S_T - \hat{F}, K_2 - \hat{F}), K_1 - \hat{F}]\,.$$

Moving the min operator contained in the max operator yields:

[78] Note that $\hat{L} \in R_+$.

$$
\begin{aligned}
f(S_T) &= \min(S_T - \hat{F}, K_2 - \hat{F}) + \max[0, K_1 - \hat{F} - \min(S_T - \hat{F}, K_2 - \hat{F})] \\
&= S_T - \hat{F} + \min[S_T - \hat{F} - S_T + \hat{F}, K_2 - \hat{F} - S_T + \hat{F}] \\
&\qquad + \max[0, K_1 - \hat{F} + \max(\hat{F} - S_T, \hat{F} - K_2)] \\
&= S_T - \hat{F} - \max(0, S_T - K_2) + \max[0, K_1 - \hat{F} + \hat{F} - S_T \\
&\qquad + \max(\hat{F} - S_T - \hat{F} + S_T, \hat{F} - K_2 - \hat{F} + S_T)] \\
&= \underbrace{S_T - \hat{F} - (S_T - K_2)^+}_{A} + \underbrace{\max[0, K_1 - S_T + \max(0, S_T - K_2)]}_{B}.
\end{aligned}
$$

Consider $S_T < K_1$ then $S_T < \hat{F}$ and $S_T < K_2$ and apply these facts to Expression B above noting the addition of Expression A (which is kept intact) to the result:

$$
\begin{aligned}
f(S_T) &= S_T - \hat{F} - (S_T - K_2)^+ + \max[0, K_1 - S_T + \underbrace{\max(0, S_T - K_2)}_{=0(S_T < K_2)}] \\
&= S_T - \hat{F} - (S_T - K_2)^+ + (K_1 - S_T)^+.
\end{aligned}
$$

To draw the graph of the payoff function use the linear form:

$$
f(S_T) = \begin{cases} K_1 - \hat{F} \text{ if } & S_T < K_1 \\ S_T - \hat{F} \text{ if } & K_1 < S_T < K_2 \\ K_2 - \hat{F} \text{ if } & S_T > K_2. \end{cases}
$$

$S_T > K_2$ could also be considered in verifying the foregoing result. The presence of $-\hat{F}$ implies negative payoffs in certain intervals, which is apparent from Figure 6.7.

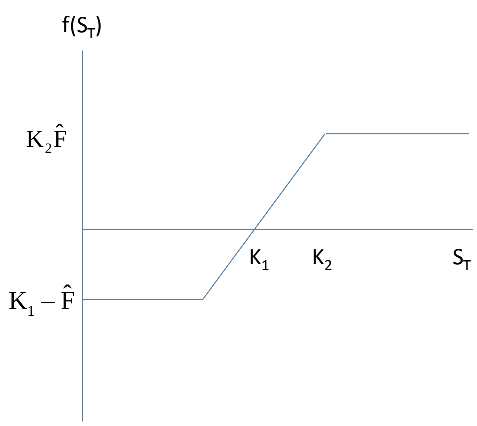


Figure 6.7: Graph Of Musiela & Rutkowski's Range Forward Function

Having applied both algorithms to Package Options (defined within the scope of this study) comprising three-component terminal payoff functions, it is natural to inquire about the effects of the application of these algorithms to the four-component case and more. In addition, it is also necessary to inquire about how our algorithm must be applied when one payoff function nests another or others. To address these issues, several examples are considered below.

6.3 Spreads

A spread is an option associated with a terminal payoff depending on the difference between two market variables (such as strike prices). The relation between these strike prices (K) are either unknown or known. If unknown, one spread's terminal payoff may nest that of another. If these relations are known, then if there are two strike prices K_1 and K_2 the possibilities are that $K_1 = K_2$, $K_1 < K_2$ and $K_1 > K_2$.

6.3.1 Butterfly I Spread

This butterfly spread is assembled with 1 call held at strike price K–x, 2 calls written at strike K and 1 call held at strike and let $K - x < K < K + x$ thus its terminal payoff function is:

$$f(S_T) = (S_T - (K - x))^+ - 2(S_T - K)^+ + (S_T - (K + x))^+ .$$

To be more specific, suppose that $K - x = K_1 < K = K_2 < K + x = K_3$ with the discriminant of $\Delta = -K_1+2K_2-K_3$ determining the payoff. Then the butterfly spread's terminal payoff will be:

$$\begin{aligned} f(S_T) &= (S_T - K_1)^+ - 2(S_T - K_2)^+ + (S_T - K_3)^+ \\ &= \max(\underbrace{S_T - K_1}_{S_T>K_1}, \underbrace{0}_{S_T<K_1}) - 2\max(\underbrace{S_T - K_2}_{S_T>K_2}, \underbrace{0}_{S_T<K_2}) + \max(\underbrace{S_T - K_3}_{S_T>K_3}, \underbrace{0}_{S_T<K_3}) . \end{aligned}$$

This yields:

$$f(S_T) = \begin{cases} 0 & \text{if} \quad S_T \le K_1 \\ S_T - K_1 & \text{if} \quad K_1 < S_T \le K_2 \\ -S_T - K_1 + 2K_2 & \text{if} \quad K_2 < S_T \le K_3 \\ -K_1 + 2K_2 - K_3 & \text{if} \quad S_T > K_3. \end{cases}$$

which is equivalent to:

Slope of	$S_T \le K_1$	$K_1 < S_T \le K_2$	$K_2 < S_T \le K_3$	$S_T \ge K_3$
$+(S_T - K_1)^+$	0	1	1	1
$-2(S_T - K_2)^+$	0	0	-2	-2
$+(S_T - K_3)^+$	0	0	0	1
Result	**0**	**1**	**-1**	**0**

Table 6.15: Tabular Approach 1 – Butterfly I Spread

Slope of	$S_T \le K_1$	$K_1 < S_T \le K_2$	$K_2 < S_T \le K_3$	$S_T \ge K_3$
0	0	0	0	0
$S_T - K_1$	0	1	1	1
$-S_T - K_1 + 2K_2$	0	0	-1	-1
$-K_1 + 2K_2 - K_3$	0	0	0	0
Result	**0**	**1**	**-1**	**0**

Table 6.16: Tabular Approach 2 – Butterfly I Spread

Unlike the Standard Options and Options and Share positions considered earlier, Tabular Approach 1 (Table 6.15) and Tabular Approach 2 (Table 6.16) are constructed in a different way but yield the same result. When there are three values of K, the terminal payoff will be 0 to maintain continuity of the payoff function.

The points of intersection are:

f_1 and f_2: (K_1, 0),

f_2 and f_3: (K_2, K_2–K_1),

f_3 and f_4: (K_3, –K_1+2K_2–K_3).

Note that:

$f(S_T) = -K_1 + 2K_2 - K_3 = 0$ or $f(S_T) = -K_1 + 2K_2 - K_3 > 0$ if $K_1 < K_2 < K_3$.

In Figure 6.8, two scenarios are possible. We omit the case of $\Delta < 0$ as it can easily be checked that this would imply losses being incurred and would therefore not be considered by an economic agent seeking maximal gains.

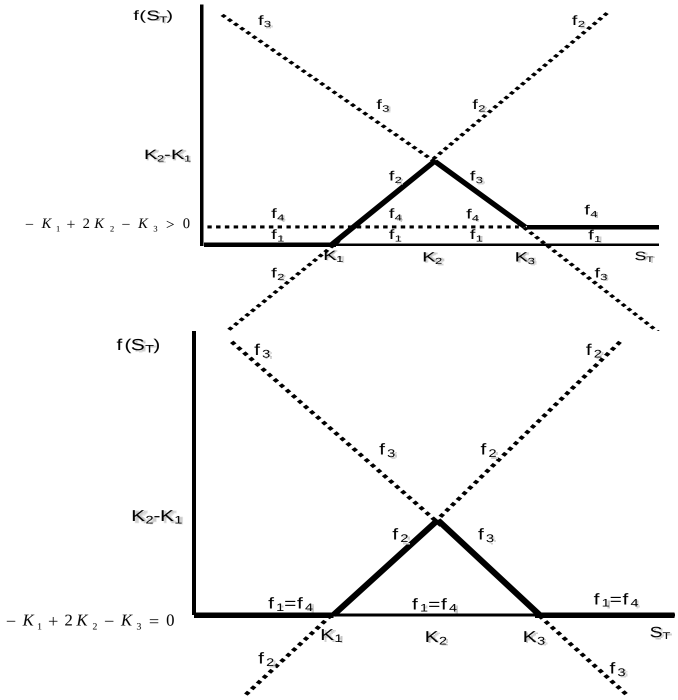


Figure 6.8: Graph Of Butterfly I Spread's Payoff Function

Panel situated at the top reveals the scenario wherein there is asymmetry while the panel situated at the bottom reveals symmetry.

For $\Delta = 0$, use Table 6.17 to obtain:

$$f(S_T) = [f_3(S_T) \wedge f_4(S_T) \wedge f_1(S_T)] \vee [f_2(S_T) \wedge f_3(S_T)] \vee [f_2(S_T) \wedge f_4(S_T)].$$

For $\Delta > 0$, Table 6.18 must be used to get:

$$f(S_T) = [f_3(S_T) \wedge f_4(S_T) \wedge f_1(S_T)] \vee [f_2(S_T) \wedge f_3(S_T)] \vee [f_2(S_T) \wedge f_4(S_T) \wedge f_1(S_T)].$$

Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c) S_a =	Minimum of functions corresponding to indices in S_a $F_a(S_T)$ =
a	**b**	**c**	**d**	**e**
$[0,K_1]$	$f_1(S_T) = 0$	$f_3(S_T),f_4(S_T),f_1(S_T)$	{3,4,1}	$f_3(S_T) \wedge f_4(S_T) \wedge f_1(S_T)$
$[K_1,K_2]$	$f_2(S_T) = S_T - K_1$	$f_3(S_T),f_2(S_T)$	{3,2}	$f_3(S_T) \wedge f_2(S_T)$
$[K_2,K_3]$	$f_3(S_T) = -S_T - K_1 + 2K_2$	$f_2(S_T),f_3(S_T)$	{2,3}	$f_2(S_T) \wedge f_3(S_T)$
$[K_3,\infty)$	$f_4(S_T) = -K_1 + 2K_2 - K_3$	$f_2(S_T),f_4(S_T)$	{2,4}	$f_2(S_T) \wedge f_4(S_T)$

Table 6.17: Algorithm Inputs – Butterfly I Spread ($\Delta = 0$ Scenario)

Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c) $S_a =$	Minimum of functions corresponding to indices in S_a $F_a(S_T) =$
a	**b**	**c**	**d**	**e**
$[0,K_1]$	$f_1(S_T) = 0$	$f_3(S_T),f_4(S_T),f_1(S_T)$	{3,4,1}	$f_3(S_T)\wedge f_4(S_T)\wedge f_1(S_T)$
$[K_1,K_2]$	$f_2(S_T) = S_T–K_1$	$f_3(S_T),f_2(S_T)$	{3,2}	$f_3(S_T)\wedge f_2(S_T)$
$[K_2,K_3]$	$f_3(S_T) = –S_T–K_1+2K_2$	$f_2(S_T),f_3(S_T)$	{2,3}	$f_2(S_T)\wedge f_3(S_T)$
$[K_3,\infty)$	$f_4(S_T) = –K_1+2K_2–K_3$	$f_2(S_T),f_4(S_T),f_1(S_T)$	{2,4,1}	$f_2(S_T)\wedge f_4(S_T)\wedge f_1(S_T)$

Table 6.18: Algorithm Inputs – Butterfly I Spread ($\Delta > 0$ Scenario)

6.3.2 Butterfly II Spread

Given strike prices satisfying $0 < K_1 < K < K_2$, write one call at K, write one put at K, hold one call at K_1 and hold one put at K_2. Thus, the ensuing option's terminal payoff is:

$$\begin{aligned} f(S_T) &= -(S_T - K)^+ - (K - S_T)^+ + (S_T - K_1)^+ + (K_2 - S_T)^+ \\ &= -\max(S_T - K,0) - \max(K - S_T,0) + \max(S_T - K_1,0) + \max(K_2 - S_T,0). \end{aligned}$$

Therefore the linear form of the same function is:

$$f(S_T) = \begin{cases} -(K - S_T) + (K_2 - S_T) = K_2 - K & \text{if} \quad S_T < K_1 \\ -(K - S_T) + (S_T - K_1) + (K_2 - S_T) = K_2 - K - K_1 + S_T & \text{if} \quad K_1 \le S_T < K \\ -(S_T - K) + (S_T - K_1) + (K_2 - S_T) = K_2 + K - K_1 - S_T & \text{if} \quad K \le S_T < K_2 \\ -(S_T - K) + (S_T - K_1) = K - K_1 & \text{if} \quad S_T > K_2. \end{cases}$$

The points of intersection in the graph of this function (Figure 6.9) are:

f_1 and f_2: $(K_1, K_2–K)$,

f_1 and f_3: $(2K–K_1, K_2–K)$,

f_2 and f_3: $(K, K_2–K_1)$,

f_3 and f_4: $(K_2, K–K_1)$.

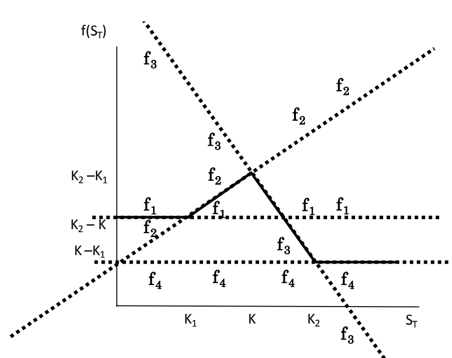


Figure 6.9: Graph Of Butterfly II Spread's Payoff Function

Using Table 6.19, taking the max and applying [LA3] yields:

$$f(S_T) = [f_3(S_T) \wedge f_1(S_T)] \vee [f_2(S_T) \wedge f_3(S_T)] \vee$$
$$[f_2(S_T) \wedge f_1(S_T) \wedge f_3(S_T)] \vee [f_2(S_T) \wedge f_1(S_T) \wedge f_4(S_T)].$$

Tedious calculations in accordance with the decomposition algorithm will yield the desired additive form.

Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c) S_a =	Minimum of functions corresponding to indices in S_a $F_a(S_T)$ =
A	**b**	**c**	**D**	**e**
$[0,K_1]$	$f_1(S_T) = K_2-K$	f_3,f_1	{3,1}	$f_3(S_T) \wedge f_1(S_T)$
$[K_1,K]$	$f_2(S_T) = S_T-K_1-K+K_2$	f_3,f_2	{3,2}	$f_2(S_T) \wedge f_3(S_T)$
$[K,K+K_2/2]$	$f_3(S_T) = -S_T-K_1+K+K_2$	f_2,f_3	{2,3}	$f_2(S_T) \wedge f_3(S_T)$
$[K+K_2/2, K_2]$	$f_3(S_T) = -S_T-K_1+K+K_2$	f_2,f_1,f_3	{2,1,3}	$f_2(S_T) \wedge f_1(S_T) \wedge f_3(S_T)$
$[K_2,\infty)$	$f_4(S_T) = K-K_2$	f_2,f_1,f_4	{2,1,4}	$f_2(S_T) \wedge f_1(S_T) \wedge f_4(S_T)$

Table 6.19: Algorithm Inputs – Butterfly II Spread

6.3.3 Butterfly III Spread

The Butterfly III Spread is assembled with one call held at K_1, one put held at K_2, one call written at the arithmetic mean of K_1 and K_2 and one put written at the same arithmetic mean with the relation between K_1 and K_2 being unknown.

When its terminal payoff function is expressed in additive form, the constituent expressions that represent porfolios of assets are difficult to evaluate for the purpose of obtaining the more tractable form. Thus, the Butterfly III Spread's terminal payoff function (defined below) is substituted into Equation 2.6 to obtain a more tractable expression as the following steps reveal:

$$f(S_T) = (S_T - K_1)^+ + (K_2 - S_T)^+ + \left\{-\left(S_T - \frac{K_1 + K_2}{2}\right)^+\right\} + \left\{-\left(\frac{K_1 + K_2}{2} - S_T\right)^+\right\}$$

$$= \frac{S_T - K_1 + 0 + |S_T - K_1 - 0|}{2} + \frac{K_2 - S_T + 0 + |K_2 - S_T - 0|}{2}$$

$$+ \frac{-\left(S_T - \frac{K_1 + K_2}{2}\right) + 0 + \left|-\left(S_T - \frac{K_1 + K_2}{2}\right) - 0\right|}{2}$$

$$+ \frac{-\left(\frac{K_1 + K_2}{2} - S_T\right) + 0 + \left|-\left(\frac{K_1 + K_2}{2} - S_T\right) - 0\right|}{2}$$

$$= \frac{-K_1 + |S_T - K_1| + K_2 + |K_2 - S_T| - S_T + \frac{K_1 + K_2}{2} + \left|\left(-S_T + \frac{K_1 + K_2}{2}\right)\right| - \frac{K_1 + K_2}{2} + S_T + \left|-\frac{K_1 + K_2}{2} + S_T\right|}{2}$$

$$= \frac{-K_1 + |S_T - K_1| + K_2 + |K_2 - S_T| + \left|\left(-S_T + \frac{K_1 + K_2}{2}\right)\right| + \left|-\frac{K_1 + K_2}{2} + S_T\right|}{2}$$

$$= \frac{-K_1 + |S_T - K_1| + K_2 + |K_2 - S_T| + 2\left|\left(-S_T + \frac{K_1 + K_2}{2}\right)\right|}{2}$$

$$= \frac{-K_1 + |S_T - K_1| + K_2 + |K_2 - S_T|}{2} + \left|\left(-S_T + \frac{K_1 + K_2}{2}\right)\right|$$

$$\equiv \frac{-K_1 + |S_T - K_1| + K_2 + |K_2 - S_T|}{2} + \left|\left(S_T - \frac{K_1 + K_2}{2}\right)\right|.$$

We can only convert the foregoing function into linear form when various scenarios are considered for the two strike prices, which in turn requires the creation of a new function in additive form. First, if $\Delta = K_1 - K_2$ let $\Delta < 0$ which yields:

$$f(S_T) = \max(S_T - K_1, 0) + \max(K_2 - S_T, 0) - \max\left(S_T - \frac{K_1 + K_2}{2}, 0\right) - \max\left(\frac{K_1 + K_2}{2} - S_T, 0\right),$$

and from this additive form the linear form of the function when $\Delta < 0$ can be obtained as follows:

$$
f(S_T) = \begin{cases}
K_2 - S_T - \frac{K_1+K_2}{2} + S_T = K_2 - \frac{K_1+K_2}{2} & \text{if} \quad S_T < K_1 \\
S_T - K_1 + K_2 - S_T - \frac{K_1+K_2}{2} + S_T = S_T - K_1 + K_2 - \frac{K_1+K_2}{2} & \text{if} \quad K_1 \le S_T < \frac{K_1+K_2}{2} \\
S_T - K_1 + K_2 - S_T - S_T + \frac{K_1+K_2}{2} = -S_T - K_1 + K_2 + \frac{K_1+K_2}{2} & \text{if} \quad \frac{K_1+K_2}{2} \le S_T < K_2 \\
S_T - K_1 - S_T + \frac{K_1+K_2}{2} = -K_1 + \frac{K_1+K_2}{2} & \text{if} \quad S_T > K_2.
\end{cases}
$$

and:

$$
f'(S_T) = \begin{cases}
0 & \text{if} \quad S_T < K_1 \\
1 & \text{if} \quad K_1 \le S_T < \frac{K_1+K_2}{2} \\
-1 & \text{if} \quad \frac{K_1+K_2}{2} \le S_T < K_2 \\
0 & \text{if} \quad S_T > K_2.
\end{cases}
$$

The points of intersection are:

f_1 and f_2: $\left(K_1, K_2 - \frac{K_1+K_2}{2}\right)$,

f_2 and f_3: $\left(\frac{(K_1+K_2)}{2}, (K_2 - K_1)\right)$,

f_3 and f_4: $\left(K_2, -K_1 + \frac{K_1+K_2}{2}\right)$,

Note that $f(S_T) = -K_1 + \frac{K_1+K_2}{2} = K_2 - \frac{K_1+K_2}{2}$.

The graph of the function when $\Delta < 0$ appears in the central panel of Figure 6.10.

Next, let $\Delta > 0$ then the additive form is:

$$f(S_T) = (S_T - K_1)^+ + (K_2 - S_T)^+ + \left\{-\left(S_T - \frac{K_1+K_2}{2}\right)^+\right\} + \left\{-\left(\frac{K_1+K_2}{2} - S_T\right)^+\right\}$$
$$= \max(S_T - K_1, 0) + \max(K_2 - S_T, 0) - \max\left(S_T - \frac{K_1+K_2}{2}, 0\right) - \max\left(\frac{K_1+K_2}{2} - S_T, 0\right).$$

In each interval, the linear form of the same function is:

$$f(S_T) = \begin{cases} K_2 - S_T - \frac{K_1+K_2}{2} + S_T = K_2 - \frac{K_1+K_2}{2} & \text{if} \quad S_T < K_2 \\ -\frac{K_1+K_2}{2} + S_T = S_T - \frac{K_1+K_2}{2} & \text{if} \quad K_2 \le S_T < \frac{K_1+K_2}{2} \\ \frac{K_1+K_2}{2} - S_T = -S_T + \frac{K_1+K_2}{2} & \text{if} \quad \frac{K_1+K_2}{2} \le S_T < K_1 \\ S_T - K_1 + \frac{K_1+K_2}{2} - S_T = -K_1 + \frac{K_1+K_2}{2} & \text{if} \quad S_T > K_1. \end{cases}$$

and:

$$f'(S_T) = \begin{cases} 0 & \text{if} \quad S_T < K_2 \\ 1 & \text{if} \quad K_2 \le S_T < \frac{K_1+K_2}{2} \\ -1 & \text{if} \quad \frac{K_1+K_2}{2} \le S_T < K_1 \\ 0 & \text{if} \quad S_T > K_1. \end{cases}$$

The points of intersection are:

f_1 and f_2: $\left(K_2, K_2 - \frac{K_1+K_2}{2}\right)$,

f_2 and f_3: $\left(\frac{K_1+K_2}{2}, 0\right)$,

f_3 and f_4: $\left(K_1, -K_1 + \frac{K_1+K_2}{2}\right)$.

Note that $f(S_T) = K_2 - \frac{K_1+K_2}{2} = -K_1 + \frac{K_1+K_2}{2}$.

The graph of the function when $\Delta < 0$ appears in the left panel of Figure 6.10.

If $\Delta = 0$, $f(S_T) = 0$ and its graph appears in the right panel of Figure 6.10. In applying the reverse decomposition algorithm, the latter case will be ignored because of the function’s zero value. However, the reverse decomposition algorithm will be applied to the nontrivial cases wherein $\Delta < 0$ and $\Delta > 0$. The inputs required for the algorithm's application in the latter cases are contained in the Table 6.20 and Table 6.21.

Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c) $S_a =$	Minimum of functions corresponding to indices in S_a $F_a(S_T) =$
a	**b**	**c**	**d**	**e**
$[0,K_2]$	$f_1(S_T) = K_2 - \frac{K_1+K_2}{2}$	$f_3(S_T),f_4(S_T),f_1(S_T)$	{3,4,1}	$f_3(S_T) \wedge f_4(S_T) \wedge f_1(S_T)$
$[K_2,K_1+K_2/2]$	$f_2(S_T) = S_T - \frac{K_1+K_2}{2}$	$f_3(S_T),f_2(S_T)$	{3,2}	$f_3(S_T) \wedge f_2(S_T)$
$[K_1+K_2/2,K_1]$	$f_3(S_T) = -S_T + \frac{K_1+K_2}{2}$	$f_2(S_T),f_3(S_T)$	{2,3}	$f_2(S_T) \wedge f_3(S_T)$
$[K_1,\infty)$	$f_4(S_T) = -K_1 + \frac{K_1+K_2}{2}$	$f_2(S_T),f_4(S_T),f_1(S_T)$	{2,4,1}	$f_2(S_T) \wedge f_4(S_T) \wedge f_1(S_T)$

Table 6.20: Algorithm Inputs – Butterfly III ($\Delta > 0$ Scenario)

Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c) $S_a =$	Minimum of functions corresponding to indices in S_a $F_a(S_T) =$
a	**b**	**c**	**d**	**e**
$[0,K_1]$	$f_1(S_T) = K_2 - \frac{K_1+K_2}{2}$	$f_3(S_T),f_4(S_T),f_1(S_T)$	{3,4,1}	$f_3(S_T) \wedge f_4(S_T) \wedge f_1(S_T)$
$[K_1, K_1+K_2/2]$	$f_2(S_T) = S_T - K_1 + K_2 - \frac{K_1+K_2}{2}$	$f_3(S_T),f_2(S_T)$	{3,2}	$f_3(S_T) \wedge f_2(S_T)$
$[K_1+K_2/2,K_2]$	$f_3(S_T) = -S_T - K_1 + K_2 + \frac{K_1+K_2}{2}$	$f_2(S_T),f_3(S_T)$	{2,3}	$f_2(S_T) \wedge f_3(S_T)$
$[K_2,\infty)$	$f_4(S_T) = -K_1 + \frac{K_1+K_2}{2}$	$f_2(S_T),f_4(S_T),f_1(S_T)$	{2,4,1}	$f_2(S_T) \wedge f_4(S_T) \wedge f_1(S_T)$

Table 6.21: Algorithm Inputs – Butterfly III ($\Delta < 0$ Scenario)

For both the $\Delta < 0$ case and its counterpart ($\Delta > 0$), take the max of column e's entries (see Table 6.20 and Table 6.21) to obtain:

$$f(S_T) = [f_3(S_T) \wedge f_4(S_T) \wedge f_1(S_T)] \vee [f_2(S_T) \wedge f_3(S_T)] \vee [f_2(S_T) \wedge f_4(S_T) \wedge f_1(S_T)].$$

However, given that the component functions have different values when $\Delta < 0$ and $\Delta > 0$ the expressions in each case will differ upon substitution of the components into the equation above.

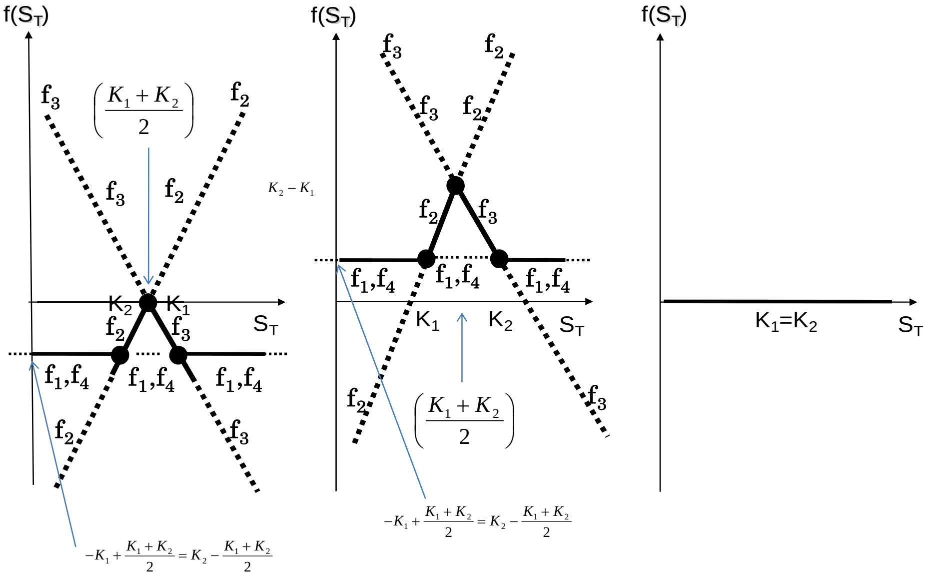


Figure 6.10: Graphs Of Butterfly III Spread's Payoff Functions

$\Delta > 0$ *(left)*, $\Delta < 0$ *(centre)*, $\Delta = 0$ *(right)*

6.3.4 Carroll & Brask's (1999) Modified Butterfly Spread

This spread is constructed by holding call options with strike prices of 10, 35, 35 and 60 and selling call options with strike prices of 20, 30, 40 and 50 therefore its terminal payoff function in additive form is:

$$\begin{aligned} f(S_T) &= (S_T - 10)^+ - (S_T - 20)^+ - (S_T - 30)^+ + (S_T - 35)^+ \\ &+ (S_T - 35)^+ - (S_T - 40)^+ - (S_T - 50)^+ + (S_T - 60)^+. \end{aligned}$$

Therefore the linear form is:

$$
f(S_T) = \begin{cases} 0 & \text{if} & S_T < 10 \\ S_T - 10 & \text{if} & 10 < S_T \le 20 \\ 10 & \text{if} & 20 < S_T \le 30 \\ -S_T + 40 & \text{if} & 30 < S_T \le 35 \\ S_T - 30 & \text{if} & 35 < S_T \le 40 \\ 10 & \text{if} & 40 < S_T \le 50 \\ -S_T + 60 & \text{if} & 50 < S_T \le 60 \\ 0 & \text{if} & S_T > 60. \end{cases}
$$

and:

$$
f'(S_T) = \begin{cases} 0 & \text{if} & S_T < 10 \\ 1 & \text{if} & 10 < S_T \le 20 \\ 0 & \text{if} & 20 < S_T \le 30 \\ -1 & \text{if} & 30 < S_T \le 35 \\ 1 & \text{if} & 35 < S_T \le 40 \\ 0 & \text{if} & 40 < S_T \le 50 \\ -1 & \text{if} & 50 < S_T \le 60 \\ 0 & \text{if} & S_T > 60. \end{cases}
$$

The derivative values can be verified by the result of Tabular Approach 1 (Table 6.22).

The equivalence of the result for Tabular Approach 2 can be easily verified.

Slope of	$S_T \le 10$	$10 < S_T \le 20$	$20 < S_T \le 30$	$30 < S_T \le 35$	$35 < S_T \le 40$	$40 < S_T \le 50$	$50 < S_T \le 60$	$S_T > 60$
$(S_T - 10)^+$	0	1	1	1	1	1	1	1
$-(S_T - 20)^+$	0	0	-1	-1	-1	-1	-1	-1
$-(S_T - 30)^+$	0	0	0	-1	-1	-1	-1	-1
$+3(S_T - K_4)^+$	0	0	0	0	2	2	2	2
$+2(S_T - 35)^+$	0	0	0	0	0	-1	-1	-1
$-(S_T - 40)^+$	0	0	0	0	0	0	-1	-1
$-(S_T - 50)^+$	0	0	0	0	0	0	0	1
Result	**0**	**1**	**0**	**-1**	**1**	**0**	**-1**	**0**

Table 6.22: Tabular Approach 1 – Carroll & Brask's (1999) Modified Butterfly Spread

The points of intersection of the graphs of this function (Figure 6.11) are:

f_1 and f_2: (10,0),

f_2 and f_3: (20,10),

f_2 and f_4: (25,15),

f_2 and f_7: (35,25),

f_3 and f_4: (30,10,)

f_4 and f_5: (35,5),

f_5 and f_6: (40,10),

f_5 and f_7: (45,15),

f_6 and f_7: (50,10),

f_7 and f_8: (60,0).

The inputs required for the reverse decomposition algorithm's application are contained in Table 6.23.

Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c) S_a =	Minimum of functions corresponding to indices in S_a $F_a(S_T)$ =
a	**b**	**c**	**d**	**e**
[0,10]	$f_1(S_T) = 0$	f_7,f_4,f_3,f_6,f_1,f_8	{7,4,3,6,1,8}	$[f_7 \wedge f_4 \wedge f_3 \wedge f_6 \wedge f_1 \wedge f_8]$
[10,20]	$f_2(S_T) = S_T-10$	f_7,f_4,f_3,f_6,f_2	{7,4,3,6,2}	$[f_7 \wedge f_4 \wedge f_3 \wedge f_6 \wedge f_2]$
[20,25]	$f_3(S_T) = 10$	f_7,f_4,f_3,f_6,f_2	{7,4,3,6,2}	$[f_7 \wedge f_4 \wedge f_3 \wedge f_6 \wedge f_2]$
[25,30]	$f_3(S_T) = 10$	f_7,f_4,f_3,f_6,f_2	{7,4,3,6,2}	$[f_7 \wedge f_4 \wedge f_3 \wedge f_6 \wedge f_2]$
[30,35]	$f_4(S_T) = -S_T + 40$	f_7,f_4,f_3,f_6,f_2	{7,4,3,6,2}	$[f_7 \wedge f_4 \wedge f_3 \wedge f_6 \wedge f_2]$
[35,40]	$f_5(S_T) = S_T-30$	f_7,f_2,f_5,f_3,f_6	{7,2,5,3,6}	$[f_7 \wedge f_2 \wedge f_5 \wedge f_3 \wedge f_6]$
[40,45]	$f_6(S_T) = 10$	f_7,f_2,f_5,f_3,f_6	{7,2,5,3,6}	$[f_7 \wedge f_2 \wedge f_5 \wedge f_3 \wedge f_6]$
[45,50]	$f_6(S_T) = 10$	f_7,f_2,f_5,f_3,f_6	{7,2,5,3,6}	$[f_7 \wedge f_2 \wedge f_5 \wedge f_3 \wedge f_6]$
[50,60]	$f_7(S_T) = -S_T + 60$	f_7,f_2,f_5,f_3,f_6	{7,2,5,3,6}	$[f_7 \wedge f_2 \wedge f_5 \wedge f_3 \wedge f_6]$
[60,∞)	$f_8(S_T) = 0$	f_2,f_5,f_3,f_6,f_1,f_8	{2,5,3,6,1,8}	$[f_2 \wedge f_5 \wedge f_3 \wedge f_6 \wedge f_1 \wedge f_8]$

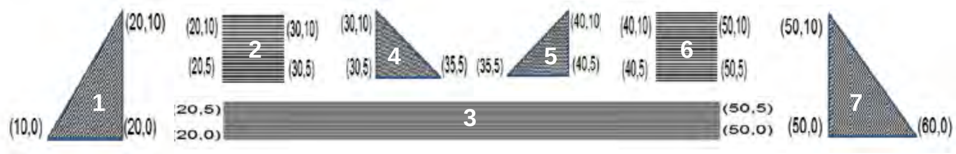
Table 6.23: Algorithm Inputs – Carroll & Brask's (1999) Modified Butterfly Spread

Taking the max and applying [LA3] yields:

$$\begin{aligned} f(S_T) &= [f_7 \wedge f_4 \wedge f_3 \wedge f_6 \wedge f_1 \wedge f_8] \vee [f_7 \wedge f_4 \wedge f_3 \wedge f_6 \wedge f_2] \\ &\quad \vee [f_7 \wedge f_2 \wedge f_5 \wedge f_3 \wedge f_6] \vee [f_2 \wedge f_5 \wedge f_3 \wedge f_6 \wedge f_1 \wedge f_8] \\ &= \max[\min(f_7, f_4, f_3, f_6, f_1, f_8), \min(f_7, f_4, f_3, f_6, f_2), \min(f_7, f_2, f_5, f_3, f_6), \min(f_2, f_5, f_3, f_6, f_1, f_8)]. \end{aligned}$$

The region of gain/loss can now be computed over [0,60], with the following methods.

PF. Over [0,60], the following plane figures which correspond to Figure 6.11, are individually assigned numbers:



The seven plane figures created above correspond to Carroll & Brask's (1999:392) building blocks a, d, f and g which are collectively assigned the value of 7.6412. We calculate the combined area as:

$$\text{Area of } 1 + \ldots + \text{Area of } 7 = 375$$

where:

$\text{Area of } 1 = 50$,

$\text{Area of } 2 = 50$,

$\text{Area of } 3 = 150$,

$\text{Area of } 4 = 12.5$,

$\text{Area of } 5 = 12.5$,

$\text{Area of } 6 = 50$,

$\text{Area of } 7 = 50$.

These results are equivalent to integrating over the interval [10,60] using:

IA1.

$$\int_{10}^{60} f(S_T)dS_T = \int_{10}^{60} \left\{ \begin{array}{l} (S_T - 10)^+ - (S_T - 20)^+ - (S_T - 30)^+ + 2(S_T - 35)^+ \\ -(S_T - 40)^+ - (S_T - 50)^+ + (S_T - 60)^+ \end{array} \right\} dS_T$$

$$= \int_{10}^{60} \left\{ \begin{array}{l} \max(S_T - 10,0) - \max(S_T - 20,0) - \max(S_T - 30,0) \\ + 2\max(S_T - 35,0) - \max(S_T - 40,0) - \max(S_T - 50,0) + \max(S_T - 60,0) \end{array} \right\} dS_T$$

$$= \int_{10}^{60} (S_T - 10)dS_T - \int_{10}^{20} 0dS_T - \int_{20}^{60} (S_T - 20)dS_T$$

$$- \int_{10}^{30} 0dS_T - \int_{30}^{60} (S_T - 30)dS_T + 2\left[\int_{10}^{35} 0dS_T + \int_{35}^{60} (S_T - 35)dS_T \right] - \int_{10}^{40} 0dS_T$$

$$- \int_{40}^{60} (S_T - 40)dS_T - \int_{10}^{50} 0dS_T - \int_{50}^{60} (S_T - 50)dS_T + \int_{10}^{60} 0dS_T$$

$$= 375.$$

IA2. Applying Equation 2.6 yields:

$$\int_{10}^{60} f(S_T)dS_T = \int_{10}^{60} \left\{ \begin{array}{l} \dfrac{S_T - 10 + 0 + |S_T - 10 - 0|}{2} + \dfrac{S_T - 20 + 0 + |S_T - 20 - 0|}{2} \\ - \dfrac{S_T - 30 + 0 + |S_T - 30 - 0|}{2} + 2\left[\dfrac{S_T - 35 + 0 + |S_T - 35 - 0|}{2}\right] \\ - \dfrac{S_T - 40 + 0 + |S_T - 40 - 0|}{2} - \dfrac{S_T - 50 + 0 + |S_T - 50 - 0|}{2} \\ + \dfrac{S_T - 60 + 0 + |S_T - 60 - 0|}{2} \end{array} \right\} dS_T = 375 .$$

IB. It can be checked that the use of the linear form of the payoff function, yields the same result more easily compared to the other three methods.

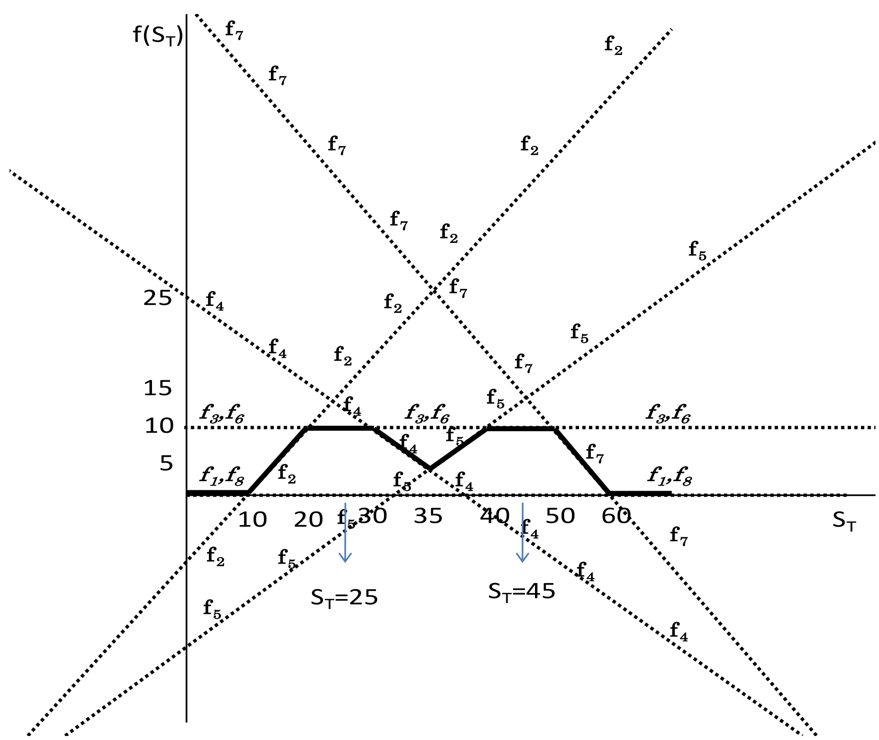


Figure 6.11: Graph Of Carroll & Brask's Modified Butterfly Spread Function

To two decimal place accuracy, our results are exactly 49.08 times greater than that obtained by Carroll & Brask's (1999) simply because these authors like Cheung & Chung (1996) err in equating the concept of payoff (in graphical from) and the Black & Scholes (1973) building block price. The price (7.6412) measures the area between the graph of the expected value of the piecewise linear payoff function and the horizontal axis over [10,60] and is analogous but different to the area between the graph of the piecewise linear payoff function (depicted in Figure 6.11) and the horizontal axis over the same interval (375).

6.3.5 Bull Spread

The bull spread consists of one call being held with strike K_1 and one call being written with strike K_2 and $K_1 < K_2$. Its terminal payoff function in additive and linear form can be expressed as:

$$f(S_T) = (S_T - K_1)^+ - (S_T - K_2)^+ = \begin{cases} 0 & \text{if} \quad S_T \le K_1 \\ S_T - K_1 & \text{if} \quad K_1 < S_T \le K_2 \\ K_2 - K_1 & \text{if} \quad S_T > K_2. \end{cases}$$

The points of intersection of the graph of this function (left panel of Figure 6.12) are:

f_1 and f_2: $(K_1,0)$,

f_2 and f_3: $(K_2,K_2–K_1)$.

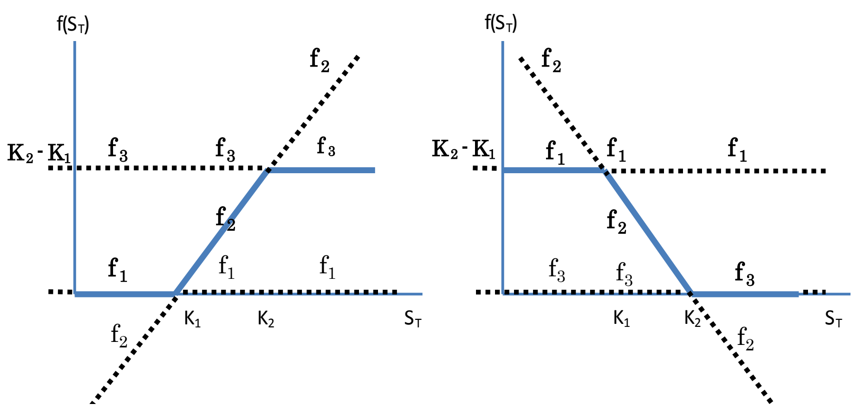


Figure 6.12: Graph Of Bull and Bear Spread Payoff Functions

Bull Spread (left) and Bear Spread (right). Bold type indicates dominant functions over a given interval.

Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c) $S_a =$	Minimum of functions corresponding to indices in S_a $F_a(S_T) =$
a	**b**	**c**	**d**	**e**
$[0,K_1]$	$f_1(S_T) = 0$	$f_3(S_T),f_1(S_T)$	{3,1}	$f_3(S_T) \wedge f_1(S_T)$
$[K_1,K_2]$	$f_2(S_T) = S_T - K_1$	$f_3(S_T),f_2(S_T)$	{3,2}	$f_3(S_T) \wedge f_2(S_T)$
$[K_2,\infty)$	$f_3(S_T) = K_2 - K_1$	$f_2(S_T),f_3(S_T)$	{2,3}	$f_2(S_T) \wedge f_3(S_T)$

Table 6.24: Algorithm Inputs – Bull Spread

Taking the max of column e entries in Table 6.24 and applying [LA3] we have:

$$f(S_T) = [f_1(S_T) \wedge f_3(S_T)] \vee [f_3(S_T) \wedge f_2(S_T)].$$

The distributive law can be applied therefore:

$$f(S_T) = [f_3(S_T) \wedge f_1(S_T)] \vee [f_2(S_T) \wedge f_3(S_T)] = f_3(S_T) \wedge [f_1(S_T) \vee f_2(S_T)].$$

Substitution of each function yields:

$$\begin{aligned} f(S_T) &= (K_2 - K_1) \wedge [0 \vee (S_T - K_1)] \\ &= \min[(K_2 - K_1), \max[0,(S_T - K_1)]] \\ &= \max[0,(S_T - K_1)] \\ &\quad + \min[(K_2 - K_1) - \max[0,(S_T - K_1)], \max[0,(S_T - K_1)] - \max[0,(S_T - K_1)]] \\ &= \max[0,(S_T - K_1)] + \min[(K_2 - K_1) - \max[0,(S_T - K_1)],0]. \end{aligned}$$

Let $S_T > K_2$ then $S_T > K_1$ then:

$$\begin{aligned} f(S_T) &= \max[0,(S_T - K_1)] + \min[(K_2 - K_1) - \max[0,(S_T - K_1)],0] \\ &= (S_T - K_1)^+ + \min[(K_2 - K_1) - (S_T - K_1)],0] \\ &= (S_T - K_1)^+ + \min[(K_2 - S_T,0] \\ &= (S_T - K_1)^+ - (S_T - K_2)^+. \end{aligned}$$

The decomposition is precluded where $S_T < K_2$ then $S_T > K_1$ or $S_T < K_1$.

6.3.6 Bear Spread

The bear spread consists of one put being held with strike K_2 and one put being written with strike K_1 and $K_1 < K_2$:

$$\begin{aligned} f(S_T) &= (K_2 - S_T)^+ - (K_1 - S_T)^+ \\ &= \max(\underbrace{K_2 - S_T}_{S_T<K_2}, \underbrace{0}_{S_T>K_2}) - \max(\underbrace{K_1 - S_T}_{S_T<K_1}, \underbrace{0}_{S_T>K_1}). \end{aligned}$$

When $K_1 < K_2$ the linear form is:

$$f(S_T) = \begin{cases} K_2 - S_T - (K_1 - S_T) & = K_2 - K_1 \quad \text{if} \quad S_T \le K_1 \\ 0 + K_2 - S_T & = K_2 - S_T \quad \text{if} \quad K_1 < S_T \le K_2 \\ 0 & \quad \text{if} \quad S_T > K_2. \end{cases}$$

The bear spread's terminal payoff diagram appears in the right panel of Figure 6.12.

Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c) $S_a =$	Minimum of functions corresponding to indices in S_a $F_a(S_T) =$
a	**b**	**c**	**d**	**e**
$[0,K_1]$	$f_1(S_T) = K_2 - K_1$	$f_2(S_T), f_1(S_T)$	{2,1}	$f_2(S_T) \wedge f_1(S_T)$
$[K_1,K_2]$	$f_2(S_T) = K_2 - S_T$	$f_1(S_T), f_2(S_T)$	{1,2}	$f_1(S_T) \wedge f_2(S_T)$
$[K_2,\infty)$	$f_3(S_T) = 0$	$f_1(S_T), f_3(S_T)$	{1,3}	$f_1(S_T) \wedge f_3(S_T)$

Table 6.25: Algorithm Inputs – Bear Spread

By taking the max in Table 6.25, applying [LA3] and the distributive law we get:

$$f(S_T) = [f_1(S_T) \wedge f_2(S_T)] \vee [f_1(S_T) \wedge f_3(S_T)] = f_1(S_T) \wedge [f_2(S_T) \vee f_3(S_T)].$$

Substitution of each function yields:

$$\begin{aligned} f(S_T) &= (K_2 - K_1) \wedge [(K_2 - S_T) \vee 0] \\ &= \min[(K_2 - K_1), \max[(K_2 - S_T),0]] \\ &= \max[(K_2 - S_T),0] \\ &\quad + \min[(K_2 - K_1) - \max[(K_2 - S_T),0], \max[(K_2 - S_T),0] - \max[(K_2 - S_T),0]] \\ &= \max[(K_2 - S_T),0] + \min[[(K_2 - K_1) - \max[(K_2 - S_T),0]],0] \\ &= \max[(K_2 - S_T),0] - \max[[(-K_2 + K_1) + \max[(K_2 - S_T),0]],0]. \end{aligned}$$

The decomposition occurs by considering $S_T < K_2$ which ensures that S_T is retained to retrieve the original function.

$$\begin{aligned} f(S_T) &= \max[(K_2 - S_T),0] - \max[(-K_2 + K_1 + K_2 - S_T,0),0] \\ &= \max[(K_2 - S_T),0] - \max[(K_1 - S_T,0),0] \\ &= (K_2 - S_T)^+ - (K_1 - S_T)^+. \end{aligned}$$

6.3.7 Big W Spread

Anderson, Barber & Keys (2006) construct a spread which consists of a Bond with price \$10 and a series of multiple call options both held and written with strikes prices expressed in USD such that $5 < 10 < 15 < 20 < 25$ which yields:

$$f(S_T) = 10 - 2(S_T - 5)^+ + 3(S_T - 10)^+ - 2(S_T - 15)^+ + 3(S_T - 20)^+ - 2(S_T - 25)^+ .$$

When B = 0, the terminal payoff function is piecewise linear, otherwise it is piecewise affine. Therefore the linear form is:

$$f(S_T) = \begin{cases} 10 & \text{if} & S_T \le 5 \\ -2S_T + 20 & \text{if} & 5 < S_T \le 10 \\ S_T - 10 & \text{if} & 10 < S_T \le 15 \\ -S_T + 20 & \text{if} & 15 < S_T \le 20 \\ 2S_T - 40 & \text{if} & 20 < S_T \le 25 \\ 10 & \text{if} & S_T > 25 \end{cases}$$

and:

$$f'(S_T) = \begin{cases} 0 & \text{if} & S_T \le 5 \\ -2 & \text{if} & 5 < S_T \le 10 \\ 1 & \text{if} & 10 < S_T \le 15 \\ -1 & \text{if} & 15 < S_T \le 20 \\ 2 & \text{if} & 20 < S_T \le 25 \\ 0 & \text{if} & S_T > 25 \end{cases}$$

The Tabular Approach 1 yields the same result (Table 6.26).

Slope of:	$S_T \le 5$	$5 < S_T \le 10$	$10 < S_T \le 15$	$15 < S_T \le 20$	$20 < S_T \le 25$	$S_T > 25$
10	0	0	0	0	0	0
$-2(S_T - 5)^+$	0	-2	-2	-2	-2	-2
$+3(S_T - 10)^+$	0	0	3	3	3	3
$-2(S_T - 15)^+$	0	0	0	-2	-2	-2
$+3(S_T - 20)^+$	0	0	0	0	3	3
$-2(S_T - 25)^+$	0	0	0	0	0	-2
Result	**0**	**-2**	**1**	**-1**	**2**	**0**

Table 6.26: Tabular Approach 1 – Big W Spread

The points of intersection for the graph of this function (Figure 6.13) are:

f_1 and f_2: $(5,10)$,

f_2 and f_3: $(10,0)$,

f_3 and f_4: $(15,5)$,

f_4 and f_5: $(20,0)$,

f_5 and f_6: $(25,10)$,

f_2 and f_4: $(0,20)$,

f_2 and f_5: $(15,-10)$,

f_3 and f_5: $(30,20)$.

The method used by Anderson, Barber & Keys (2006), like Lefèvre (1873) in constructing the graph omits the consideration of each component of the function because it utilises the graphs of each separate payoff function and "approximates" the entire function.

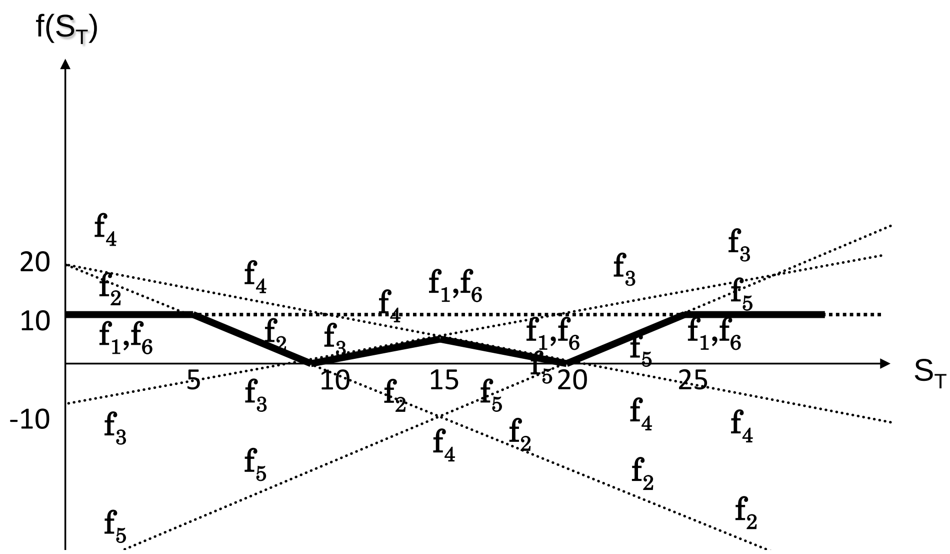


Figure 6.13: Graph Of Big W Spread's Payoff Function

Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c) $S_a =$	Minimum of functions corresponding to indices in S_a $F_a(S_T) =$
a	**b**	**c**	**d**	**e**
[0,5]	$f_1(S_T) = 10$	f_1,f_2,f_4,f_6	{1,2,4,6}	$f_1(S_T) \wedge f_2(S_T) \wedge f_4(S_T) \wedge f_6(S_T)$
[5,10]	$f_2(S_T) = -2S_T+20$	f_1,f_2,f_4,f_6	{1,2,4,6}	$f_1(S_T) \wedge f_2(S_T) \wedge f_4(S_T) \wedge f_6(S_T)$
[10,15]	$f_3(S_T) = S_T-10$	f_1,f_6,f_4,f_3	{1,6,4,3}	$f_1(S_T) \wedge f_6(S_T) \wedge f_4(S_T) \wedge f_3(S_T)$
[15,20]	$f_4(S_T) = -S_T+20$	f_1,f_6,f_4,f_3	{1,6,4,3}	$f_1(S_T) \wedge f_6(S_T) \wedge f_4(S_T) \wedge f_3(S_T)$
[20,25]	$f_5(S_T) = 2S_T-40$	f_1,f_3,f_5,f_6	{1,3,5,6}	$f_1(S_T) \wedge f_3(S_T) \wedge f_5(S_T) \wedge f_6(S_T)$
[25,∞)	$f_6(S_T) = 10$	f_1,f_3,f_5,f_6	{1,3,5,6}	$f_1(S_T) \wedge f_3(S_T) \wedge f_5(S_T) \wedge f_6(S_T)$

Table 6.27: Algorithm Inputs – Big W Spread

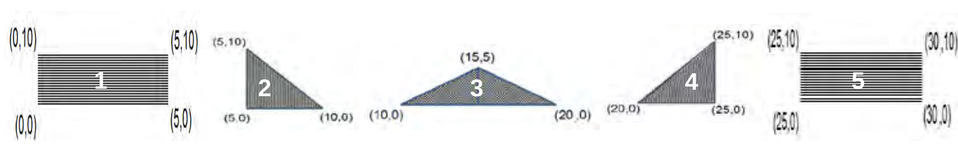
Taking the max of column e entries in Tabl 6.27 and applying [LA3] yields:

$$f(S_T) = [f_1(S_T) \wedge f_2(S_T) \wedge f_4(S_T) \wedge f_6(S_T)] \vee [f_1(S_T) \wedge f_6(S_T) \wedge f_4(S_T) \wedge f_3(S_T)] \\ \vee [f_1(S_T) \wedge f_3(S_T) \wedge f_5(S_T) \wedge f_6(S_T)].$$

The additive form can be obtained after some tedious manipulation.

Next, compute the region of gain/loss over [0,30] with the following methods:

PF. Each of the following plane figures corresponding to Figure 6.13, have been assigned numbers:



Therefore:

$$\text{Area of } 1 + ... + \text{Area of } 5 = 50 + 25 + 25 + 25 + 50 = 175$$

where:

$$\text{Area of } 1 = 5(10) = 50$$

$$\text{Area of } 2 = \frac{1}{2}(10-5)(10-0) = 25$$

$$\text{Area of } 3 = \frac{1}{2}(20-10)(5-0) = 25$$

$$\text{Area of } 4 = \frac{1}{2}(25-20)(10-0) = 25$$

$$\text{Area of } 5 = (5)(10) = 50.$$

IA1.

$$\int_0^{30} f(S_T)\,dS_T = \int_0^{30} [10 - 2(S_T - 5)^+ + 3(S_T - 10)^+ - 2(S_T - 15)^+ + 3(S_T - 20)^+ - 2(S_T - 25)^+]\,dS_T$$

$$= \int_0^{30} 10\,dS_T - 2\int_0^{30} \max(S_T - 5,0)\,dS_T + 3\int_0^{30} \max(S_T - 10,0)\,dS_T$$

$$-2\int_0^{30} \max(S_T - 15,0)\,dS_T + 3\int_0^{30} \max(S_T - 20,0)\,dS_T - 2\int_0^{30} \max(S_T - 25,0)\,dS_T$$

$$= \int_0^{30} B dS_T - 2\left[\int_0^{5} 0\,dS_T + \int_5^{30} (S_T - 5)\,dS_T\right]$$

$$+3\left[\int_0^{10} 0\,dS_T + \int_{10}^{30} (S_T - 10)\,dS_T\right] - 2\left[\int_0^{15} 0\,dS_T + \int_{15}^{30} (S_T - 15)\,dS_T\right]$$

$$+3\left[\int_0^{20} 0\,dS_T + \int_{20}^{30} (S_T - 20)\,dS_T\right] - 2\left[\int_0^{25} 0\,dS_T + \int_{25}^{30} (S_T - 25)\,dS_T\right]$$

$$= 175.$$

IA2.

$$\int_0^{30} f(S_T)\,dS_T = \int_0^{30} [10 - 2(S_T - 5)^+ + 3(S_T - 10)^+ - 2(S_T - 15)^+ + 3(S_T - 20)^+ - 2(S_T - 25)^+]\,dS_T$$

$$= \int_0^{30} 10\,dS_T - 2\int_0^{30} \frac{S_T - 5 + 0 - |S_T - 5 - 0|}{2}\,dS_T + 3\int_0^{30} \frac{S_T - 10 + 0 - |S_T - 10 - 0|}{2}\,dS_T$$

$$-2\int_0^{30} \frac{S_T - 15 + 0 - |S_T - 15 - 0|}{2}\,dS_T$$

$$+3\int_0^{30} \frac{S_T - 20 + 0 - |S_T - 20 - 0|}{2}\,dS_T - 2\int_0^{30} \frac{S_T - 25 + 0 - |S_T - 25 - 0|}{2}\,dS_T$$

$$
= \int_0^{30} 10\, dS_T + \frac{1}{2} \left\{ \begin{array}{l} -2\left[\int_0^{30} (S_T - 5) - \int_0^{30} | S_T - 5 | \right] + 3\left[\int_0^{30} (S_T - 10) - \int_0^{30} | S_T - 10 | \right] \\ -2\left[\int_0^{30} (S_T - 15) - \int_0^{30} | S_T - 15 | \right] \\ +3\left[\int_0^{30} (S_T - 20) - \int_0^{30} | S_T - 20 | \right] - 2\left[\int_0^{30} S_T - 25 - \int_0^{30} | S_T - 25 | \right] \end{array} \right\} dS_T
$$

$$
= \int_0^{30} 10\, dS_T + \frac{1}{2} \left\{ \begin{array}{l} -2\left[\int_0^{30} (S_T - 5) - \int_0^{5} (5 - S_T) - \int_5^{30} (S_T - 5) \right] \\ +3\left[\int_0^{30} (S_T - 10) - \int_0^{10} (10 - S_T) - \int_{10}^{30} (S_T - 10) \right] \\ -2\left[\int_0^{30} (S_T - 15) - \int_0^{15} (15 - S_T) - \int_{15}^{30} (S_T - 15) \right] \\ +3\left[\int_0^{30} (S_T - 20) - \int_0^{20} (20 - S_T) - \int_{20}^{30} (S_T - 20) \right] \\ -2\left[\int_0^{30} S_T - 25 - \int_0^{25} (25 - S_T) - \int_{25}^{30} (S_T - 25) \right] \end{array} \right\} dS_T
$$

$$
= 175.
$$

IB. Using the linear form adjusting for [0,30] we obtain:

$$
\int_0^{30} f(S_T)\, dS_T = \int_0^{5} 10\, dS_T + \int_5^{10} (20 - 2S_T)\, dS_T + \int_{10}^{15} (S_T - 10)\, dS_T
$$

$$
\ldots + \int_{15}^{20} (20 - S_T)\, dS_T + \int_{20}^{25} (2S_T - 40)\, dS_T + \int_{25}^{30} 10\, dS_T = 175
$$

Given relatively many components of the function, the advantage of using the IB method in this case is conspicuous like it was for the Carroll & Brask’s Modified Butterfly Spread considered earlier. It can be easily checked that the area computed (175) would differ to that obtained from a suitable modification of the Cheung & Chung (1996) and Carroll & Brask (1999) building block approach.

6.4 Combinations

The positions discussed in the sequel contain both call and put options on the same asset. In order to introduce the strangle option’s terminal payoff function which nests other functions consisting of fewer pieces, the strap, strip and straddle are first introduced (corresponding payoff functions appear in Figure 6.14). Note that for the combinations considered in this section, the table of algorithm inputs will be omitted for to preclude repetition.

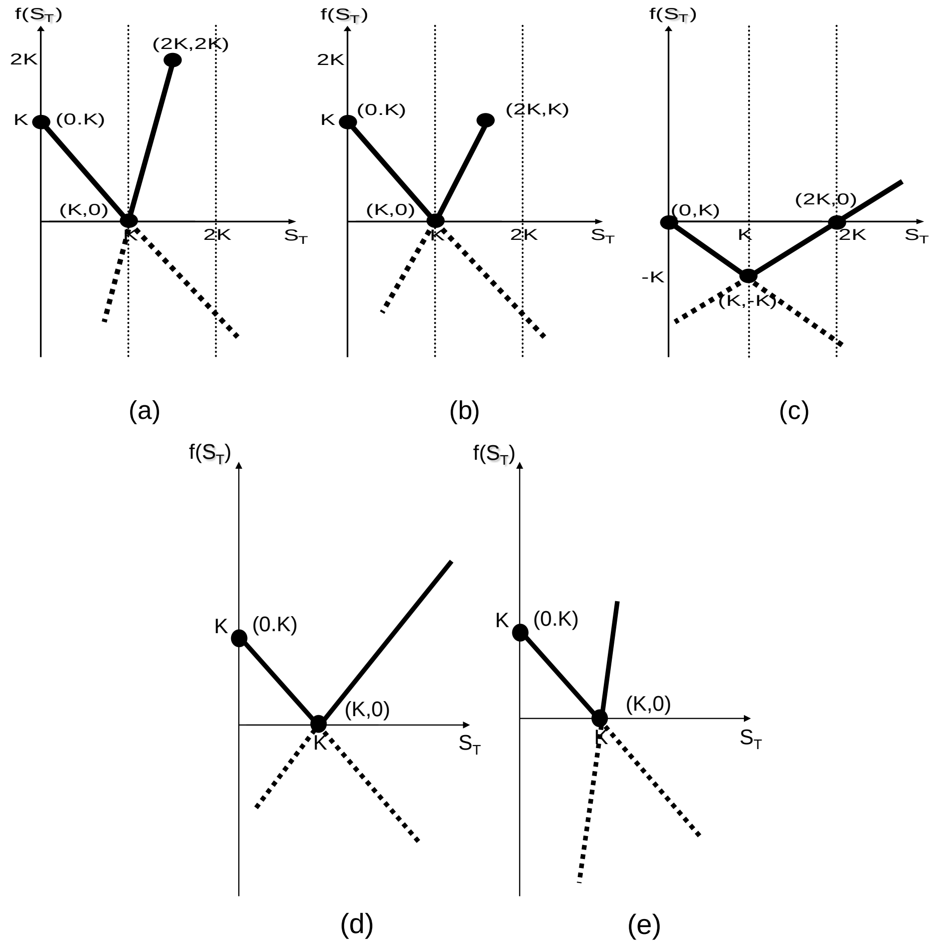


Figure 6.14: Graphs Of Five Combination Payoff Functions

Strap (Panel a), Strip I (Panel b), Straddle I (Panel c), Straddle II (Panel d), Strip II (Panel e).

6.4.1 Strap

One put held and two call options held both at K yields the terminal payoff function in additive form of:

$$
\begin{aligned}
f(S_T) &= 2(S_T - K)^+ + (K - S_T)^+ \\
&= 2\left(\frac{S_T - K + 0 + |S_T - K - 0|}{2}\right) + \frac{K - S_T + 0 + |K - S_T - 0|}{2} \\
&= S_T - K + |S_T - K| + \frac{K - S_T + |K - S_T|}{2} \\
&= \frac{1}{2}(S_T - K) + |S_T - K| + \frac{|K - S_T|}{2}.
\end{aligned}
$$

Thus the linear form is:

$$
f(S_T) = \begin{cases} K - S_T \text{ if } S_T < K \\ 2(S_T - K) \text{ if } S_T > K. \end{cases}
$$

Define $f_1(S_T) = K - S_T$ and $f_2(S_T) = 2(S_T - K)$ to obtain a point of intersection at (K,0) in Panel a of Figure 6.14. Thus:

$$
\begin{aligned}
f(S_T) &= f_1(S_T) \vee f_2(S_T) \\
&= (K - S_T) \vee 2(S_T - K) \\
&= \frac{K - S_T + 0 + |K - S_T|}{2} + 2\left\{\frac{S_T - K + 0 + |S_T - K - 0|}{2}\right\} \\
&= (K - S_T)^+ + 2(S_T - K)^+.
\end{aligned}
$$

Equation 2.6 was applied in reaching the last step.

6.4.2 Strip I

One call held and two put options held both at K yield a terminal payoff function in additive and linear form of:

$$
f(S_T) = (S_T - K)^+ + 2(K - S_T)^+ = \begin{cases} 2(K - S_T) \text{ if } S_T < K \\ (S_T - K) \text{ if } S_T > K. \end{cases}
$$

Set $f_1(S_T) = 2(K - S_T)$ and $f_2(S_T) = S_T - K$ to obtain a point of intersection is (K,0). The graph of Strip I appears in Panel b of Figure 6.14. Application of the algorithm's remaining steps yields:

$$\begin{aligned} f(S_T) &= f_1(S_T) \vee f_2(S_T) \\ &= 2(K - S_T) \vee (S_T - K) \\ &= 2\left\{\frac{K - S_T + 0 + |K - S_T - 0|}{2}\right\} + \left\{\frac{S_T - K + 0 + |S_T - K - 0|}{2}\right\} \\ &= 2(K - S_T)^+ + (S_T - K)^+. \end{aligned}$$

Again, Equation 2.6 was applied in reaching the last step.

6.4.3 Strip II

The Strip II comprises one put held and two call options held both at K, which yields the following terminal payoff function in additive and linear form:

$$f(S_T) = (K - S_T)^+ + 2(S_T - K)^+ = \begin{cases} K - S_T \text{ if } S_T < K \\ 2(S_T - K) \text{ if } S_T > K. \end{cases}$$

The point of intersection of the component functions is (K,0) in Panel e of Figure 6.14. When the remaining steps of the algorithm are applied, the following DNF is obtained:

$$f(S_T) = f_1(S_T) \vee f_2(S_T) = (K - S_T) \vee 2(S_T - K)\,.$$

6.4.4 Straddle I

The Straddle I consists of one share short sold and two call options held (at strike K) with a terminal payoff function of:

$$f(S_T) = -S_T + (S_T - K)^+ + (S_T - K)^+ = -S_T + 2(S_T - K)^+ = |S_T - K| - K \equiv |K - S_T| - K\,,$$

or in linear form:

$$f(S_T) = \begin{cases} -S_T \text{ if } S_T < K \\ S_T - 2K \text{ if } S_T > K. \end{cases}$$

The point of intersection is (K, -K) in Panel c of Figure 6.14. It can easily be verified that the DNF is:

$$f(S_T) = f_1(S_T) \vee f_2(S_T) = -S_T \vee (S_T - 2K) = -K + |S_T - K|.$$

6.4.5 Straddle II

The Straddle II comprises one call held and one put held (at strike K). Its terminal payoff function in additive and linear form is:

$$f(S_T) = (S_T - K)^+ + (K - S_T)^+ = \begin{cases} K - S_T \text{ if } S_T < K \\ S_T - K \text{ if } S_T > K. \end{cases}$$

The point of intersection is (K,0) in Panel d of Figure 6.14. It can be checked that the DNF is:

$$f(S_T) = f_1(S_T) \vee f_2(S_T) = (K - S_T) \vee (S_T - K)\,.$$

6.4.6 Strangle

The strangle consists of one call held at K_1 along with long one put at K_2 with an unknown relation between K_1 and K_2 with the terminal payoff function being:

$$f(S_T) = (S_T - K_1)^+ + (K_2 - S_T)^+\,.$$

Given that one strike price is greater than another and vice versa, in addition to being equal to its counterpart, three scenarios must be considered which implies three applications of the algorithm. Let $\Delta = K_2 - K_1$ then in the first scenario $\Delta > 0$, the terminal payoff function will consist of:

$$f(S_T) = \begin{cases} K_2 - S_T & \text{if} \quad S_T < K_1 \\ K_2 - K_1 & \text{if} \quad K_1 < S_T < K_2 \\ S_T - K_1 & \text{if} \quad S_T > K_2. \end{cases}$$

Thus:

$$f'(S_T) = \begin{cases} -1 & \text{if} \quad S_T < K_1 \\ 0 & \text{if} \quad K_1 < S_T < K_2 \\ 1 & \text{if} \quad S_T > K_2. \end{cases}$$

The graph of the function when $\Delta > 0$ appears in the left panel of Figure 6.15. The points of intersection of the component functions when $\Delta > 0$ are:

f_1 and f_2: $(K_1, K_2–K_1)$,

f_1 and f_3: $\left(\frac{K_1 + K_2}{2}, \frac{K_2 - K_1}{2}\right)$,

f_2 and f_3: $(K_2, K_2–K_1)$.

Note that the option's holder seeks substantial deviations in S_T over $[K_1, K_2]$.

Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c) $S_a =$	Minimum of functions corresponding to indices in S_a $F_a(S_T) =$
a	**b**	**c**	**d**	**e**
$[0,K_1]$	$f_1(S_T) = K_2-S_T$	$f_1(S_T)$	{1}	$\wedge f_1(S_T)$
$[K_1,K_1+K_2/2]$	$f_2(S_T) = K_2-K_1$	$f_2(S_T)$	{2}	$\wedge f_2(S_T)$
$[K_1+K_2/2,K_2]$	$f_2(S_T) = K_2-K_1$	$f_2(S_T)$	{2}	$\wedge f_2(S_T)$
$[K_2, \infty)$	$f_3(S_T) = S_T-K_1$	$f_3(S_T)$	{3}	$\wedge f_3(S_T)$

Table 6.28: Algorithm Inputs – Strangle ($\Delta > 0$ Scenario)

Using Table 6.28, taking the max when $\Delta > 0$ yields:

$$f(S_T) = f_1(S_T) \vee f_2(S_T) \vee f_3(S_T)\,.$$

If the holder seeks an increase in S_T such that it is greater than K_1 or a decrease such that it is less than K_2, let $\Delta < 0$:

$$f(S_T) = \begin{cases} K_2 - S_T & \text{if} \quad S_T < K_2 \\ 0 & \text{if} \quad K_2 < S_T < K_1 \\ S_T - K_1 & \text{if} \quad S_T > K_1. \end{cases}$$

The graph of the function when $\Delta < 0$ appears in the central panel of Figure 6.15. The values of the slope in each interval for $\Delta < 0$ is identical to the $\Delta > 0$ case. The points of intersection of the component functions when $\Delta < 0$ are:

f_1 and f_2: $(K_2,0)$,

f_1 and f_3: $\left(\frac{K_2+K_1}{2}, \frac{K_2-K_1}{2}\right)$ since $K_2 < K_1$ implies that $f(S_T) = \frac{K_2-K_1}{2} < 0$,

f_2 and f_3: $(K_1, 0)$.

Using Table 6.29, taking the max for $\Delta < 0$ yields:

$$f(S_T) = f_1(S_T) \vee f_2(S_T) \vee f_3(S_T)\,.$$

Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c) S_a =	Minimum of functions corresponding to indices in S_a $F_a(S_T)$ =
a	**b**	**c**	**d**	**e**
$[0,K_1]$	$f_1(S_T) = K_2\text{-}S_T$	$f_1(S_T)$	{1}	$\wedge f_1(S_T)$
$[K_1,K_1+K_2/2]$	$f_2(S_T) = 0$	$f_2(S_T)$	{2}	$\wedge f_2(S_T)$
$[K_1+K_2/2,K_2]$	$f_2(S_T) = 0$	$f_2(S_T)$	{2}	$\wedge f_2(S_T)$
$[K_2, \infty)$	$f_3(S_T) = S_T\text{-}K_1$	$f_3(S_T)$	{3}	$\wedge f_3(S_T)$

Table 6.29: Algorithm Inputs – Strangle ($\Delta < 0$ Scenario)

Finally, if $\Delta = 0$ and either of these values are substituted into $f(S_T)$, $f(S_T) = |S_T - K_1|$ thus:

$$f(S_T) = \begin{cases} K_1 - S_T \text{ if } S_T < K_1 \\ S_T - K_1 \text{ if } S_T \geq K_1 \end{cases}$$

which is a long straddle's terminal payoff:

$$f'(S_T) = \begin{cases} -1 \text{ if } S_T < K_1 \\ \ \ 1 \text{ if } S_T \geq K_1 \end{cases}$$

$(K_1,0)$ is the point of intersection in the right panel of Figure 6.15. It can be confirmed that:

$$f(S_T) = f_1(S_T) \vee f_2(S_T) = (K_1 - S_T) \vee (S_T - K_1) = |K_1 - S_T| = |S_T - K_1|,$$

using Equation 2.6. However, it is easier to use the fact that $|a| = \max(a,-a)$.

Effectively, two of the terminal payoff functions represent strangle payoffs whilst the third represents a long straddle's payoff.

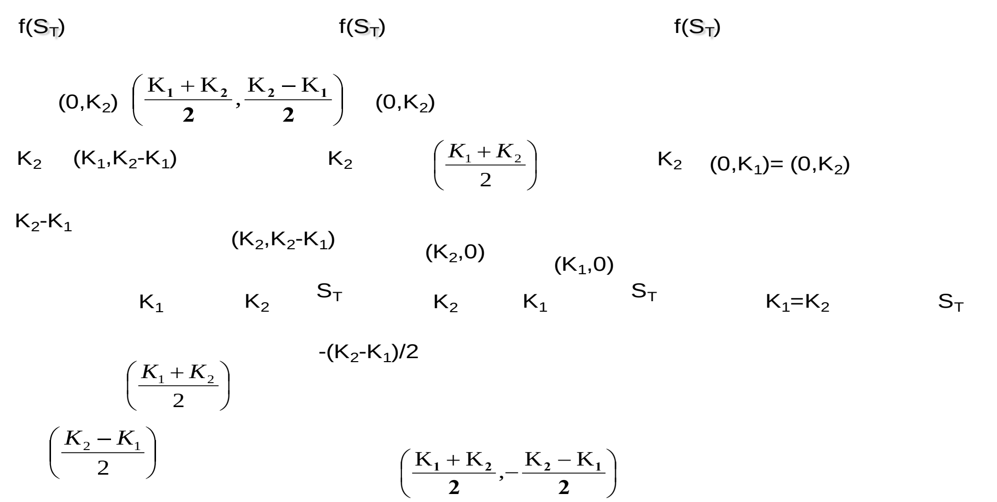


Figure 6.15: Graphs Of Terminal Option Payoff Functions

Strangle $\Delta > 0$ *(left) to Strangle* $\Delta < 0$ *(centre) and Long Straddle* $\Delta = 0$ *(right)*

6.4.7 American Strangle

The terminal payoff function of an American strangle in additive and linear forms assuming that $K_1 < K_2$ is:

$$A_{a_1,a_2}(S) = f(S) = (K_1 - S)^+ + (S - K_2)^+ = \begin{cases} K_1 - S & \text{if} \quad S < K_1 \\ 0 & \text{if} \quad K_1 \le S \le K_2 \\ S - K_2 & \text{if} \quad S > K_2 \end{cases},$$

where:

A(.) = terminal payoff expressed in units of currency

S = share’s price ($0 < S < \infty$)

K_1 = strike price of put held option

K_2 = strike price of call held option

a_1 = early exercise boundary (put side, west of K_1)

a_2 = early exercise boundary (call side, east of K_2)

Since the option may be exercised before expiry, three so-called “smooth-pasting” conditions ensure that the option’s terminal payoff and its first derivative with respect to S are continuous and the market is arbitrage-free. Given that there may be early exercise, the authors define two intervals consisting of a continuation region situated between a_1 and a_2 and a stopping region situated west of a_1 and east of a_2.

The points of intersection of the graph of this function (Figure 6.16) are:

f_1 and f_2: $(K_1,0)$

f_1 and f_3: $\left(\frac{K_1+K_2}{2},\frac{K_1-K_2}{2}\right)$

f_2 and f_3: $(K_2,0)$.

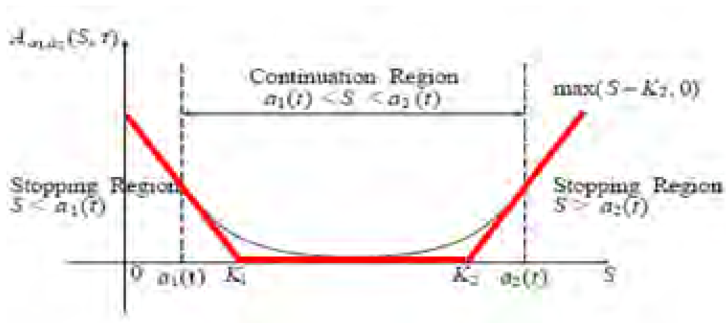


Figure 6.16: Graph Of American Strangle's Payoff Function

Source: Adapted from Chiarella & Ziogas (2005:34)

The sets corresponding to each interval appear in Table 6.30.

Interval	f(a) =	Function(s) greater than or equal to f(a)	Set containing indices corresponding to functions identified in (c) S_a =	Minimum of functions corresponding to indices in S_a $F_a(S)$ =
a	**b**	**c**	**d**	**e**
$[0,K_1]$	$f_1(S) = K_1$-S	$f_2(S),f_1(S)$	{1}	$\wedge f_1(S)$
$[K_1,K_1+K_2/2]$	$f_2(S) = 0$	$f_2(S),f_1(S)$	{2,1}	$f_2(S)\wedge f_1(S)$
$[K_1+K_2/2,K_2]$	$f_2(S) = 0$	$f_2(S),f_3(S)$	{2,3}	$f_2(S)\wedge f_3(S)$
$[K_2, \infty)$	$f_3(S) = S-K_2$	$f_1(S),f_3(S)$	{3}	$\wedge f_3(S)$

Table 6.30: Algorithm Inputs – American Strangle

Taking the max of column e entries in Table 6.30, applying [LA3], and noting that $\wedge f(S) = f(S)$ yields:

$$\begin{aligned} f(S) &= f_1(S)\vee\left[f_2(S)\wedge f_3(S)\right]\vee f_3(S) \\ &= (K_1-S)\vee[0\wedge(S-K_2)]\vee(S-K_2). \end{aligned}$$

In this case, DNF applies to the continuous piecewise linear function represented by the red line in Figure 6.16. If the non-linear component function was considered (shaded black) instead of the linear component function (shaded red and traversing [K_1,K_2]), then the foregoing DNF must be discarded.

The additive form can be obtained in a similar manner to previous examples.

CHAPTER 7. CONCLUSION

The problem of systematically converting an option payoff function of one variable in graphical form and in linear form or additive form (or both) into DNF, has been solved via the application of the reverse decomposition algorithm. The advantage of using the DNF representation is that all continuous piecewise linear and affine option payoff functions of a single variable have a unique and concise form. The algorithms devised in this study can be effectively applied by researchers or shareholder activists (or both) seeking a better understanding of certain non-standard option transactions.

To obtain DNF from the graph of a given function, we demonstrated that the same graph may be drawn using either the slopes of component functions or superimposition of one component function on another. While these methods yield the same graph, calculation of slopes is more reliable than superimposition. We identified a slight difference in slope-based methods as apparent in Tabular Approach 1 and Tabular Approach 2.

The associated problem of converting a function in DNF into additive form is achieved by the application of the decomposition algorithm. In obtaining the additive form, the linear and graphical form can be easily retrieved. Clearly, all four representations yield the same output result.

Economic historians must resolve the issue of who first used DNF, additive, linear and graphical forms in representing the payoffs to piecewise linear or affine functions of one variable. Neglected non-English language treatises and articles published before the twentieth century might contain the information requisite for such a resolution.

An important part of the reverse decomposition algorithm consisted of computing the points at which hyperplanes intersect. The importance of this computation arises from the fact that over a given interval of the domain, two components of some piecewise linear or affine functions intersect. If this fact was ignored, the DNF representation may lead to the wrong expression. It would be useful to investigate why the intersection of hyperplanes is of considerable significance in obtaining the correct DNF of a given function.

When the reverse decomposition algorithm was applied to standard options, we demonstrated that several max and min operator concepts (and their vector lattice counterparts) were applicable to option pricing. First, duality between the purchase and sale of a standard option is analogous to the duality between a max and min operator. Second, the use of the vector lattice propositions (VLP2, VLP4 and VLP12) to scale a function of a single variable (underlying asset) constructed with max or min operators, represents multiple purchases or short sales of certain standard options. Third, the dominance of a standard option's payoff function over its profit and loss function is the consequence of the addition or subtraction of a constant from a max operator. The economic meaning of such dominance arises from the payment of premia when call and put options are held.

The collection of lattice propositions and vector lattice propositions proved to be particularly useful when the decomposition algorithm was applied, compared to its reverse decomposition algorithm counterpart. Despite the use of different lattice or vector lattice propositions in converting a function in DNF to additive form, the same output results will be obtained. However, from a computational perspective, additive form can be attained more easily with the use of some propositions than others. It was also demonstrated that for a Collar option's payoff function, the Boolean lattice operations + and . of [BLP1] replaced the Boolean notation for min and max in applying the decomposition algorithm. The expressions were identical.

Systematic representation of a finite number of components of option payoff functions in DNF can be extended to higher dimensions as Ovchinnikov (2002) and Aliprantis, Harris & Tourky (2006) have demonstrated with the hyperplane arrangement. However, Aliprantis, Harris & Tourky (2006) concede that their algorithm for obtaining a sup-inf representation (analogous to the decomposition algorithm developed in this study) is "rudimentary". Another matter that requires further investigation with regard to the work of Ovchinnikov (2002) and Aliprantis, Harris & Tourky (2006) among others, is that distributivity cannot be used to compress a function of a single variable in DNF in higher dimensions.

It would be useful to commence higher-dimensional analysis of option pricing with a study of Riesz estimators. Aliprantis Harris & Tourky (2007) suggest that such estimators are equivalent to generalised options, or recursively, compound options (options on options).[79] Moreover, use of the combinatorial approach of Ovchinnikov (2010) in the aforementioned context would represent a significant contribution to the option pricing literature.

Given that the state variable, Chua & Deng's (1988) canonical form and Choquet integral are well-known alternative representations of piecewise linear and affine functions, it would be useful to devise algorithms that permit the representation of option payoff functions in such forms in addition to the forms studied herein. However, caution is required in using Chua & Deng's (1988) canonical form because Sakurai & Murofushi (2004) state that a limited class of piecewise linear functions possess such form. The Choquet integral and state variable representations on the other hand, apply to all piecewise linear functions. It would be useful to investigate which of these representations is superlative using criteria including applicability to all piecewise linear functions, conciseness, uniqueness of representation and computational complexity.

Hitherto, we have referred to continuous piecewise linear and affine functions. The foregoing analyses would be complicated by the introduction of one or all of the following elements in n-dimensions. First, it is possible that both S_T and K are stochastic as suggested by Fisher (1978) and Blenman & Clark (2005). Second, discontinuity in all dimensions is possible. Third, non-linearity in all dimensions is often observed. The first example is of fixed-income securities because Andersen & Piterbarg (2007) state that the terminal payoffs to fixed-income securities are "super-linear" or grow at a quicker rate than linear. The second example arises from Jewson, Brix & Ziehmann (2005) mentioning that the ostensibly terminal payoff functions to weather derivatives, can sometimes be constructed with piecewise non-linear functions.[80] Brown, Huijsmans & de

[79] Geske (1979) is the first to assign values to compound options.
[80] Jewson, Brix & Ziehmann (2005:25) state that:

Pagter (1990) use the theory of f-algebras which are lattice-ordered rings with a special condition[81] to model asset payoffs which are equal to the product of payoffs to call options, implying continuous but non-linear piecewise linear payoff functions. This matter is also considered by Zhu (2000) and Esser (2003).

Max and min operators have also been applied to the field of economic dynamics with Granger & Hyung (2006) introducing m-m processes with applications to systems of iterated maps, known as difference equations. It is possible that in certain circumstances, asset prices can be modeled using this approach.

The algorithms devised herein were applied to only a portion of the total known options which are associated with continuous piecewise linear and affine terminal payoff functions. It would be useful to create a compendium of such transactions wherein the foregoing algorithms are applied.

In this study, DNF was used but it would be useful to investigate whether CNF representation is possible. If so, it would be useful to obtain a formula that admits conversion of CNF into DNF and vice versa. The concept of the Indictor Function was considered superficially. It is our belief that this concept has an important role in the construction of the DNF representation for functions of one variable. It would be useful to investigate whether the algorithm constructed in this study could be applied to the piecewise linear functions drawn by Malkiel & Quandt (1969). Obtaining the succinct forms for representing piecewise non-linear functions which are analogous to the Boolean normal forms for piecewise linear functions would be a useful contribution to Mathematics. Such forms could be applied to, for example, options endowed with non-linear terminal payoff functions, which are being traded to a greater extent.

> "[w]ind power production, for example, depends on the cube of wind speed. Thus one way to structure a hedge for a wind power plant would be to define a swap or an option based on wind speed, with a cubic polynomial for the pay-off function (although a more common way to deal with this situation would be to define the index using a cubic polynomial and use a standard piecewise linear pay-off".

[81] Huijsmans (1990) contains a comprehensive discussion of f-algebras.

Syntax of the algorithms created in this study must be constructed in an appropriate programming language. Electronic implementation would be particularly useful when n-dimensional problems are considered. The algorithm's code could be appended to the compendium mentioned above. However, swift obsolescence of certain programming languages, which is expected to persist, should be considered in defining the scope of such studies, given the time lag between the writing and publication of an article and readers gaining a mastery of theoretical content to write such code.

The area of gain or loss was quantified precisely in several ways. Of particular significance is that this study has collected approaches to integration in a lattice context. It is shown that the IB method which relies on functions being written in linear form yields the area required with the least computation, compared to IA1 and IA2 which are associated with the additive form. As the number of components of a continuous piecewise linear function increases and it is necessary to compute the area thereof, it becomes crucial to write functions in linear form so that the IB method can be applied to benefit from the ease of computation discussed above. We have used the foregoing ideas concerning integration to demonstrate that the concept of an option's price obtained from the Black & Scholes (1973) building block approach of Cheung & Chung (1996) and Carroll & Brask (1999) differs considerably to that of its terminal payoff. The use of the graphical form of an option's payoff function complements the linear form in the sense that limits of integration can be set more easily than in the absence of a diagram. Extensions of the concepts discussed herein to n-dimensions must include integration.

Such analysis can be extended to an infinite number of components in n-dimensions. However, the exercise would be feasible for mathematicians rather than financial economists given the absence of option payoff functions containing an infinite number of components in n-dimensions. If mathematicians study piecewise linear functions consisting of an infinite number of components in n-dimensions, then a better understanding can be gained of functions comprising a finite number of components in lower dimensions.

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APPENDICES

Appendix A: Writing Option Positions With Vector Notation

An interesting feature of Kruizenga (1964a) and Boness (1964) is that vector notation is used in studying the conversion of a put position into a call position because the authors consider the presence of graphs in their analysis to be of secondary importance. Malkiel & Quandt (1969) also use vector notation for the same purpose as their predecessors.

Malkiel & Quandt (1969) use 2×1 vectors to determine an investor's position in sixteen strategies related to stocks. To be specific, the investor's position refers to how the investor is affected by changes in stock prices. Of the aforementioned strategies, a call option held is represented by the vector $\begin{bmatrix} +1 \\ 0 \end{bmatrix}$ wherein the upper element denotes the extent to which an investor gains, loses or neither gains nor loses if the stock price increases, which is represented by +1,-1 and 0 respectively. The vector's lower element denotes the extent of the gain, loss, or neither gain nor loss when the stock's price decreases, with the numeric values meaning the same as before. Note that the call position held can be re-written succinctly as $(+1,0)^T$ with the left (right)-most number corresponding to the upper (lower) element, thus preserving the foregoing interpretation and superscript T denoting the transpose of a vector. Using such notation, the corresponding call written is $(-1,0)^T$, a put option held is $(0,+1)^T$ and a put written is $(0,-1)^T$.

Malkiel & Quandt (1969) consider more complex strategies. For example, the simultaneous purchase of a stock and writing of a call option yields a net position vector of $(+1,-1)^T + (-1,0)^T = (0,-1)^T$. The investor's net loss occurs only if the stock price drops to an extent greater than the call option premium net of transactions costs such as commissions. Malkiel & Quandt (1969) draw linear and piecewise linear payoff functions to the sixteen positions considered wherein the dependent variable is the payoff (return per dollar of investment) and independent variable is the stock price change (increment or decrement if there is a change). This approach differs to the one used in this study - in the graphs considered, the independent variable is the terminal stock price. Thus, the direction vector technique differs to the Kruizenga (1964a) and Malkiel & Quandt (1969) vector technique.

Contrary to Kruizenga (1964a), Malkiel & Quandt's (1969) use of vectors and graphs imply that graphs need not be supplanted by the use of vectors.

Appendix B: Piecewise Linear Option Payoff Functions and Option Prices

In computing the price of a European-style call option (which is a distinct concept to the same call option's terminal payoff), lattice theory and probability theory must be combined. There are many ways of deriving the original Black & Scholes (1973) call price. These methods can be classified as the *call replication method* which was first suggested by Merton (1977) (which itself has several variants) and the *bond replication method*, which was the original approach used by Black & Scholes (1973). The latter comprises numerous variants as well.

Consider Panel a Figure B1 which depicts the terminal payoff to a European call option and the purchase of a single unit of stock discussed in Chapter 5. Now superimpose the Probability Density Function (PDF) of the normal distribution on the payoff diagram (Panel b) to reflect the assumption that rates of change in the stock's price follow the normal distribution at most frequencies (the stock's price is assumed to follow the log-normal distribution). The peak of the PDF having a centre at K represents the idea that in an efficient market an asset's buyer can lose or gain with equal probability so if a call option is held, the probability that it will expire out of the money is equal to the probability that it will expire in the money (50%).

Now the average payoff or call option price must be found. Consider the Standard European call option's payoff for a holder with a stochastic underlying asset price on the date of the option's expiry $S_T > 0$ and a deterministic option exercise price $K > 0$:

$$\max(S_T - K, 0) = \max(S_T, K) - K\,.$$

If S_T was deterministic instead of stochastic (with still K being deterministic) and the max operator above was integrated with respect to S, assuming the upper and lower limits of integration were neither positive nor negative infinity, the area beneath the piecewise linear function obtained from such an operator can be found. However, with S_T being stochastic and the upper limit of integration being positive infinity, such an integral will never converge.

For example given $\int_K^{+\infty} \max(S_T - K,0) = \int_K^{+\infty} \max(S_T, K) - K$ suppose S_T = 60 and K = 50 then $\int_{50}^{+\infty} \max(60-50,0) = \int_{50}^{+\infty} \max(10,0)$ which no solution as the integral does not converge[82]. Thus, an approximation must be found.

The only way in which the area can be computed in such circumstances is by taking the expected value of the aforementioned max operator (i.e. Emax(a,b) where E is the expectations operator, a is a random variable and b is a constant), multiplying by the normal PDF and integrating from K to ∞ (where the option holder gains).[83] This yields the Black & Scholes (1973) call option price or its extrinsic value:

$$c_{EV} = e^{-rT}\hat{E}[(S_T - K)^+] = e^{-rT}[S_0 e^{rT} N(d_1) - K\ N(d_2)],$$

where $d_1 = \dfrac{\ln\left(\dfrac{S_0}{K}\right) + \left(r + \dfrac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$, $\hat{E}$ denotes the expectation operator in a risk-neutral world, S is a log-normally distributed share price, r is the risk-free rate of interest, σ is the standard deviation of ln(S), K is the strike price, N(.) denotes the cumulative distribution function for a standardised normal distribution and 0 and T denote the date on which the call option is struck and expires respectively.

Thus, the continuous piecewise and linear terminal payoff function which is represented by the bold line in Panel c, becomes the dashed non-linear curve in the same panel. For a European style call, a single non-linear function only can be drawn given that early exercise is precluded (there would be multiple curves if the option was American). It is apparent from Panel c that the call option's price dominates and estimates its payoff.

[82] The same applies to the corresponding profit and loss function.
[83] Hull (2003) provides the exact formulae.

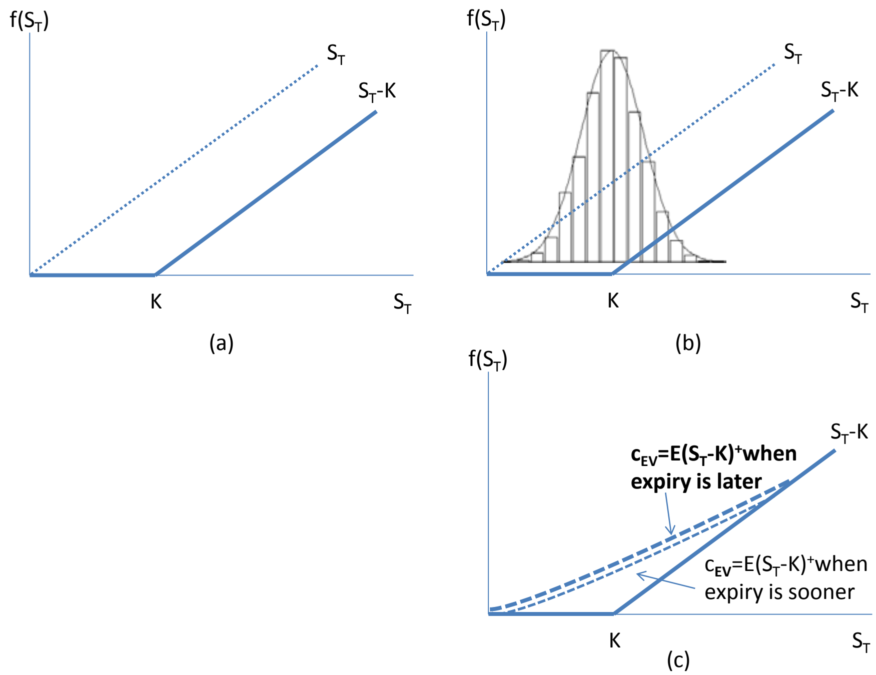

Figure B1: PDF Approach To Obtaining a European-Style Call Option's Price

Several approaches supersede the use of the PDF in option pricing. Zhu (2000) uses the Characteristic Function (CF) approach. Borokov & Novikov (2002) state that the expectation of the terminal payoff of the European call option and therefore its price can be obtained by integrating the corresponding moment generating function with a given weight. Note that these authors use the alphabetic and symbolic representation of the positive form in writing the call option's payoff function.

Appendix C: Properties Of Characteristic Functions and Max and Min Operators

Characteristic Functions or Indicator Functions are functions whose outputs are the two values of 0 and 1. The equation in Part 2 of Note 4.2.19 can be expressed using the Indicator Function. Moreover, note that the term Indicator Function and Characteristic Function are often considered synonymous. Moreover, the cognate Heaviside, Signum, Ramp and Dirac Delta functions are now defined.

Definition. 1 [Folland] Given a set A which is a subset of a set X, the *Indicator Function* of A is:

$$I_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \in X \setminus A. \end{cases}$$

Baaquie (2004) represents the product of Equation 2.8 and the Indicator Function (I) as:

$$a^{+}I = \max(a,0)I \,.$$

Another related application of $I_A(x)$ is to a variation of the equation in Part 2 of Note 4.2.19 in R. with the special case of α yielding:

$$[(a-b)^{+}]^{0} = (a-b)^{0} = I(a > b)\,.$$

Sometimes the notation $a_{-}^{p} = (a)_{-}^{p} = (a^{-})^{p}$ where a^{-} = max[-a,0] are encountered. It means that if a = 4 and p = 2 then $4_{-}^{2} = (4^{-})^{2} = [\max\{-4,0\}]^{2} = 0$. Similarly,

$$a_{+}^{p} = (a)_{+}^{p} = (a^{+})^{p} = \begin{cases} a^{p} & \text{if} \quad a \ge 0 \\ 0 & \text{if} \quad a < 0. \end{cases}$$

Remark. 2

i. Equation 2.6 is retrieved when $\alpha = 1$ and the graph of |a-b| can be drawn.

ii. Similarly, $|a-b|^{\alpha}$ is not amenable either to the application of Pascal's Triangle or the Binomial Formula unless $|a-b|^{\alpha} = (a-b)^{\alpha}$ is used. This is a specific case of the modulus of a variable or the sum of several variables because the power of max and min operators' origin is $|a^{\alpha}| = a^{\alpha}$ if a $\geq$ 0 where $a \in R$ and $\alpha \ge R$ if $b \in R$ which implies that the definition of the max operator can also be written as:

$$\max[a,b]^{1} = \left(\frac{a+b+|a-b|}{2}\right),$$

which is the same as Equation 2.6.

Consider $\alpha = 2$ and use Pascal's Triangle instead of the binomial formula then with the foregoing case being the same as Equation 2.6:

$$\max[a,b]^2 = \left(\frac{a+b+|a-b|}{2}\right)^2 = \frac{(a+b)^2+|a-b|^2}{2^2} = \frac{a^2+2ab+b^2+|a-b|^2}{4},$$

$$\max[a,b]^3 = \left(\frac{a+b+|a-b|}{2}\right)^2 = \frac{(a+b)^3+|a-b|^3}{2^3} = \frac{a^3+3a^2b+3ab^2+b^3+|a-b|^3}{8},$$

and so on for higher powers. Since the generating function of n! is:

$$\sum_{n=0}^{\infty} \frac{n!\,t^n}{n!} = \frac{1}{1-t},$$

where $n! = \Gamma(n+1)$ and where, in turn, Γ denotes the gamma function or $\Gamma(n) = (n-1)!$ it can be shown that if n! is differentiable:

$$\frac{d!}{d_n}(n!) = \Gamma(n+1)\Psi^{(0)}(n+1),$$

with $\psi^{(n)}$ being the n^{th} order derivative of the trigamma function. If n! is differentiable and the max operator of Equation 2.8 is differentiable, then the slopes of the pieces of each ensuing function can be obtained via differentiation.

The graph of n! and therefore the expression above can be drawn. The negative binomial formula discussed by Riley, Hobson & Bence (2006), applies to the case of $\alpha < 1$. Fowler (1996) states that the factorial function can be graphed if it is defined as a piecewise linear function.

iii. Multiplication of max operators on the real line implies that :

$$[(a-b)^+]^1[(a-b)^+]^1 = [(a-b)^+]^2 = (a-b)^2$$

$$[(a-b)^+]^2[(a-b)^+]^1 = [(a-b)^+]^1[(a-b)^+]^1[(a-b)^+]^1 = [(a-b)^+]^3 = (a-b)^3.$$

The extension to higher dimensions is omitted for the purpose of this chapter's exposition.

iv. In this calculus, the following property applies:

$$\max[(a-b),0]^{\alpha} \neq -b + \max[a,b]^{\alpha}$$
$$\max[(a-b),0]^{\alpha} \neq -b^{\alpha} + \max[a,b]^{\alpha}.$$

v. For three real numbers a,b and c, considering that the formula for max (or min) can only be applied to pairs, as mentioned earlier, the binomial rather than multinomial formula expression applies to $\max[(a-b-c),0]^{\alpha}$.

vi. when $\alpha = 1$ a piecewise linear function can be sketched but as soon as powers greater than 1 appear, the functions are piecewise non-linear.

vii. The binomial coefficient can be sketched in three dimensions (Wolfram, 2011).

viii. In n-dimensional Euclidean space the binomial formula (including coefficients) can be applied to matrices.

A representation of the definition of the max operator which comprises the following functions in R is also possible.

Definition. 3 [Riley, Hobson & Bence] The Heaviside or unit step function H(a) is

$$H(a) = \begin{cases} 1 & \text{if} \quad a > 0 \\ 0 & \text{if} \quad a < 0 \end{cases}.$$

Notice the omission of a value for H(a) when a = 0, in which case H(a) is assigned the value of ½. Using H(a), Equation 2.8 can be defined as a^{+} = max(a,0) = a.H(a).

Note. **4** The Heaviside function can appear as the power of a max operator such as in the expression:

$$(t-s)_{+}^{H-\frac{1}{2}} = (t-s)_{+}^{H-\frac{1}{2}} = ([t-s]^{+})^{H-\frac{1}{2}} = \begin{cases} (t-s)^{H-\frac{1}{2}} & \text{if} \quad (t-s) > 0 \\ 0 & \text{if} \quad (t-s) < 0. \end{cases}$$

In terms of notation, the positive sign is attached as a subscript and then as a superscript.

Definition. 5 [Morrison] The analytical definition of the Dirac Delta or unit impulse function δ(a) is $\delta(a) = 0 \;; a \neq 0$ and $\int_{-\infty}^{\infty} \delta(a)\ da = 1$.

Note. 6 [Riley, Hobson & Bence] Differentiation of H(a) with respect to a yields δ(a) therefore integration of the Dirac delta function yields H(a).

Definition. 7 [Morrison] The signum function sign (a) is

$$\text{sign(a)} = \begin{cases} 1 \text{ if } a \geq 0 \\ -1 \text{ if } a < 0 \end{cases}$$

$$= \frac{a}{|a|} \text{ if } a \neq 0$$

implying that a = [sign(a)].|a| where |a| is the modulus of a. H(a) is obtained from the signum function of a (sign a)

$$H(a) = \frac{1}{2}[1 + \text{sign}(a)]$$

If a^+ is expressed using the H(a), sign(a), or δ(a), differentiate such a function with respect to a.

We demonstrate with a simple numerical example.

Example. 8

$$7^+ = \max(7,0) = 7 \vee 0 = 7H(7)$$

Since H(7) = 1

$$\text{sign}(7) = \begin{cases} 1 & \text{if} \quad 7 \geq 0 \\ -1 & \text{if} \quad 7 < 0 \end{cases} = \frac{7}{|7|} = 1$$

where:

$$H(7) = \begin{cases} 1 & \text{if} \quad 7 \geq 0 \\ 0 & \text{if} \quad 7 < 0 \end{cases}$$

Thus $7 = \underbrace{\text{sign}(7)}_{=1}.|7| = 7$.

If $H(a) = \dfrac{1 + \text{sign}(a)}{2}$ and a=7 then $H(7) = \dfrac{1 + \text{sign}(7)}{2} = 1$. Thus $7^+ = 7(1) = 7$.

The ramp function may be defined analytically in several ways, with one of these being discussed in the sequel.

Definition. 9 The *ramp function* R(a) is:

$$R(a) = \begin{cases} a & \text{if} \quad a \geq 0 \\ 0 & \text{if} \quad a < 0. \end{cases}$$

for real-valued a.

The arithmetic mean of the input and its modulus is

$$R(a) = \frac{a + |a|}{2} \text{………………..(D1)}$$

Equation D1 is obtained from Equation 2.6, with b = 0.

The Heaviside function multiplied by the input is the ramp function:

$$R(a) = a\,H(a).$$

If $a \neq 0$ taking the first derivative of the ramp function yields the Heaviside function:

$$R'(a) = H(a).$$

The Heaviside function H(a) is related to the indicator function of the subset a of set X denoted by $I_a : X \to (0,1)$ where I is the Indicator function[84] with:

$$I_a(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A. \end{cases}$$

If a is a random variable, using the indicator function Equation 2.8 can be represented by:

$$a^+ = \max(a,0) = a.I_{\{a>0\}}.$$

The connection between max and min operators on the one hand and indicator functions on the other become apparent in the following series of definitions.

Definition. 10 [Nelson] Let $a \subset X$ and $b \subset X$ then (real numbers):

1. $I_{a \cap b} = \min(I_a, I_b) = I_a I_b = I_a \wedge I_b$.

2. $I_{a \cup b} = \max(I_a, I_b) = I_a + I_b - I_a I_b = I_a \vee I_b$.

---

[84] The indicator variable can be denoted by the following symbols:
i) blackboard (**1**, **I**); and
ii) calligraphic $\|$.

3. $\max\{I_{a_1}(x),\ldots,I_{a_n}(x)\} = I_{a_1 \cup a_2 \cup \ldots \cup a_n}(x) = I_{a_1}(x) \vee I_{a_n}(x)$.

We note that the sum of a continuous piecewise linear function and a discontinuous step function yields a discontinuous piecewise affine function:

$$(x-a)^+ + I_{x>a} = \begin{cases} 0 & \text{if} \quad x \le a \\ x-a+1 & \text{if} \quad x \ge a. \end{cases}$$

A useful check of whether a function is continuous is to observe whether on each interval there is only one function. If not, then there will be a discontinuity. The function discussed in Appendix F is a case in point.

Morrison (1994) states that functions with discontinuities can be differentiated, if generalized rather than ordinary functions are considered.

Appendix D: Powered Options

Esser (2003) describes the Symmetric Powered Option's terminal with $a \in N$:

$$f(S_T) = [\max(S_T - K,0)]^a = (S_T - K)^a I(S_T > K) = \sum_{j=0}^{a} \binom{a}{j} S_T^{a-j} (-K)^j I(S_T > K) .$$

For even exponents a = 2L with L being a natural number:

$$f(S_T) = (S_T - K)^{2L} = \sum_{j=0}^{2L} \binom{2L}{j} S_T^{2L-j} (-K)^j .$$

But for odd exponents $SPowPA_T = PowPA_T$.

In the event that sums of Powered Options are considered, the following are examples of two max functions raised to different powers:

$$\begin{aligned} f(S_T) &= [\max(S_T - K,0)] + [\max(S_T - K,0)]^2 \\ &= (S_T - K) + (S_T - K)^2 I(S_T > K) = (S_T - K) + \sum_{j=0}^{2} \binom{2}{j} S_T^{2-j} (-K)^j I(S_T > K) \end{aligned}$$

$$\begin{aligned} f(S_T) &= [\max(S_T - K,0)] + [\max(S_T{}^2 - K,0)]^2 \\ &= (S_T - K) + (S_T{}^2 - K)^2 I(S_T^2 > K). \end{aligned}$$

In addition:

$$\begin{aligned} f(S_T) &= [\max(S_T - K,0)]^2 + [\max(S_T - K,0)]^2 \\ &= 2\left\{(S_T - K)^2 I(S_T > K)\right\} \\ &= 2\sum_{j=0}^{2} \binom{2}{j} S_T^{2-j} (-K)^j I(S_T > K). \end{aligned}$$

Many other variations are also valid.

Appendix E: Digital Options

A binary option or digital option is an option which yields either an asset at expiry, or nothing depending on whether the option expires in-the-money. Digital options' payoff functions are discontinuous in the underlying asset price, rendering the reverse decomposition algorithm inapplicable. For a digital call option with strike K at time T, its payoff is a Heaviside function:

For a digital call option with strike K at time T, the payoff is a Heaviside function:

$$f(S_T) = H(S_T - K) = \begin{cases} 0 & \text{if} \quad S_T \le K \\ 1 & \text{if} \quad S_T > K. \end{cases}$$

Similarly, the digital put option can be expressed as

$$f(S_T) = H(S_T - K) = \begin{cases} 1 & \text{if} \quad S_T \le K \\ 0 & \text{if} \quad S_T > K. \end{cases}$$

One example of a binary option is a gap option. For a gap option, the terminal payoff is computed by comparing the asset price (S_T) to a "trigger price" (K_2) that differs to a strike price of K_1.

A gap call option's terminal payoff function is expressed as:

$$f(S_T) = \begin{cases} 0 & \text{if} \quad S_T \le K_2 \\ S_T - K_1 & \text{if} \quad S_T > K_2. \end{cases}$$

while the gap put's terminal payoff function is expressed as:

$$f(S_T) = \begin{cases} K_1 - S_T & \text{if} \quad S_T < K_2 \\ 0 & \text{if} \quad S_T \ge K_2. \end{cases}$$

K_2 is a "trigger" value which determines whether the holder of either option will receive a non-negative amount. Both are examples of options associated with discontinuous terminal payoff functions when $K_1 \ne K_2$ (Figure E1), which implies that the algorithm devised in this study must be discarded. However, when $K_1 = K_2$ gap call and gap put option terminal payoff functions are equivalent to European-style call and put option terminal payoff functions. Naturally, in the latter case the function can be written in DNF.

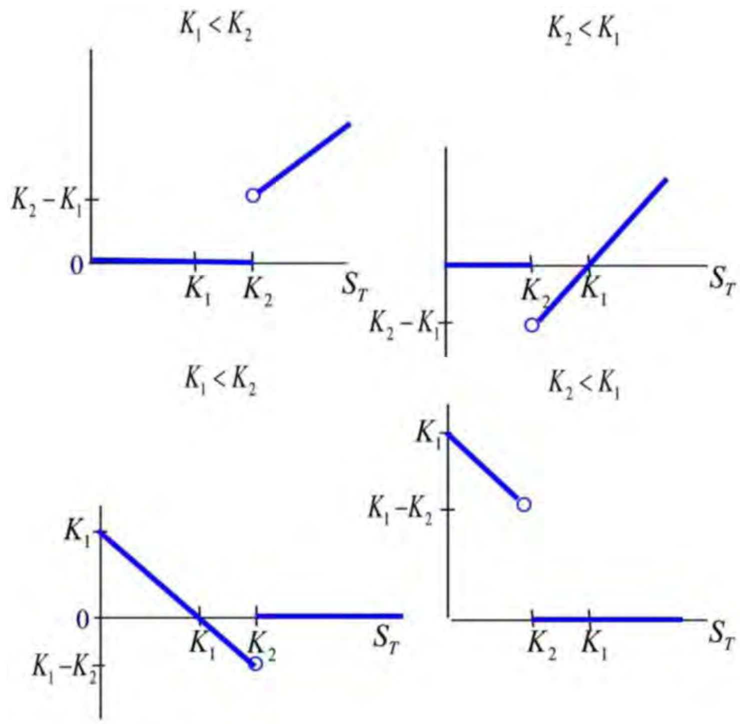

Figure E1: Graphs Of Gap Call and Gap Put Option Payoff Functions

The gap call appears in the top panels whereas the gap put appears in the bottom panels. Source: Finan (2011:302-303)

More examples of digital options are studied by Nelken (1996) and Hull (2003) among others.

Appendix F: Max Operators Comprising Intervals Of Terminal Payoff Functions

Khil & Suh (2010) consider the Knock-in Knock-out (wKIKO) option held by some South Korean companies in order to prevent cash (denominated in foreign currencies) held by these companies from losing value when the South Korean Won (KRW) appreciated (in anticipation of such cash being repatriated to South Korea), thus rewarding holders if the KRW appreciated to a certain extent. If the KRW depreciated, holders of this contract lost cash.

This contract comprised a series of barriers – a low barrier (K_{low}) which denotes the knock-out event for the KIKO option comprising both call and put options and the high barrier (K_{high}) denotes the knock-in event for the call option with the strike price K taking on such values that $K < K_{high}$ but $K < K_{low}$ or $K > K_{low}$. The window KIKO option (wKIKO) consists in the notional amount of USD 1.00 at each valuation date which yields:

$$f(S_{T_i}, K, K_{high}, K_{low}, \rho) = \begin{cases} 0 & \text{if} \quad \underline{S}_{(T_{i-1},T_i]} \leq K_{low} \\ -\rho[(S_{T_i} - K)^+] & \text{if} \quad \underline{S}_{(T_{i-1},T_i]} > K_{low}, \bar{S}_{(T_{i-1},T_i]} \geq K_{high} \\ (K - S_{T_i})^+ & \text{if} \quad \text{otherwise.} \end{cases}$$

with S_{Ti} = KRW/USD denoting a value of the exchange rate at the i^{th} valuation date T_i and ρ being the number of call options. In addition:

$$\underline{S}_{(T_{i-1},T_i]} \equiv \min\{S_t \mid t \in (T_{i-1}, T_i]\}$$

$$\bar{S}_{(T_{i-1},T_i]} \equiv \max\{S_t \mid t \in (T_{i-1}, T_i].$$

The aggregation of valuation date i terminal payoff functions will yield the option's entire terminal payoff.

The foregoing payoff function represents a special case of max expressions corresponding to intervals, which means that the algorithm used to draw the piecewise linear terminal payoff function must be adjusted, as mentioned earlier. In addition, the linear function comprising three main regions $[0, K_{low}]$, $[K_{low}, K_{high}]$ which represents the terminal payoff associated with holding a put option and $[K_{high}, \infty)$ thus the foregoing terminal payoff function can be adjusted to:

$$f(S_{T_i}, K, K_{high}, K_{low}, \rho) = \begin{cases} 0 & \text{if} \quad \underline{S}_{(T_{i-1},T_i]} \le K_{low} \\ -\rho[(S_{T_i} - K)^+] & \text{if} \quad \underline{S}_{(T_{i-1},T_i]} > K_{low}, \bar{S}_{(T_{i-1},T_i]} \ge K_{high} \\ (K - S_{T_i})^+ & \text{if} \quad \underline{S}_{(T_{i-1},T_i]} > K_{low}, \bar{S}_{(T_{i-1},T_i]} \le K_{high}. \end{cases}$$

It is known that $K < K_{low}$ or $K > K_{low}$. Suppose that $K > K_{low}$ then:

$$f(S_{T_i}, K, K_{high}, K_{low}, \rho) = \begin{cases} 0 & \text{if} \quad \underline{S}_{(T_{i-1},T_i]} \le K_{low} \\ -\rho[(S_{T_i} - K)^+] = \begin{matrix} -\rho\left\{\dfrac{S_{T_i} - K + 0 + |S_{T_i} - K - 0|}{2}\right\} = -\rho\left\{\dfrac{S_{T_i} - K + K - S_{T_i}}{2}\right\} = 0 & \text{if} \quad S_{T_i} < K \\ -\rho\left\{\dfrac{S_{T_i} - K + 0 + |S_{T_i} - K - 0|}{2}\right\} = -\rho\left\{\dfrac{S_{T_i} - K + S_{T_i} - K}{2}\right\} = -\rho[(S_{T_i} - K) & \text{if} \quad S_{T_i} > K \end{matrix} & \text{if} \quad \underline{S}_{(T_{i-1},T_i]} > K_{low}, \bar{S}_{(T_{i-1},T_i]} \ge K_{high} \\ (K - S_{T_i})^+ = \begin{matrix} \dfrac{K - S_{T_i} + 0 + |K - S_{T_i} - 0|}{2} = \dfrac{K - S_{T_i} + K - S_{T_i}}{2} = K - S_{T_i} & \text{if} \quad S_{T_i} < K \\ \dfrac{K - S_{T_i} + 0 + |K - S_{T_i} - 0|}{2} = \dfrac{K - S_{T_i} + S_{T_i} - K}{2} = 0 & \text{if} \quad S_{T_i} > K \end{matrix} & \text{if} \quad \underline{S}_{(T_{i-1},T_i]} > K_{low}, \bar{S}_{(T_{i-1},T_i]} \le K_{high}. \end{cases}$$

which can be re-written as components of the entire function:

$$f(S_{T_i}, K, K_{high}, K_{low}, \rho) = \begin{cases} 0 & \text{if} \quad \underline{S}_{(T_{i-1},T_i]} \le K_{low} \\ -\rho[(S_{T_i} - K)^+] = \begin{cases} N/A & \text{if} \quad S_{T_i} < K \\ -\rho(S_{T_i} - K) & \text{if} \quad S_{T_i} > K \end{cases} & \text{if} \quad \underline{S}_{(T_{i-1},T_i]} > K_{low}, \bar{S}_{(T_{i-1},T_i]} \ge K_{high} \\ (K - S_{T_i})^+ = \begin{cases} K - S_{T_i} & \text{if} \quad S_{T_i} < K \\ 0 & \text{if} \quad S_{T_i} > K \end{cases} & \text{if} \quad \underline{S}_{(T_{i-1},T_i]} > K_{low}, \bar{S}_{(T_{i-1},T_i]} \le K_{high}. \end{cases}$$

Differentiation yields:

$$f'(S_{T_i}) = \begin{cases} 0 & \text{if} \quad \underset{-(T_{i-1},T_i]}{S} \le K_{low} \\ \begin{cases} 0 & \text{if} \quad S_{T_i} < K \\ -\rho & \text{if} \quad S_{T_i} > K \end{cases} \Bigg\{ & \text{if} \quad \underset{-(T_{i-1},T_i]}{S} > K_{low}, \bar{S}_{(T_{i-1},T_i]} \ge K_{high} \\ \begin{cases} -1 & \text{if} \quad S_{T_i} < K \\ 0 & \text{if} \quad S_{T_i} > K \end{cases} \Bigg\{ & \text{if} \quad \underset{-(T_{i-1},T_i]}{S} > K_{low}, \bar{S}_{(T_{i-1},T_i]} \le K_{high}. \end{cases}$$

The slope of the $S_{Ti} < K_{low}$ is unambiguous. Consider the $S_{Ti} > K$ regions. The component of the function corresponding to a terminal payoff greater than K_{high} represents the continuation of the function from the region where $\underset{-(T_{i-1},T_i]}{S} > K_{low}, \bar{S}_{(T_{i-1},T_i]} \ge K_{high}$. Khil & Suh (2010) offer a related example of $K > K_{low}$.

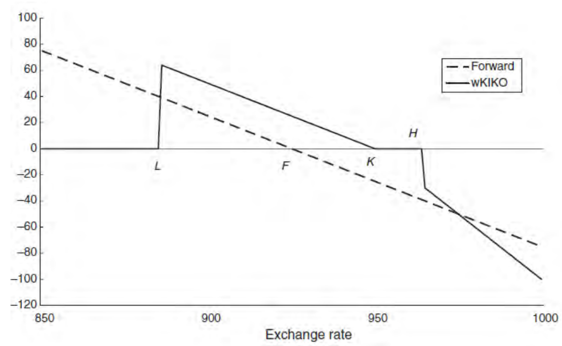

Figure F1: Graph Of Khil & Suh's wKIKO and FX Forward Payoff Function
Payoffs of window knock-in knock-out (wKIKO) option and FX forward: K_{low} *= L = 885, K = 950,* K_{high} *= H = 965, h = 2, and F (forward exchange rate) = 925.*
Source: Khil & Suh (2010:32)

In Figure F1, for $K > K_{low}$, S = K tentatively serves as a "line of symmetry", given that a pair of pieces are situated on either side of K. This can be verified by inspection of the modulus expression comprising the max operators contained in each interval.

The algorithms described in this study is inapplicable to this case because multiple outputs occur for a given input in K_{low} and K_{high}, which is represented by discontinuity of the graph at these points.

Appendix G: Index Currency Option (ICON)

The example of the inverse of a variable is a simple non-linear example of a function which cannot be written in DNF. Hull (2003:14) mentions that Bankers Trust[85] released a bond called the Index Currency Option Note (ICON) on behalf of the Long Term Credit Bank of Japan during the 1980s. An ICON consists of an amount received by the holder on the date of maturity in 1995, which depends on a foreign exchange rate. Two exchange rates, K_1 and K_2 with $K_1 > K_2$, are considered. If, on the ICON's date of maturity the exchange rate was greater than K_1, the ICON holder received the complete face value, but if it was less than K_2, nothing was received. Between K_2 and K_1, a proportion of the complete face value was received. In the foregoing deal, if S_T = JPY/USD was greater than JPY/USD 169 at maturity, holders received USD 1000 but if JPY/USD was less than 169, holders received:

$$1000 - \max\left[0{,}1000\left(\frac{169}{S_T} - 1\right)\right].$$

Finally, when JPY/USD was less than 84.5, holders received nothing at maturity.

When drawing the terminal payoff diagram, it is necessary to consider that the inverse of the underlying asset contained in the max operators comprise the ICON's terminal payoff function. Denote spot exchange rates by:

S_T = spot exchange rate at time T (date of ICON's expiry)

K_1 = JPY/USD 169

K_2 = JPY/USD 84.5

Note that $K_1 > K_2$**.**

If:

$S_T > K_1$, bond holders receive complete face value

$S_T < K_2$, bond holders receive nothing

$K_1 \geq S_T \geq K_2$, bond holders receives partial face value

If $84.5 \leq S_T \leq 169$ the ICON's terminal payoff is $f(S_T) = 2000 - \dfrac{169000}{S_T}$.

[85] Now a subsidiary of Deutsche Bank.

The ICON's payoff effectively comprises the sum of the terminal payoff derived from

- a regular bond worth USD 1,000 at T
- the writing of call options to buy JPY 169000 with an exercise price of 1/169 and
- the holding of call options to buy JPY 169000 with an exercise price of 1/84.5.

The procedure for converting the inverse of a given value appears below:

$$\frac{1}{169} > \frac{1}{S_T} \Rightarrow S_T > 169$$
$$\frac{1}{169} < \frac{1}{S_T} \Rightarrow S_T < 169$$

$$\frac{1}{S_T} > \frac{1}{84.5} \Rightarrow S_T < 84.5$$
$$\frac{1}{S_T} < \frac{1}{84.5} \Rightarrow S_T > 84.5.$$

Using a series of European-style put options and call options the option's terminal payoff is:

$$f(S_T) = 1000 + 169{,}000 \min\left\{\frac{1}{K_1} - \frac{1}{S_T}, 0\right\} + 169{,}000 \max\left\{\frac{1}{S_T} - \frac{1}{K_2}, 0\right\}.$$

Substitution of the relevant values of K yields:

$$f(S_T) = 1000 + 169{,}000 \min\left\{\frac{1}{169} - \frac{1}{S_T}, 0\right\} + 169{,}000 \max\left\{\frac{1}{S_T} - \frac{1}{84.5}, 0\right\}.$$

Since terminal payoffs are conventionally written with max functions, convert min the min expression of the above function to its max counterpart by using $\min\{a, b\} = -\max\{-a, -b\}$ therefore:

$$169{,}000 \min\left\{\frac{1}{169} - \frac{1}{S_T}, 0\right\} = -169{,}000 \max\left\{-\frac{1}{169} + \frac{1}{S_T}, 0\right\}.$$

At this stage, it is easier to write the intervals as the inverse of S_T:

$$f(S_T) = 1000 - 169{,}000\left\{\underbrace{-\frac{1}{169}+\frac{1}{S_T}}_{\frac{1}{S}>\frac{1}{169}}\right\}^{+} + 169{,}000\left\{\underbrace{\frac{1}{S_T}-\frac{1}{84.5}}_{\frac{1}{S}>\frac{1}{84.5}}\right\}^{+}$$

$$= 1000 - 169{,}000\max\left\{\underbrace{-\frac{1}{169}+\frac{1}{S_T}}_{\frac{1}{S}>\frac{1}{169}},\ \underbrace{0}_{\frac{1}{S}<\frac{1}{169}}\right\} + 169{,}000\max\left\{\underbrace{\frac{1}{S_T}-\frac{1}{84.5}}_{\frac{1}{S}>\frac{1}{84.5}},\ \underbrace{0}_{\frac{1}{S}<\frac{1}{84.5}}\right\}.$$

The functions corresponding to each interval are:

$$f(S_T) = \begin{cases} 1000 - 0 + 0 & \text{if} \quad \dfrac{1}{S_T} < \dfrac{1}{169} \\ 1000 - 169{,}000\left(-\dfrac{1}{169}+\dfrac{1}{S_T}\right) = 2000 - \dfrac{169{,}000}{S_T} & \text{if} \quad \dfrac{1}{169} \le \dfrac{1}{S_T} \le \dfrac{1}{84.5} \\ 1000 + 1000 - \dfrac{169{,}000}{S_T} + \dfrac{169{,}000}{S_T} - 2000 = 0 & \text{if} \quad \dfrac{1}{S_T} > \dfrac{1}{84.5}. \end{cases}$$

Since it is conventional to express intervals in relation to S_T:

$$f(S_T) = \begin{cases} 1000 - 0 + 0 & \text{if} \quad S_T > 169 \\ 1000 - 169{,}000\left(-\dfrac{1}{169}+\dfrac{1}{S_T}\right) = 2000 - \dfrac{169{,}000}{S_T} & \text{if} \quad 84.5 \le S_T \le 169 \\ 1000 + 1000 - \dfrac{169{,}000}{S_T} + \dfrac{169{,}000}{S_T} - 2000 = 0 & \text{if} \quad S_T < 84.5. \end{cases}$$

Thus:

$$f'(S_T) = \begin{cases} 0 & \text{if} \quad \dfrac{1}{S_T} < \dfrac{1}{169} \\ \left(+\dfrac{169{,}000}{S_T^2}\right) & \text{if} \quad \dfrac{1}{169} \le \dfrac{1}{S_T} \le \dfrac{1}{84.5} \\ 0 & \text{if} \quad \dfrac{1}{S_T} > \dfrac{1}{84.5}. \end{cases}$$

Given the presence of $1/S_T$, differentiation with respect to S_T yields a continuous piecewise-defined function comprising a non-linear component and linear components (Figure G6).

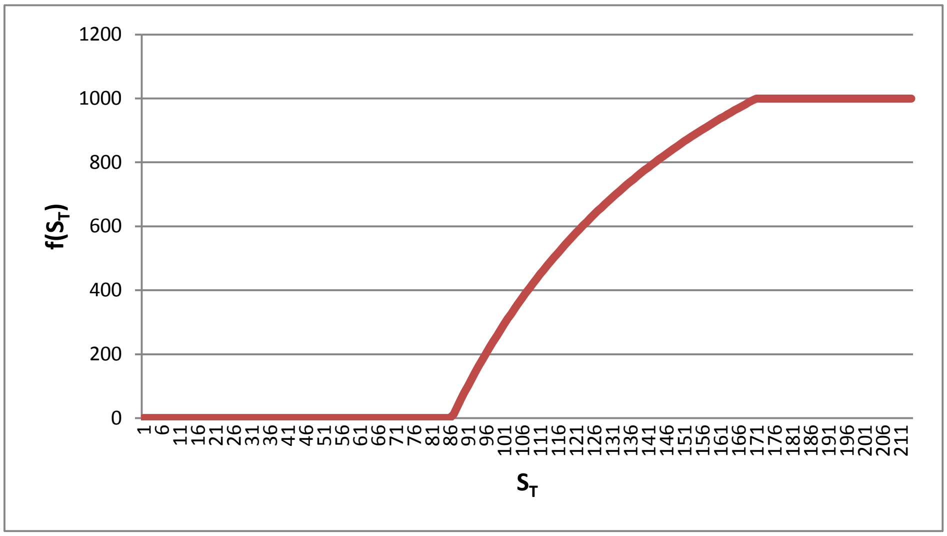

Figure G1: Graph Of ICON Payoff Function

Appendix H: Multi-Strike Options

Options with terminal payoffs depending linearly on N assets include the Multi-strike Call Option with a terminal payoff function of:

$$f\left(\left(S_1^T, S_2^T, .., S_N^T\right)\right) = \max\left[S_1^T - K_1, S_2^T - K_2, .., S_N^T - K_N, 0\right].$$

The put option's payoff function is obtained similarly.

A similar transaction is called the option on the max or min of N assets. The call option on the maximum of N risky assets yields the best of N risky or cash less the cash amount K with a payoff function of:

$$f\left(\left(S_1^T, S_2^T, .., S_N^T\right)\right) = \max\left[\max\left(S_1^T, S_2^T, .., S_N^T\right) - K, 0\right].$$

This function can be decomposed in the following way:

$$\begin{aligned} f\left(\left(S_1^T, S_2^T, .., S_N^T\right)\right) &= \max\left(S_1^T, S_2^T, .., S_N^T\right) - K \\ &\quad + \max\left[\max\left(S_1^T, S_2^T, .., S_N^T\right) - K - \left(\max\left(S_1^T, S_2^T, .., S_N^T\right) - K\right), 0 - \max\left(S_1^T, S_2^T, .., S_N^T\right) - K\right] \\ &= \max\left(S_1^T, S_2^T, .., S_N^T\right) - K. \end{aligned}$$

The result is reached by assuming that for the $0 - \max\left(S_1^T, S_2^T, .., S_N^T\right) - K$ part, a negative value is obtained because at least one of the terminal risky asset prices will be greater than or equal to K therefore the max of the max expression equals 0. A call option on the min is obtained similarly. For a recent related example, see DeTemple, Feng & Tian's (2003) discussion of an American-style call min-option.

For a put option on the min of N risky assets the terminal payoff function is:

$$f\left(\left(S_1^T, S_2^T, .., S_N^T\right)\right) = \max\left[K - \min\left(S_1^T, S_2^T, .., S_N^T\right), 0\right].$$

The decomposition of this function is obtained by following a similar procedure.

The graphical form of these multivariate option payoff functions consists of a series of hyperplanes which intersect.

The prices of such options can be obtained in several ways. For example, Stulz (1982) assigns values to multivariate options applying max and min operators containing more than three components.