THE HAMILTON-JACOBI THEORY IN GENERAL RELATIVITY THEORY AND CERTAIN PETROV TYPE D METRICS

Thesis

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CONTENTS

INTRODUCTION

The discovery of new solutions to Einstein's field equations has long been a problem in General Relativity. However due to new techniques of Newman and Penrose [1], Carter [2] and others there has been a considerable proliferation of new solutions in recent times. Consequently a new problem has arisen. How are we to interpret the new solutions physically? The tools available, despite a spate of papers in the past fifteen years, remain inadequate although often sophisticated.

Any attempts at physical interpretations of metrics are beset with difficulties. There is always the possibility that two entirely different physical pictures will emerge. For example a direct approach would be to attempt an "infilling" of the metric, that is, an extension of the metric into the region occupied by the gravitating matter. However even for the Kerr [1] metric the infilling is by no means unique, in fact a most natural "infilling" turns out to be unphysical (Israel [1]). Yet few people would doubt the physical significance of the Kerr metric.

Viewed in this light our attempt to discuss, among other things, the physical interpretation of type D metrics is slightly ambitious. However the problems with regard to this type of metric are not as formidable as for most of the other metrics, since we have been able to integrate the geodesic equations. Nevertheless it is still not possible to

produce/...

produce complete answers to all the questions posed.

After a chapter on Mathematical preliminaries the study divides naturally into four sections. We start with an outline of the Hamilton-Jacobi theory of Rund [1] and then go on to show how this theory can be applied to the Carter [2] metrics. In the process we lay a foundation in the calculus of variations for Carter's work. This leads us to the construction of Killing tensors for all but one of the Kinnersley [1] type D vacuum metrics and the Carter [2] metrics which are not necessarily vacuum metrics. The geodesic equations, for these metrics, are integrated using the Hamilton-Jacobi procedure.

The remaining chapters are devoted to the Kinnersley [1] type D vacuum metrics. We omit his class I metrics since these are the Schwarzschild metrics, and have been studied in detail before. Chapter three is devoted to a general study of his class II a metric, a generalisation of the Kerr [1] and NUT (Newman, Tamburino and Unti [1]) metrics. We integrate the geodesic equations and discuss certain general properties: the question of geodesic completeness, the asymptotic properties, and the existence of Killing horizons. Chapter four is concerned with the interpretation of the new parameter 'ℓ', that arises in the class II a and NUT metrics. This parameter was interpreted by Demianski and Newman [1] as a magnetic monopole of mass. Our work centers on the possibility of obtaining observable effects from the presence of 'ℓ'. We have been able to show that its presence is observable, at

least/....

least in principle, from a study of the motion of particles in the field. In the first place, if ℓ is comparable to the mass of the gravitating system, a comparatively large perihelion shift is to be expected. The possibility of anomalous behaviour in the orbits of test particles, quite unlike anything that occurs in a Newtonian or Schwarzschild field, also arises.

In the fifth chapter the Kinnersley class IV metrics are considered. These metrics, which in their simplest form have been known for some time, present serious problems and no interpretations have been suggested. Our discussion is essentially exploratory and the information that does emerge takes the form of suggestions rather than conclusions. Intrinsically the metrics give the impression that interesting results should be obtainable since they are asymptotically flat in certain directions. However the case that we have dealt with does not appear to represent a radiation metric.

There are two appendices. The first is devoted to the reduction of the geodesic equations for the six Kinnersley [1] class II metrics to quadratures. In the second we consider the possibility of generating new solutions of Einstein's equations by applying conformal transformations to Carter type metrics.

References to equations are given in the form (a.b.c), where a denotes the chapter number, b the section number and c the equation number. If the equation is in the same chapter the reference is given as (b.c) with the chapter number omitted. References are given as "Author [a]" and details of the publication are given in the reference list.

The list of references given at the end is by no means intended as an extensive bibliography. Further references can be found in the literature quoted.

CHAPTER I

PRELIMINARIES

Before we attempt to discuss the problems that interest us, it is necessary to establish a number of definitions and an agreed notation which, once defined, will be used subsequently without reference.

§1.1 Notation and Mathematical Background.

For our purposes a space-time is defined to be a 4-dimensional connected, orientable (Hausdorff) differentiable manifold M of class C^∞ without boundary, together with a Lorentzian metric ds^2 on M. The signature of ds^2 will be taken to be (-2). In a coordinate neighbourhood G of M, with local coordinates $\{x^i\}$[1], the covariant components of the metric will be denoted by $g_{ij}(x^k) = g_{ji}(x^k)$ and the contravariant components by g^{ij}.

The curvature tensor is defined via the Ricci identities

1.1 $$X_{j;k;m} - X_{j;m;k} = R^i{}_{jkm}X_i ,$$

for/.....

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(1) We will use Latin indices $i,j,k,\dots = 1,2,3,4$ throughout. Where necessary Greek indices $\alpha,\beta,\gamma,\dots = 1,2,3$ will be used.

for an arbitrary vector field X_i, where a semi-colon denotes a covariant derivative in terms of the Christoffel Symbols associated with the tensor g_{ij}. A partial derivative will be denoted by a comma thus

$$\frac{\partial X_i}{\partial x^k} = X_{i,k} .$$

The curvature tensor determines the Ricci tensor

1.2 $$R_{jk} = R^i{}_{jki} ,$$

the curvature invariant

1.3 $$R = R^i{}_i$$

and the Weyl tensor,

1.4 $$C^h{}_{ijk} = R^h{}_{ijk} + \frac{1}{n-2}(\delta^h_j R_{ik} - \delta^h_k R_{ij} + g_{ik} R^h{}_j - g_{ij} R^h{}_k) ,$$

where δ^h_j is the Kronecker delta.

A geodesic in G is a solution $x^i = x^i(t)$ of the second order differential equations

1.5 $$\frac{d^2x^i}{dt^2} + \Gamma^i_{kj}\frac{dx^k}{dt}\frac{dx^j}{dt} = 0 .$$

The parameter t is called an affine parameter and the affine distance between two points p and q on the geodesic is given by

1.6 $$t_{pq} = \int_p^q dt .$$

A/........

A geodesic γ will be said to be incomplete if for any point p on γ the affine length of the geodesic from p in either or both directions is finite. On a non-null curve t_{pq} is related by a linear transformation to the proper time between p and q.

The structure imposed on V, up to now, has been purely geometric. The physical content of the theory enters through the Einstein field equations for ds^2, namely

1.7 $$R_{ij} - \tfrac{1}{2}g_{ij}R - \lambda R = kT_{ij} ,$$

where k is the gravitational constant and λ is the cosmological constant. The tensor field T_{ij} describes physical fields other than gravitation, and thus equation (1.7) relates the physical content of the region to its geometry (1). A solution to the equations (1.7) is referred to as a solution to the Einstein-Maxwell equations if T_{ij} is the energy momentum tensor for an electromagnetic field.

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(1) Both the interpretations of the field equations (1.7) suggested by Synge [1] are used in this thesis: for the Carter [1] metrics a tensor T_{ij} is proposed and then the equations (1.7) are solved, and in Appendix B we allow the T_{ij} to be determined by the equations (1.7) and attempt to transform to a physically reasonable problem.

A device that we will use extensively is the Newman-Penrose null tetrad [1] of vectors; ℓ^j, n^j, m^j and $\bar{m}^j$, where ℓ^j and n^j are real null vectors [2] and m^j and $\bar{m}^j$ (the components of $\bar{m}^j$ are complex conjugates of the components of m^j) are complex null vectors. The vector m^j can be defined in terms of a pair of real, orthogonal, space-like vectors a^j and b^j by

1.8 $$m^j = \frac{1}{\sqrt{2}}(a^j + ib^j) ,$$

where $i = \sqrt{-1}$. The system of vectors ℓ^j, n^j, m^j and $\bar{m}^j$ determines a complex anholonomic coordinate system.

It is convenient to introduce the tetrad notation:

1.9 $$h_a{}^j = \ell^j\delta^1_a + n^j\delta^2_a + m^j\delta^3_a + \bar{m}^j\delta^4_a ,$$

where the indices (labels) $a, b, c, \ldots = 1, 2, 3, 4$ are called tetrad indices. We can now define the tetrad components of the metric tensor by

1.10/.....

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(1) We will refer to these vectors as the Newman-Penrose null tetrad although the ideas were introduced in Sachs [1].

(2) Given a vector with components k^i we will use the terminology "k^i is null" if $k^2 = g_{ij}k^ik^j = 0$, "k^i is timelike" if $k^2 > 0$ and "k^i is spacelike" if $k^2 < 0$.

1.10 $$\eta_{ab} = g_{ij}h_a{}^i h_b{}^j ,$$

which, under the orthogonality condition imposed by Newman and Penrose [1], yields

$$\eta_{ab} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

The invariant η_{ab} and its inverse η^{ab} are used to raise and lower the tetrad indices. With the machinery at our disposal it is possible to define the tetrad components of any tensor, but for our purposes it suffices that the tetrad components of any covariant tensor can be defined by analogy with the definition (1.10).

From the definition (1.10) it follows that the contravariant components of the metric tensor are given, in terms of the tetrad, by

$$g^{ij} = \eta^{ab}h_a{}^i h_b{}^j ,$$

or,

1.11 $$g^{ij} = \ell^i n^j + n^i \ell^j - m^i \bar{m}^j - \bar{m}^i m^j .$$

Certain tetrad components of the Weyl tensor are important since they can be employed in the classification of the Weyl tensor;

1.12/.....

1.12 $$\Psi_0 = -C_{ijk\ell}\, \ell^i m^j \ell^k m^\ell ,$$

1.13 $$\Psi_1 = -C_{ijk\ell}\, \ell^i n^j \ell^k m^\ell ,$$

1.14 $$\Psi_2 = -\tfrac{1}{2} C_{ijk\ell} (\ell^i n^j \ell^k n^\ell - \ell^i n^j m^k \bar{m}^\ell) ,$$

1.15 $$\Psi_3 = -C_{ijk\ell}\, n^i \ell^j n^k \bar{m}^\ell ,$$

1.16 $$\Psi_4 = -C_{ijk\ell}\, n^i \bar{m}^j n^k \bar{m}^\ell .$$

We will call these the Newman-Penrose invariants.

The orthogonality conditions imposed on the tetrad only define it up to a six parameter restricted Lorentz group given by Newman and Janis [1] as;

1.17 $$\tilde{\ell}^j = \ell^j ,$$

1.18 $$\tilde{m}^j = m^j + a\ell^j ,$$

1.19 $$\tilde{n}^j = n^j + a\bar{m}^j + \bar{a}m^j + a\bar{a}\ell^j ,$$

where a is an arbitrary complex function;

1.20 $$\tilde{\ell}^j = \lambda \ell^j ,$$

1.21 $$\tilde{n}^j = \lambda^{-1} n^j ,$$

1.22 $$\tilde{m}^j = e^{i\phi} m^j ,$$

where λ and ϕ are real functions;

1.23 $$\tilde{\ell}^j = \ell^j + b\bar{m}^j + \bar{b}m^j + b\bar{b}n^j ,$$

1.24/.....

1.24 $$\tilde{m}^j = m^j + bn^j ,$$

1.25 $$\tilde{n}^j = n^j ,$$

where b is an arbitrary complex function.

Under these transformations the Newman-Penrose invariants transform as follows;

(a) for the transformations (1.17), (1.18) and (1.19),

1.26 $$\begin{cases} \tilde{\Psi}_0 = \Psi_0 , \\ \tilde{\Psi}_1 = \Psi_1 + \bar{a}\Psi_0 , \\ \tilde{\Psi}_2 = \Psi_2 + 2\bar{a}\Psi_1 + \bar{a}^2\Psi_0 , \\ \tilde{\Psi}_3 = \Psi_3 + 3\bar{a}\Psi_2 + 3\bar{a}^2\Psi_1 + \bar{a}^3\Psi_0 , \\ \tilde{\Psi}_4 = \Psi_4 + 4\bar{a}\Psi_3 + 6\bar{a}^2\Psi_2 + 4\bar{a}^3\Psi_1 + \bar{a}^4\Psi_0 , \end{cases}$$

(b) for the transformations (1.20), (1.21) and (1.22)

1.27 $$\begin{cases} \tilde{\Psi}_0 = \lambda^2 e^{2i\phi}\Psi_0 , \\ \tilde{\Psi}_1 = \lambda e^{i\phi}\Psi_1 , \\ \tilde{\Psi}_2 = \Psi_2 , \\ \tilde{\Psi}_3 = \lambda^{-1} e^{-i\phi}\Psi_3 , \\ \tilde{\Psi}_4 = \lambda^{-2} e^{-2i\phi}\Psi_4 , \end{cases}$$

and

(c)/.......

(c) for the transformations (1.23), (1.24) and (1.25),

1.28
$$\begin{cases} \tilde{\Psi}_0 = \Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4 , \\ \tilde{\Psi}_1 = \Psi_1 + 3b\Psi_2 + 3b^2\Psi_3 + b^3\Psi_4 , \\ \tilde{\Psi}_2 = \Psi_2 + 2b\Psi_3 + b^2\Psi_4 , \\ \tilde{\Psi}_3 = \Psi_3 + b\Psi_4 , \\ \tilde{\Psi}_4 = \Psi_4 . \end{cases}$$

The space-times we consider are not flat and therefore at least one of the Newman-Penrose invariants must be non zero. Thus from the transformations (1.26) and (1.28) it follows that we can always choose a tetrad so that Ψ_4 and Ψ_0 are both non zero, and consequently there is no loss in generality if we assume them non zero. From the transformations (1.26) it is clear that by an appropriate choice of $\bar{a}$ we can make $\Psi_4 = 0$. The required function $\bar{a}$ will be a solution of the quartic

1.29
$$\Psi_4 + 4\bar{a}\Psi_3 + 6\bar{a}^2\Psi_2 + 4\bar{a}^3\Psi_1 + \bar{a}^4\Psi_0 = 0 .$$

Each vector $\tilde{n}^j$ corresponding to a root of (1.29) is called a principal null vector. Since $\Psi_0 \neq 0$ there will be four such principal null vectors, not necessarily all different.

We may use the equation (1.29) to define the Petrov types. If equation (1.29) has

(i) four distinct roots - the Weyl tensor is of Petrov Type I,

(ii)/.....

(ii) two single roots and one repeated root - the Weyl tensor is of Petrov Type II,

(iii) one single root and three coincident roots - the Weyl tensor is of Petrov Type III

(iv) four coincident roots - the Weyl tensor is of Petrov Type N,

(v) two repeated roots - the Weyl tensor is of Petrov Type D.

It can be shown (see for instance Newman and Janis [1]) that $\tilde{l}^j$ and $\tilde{n}^j$ may be chosen as the principal null vectors for a Petrov Type D Weyl tensor and then $\tilde{\Psi}_0 = \tilde{\Psi}_1 = \tilde{\Psi}_3 = \tilde{\Psi}_4 = 0$ and only $\tilde{\Psi}_2$ is non zero. In the subsequent discussion we will loosely refer to metrics as being of Petrov Type D, meaning in each case that the corresponding Weyl tensor is of that type.

The fact that the metric g_{ij} is only determined locally poses problems, since the "extent" of the manifold M has to be deduced from the metric (Geroch [1]). The main problem is the occurence of "singularities" in the metric. Since even a definition of a "singularity" has not yet been satisfactorily stated (Geroch [1]), the problems are considerable. However we will confine ourselves to what seems to be the most useful approach and define a space-time to be non-singular if all the geodesics are complete, in the sense that, given any point p on a geodesic γ, the affine parameter must be unbounded in either direction from p.

A criterion for an incomplete geodesic has been estab-

lished/...

lished by Walker [1] who has shown that a geodesic is not extendible if a C^{∞} scalar field becomes unbounded along it. This type of singularity is the only one that we will consider in the subsequent discussion. In some cases, where we require it, we will show how the effect of a spurious singularity, the result of an unfortunate choice of coordinates, can be removed.

Once the geodesics for a metric have been obtained it is possible to find the so called g-boundary for a singularity (Geroch [2]). This boundary does yield the possibility of characterising a singularity locally but, as mentioned by Geroch [1], it does not lead to a unique definition. The b-boundary of Schmidt [1] may be preferable but, as yet, it is difficult to compare the two types of boundary as Schmidt's technique has not been sucessfully applied to even the Schwarzschild or Riessner-Nordström metrics. Until the concept of a singular boundary has been developed sufficiently there is little that can be contributed by computing the g-boundaries for all Petrov type D metrics, although the computation, using our results, would be relatively simple.

CHAPTER II

THE HAMILTON-JACOBI THEORY

In this chapter we will discuss some aspects of the Hamilton-Jacobi theory in General Relativity, and then go on to show how the approach of Rund [1], [2] and [3] to the theory can be used to relate the work of Penrose and Walker [1] on Killing Tensors, for Petrov type D metrics, to the work of Carter [1] on metrics which lead to separable Hamilton-Jacobi equations.

§2.1 Some Remarks on the Hamilton-Jacobi Theory in General Relativity.(1)

There are various ways in which the geodesic equations in a space-time can be approached. In Chapter one we defined the geodesics to be the curves which satisfy the equations (1.1.5). Alternatively we could have defined the geodesics as the extremals of a suitably defined Lagrangian, in which case the Euler-Lagrange equations will correspond to our equation (1.1.5). Whichever way we approach the equations they usually prove difficult to integrate. This situation is most un-

fortunate/...

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(1) The essential results in this chapter, that is in §2.1, §2.2 and §2.3, are to be published in Matravers [1].

fortunate because, as we have shown, much of the work on singularities is based on a knowledge of the geodesics (see §1.1 and the references there). Also, in experimental tests of the predictions of General Relativity, the motion of relatively small bodies is compared with the expected behaviour based on a study of the timelike geodesics. The classic instance of such a comparison is the work on the perihelion shift of Mercury.

Under these circumstances it is natural that some authors, Vanstone [1], Sen [1], Carter [1] and [2] and Fock [1], should have explored the possibility of using an analogue of the Hamilton-Jacobi theory of classical mechanics to integrate the geodesic equations. However these authors do not agree on the choice of formalism, and therefore we will examine what we would expect from such a formalism:

(i) a set of equations of motion, preferably in the form of Euler-Lagrange equations or Hamilton's Canonical Equations;

(ii) a first order partial differential equation, the complete integrals of which yield solutions to the equations of motion.

In addition the four space-time coordinates should be treated symmetrically, if possible, and the whole formalism should arise from a problem in the calculus of variations. We do not require that the Hamiltonian be identifiable with the energy, since one would expect the energy to be associated with the momenta in a symmetric way. Also, we are not

concerned/...

concerned with the use of the canonical equations and Hamilton-Jacobi formalism in quantisation, since, in view of the new role of the Hamiltonian, it is quite clear that the quantisation procedure of non-relativistic mechanics cannot be applied to a relativistic formalism without great care. This aspect is discussed for the special relativistic case in Rund [1].

We are now in a position to comment on some of the suggested formalisms. Sen [1] has attempted to retain the characteristics of the non-relativistic Hamilton-Jacobi equation and thus does not treat the coordinates in a homogeneous way. Therefore his formalism does not satisfy our requirements.

The approach of Carter [1] and [2] is similar in many ways to the five dimensional formulation of Kalman [1]. As we will show here, the introduction of a five dimensional formulation is not necessary to achieve all the results that Carter obtains and the interpretation of the work is much simpler. Rund [3] has discussed the five dimensional scheme of Kalman [1] and has shown that the formalism that we will use has advantages over that particular scheme. The ideas of Fock [1] on the Hamilton-Jacobi formalism are essentially the same as those used here for null geodesics but his interpretation and nomenclature differ considerably. He calls the fourth component of the momentum the Hamiltonian and does not attempt to suggest a canonical procedure.

§2.2/.....

§2.2 The Hamilton-Jacobi Equation.

We will now briefly outline the Hamilton-Jacobi theory of Rund [1], [2] and [3]. This formalism satisfies all our criteria given in §2.1. The final equations that we will obtain are the same as those of Rund [1] although they are expressed in a slightly different form. The form that we use has the slight advantage that the Hamilton-Jacobi equation for a charged particle in an Einstein-Maxwell field corresponds to that obtained by Vanstone [1]. Therefore it becomes apparent that the equations of motion for charged particles bear the same formal relation to those for uncharged particles as in the non-relativistic case.

Let $\{x^i\}$, $i,j = 1,2,3,4$ be local coordinates in a region G of space-time with metric $ds^2 = g_{ij}dx^i dx^j$, where $\det|g_{ij}| < 0$ and $g_{ij} = g_{ij}(x^k) \in C$. We take the motion of a particle to be determined by the geodesics of the variational problem with Lagrangian

2.1 $$L = (g_{ij}x'^i x'^j)^{\frac{1}{2}}, \qquad x'^i = \frac{dx^i}{d\tau},$$

where τ is an arbitrary parameter. For the present we assume $L \neq 0$.

Following Rund [1], we define the canonical momenta by:

2.2 $$y_i = L\frac{\partial L}{\partial x'^i} = \frac{1}{2}\frac{\partial L^2}{\partial x'^i},$$

and thence, since $\det|g_{ij}| \neq 0$, we solve for

2.3 $$x'^i = \chi^i(x^j, y_j) .$$

Again following Rund, the Hamiltonian is defined by

2.4 $$H^2(x^j, y_j) = L^2(x^j, \chi^j(x^i, y_i))$$

2.5 $$= g^{ij} y_i y_j ,$$

and the Hamilton-Jacobi equation is

2.6 $$H\left(x^i, \frac{\partial S}{\partial x^i}\right) = \pm 1 ,$$

where $S(x^i) = \Sigma$ is a one-parameter family of hypersurfaces which cover the region G simply. The sign used in equation (2.6) must coincide with that of L. (See equation (2.9) below). The congruence of curves determined by

2.7 $$y_i = \frac{\partial S}{\partial x^i}$$

satisfies the Euler-Lagrange equations,

2.8 $$\frac{d}{d\tau}\left(\frac{\partial L}{\partial x'^i}\right) - \frac{\partial L}{\partial x^i} = 0 ,$$

provided that the parameter τ be chosen such that:

2.9 $$L = H = \pm 1 .$$

The sign in equations (2.6) and (2.9) is fixed if we choose $\tau = s$ to be the proper time s. We now put

2.10 $$F = \tfrac{1}{2} L^2 .$$

Since/....

Since $\frac{dF}{ds} = 0$, the Euler-Lagrange equations become

2.11 $$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{x}^i}\right) - \frac{\partial F}{\partial x^i} = 0 \ , \qquad \left(\dot{x}^i = \frac{dx^i}{ds}\right)$$

with

2.12 $$y_i = \frac{\partial F}{\partial \dot{x}^i} \ .$$

Clearly, the null geodesics, for which $L = 0$, cannot be handled in the above manner. However, results analogous to the above equations do hold. Rund [1], p.318 f, shows that the orthogonal trajectories of the family of hypersurfaces $S(x^i) = \Sigma$, which cover G simply and for which $H^2\left(x^i, \frac{\partial S}{\partial x^i}\right) = 0$, satisfy the Euler-Lagrange equations (2.11).

Thus we are left with the conclusion that, in the region G of space-time, the geodesics satisfy

2.13 $$g^{ij}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^j} = \lambda \ , \qquad \lambda = 1, \ -1 \ \text{ or } \ 0 \ ,$$

and the equations (2.11). The y_i are given in terms of x^i by the equations (2.7) or by equations (2.2) in terms of x^i and $\dot{x}^i$. The value of λ depends on whether the geodesics are timelike, spacelike or null. We regard the equations (2.11), (2.13) with (2.7) and (2.12) as the generalisation of the Hamilton-Jacobi theory of classical mechanics. Note that the complete set of equations does not arise from a parameter-invariant problem in the calculus of variations. Although this is undesirable, the choice of parameter for the null geodesics cannot be arbitrary and it is therefore unavoidable. For timelike and spacelike geodesics, however, the equations do arise from a parameter-invariant problem.

The Hamilton-Jacobi theory we have introduced can be used to integrate the geodesic equations for all the Carter [1] metrics. The method is as follows.

All these metrics depend on only two of the four coordinates, let us say x^1 and x^2. Thus there exist two first integrals:

2.14 $$y_3 = \frac{\partial F}{\partial \dot{x}^3} = k_1 ,$$

2.15 $$y_4 = \frac{\partial F}{\partial \dot{x}^4} = k_2 .$$

Thus from the equations (2.7) we have

2.16 $$\frac{\partial S}{\partial x^3} = k_1 ,$$

2.17 $$\frac{\partial S}{\partial x^4} = k_2 .$$

In addition, once appropriate coordinates have been found, the metrics that we are considering take the form:

2.18 $$g^{ij} = \frac{1}{U} \begin{bmatrix} a_1^{11} & 0 & 0 & 0 \\ 0 & a_2^{22} & 0 & 0 \\ 0 & 0 & a_1^{33} + a_2^{33} & a_1^{34} + a_2^{34} \\ 0 & 0 & a_1^{43} + a_2^{43} & a_1^{44} + a_2^{44} \end{bmatrix}$$

where the suffixes 1 and 2 denote that the quantity in question is a function of x^1 or x^2 only, and $U = U(x^1, x^2)$.

Thus/.....

Thus the Hamilton-Jacobi equation becomes:

$$a_1^{11}\left(\frac{\partial S}{\partial x^1}\right)^2 + a_2^{22}\left(\frac{\partial S}{\partial x^2}\right)^2 + (a_1^{33} + a_2^{33})\left(\frac{\partial S}{\partial x^3}\right)^2 + 2(a_1^{34} + a_2^{34})\left(\frac{\partial S}{\partial x^3}\frac{\partial S}{\partial x^4}\right) + (a_1^{44} + a_2^{44})\left(\frac{\partial S}{\partial x^4}\right)^2 = \lambda U , \tag{2.19}$$

which is separable if $U = U_1(x^1) + U_2(x^2)$ or $\lambda = 0$, in which case it has a solution of the form

$$S = S_1(x^1) + S_2(x^2) + k_1 x^3 + k_2 x^4 . \tag{2.20}$$

The first case, where U can be reduced to a sum of terms in x^1 and x^2, arises with all Carter metrics. Of course $\lambda = 0$ for all null geodesics. The generalisation to cases where U is not separable has not previously been considered, and these will be referred to subsequently as Generalised Carter metrics.

Once S has been obtained in the form (2.20), the equations (2.16) and (2.17) together with equations (2.7) and (2.12) for y_1 and y_2 yield a set of four ordinary differential equations which can be reduced to quadratures because of the simple form of the metric.

The particular form (2.20) of the solution to the Hamilton-Jacobi equation is not surprising since, for the Carter metrics, the formalisms of Carter and Rund agree although the interpretation is not the same. For instance, unlike the situation in the Carter approach where the solutions $S(x^i)$ have no apparent significance (Carter [2]) the equation (2.13) has an immediate geometric interpretation:

according/..

according to the choice of λ the solutions yield one-parameter families of null, spacelike or timelike hypersurfaces transversal to the corresponding geodesics. Thus the separability of the solutions implies that, for these metrics, there exist coordinate systems in which families of null, timelike and spacelike hypersurfaces transversal to the geodesics take the form

2.21 $$S_1(x^1) + S_2(x^2) + k_1 x^3 + k_2 x^4 = \Sigma \;,$$

where Σ is a constant over each surface.

If the motion of a charged particle in an Einstein-Maxwell field (with electromagnetic potential given by ϕ_i) is considered, then the Hamilton-Jacobi equation can be written as

2.22 $$g^{ij}\left(\frac{\partial S}{\partial x^i} - e\phi_i\right)\left(\frac{\partial S}{\partial x^j} - e\phi_j\right) = 1 \;,$$

in suitable units (Vanstone [1]). The charge on the particle is e. If ϕ_i is chosen according to the prescription of Carter [1] and the metric g^{ij} is a Carter separable solution to the Einstein Equations, then the equation (2.22) separates and has a solution of the form (2.21). This solution differs from the type of solution obtained in Carter [1] and [2] in the same way as for the uncharged particle. However, unlike the case of an uncharged particle and for the Carter formalism, the geodesic equations cannot be obtained as simple first order ordinary differential equations from the momenta. This problem is merely formal however, as the geodesic equations can be reduced to quadratures; we use the solution of equation

(2.22)/...

(2.22) and the fact that the derivatives with respect to the integration constants are constant. The first order ordinary differential equations required can then be found by differentiating these integrated forms of the equations, as is done in Vanstone [1].

§2.3 Killing Tensors for the Carter Metrics.

Killing tensors (1) and Conformal Killing tensors are defined by Penrose and Walker [1] as follows. If each parallelly-propagated tangent vector t^i along the geodesics satisfies

$$a_{ij}t^i t^j = \text{constant} \tag{3.1}$$

for some tensor a_{ij} symmetric in i and j, then a_{ij} is called a Killing tensor of valence two. If equation (3.1) holds only along the null geodesics, then a_{ij} is called a Conformal Killing tensor of valence two. We note that, if a_{ij} is a Conformal Killing tensor, then

$$a_{ij} + \phi g_{ij} , \tag{3.2}$$

where ϕ is an arbitrary function of the coordinates, is also a Conformal Killing tensor, since along the null geodesic

3.3/......

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(1) These tensors are not the same as the Yano-Killing tensors (Yano [1]). A Killing tensor which is also a Yano-Killing tensor vanishes identically.

3.3 $$(a_{ij} + \phi g_{ij})t^i t^j = a_{ij}t^i t^j = \text{constant} .$$

We will now show that all Carter metrics admit Killing tensors and that all Generalised Carter metrics admit Conformal Killing tensors.

As we have already seen, the equation (2.13) has a solution of the form (2.20) if $U = U_1 + U_2$. Thus equation (2.13) can be separated into:

3.4 $$a_1^{ij} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^j} - \lambda U_1 = K$$

and

3.5 $$a_2^{ij} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^j} - \lambda U_2 = -K ,$$

where non-constant terms in the first equation depend only on x^1; those in the second equation depend only on x^2.

These equations can be written as one if we use the notation:

3.6 $$a_{(r)}^{ij} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^j} - \lambda U_{(r)} = (-1)^{r+1}K , \qquad r = 1,2 .$$

If we substitute for the $\frac{\partial S}{\partial x^i}$ in the equations (3.6) using the equations (2.7) and then use the equations (2.12) to express the y_i as functions of the $\dot{x}^i$, we obtain:

3.7 $$a_{(r)}^{ij} g_{ki} g_{mj} \dot{x}^k \dot{x}^m - \lambda U_{(r)} = (-1)^{r+1}K .$$

If we introduce

3.8 $$M_{(r)km} = a^{ij}_{(r)} g_{ki} g_{mj} ,$$

and recall the definition of λ,

3.9 $$g_{ij} \dot{x}^i \dot{x}^j = \lambda ,$$

equation (3.7) becomes

3.10 $$\left(M_{(r)ij} - U_{(r)} g_{ij}\right) \dot{x}^i \dot{x}^j = (-1)^{r+1} K ,$$

which shows that $\left(M_{(r)ij} - U_{(r)} g_{ij}\right)$ is a Killing tensor.

Thus we have:

Theorem 1. All Carter [1] metrics admit Killing tensors of the form

3.11 $$M_{(r)ij} - U_{(r)} g_{ij} ,$$

where $r = 1, 2$.

If we consider the Generalised Carter metrics for which U is not separable, then, for null geodesics, equations similar to (3.4) and (3.5) hold with $\lambda = 0$. Thus one finds very simply that $M_{(r)ij}$ is a Conformal Killing tensor for the Generalised Carter metrics. In view of the tensor defined by (3.2), we have:

Theorem 2. All Generalised Carter metrics admit Conformal Killing tensors of the form

3.12 $$a^{mk}_{(r)} g_{mi} g_{kj} + \phi g_{ij} ,$$

where ϕ is an arbitrary function of the coordinates.

These/....

These theorems provide a link between the work of Carter [1], on metrics which yield separable Hamilton-Jacobi equations, and Penrose and Walker [1], on the existence of Killing tensors and Conformal Killing tensors for type D metrics. In particular Theorem 2 relates to the fact that, although the Kinnersley [1] class III a metric is not a Carter [1] metric, the null geodesics can still be integrated using the Hamilton-Jacobi formalism.

We also note that these two theorems support, at least in part, the conjecture of Kinnersley and Walker [1], given in a footnote on page 1367, that Killing tensors exist for all type D solutions (since the Carter metrics are type D solutions to the Einstein-Maxwell equations).

The approach via the Killing tensors has the advantage of not depending on the coordinate system. However, once appropriate coordinates have been found, our method yields more information.

Kinnersley [1] gives an exhaustive set of Petrov type D vacuum metrics and we show that all the ones that we consider are Generalised Carter metrics if not Carter metrics. The only case that we do not consider is the Kinnersley class III b metric which involves elliptic functions and looks less interesting from the physical viewpoint. However the work of Penrose and Walker [1] suggests that it will probably be possible to find a set of coordinates in which it is a Generalised Carter metric.

CHAPTER III

THE KINNERSLEY CLASS II a OR NEWMAN-DEMIANSKI METRIC

§3.1 The Metric.

This metric was first found by Demianski and Newman [1] using the "complex derivation" technique that Newman and Janis [1] developed to "derive" the Kerr [1] metric. Unfortunately the Demianski and Newman [1] paper is marred by misprints, and therefore difficult to use. Later Kinnersley [1] found the same metric, in different coordinates, using the Newman-Penrose formalism, demonstrating quite clearly the value of the formalism since he has been able to obtain all the Petrov type D vacuum metrics. Again, the work is marred by errors; the tetrad from which Kinnersley obtains the metrics does not satisfy the conditions that he presupposes. However, the metrics have been confirmed as type D vacuum metrics by D'Inverno and Russell-Clark [1].

Although no details are known on when and why the Newman and Janis [1] complex transformation technique works, it provides a useful tool for finding the Newman-Penrose [1] tetrad. Of course, in general, it is as difficult to find the complex transformation as it is to find the tetrad by direct methods. However, for the Kinnersley II a metric, the Newman-Janis technique proves the simpler approach. The following argument leading to the Newman-Penrose tetrad is

not/......

not meant as a derivative of the metric.[1]

The complexification process is applied to the Schwarzschild metric,

1.1 $$ds^2 = \left(1 - \frac{2m_o}{r}\right) dt^2 - \left(1 - \frac{2m_o}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\phi^2) ,$$

after we introduce the retarded time coordinate,

1.2 $$u = t - r \, 2m_o \log(r - 2m_o) ,$$

to obtain the Eddington [1] form of the metric,

1.3 $$ds^2 = \left(1 - \frac{2m_o}{r}\right) du^2 + 2 \, du \, dr - r^2(d\theta^2 + \sin^2\theta \, d\phi^2) .$$

The Newman-Penrose [1] tetrad for this metric is

1.4 $$\ell^j = \delta_2^j ,$$

1.5 $$n^j = \delta_1^j - \frac{1}{2}\left(1 - \frac{2m_o}{r}\right) \delta_2^j ,$$

1.6 $$m^j = (\sqrt{2} \, r)^{-1}(\delta_3^j + i \operatorname{cosec} \theta \delta_4^j) ,$$

1.7 $$\bar{m}^j = (\sqrt{2} \, r)^{-1}(\delta_3^j + i \operatorname{cosec} \theta \delta_4^j) ,$$

where/.....

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(1) The tetrad for the Newman-Demianski metric can be obtained in the same way, but unfortunately the complexification technique has to be different from the one we use to get the Kinnersley II a metric and therefore it proves of little use in relating the two coordinate systems, and will not be discussed.

where $i = \sqrt{-1}$. This tetrad has the property that, if the contravariant components of the metric tensor in equation (1.3) are denoted by g^{ij} with

1.8 $$x^1 = u, \quad x^2 = r, \quad x^3 = \theta, \quad x^4 = \phi \,,^{(1)}$$

then,

1.9 $$g^{ij} = \ell^i n^j + \ell^j n^i - m^i \bar{m}^j - m^j \bar{m}^i \,.$$

Before we go on with the complexification procedure it is convenient to write the tetrad (1.4), (1.5), (1.6) and (1.7) in differential operator form;

1.4a $$\underset{\sim}{\ell} = \frac{\partial}{\partial r} \,,$$

1.5a $$\underset{\sim}{n} = \frac{\partial}{\partial u} - \frac{1}{2}\left(1 - \frac{2m_o}{r}\right) \frac{\partial}{\partial r} \,,$$

1.6a $$\underset{\sim}{m} = (\sqrt{2}\, r)^{-1}\left(\frac{\partial}{\partial \theta} + i \operatorname{cosec} \theta \frac{\partial}{\partial \phi}\right) ,$$

1.7a $$\underset{\sim}{\bar{m}} = (\sqrt{2}\, r)^{-1}\left(\frac{\partial}{\partial \theta} - i \operatorname{cosec} \theta \frac{\partial}{\partial \phi}\right) .$$

The reason for preferring this form is that it makes the subsequent complexification procedure clearer. We replace m_o by $m - i\ell$ where m is a real constant and not to be confused with the vector m, and allow r and u to take complex values. The tetrad (1.4a), (1.5a), (1.6a) and (1.7a)

is/......

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(1) We will use either of the coordinates given in (1.8) according to which is most appropriate in the following discussion.

is replaced by the new tetrad, which is arbitrarily defined to resemble the original form;[1]

1.10 $$\underset{\sim}{\ell} = \frac{\partial}{\partial r} ,$$

1.11 $$\underset{\sim}{n} = \frac{\partial}{\partial u} - \frac{1}{2}\left(1 - \frac{m_o}{r} - \frac{\bar{m}_o}{\bar{r}}\right) \frac{\partial}{\partial r} ,$$

1.12 $$\underset{\sim}{m} = (\sqrt{2}\ \bar{r})^{-1}\left(\frac{\partial}{\partial \theta} + i \text{ cosec } \theta \frac{\partial}{\partial \phi}\right) ,$$

1.13 $$\underset{\sim}{\bar{m}} = (\sqrt{2}\ r)^{-1}\left(\frac{\partial}{\partial \theta} - i \text{ cosec } \theta \frac{\partial}{\partial \phi}\right) ,$$

where a bar indicates the complex conjugate. The coordinates used in the tetrad (1.10), (1.11), (1.12), (1.13) are projected back into the real domain by the transformations:

1.14 $$u' = u - 2i\ell \log(\text{cosec } \theta) - ia \cos \theta$$

1.15 $$r' = r - i(\ell - a \cos \theta)$$

1.16 $$\theta' = \theta$$

1.17 $$\phi' = \phi$$

where u', r', θ' and ϕ' are real. The transformations (1.14) and (1.15), in effect, define the complexification of the r and u.

Before/...

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(1) This arbitrariness poses essential difficulties for any theoretical discussion of the technique.

Before we apply the transformation, we note that

1.18 $$1 - \frac{m}{r} - \frac{\bar{m}}{\bar{r}} = (r\bar{r} - m_0\bar{r} - \bar{m}_0 r)(r\bar{r})^{-1}$$

$$= (r'^2 - 2mr' - \ell^2 + a^2\cos^2\theta)(r\bar{r})^{-1}$$

$$= (\Delta' - a^2\sin^2\theta)(r\bar{r})^{-1}$$

where

1.19 $$\Delta' = r'^2 - 2mr' - \ell^2 + a^2 .$$

It is also convenient to define,

1.20 $$\rho = r' + i(\ell - a\cos\theta) ,$$

then

1.21 $$\rho^2 = \bar{\rho}\rho = r'^2 + (\ell - a\cos\theta)^2 .$$

In the transformations (1.14) and (1.15) we regard u' as a real valued function of u and θ, and r' as a real valued function of r and θ. Thus we have,

1.22 $$\left(\frac{\partial u'}{\partial u}\right)_\theta = 1 ,$$

1.23 $$\left(\frac{\partial u'}{\partial \theta}\right)_u = i(a\sin\theta + 2\ell\cot\theta) ,$$

1.24 $$\left(\frac{\partial r'}{\partial r}\right)_\theta = 1 ,$$

1.25 $$\left(\frac{\partial r'}{\partial \theta}\right)_r = -ia\sin\theta ,$$

where the suffixes indicate which variables are held constant. Note that in equations (1.23) and (1.24) the complex variables r and u are held constant. After the transformation the tetrad becomes

1.26 $$\underset{\sim}{\ell} = \frac{\partial}{\partial r}$$

1.27 $$\underset{\sim}{n} = \frac{\partial}{\partial u'} - \tfrac{1}{2}\rho^{-2}(\Delta - a^2\sin^2\theta')\frac{\partial}{\partial r'}$$

1.28 $$\underset{\sim}{m} = (\sqrt{2}\rho)^{-1}\Big\{i(a\sin\theta' + 2\ell\cot\theta')\frac{\partial}{\partial u'} - ia\sin\theta'\frac{\partial}{\partial r'} + \frac{\partial}{\partial\theta'} + i\,\mathrm{cosec}\,\theta'\frac{\partial}{\partial\phi'}\Big\}$$

1.29 $$\underset{\sim}{\bar{m}} = (\sqrt{2}\rho)^{-1}\Big\{-i(a\sin\theta' + 2\ell\cot\theta')\frac{\partial}{\partial u'} + ia\sin\theta'\frac{\partial}{\partial r'} + \frac{\partial}{\partial\theta'} - i\,\mathrm{cosec}\,\theta'\frac{\partial}{\partial\phi'}\Big\} .$$

As no confusion can arise we will drop the primes and compute the metric from the new tetrad (1.26), (1.27), (1.28) and (1.29) using equations (1.9). We get;

1.30 $$g^{11} = -\rho^{-2}(a\sin\theta + 2\ell\cot\theta)^2 ,$$

1.31 $$g^{12} = \rho^{-2}(r^2 + \ell^2 + a^2) ,$$

1.32 $$g^{13} = 0 ,$$

1.33 $$g^{14} = -\rho^{-2}\mathrm{cosec}\,\theta(a\sin\theta + 2\ell\cot\theta) ,$$

1.34 $$g^{22} = -\rho^{-2}\Delta ,$$

1.35 $$g^{23} = 0 ,$$

1.36 $$g^{24} = a\rho^{-2} ,$$

1.37 $$g^{33} = -\rho^{-2} ,$$

1.38 $$g^{34} = 0 , \quad ,$$

1.39 $$g^{44} = -\rho^{-2}\mathrm{cosec}^2\theta ,$$

from/.....

from which the covariant components are:

1.40 $$g_{11} = \rho^{-2}(\Delta - a^2\sin^2\theta) ,$$

1.41 $$g_{12} = 1 ,$$

1.42 $$g_{13} = 0 ,$$

1.43 $$g_{14} = -2\rho^{-2}\ell \cos\theta\Delta + 2\rho^{-2}a \sin^2\theta(mr + \ell^2) ,$$

1.44 $$g_{24} = -a \sin^2\theta - 2\ell \cos\theta ,$$

1.45 $$g_{33} = -\rho^2 ,$$

1.46 $$g_{44} = \rho^{-2}\Delta(a \sin^2\theta + 2\ell \cos\theta)^2 - \rho^{-2}\sin^2\theta(r^2 + \ell^2 + a^2)^2 ,$$

1.47 $$g_{22} = g_{23} = g_{34} = 0 .$$

The metric tensor g_{ij} defined by equations (1.40) to (1.47) is the class II a metric, its determinant

1.48 $$g = \det|g_{ij}| ,$$

is

1.49 $$g = -\rho^4\sin^2\theta ,$$

which seems to imply that there will be no difficulties with the signature of the metric near $\theta = 0$ or $\theta = \pi$, at least in appropriate coordinates. This property will be discussed again when we have an asymptotic form for the metric.

Certain properties of the class II a metric have been determined. Demianski and Newman [1] and Kinnersley [1] have shown that it is a Petrov type D metric with

1.50/.....

1.50 $$\Psi_2 = \frac{m + i\ell}{\rho^3} .$$

Also they have pointed out that the metric becomes the Kerr metric (Kerr [1]) if $\ell = 0$, and the NUT metric (Newman, Tamburino and Unti [1]) if $a = 0$. If $a = 0$ and $\ell = 0$ it is obvious from the tetrads that it reverts to the Schwarzschild metric. Thus, with the appropriate choice of ℓ, m and a, we get the NUT, Kerr or Kinnersley IIa tetrads. Accordingly m and a will be interpreted as in the case of the Kerr metric: m is taken as the mass of the source and ma as its angular momentum. This interpretation is discussed in detail in Carter [2] where detailed references to earlier work are given. Demianski and Newman [1] have endeavoured to interpret the ℓ by using the analogy of a magnetic monopole in electrodynamics. This idea led them to call ℓ the "magnetic monopole of mass" in the case where $\ell^2 > a^2$. The interpretation agrees with the definition of a magnetic monopole of mass (we will drop the inverted commas) given by Bergmann and Sachs [1] and used by Newman and Janis [1] although Bergmann and Sachs maintain, on the basis of a linearised model, that such a monopole cannot exist. The contradiction that arises may be due to the assumptions made by Bergmann and Sachs and by Newman and Janis. In both papers it is assumed that the metric is asymptotically flat and that certain coordinate conditions are satisfied. The fact that the NUT metric is not asymptotically flat suggests that the interpretation of ℓ as a monopole of any sort may be unreasonable and that perhaps the metric should be regarded as a cosmological solution as

suggested/....

suggested by Misner [1]. We will be able to add to these remarks when we have discussed the equatorial orbits of particles.

Although there are quite a number of papers that deal with the Kerr and NUT metrics separately there appear to be only two papers that are devoted to the class II a metric. These are by Demianski and Newman [1] and Kinnersley [1]. It is possible that the reason for this is the somewhat pathological behaviour of the NUT metric. However, some of the difficulties that occur with the NUT metric are less serious when it is combined with the Kerr metric thus making the combined metric more interesting.

If $\ell^2 \leqslant a^2$ then there is a singularity in the metric at

1.51 $$\rho^2 = 0 ,$$

which does not occur if $\ell^2 > a^2$ since then ρ^2 cannot vanish. The singularity (1.51) carries through to the Newman-Penrose invariants since Ψ_2, given in equation (1.50), is singular at $\rho^2 = 0$, and thus it is an "essential singularity" in the terminology of Misner [1]. This means that it cannot be removed by a coordinate transformation. We note that this singularity resembles the one that occurs in the Kerr metric in that it has a ring structure, since equation (1.51) implies that

1.52 $$r^2 = 0$$

and

1.53 $$\cos\theta = \frac{\ell}{a} .$$

The/.....

However, the ring is "tilted" from its position in the case of a Kerr metric. The nature of this singularity will be discussed again when we have integrated the geodesic equations.

§3.2 Killing Horizons.

The class II a metric is a generalisation of the Schwarzschild metric and, as we have seen, has a singularity if $\ell^2 \leq a^2$. Therefore it is natural to enquire whether a generalisation of the Schwarzschild horizon at $r = 2m_0$ exists. Before we attempt to discuss the generalisation it is best to review briefly the main characteristics of the Schwarzschild horizon. The metric in the form (1.3) has a Killing vector

2.1 $$\alpha^i = \delta^i_1 ,$$

which becomes null on the null hypersurface

2.2 $$r = 2m_0 .$$

Also the Killing vector α^i is timelike as $r \to \infty$. The generalisation of these ideas has been discussed by Carter [4] and Vishveshwara [1]. Carter's work led him to introduce the idea of a Killing Horizon which he defined as "a null surface whose generating null vector can be normalised so as to coincide with a Killing field". Thus the Killing Horizon does not carry all the characteristics of the Schwarzschild horizon since the Killing vector field may not be timelike as $r \to \infty$. In an attempt to generalise this idea Vishveshwara defined

stationary/..

stationary "observers" and "sources" for a stationary metric as those observers and sources with 4-velocities which satisfy

2.3 $$u^i = e^{-\psi}\gamma^i ,$$

2.4 $$u^i u_i = -1 ,$$

where γ^i is a timelike Killing vector field. For such observers and sources he has shown that the red shift for a light ray is given by

2.5 $$\frac{\nu_o}{\nu_s} = \frac{(-\gamma_i\gamma^i)_s^{\frac{1}{2}}}{(-\gamma_i\gamma^i)_o^{\frac{1}{2}}} ,$$

where the suffixes refer to the source and observer and ν is the frequency of the light ray. In terms of equation (2.5) the horizon $r = 2m_0$ for the Schwarzschild metric is called an infinite red shift horizon by Vishveshwara. (Condition (2.4) is of course not satisfied on the surface given by equation (2.2)).

We are now in a position to examine the class II a metric. The Killing vectors are

2.6 $$\xi^i = \delta^i_1 ,$$

and

2.7 $$\zeta^i = \delta^i_4 .$$

The vector ξ^i is timelike as $r \to \infty$ and becomes null where

2.8 $$g_{11} = \Delta - a^2 \sin^2\theta = 0 ,$$

that/.....

that is, on the surface

2.9 $$r^2 - 2mr - \ell^2 + a^2\cos^2\theta = 0 ,$$

where we have substituted from the definition (1.19) and dropped the prime. The surface described by equation (2.9) is not null unless $a = 0$. Thus, as with the Kerr metric but not the NUT or Schwarzschild metrics for which $a = 0$, the Killing vector field ξ^i becomes null on a non-null hypersurface. From equation (2.5) and the fact that the class II a metric is stationary it follows that the surface (2.9) is the Visveshwara red shift horizon and does not coincide with the Killing horizon, unless $a = 0$, since it is not null. We are thus forced to look for the Killing horizons by different methods. Experience with the Kerr and NUT metrics suggests the surface

2.10 $$\Delta = r^2 - 2mr - \ell^2 + a^2 = 0 ,$$

which is null. The proof that it is null follows easily from equation (1.34). The surface defined by equation (2.10) corresponds to the values

2.11 $$r = r_+ = m + \sqrt{m^2 - a^2 + \ell^2} ,$$

and

2.12 $$r = r_- = m - \sqrt{m^2 - a^2 + \ell^2}$$

of r, and will only exist if $m^2 + \ell^2 \geq a^2$. The normal to the surface (2.10) is given by

2.13 $$k_j = \delta^2_j .$$

The/......

The contravariant components of k_j are:

2.14 $$k^j = g^{ij}\delta^2_j$$
$$= \rho^{-2}[(r^2 + a^2 + \ell^2)\delta^j_1 - \Delta\delta^j_2 + a\delta^j_4]$$

where we have used the components of the metric given in (1.31), (1.34) and (1.36). On the surface (2.10), r takes the values r_+ and r_-. Thus the null generator of the surface $r = r_+$ is

2.15 $$k^j_+ = \rho^{-2}\{(r^2_+ + a^2 + \ell^2)\xi^j + a\zeta^j\},$$

and for $r = r_-$ it is

2.16 $$k^j_- = \rho^{-2}\{(r^2_- + a^2 + \ell^2)\xi^j + a\zeta^j\}.$$

Apart from the factor ρ^{-2} these coincide with the Killing vectors

2.17 $$\lambda^j_+ = (r^2_+ + a^2 + \ell^2)\xi^j + a\zeta^j,$$

2.18 $$\lambda^j_- = (r^2_- + a^2 + \ell^2)\xi^j + a\zeta^j$$

respectively.

Thus the surfaces $r = r_+$ and $r = r_-$ are Killing horizons for the class II a metric. These two Killing horizons correspond to the ones that occur for the Kerr metric if $\ell = 0$ and reduce to the Schwarzschild horizon $r = 2m_0$ and $r = 0$ if $a = \ell = 0$. The Vishveshwara red shift horizon (2.9), for the larger value of r, lies outside the Killing horizons at all points except where $\cos\theta = 1$. The Killing

vectors/..

vectors defined in (2.17) and (2.18) will now be shown to be spacelike as $r \to \infty$. We put

2.19 $$R_+ = r_+^2 + a^2 + \ell^2 ,$$

2.20 $$R_- = r_-^2 + a^2 + \ell^2 ,$$

2.21 $$\lambda_+^2 = g_{ij} \lambda_+^i \lambda_+^j ,$$

2.22 $$\lambda_-^2 = g_{ij} \lambda_-^i \lambda_-^j ,$$

or in terms of an obvious notation,

2.23 $$\lambda_\pm^2 = g_{ij} \lambda_\pm^i \lambda_\pm^j .$$

After substituting from the metric (1.40 to 1.47) we find that

2.24 $$\lambda_\pm^2 = \rho^{-2}[\Delta\rho_\pm^4 - a^2 \sin^2\theta (R_\pm - R)^2] ,$$

where

2.25 $$\rho_\pm = r_\pm^2 + (\ell - a \cos\theta)^2 ,$$

2.26 $$R = r^2 + a^2 + \ell^2 ,$$

and Δ is defined in equation (1.19) with the primes dropped. As $r \to \infty$ the R^2 term in equation (2.24) dominates and the Killing vectors $\lambda_\pm^j$ are spacelike except when $\theta = 0$. Note that if $a = 0$ then, as expected, the Killing vectors are timelike as $r \to \infty$.

If u is set constant then the metric on a Killing horizon is

2.27 $$ds^2 = -\rho^2 [d\theta^2 + \rho^{-4}(r^2 + \ell^2 + a^2)\sin^2\theta \, d\phi^2] ,$$

where/....

where r is constant. The topology of this surface will not be investigated here. Such an investigation will be worthwhile but involves a discussion of the behaviour of the metric near $\theta = 0$ and $\theta = \pi$. We have deliberately not considered the problems associated with these neighbourhoods of θ for reasons that will be made clear when we consider the asymptotic properties of the metric.

Since most of the discussion of black holes has included the assumption that the space-time be at least asymptotically flat (Hawking [1], Carter [5]), we will examine the asymptotic properties of the class II a metric.

§3.3 Asymptotic Properties of the Kinnersley Class II a Metric.

The asymptotic properties of the class II a metric are most easily studied if we bring it into the Boyer-Lindquist [1] form. For this purpose we introduce the notation

3.1 $$H = a \sin^2\theta + 2\ell \cos\theta ,$$

3.2 $$\Delta = r^2 - 2mr - \ell^2 + a^2 ,$$

3.3 $$R = r^2 + \ell^2 + a^2 ,$$

3.4 $$\rho^2 = r^2 + (\ell - a\cos\theta)^2 = R - aH ,$$

in terms of which the Kinnersley form of the class II a metric becomes;

3.5/......

$$3.5 \qquad ds^2 = -\rho^2 d\theta^2 + 2\,du\,dr + \rho^{-2}\Delta[du^2 + H^2 d\phi^2 - 2H\,du\,d\phi - 2\rho^2 H\Delta^{-1} dr\,d\phi] - \rho^{-2}\sin^2\theta(R\,d\phi - a\,du)^2 ,$$

or, if we complete the square,

$$3.6 \qquad ds^2 = -\rho^2 d\theta^2 - \rho^2\Delta^{-1}dr^2 + \rho\Delta[du - R\Delta^{-1}dr - H\,d\phi + a\,H\Delta^{-1}\,dr]^2 - \rho^{-2}\sin^2\theta(R\,d\phi - a\,du)^2 .$$

We now make a singular coordinate transformation, which is a generalisation of one introduced by Boyer and Lindquist [1],

$$3.7 \qquad dt = du - R\Delta^{-1}dr ,$$

$$3.8 \qquad d\phi' = d\phi - a\Delta^{-1}dr .$$

The metric then takes the form

$$3.9 \qquad ds^2 = -\rho^2 d\theta^2 - \rho^2\Delta^{-1}dr^2 + \rho^{-2}\Delta[dt - H\,d\phi]^2 - \rho^{-2}\sin^2\theta[a\,dt - R\,d\phi]^2 ,$$

where we have dropped the prime on the ϕ. The coordinates we are now using resemble the coordinates in which the Schwarzschild metric takes the form (1.1). This is immediately obvious if we put $a = \ell = 0$, when the metric (3.9) reduces to the metric (1.1). The singular transformation (3.7), (3.8) has had the effect of introducing a singularity, resembling the Schwarzschild singularity at $r = 2m$, into the metric. Thus the new coordinates and form of the metric cannot be used in the neighbourhood of the Killing horizons but there are no difficulties for large values of r. We can write

3.10/.....

3.10 $$\rho^{-2}\Delta = \left(1 - \frac{2m}{r}\right) + O(r^{-2}) ,$$

3.11 $$\rho^{2}\Delta^{-1} = \left(1 + \frac{2m}{r}\right) + O(r^{-2}) ,$$

3.12 $$\rho^{-2} = O(r^{-2}) ,$$

3.13 $$\rho^{-2}R = 1 + O(r^{-2}) ,$$

and

3.14 $$R^2\rho^{-2} = (\rho^2 + aH)^2\rho^{-2} = \rho^2 + 2aH + O(r^{-2}) ,$$

for large values of r. In terms of the asymptotic forms defined in (3.10), (3.11), (3.12), (3.13) and (3.14) the metric (3.9) becomes:

3.15 $$\begin{aligned} ds^2 = & -\left(1 + \frac{2m}{r}\right)dr^2 - \rho^2\left[d\theta^2 + \left\{1 + \left(2aH - \left(1 - \frac{2m}{r}\right)\sin^{-2}H^2\right)\rho^{-2}\right\}\sin^2\theta\, d\phi^2\right] \\ & + \left(1 - \frac{2m}{r}\right)dt^2 - \left(4\ell\cos\theta - \frac{4mH}{r}\right)dt\, d\phi \\ & + O(r^{-2}) \end{aligned}$$

also for large values of r. Now Bonnor [1] has suggested

3.16 $$\begin{aligned} ds^2 = & -\left(1 + \frac{2m}{r}\right)dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \\ & + \left(1 - \frac{2m}{r}\right)dt^2 + \frac{2h}{r}\sin^2\theta\, d\phi\, dt \end{aligned}$$

as the asymptotic form of the metric for a rotating mass, with angular momentum h, situated at the origin. From the similarity between the metrics (3.15) and (3.16) one is tempted to conclude, as do Bonnor [1] and Sackfield [1], for the NUT metric, that the class II a metric represents some form of

rotating/...

rotating source. However the "rotation" is different from a simple rotating sphere, which led Bonnor and Sackfield to consider "a semi-infinite rotating spike" (Sackfield [1]) as the source for the NUT metric. The similarities suggest that a corresponding interpretation may hold for the class II a metric. This idea will not be followed up here. However, as in §3.1, we note that extrapolation from linearised theories applied to asymptotically flat space-times can be misleading when applied to the class II a metric.

If we consider only terms of zero order in r^{-1} then the metric (3.15) becomes

3.17
$$ds^2 = -dr^2 - \rho^2[d\theta^2 + \{1 - (4\ell \cot^2\theta - a^2\sin^2\theta)\rho^{-2}\} \sin^2\theta \; d\phi^2] + dt^2 + 4\ell \cos\theta \; dt\, d\phi + O(r^{-1}) \;,$$

which suggests that the class II a metric is not asymptotically flat since as $r \to \infty$, the metric does not tend to

3.18
$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta \; d\phi^2) \;.$$

We now wish to consider the asymptotic form (3.17) of the metric in more detail. A cursory look at the metric suggests that its signature will change for θ nearly zero. This is not an altogether unexpected possibility since the signature of the NUT metric has a singularity at either $\theta = 0$ or $\theta = \pi$ (Misner [1]). Examination of the Boyer-Lindquist form of the class II a metric (3.9) and the determinant g given in (1.49), show that coordinates exist in which the θ

singularity can be reduced to a form typical of polar coordinates. However, even in the case of the NUT metric more than one approach to the θ singularity has been proposed (Misner [1], Bonnor [1]), and thus, since we are primarily interested in the geodesics on the equatorial plane i.e. on the surface $\theta = \frac{\pi}{2}$, we will not discuss it further, except to note that the problem will have to be resolved before a worthwhile investigation of the asymptotic form of surface r and t constant can be completed. However it would appear that the reason why the class II a metric does not arise in the Wald [1] classification is that it is not asymptotically flat.

§3.4 Integration of the Geodesic Equations.

We can now consider one of the attractive features of the class II a metric. Examination of the Boyer-Lindquist form in equation (3.9) reveals that it satisfies Carter's separability conditions and therefore the geodesic equations can be brought to quadratures by the Hamilton-Jacobi procedure described in Chapter two. This is an extremely important result since it means that we have additional methods at our disposal for finding an interpretation of the parameter ℓ.

The equations of the geodesics are taken to be given by $\theta = \theta(s)$, $r = r(s)$, $\phi = \phi(s)$, $t = t(s)$ where s is a parameter chosen according to the prescription in Chapter two. The Lagrangian follows from equation (3.9), 4.1 $$L^2 = [-\rho^2\dot{\theta}^2 - \rho^2\Delta^{-1}\dot{r}^2 + \rho^{-2}\Delta(\dot{t} - H\dot{\phi})^2 - \rho^{-2}\sin^2\theta(a\dot{t} + R\dot{\phi})^2] ,$$

where H, Δ and R are defined in equations (3.1), (3.2) and (3.3), and where a dot indicates the derivative with respect to s. The function F in (2.2.10) is defined by

4.2 $$F = \tfrac{1}{2}L^2 .$$

From the Euler-Lagrange equation (2.2.11) we immediately get the first integrals

4.3 $$y_t = -E ,$$

4.4 $$y_\phi = \Phi ,$$

where E and Φ are constants. From equation (2.2.12) and equations (4.1) and (4.2):

4.5 $$y_t = \frac{\partial F}{\partial t} = \rho^{-2}(\Delta - a^2\sin^2\theta)\dot{t} + \rho^{-2}(aR\sin^2\theta - H\Delta)\dot{\phi} ,$$

4.6 $$y_r = -\rho^2\Delta^{-1}\dot{r} ,$$

4.7 $$y_\theta = -\rho^2\dot{\theta} ,$$

4.8 $$y_\phi = \rho^{-2}(H^2\Delta - R^2\sin^2\theta)\dot{\phi} + \rho^{-2}(aR\sin^2\theta - H\Delta)\dot{t} .$$

The contravariant components of the Boyer-Lindquist form of the metric are

4.9/......

$$4.9 \quad g^{ij} = \begin{bmatrix} -\rho^{-2} & 0 & 0 & 0 \\ 0 & -\rho^{-2} & 0 & 0 \\ 0 & 0 & -\rho^{-2}\left(\frac{H^2}{\sin^2\theta} - \frac{R^2}{\Delta}\right) & \rho^{-2}\left(\frac{aR}{\Delta} - \frac{H}{\sin^2\theta}\right) \\ 0 & 0 & \rho^{-2}\left(\frac{aR}{\Delta} - \frac{H}{\sin^2\theta}\right) & -\rho^{-2}\left(\frac{1}{\sin^2\theta} - \frac{a^2}{\Delta}\right) \end{bmatrix}$$

where $t = x^1$, $r = x^2$, $\theta = x^3$ and $\phi = x^4$. Thus the Hamilton-Jacobi equation (2.2.19) becomes

$$4.10 \quad -\rho^{-2}\left(\frac{\partial S}{\partial\theta}\right)^2 - \rho^{-2}\Delta\left(\frac{\partial S}{\partial r}\right)^2 - \rho^{-2}\left(\frac{H^2}{\sin^2\theta} - \frac{R^2}{\Delta}\right)\left(\frac{\partial S}{\partial t}\right)^2$$
$$+ 2\rho^{-2}\left(\frac{aR}{\Delta} - \frac{H}{\sin^2\theta}\right)\frac{\partial S}{\partial t}\frac{\partial S}{\partial\phi} - \rho^{-2}\left(\frac{1}{\sin^2\theta} - \frac{a^2}{\Delta}\right)\left(\frac{\partial S}{\partial\phi}\right)^2$$
$$= \lambda ,$$

where $\lambda = +1, -1$ or 0. If we substitute from equations (4.3) and (4.4), using equation (2.2.7), then equation (4.10) becomes

$$4.11 \quad \left\{\left(\frac{\partial S}{\partial\theta}\right)^2 + \frac{H^2}{\sin^2\theta}E^2 - 2\frac{H^2}{\sin^2\theta}E\Phi + \frac{1}{\sin^2\theta}\Phi^2 - \lambda aH\right\}$$
$$+ \left\{\Delta\left(\frac{\partial S}{\partial r}\right)^2 - \frac{R^2}{\Delta}E^2 + 2\frac{aR}{\Delta}E\Phi - \frac{a^2}{\Delta}\Phi^2 + \lambda R\right\} = 0 .$$

As we have already discussed in Chapter two this equation is separable and has a solution of the form (2.2.20), that is

$$4.12 \qquad S = -Et + S_1(r) + S_2(\theta) + \Phi\phi .$$

Thus we can obtain the functions $S_1(r)$ and $S_2(\theta)$ by integrating the pair of equations,

4.13/.....

4.13 $$\left(\frac{dS_2}{d\theta}\right)^2 + \frac{H^2}{\sin^2\theta} E^2 - 2\frac{H}{\sin^2\theta} E\Phi - \frac{1}{\sin^2\theta}\Phi^2 - \lambda aH = K ,$$

4.14 $$\Delta\left(\frac{dS_1}{dr}\right)^2 - \frac{R^2}{\Delta} E^2 + 2\frac{aR}{\Delta}\Phi E - \frac{a^2}{\Delta}\Phi^2 + \lambda R = -K .$$

If we put

4.15 $$X = \lambda aH + K - \frac{1}{\sin^2\theta}(HE - \Phi)^2 ,$$

4.16 $$\Delta^{-2}Q = \Delta^{-2}(RE - a\Phi)^2 - (\lambda R + K)\Delta^{-1} ,$$

then from equation (2.2.7)

4.17 $$y_\theta = \pm\sqrt{X} ,$$

4.18 $$y_r = \pm\sqrt{\frac{Q}{\Delta^2}} ,$$

where either sign may be chosen. Then from equations (4.6) and (4.7) we have

4.19 $$\rho^2\dot{\theta} = \pm\sqrt{X} ,$$

4.20 $$\rho^2\dot{r} = \pm\sqrt{Q} .$$

Also from equations (4.3), (4.4), (4.5) and (4.8) we have

4.21 $$\rho^2\dot{\phi} = \left(\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right)\Phi + \left(\frac{H}{\sin^2\theta} - \frac{aR}{\Delta}\right)E ,$$

4.22 $$\rho^2\dot{t} = \left(\frac{aR}{\Delta} - \frac{H}{\sin^2\theta}\right)\Phi + \left(\frac{H^2}{\sin^2\theta} - \frac{R^2}{\Delta}\right)E ,$$

4.23 $$\rho^2\dot{\theta} = \pm\sqrt{\lambda aH + K - \frac{1}{\sin^2\theta}(HE - \Phi)^2} ,$$

4.24/.....

4.24 $$\rho^2 \dot{r} = \pm\sqrt{(RE - a\Phi)^2 - (\lambda R + K)\Delta}\ ,$$

where we have written the equations for $\dot{\theta}$ and $\dot{r}$ out in full for future reference.

We will end this chapter with a discussion of the behaviour of the geodesics near the pseudo-singularity $\Delta = 0$ and the singularity $\rho^2 = 0$. We remark that it is already clear from the form of the metric given in equation (3.5) that the geodesics can be extended through the Killing horizons. The whole problem of geodesic completeness will not be discussed since, as before, we will avoid the difficulties associated with the neighbourhood of points where $\sin^2\theta$ is small. The reason for avoiding this region was made clear in §3.3.

The problem resolves itself into two possible cases;

(i) $\Delta = 0$ and $m^2 + \ell^2 - a^2 = 0$

(ii) $\Delta = 0$ and $m^2 + \ell^2 - a^2 > 0$ [1].

We will consider case (i) first and show that the results may be extended to case (ii) later. The technique is formally the same as that devised by Carter [2] for the Kerr metric. The alterations required are, to a large extent, covered by the choice of notation given in equations (3.1), (3.2), (3.3) and (3.4).

The/......

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(1) If $m^2 + \ell^2 - a^2 < 0$ there is no Killing horizon.

The condition (i) yields a Killing horizon at the double root $r = m$ and therefore the space-time is divided into two regions; I in which $r < m$, and II in which $r > m$. Consider the transformations:

4.25 $$dt = du - R\Delta^{-1}dr ,$$

4.26 $$d\phi = dy - a\Delta^{-1}dr ,$$

4.27 $$r = r ,$$

4.28 $$\theta = \theta ,$$

and

4.29 $$dt = -dw + R\Delta^{-1}dr ,$$

4.30 $$d\phi = -d\tilde{y} + a\Delta^{-1}dr ,$$

4.31 $$r = r ,$$

4.32 $$\theta = \theta .$$

On substituting from either of these sets into the Boyer-Lindquist form of the metric (3.9), formally the same Kinnersley form of the metric results. A transformation from one Kinnersley form of the metric to the other is given by

4.33 $$du + dw = 2R\Delta^{-1}dr ,$$

4.34 $$dy + d\tilde{y} = 2a\Delta^{-1}dr ,$$

4.35 $$r = r ,$$

4.36 $$\theta = \theta .$$

These/....

These two coordinate systems give rise to two distinct extensions, which can be performed alternately, giving rise to an extended manifold consisting of overlapping patches of (u, r, θ, ϕ) and (w, r, ϑ, ϕ) coordinates. This process has been more than adequately described in Carter [2] and Walker [1] (1). There still remains a problem with the geodesics, as we integrated the equations in Boyer-Lindquist coordinates not the Kinnersley type. It is a pecularity of the Kerr metric that, despite its seemingly complicated form, the Hamilton-Jacobi equation proves to be integrable even in what Carter [2] calls Kerr-Newman coordinates, which correspond to the coordinates in which Kinnersley [1] gives the class II a metric. This fact apparently led Carter to the investigation which resulted in the paper we discussed in Chapter two (Carter [1]).

If we employ the transformation (4.25), (4.26), (4.27) and (4.28) then the geodesic equations (4.21) and (4.22) become;

4.37 $$\rho^2 \dot{u} = \frac{H}{\sin^2\theta}\left(\frac{H}{\sin^2\theta} E - \Phi\right) - \frac{R}{\Delta}(P - Q'),$$

and

4.38 $$\rho^2 \dot{y} = \frac{1}{\sin^2\theta}\left(\frac{H}{\sin^2\theta} E - \Phi\right) - \frac{R}{\Delta}(P - Q'),$$

where

4.39 $$P = RE - a\Phi,$$

and

4.40/.....

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(1) Clear diagrammatic representations of the process are given in Carter [3] and in Walker [1].

4.40 $$Q' = \pm\sqrt{Q} = \pm\sqrt{(RE - a\Phi)^2 - (\lambda R + K)\Delta}$$
$$= \pm\sqrt{P^2 - (\lambda R + K)\Delta}\ ^{(1)}.$$

Similarly if we use transformations (4.29), (4.30), (4.31) and (4.32) then the equations (4.21), (4.22) become

4.41 $$\rho^2 \dot{w} = -\frac{H}{\sin^2\theta}\left(\frac{H}{\sin^2\theta} E - \Phi\right) + \frac{R}{\Delta}(P + Q') ,$$

4.42 $$\rho^2 \dot{\tilde{y}} = -\frac{1}{\sin^2\theta}\left(\frac{H}{\sin^2\theta} E - \Phi\right) + \frac{R}{\Delta}(P + Q') ,$$

where we have again used equation (4.39) and (4.40).

If $P \neq 0$ at $r = m$ where $\Delta = 0$, then in the neighbourhood of $r = m$, we have

4.43 $$\sqrt{Q} = P\left[1 - (\lambda R + K)\frac{\Delta}{P^2}\right]^{\frac{1}{2}} ,$$

from equation (4.40). For values of r near enough to $r = m$,

4.44 $$\sqrt{Q} = P\left[1 - \frac{\Delta}{2P^2}(\lambda R + K) + O\left(\frac{\Delta}{P^2}\right)^2\right] ,$$

so that $\frac{R}{\Delta}(P - \sqrt{Q})$ converges to $\frac{R}{2P^2}(\lambda R + K)$.
Comparison with equations (4.37), (4.38), (4.41) and (4.42) shows that, by an appropriate choice of coordinates, which will depend on the choice of sign in equation (4.40), we do not get a singularity in the geodesic equations at $\Delta = 0$, if $P \neq 0$.

If/......

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(1) Note that the case $\lambda = K = 0$ cannot arise because the right hand side of equation (4.23) goes complex, so that we do not get the case $|P| = |Q'|$ everywhere.

If $P = 0$ and $\Delta = 0$ then, since Δ has a double root at $r = m$, Q must have a double root at $r = m$. Thus we can write

$$Q = (r - m)^2 f(r) \ , \tag{4.45}$$

where we will assume that $f(r)$ is non zero and bounded in a neighbourhood of $r = m$. Then, in terms of differentials, equation (4.20) yields;

$$ds = \frac{\rho^2 dr}{\sqrt{Q}} \ , \tag{4.46}$$

where we have taken the case with positive sign for convenience. The case with the negative sign works similarly. If we substitute from equation (4.45) in equation (4.46) and integrate from $r = r_o > m$, with $s = 0$ at r_o,

$$S_{r_1} = \int_{r_1}^{r_o} \frac{\rho^2 dr}{(r - m)\sqrt{f(r)}} \ , \tag{4.47}$$

for $m < r_1 < r_o$. If we take the minimum value of $\dfrac{\rho^2}{\sqrt{f(r)}}$ to be L then

$$\begin{aligned} S_{r_1} &> L \int_{r_1}^{r_o} \frac{dr}{(r - m)} \\ &= \{ L(\log|r_o - m| - \log|r_1 - m|) \} \end{aligned} \tag{4.48}$$

and thus $S_{r_1} \to +\infty$ as $r_1 \to m$ and the geodesic from r_o is complete in the direction of the Killing horizon. If $\rho^2 = 0$ the method fails, as expected, since $\rho^2 = 0$ is a singularity. If $f(m) = 0$ then equation (4.46) becomes

$$Q = (r - m)^3 g(r) \tag{4.49}$$

or/.......

or

4.50 $$Q = k(r - m)^4$$

where k is a constant. In both cases the method yields complete geodesics in the direction of the Killing horizon.

We now consider case (ii). In place of the single Killing horizon in case (i) we have two values $r = r_+$ and $r = r_-$ of r, for which $\Delta = 0$. However, in a similar way to case (i), if $P \neq 0$, overlapping patches of Kinnersley co-ordinates can be used to enable us to continue the geodesics through each Killing horizon. The process is the same as that used by Carter [2] and will not be described here. Again illustrations of the process are given in the references cited in the footnote on page 48.

If $P = 0$ then the occurrence of two different roots to $\Delta = 0$ poses problems, since we are no longer assured that Q will have a double root, and therefore that s is unbounded. Thus we should consider the case where both u and w diverge to $+\infty$ or $-\infty$ together. The method that we will use to overcome this problem is again based on that used by Carter [2] for the Kerr metric. The notation introduced in (3.1), (3.2), (3.3) and (3.4) makes the similarities between the two problems very easy to see. It is convenient at this stage to label the regions $r > r_+$ as I, $r_- < r < r_+$ as II and $r < r_-$ as III. Essentially the solution requires a further extension of the class IIa metric. It incorporates both the previous extensions and is done by two coordinate transformations, one covering the regions I and II and the other covering the regions II and III. Since the transformations are similar only the first will be discussed here. The transformation that we use is a generalisation of the one used by Carter [2] for the Kerr metric. It is given by:

4.51 $$x = \alpha e^{\nu_+ (t + \int \frac{R}{\Delta} dr)} ,$$

4.52 $$y = \beta e^{\nu_+ (-t + \int \frac{R}{\Delta} dr)} ,$$

4.53 $$\phi_+ = \phi - aR_+^{-1} t ,$$

4.54 $$\theta = \theta ,$$

where

4.55 $$\nu_+ = \tfrac{1}{2} R_+^{-1} (r_+ - r_-) ,$$

4.56 $$R_+ = r_+^2 + \ell^2 + a^2 ,$$

and α and β are +1 or -1 such that

4.57 $$xy > 0 , \quad \text{if} \quad r > r_+ ,$$

4.58 $$xy < 0 , \quad \text{if} \quad r_- < r < r_+ .$$

Before discussing these inequalities in more detail we need an expression for xy in terms of r. We have

4.59 $$2 \int \frac{R}{\Delta} dr = 2r + \nu_+^{-1} \log |r - r_+| + \nu_-^{-1} \log |r - r_-| ,$$

where

4.60 $$\nu_- = \tfrac{1}{2} R_-^{-1} (r_- - r_+) ,$$

4.61 $$R_- = r_-^2 + \ell^2 + a^2 ,$$

and/......

and thus,

4.62 $$xy = (r - r_+)G^{-1}(r) ,$$

where

4.63 $$G(r) = e^{-2\nu_+ r}\left|r - r_-\right|^{R_-/R_+} .$$

The region I is mapped into the region $xy > 0$, region II is mapped into the region $xy < 0$ and the Killing horizon $r = r_+$ is mapped into a bifurcate Killing horizon, in the sense that $x = 0,\ y \neq 0;\ x \neq 0,\ y = 0$ and $x = 0,\ y = 0$ all correspond to the Killing horizon $r = r_+$. (Boyer [1] defines a bifurcate Killing horizon as the union of at least two intersecting Killing horizons.) Also r is an analytic function of x and y. The metric becomes

4.64 $$ds^2 = -\rho^2 d\theta^2 - \rho^2 \frac{\Delta}{4R^2\nu_+^2}\left(\frac{dx}{x} + \frac{dy}{y}\right)^2$$
$$+ \rho^{-2}\Delta\left[\frac{1}{2\nu_+}\left(1 - \frac{aH}{R_+}\right)\left(\frac{dx}{x} - \frac{dy}{y}\right) - H\ d\phi_+\right]^2$$
$$- \rho^{-2}\sin^2\theta\left[\frac{a}{2\nu_+}\left(1 - \frac{R}{R_+}\right)\left(\frac{dx}{x} - \frac{dy}{y}\right) - R\ d\phi_+\right]^2 .$$

If we now use the result

4.65 $$\frac{\rho^4}{R^2} - \frac{\rho_+^4}{R_+^2} = \frac{aH(r - r_+)}{RR_+}\left[\frac{\rho^2}{R^2} + \frac{\rho_+^2}{R_+^2}\right]$$

and the equation (4.62), the metric becomes

4.66 $$ds^2 = -\rho^2 d\theta^2 - \frac{G(r - r_-)}{4\rho^2\nu_+^2 R_+}\left[\frac{aH}{R}\left(\frac{\rho^2}{R} + \frac{\rho_+^2}{R_+}\right)\right.$$
$$\left. + \frac{a\sin^2\theta}{R_+(r - r_-)}\right](y^2dx^2 + x^2dy^2)$$

$$- \frac{G(r - r_-)}{2\rho^2\nu_+^2} /\ldots$$

$$- \frac{G(r - r_-)}{2\rho^2 \nu_+^2} \left[\frac{\rho^4}{R^2} + \frac{\rho_+^4}{R_+^2} - \frac{a^2 \sin^2\theta}{R_+^2} \left(\frac{r - r_+}{r - r_-} \right) \right] dx\, dy$$

$$- \frac{G(r - r_-)}{\rho^2 \nu_+} \left[\frac{H\rho_+^2}{R_+} + \frac{Ra \sin^2\theta}{R_+(r - r_-)} \right] (y\, dx - x\, dy)\, d\phi_+$$

$$+ \rho^{-2}(\Delta H^2 - R \sin^2\theta)\, d\phi_+^2 .$$

This metric is regular at $x = 0$ and $y = 0$ and thus at $r = r_+$. Thus even if $P = 0$, and u and w diverge together, we can still obtain a regular metric in the neighbourhood of the Killing horizon and therefore the geodesics can be continued through it.

Finally we note that if a geodesic is to reach the singularity $\rho^2 = 0$, if it exists, then the possible values of the parameters a, ℓ, Φ, K and E are restricted. In the first place $\rho^2 = 0$ implies that $\ell^2 \leqslant a^2$ since we must have

4.67 $$\cos\theta = \frac{\ell}{a}$$

and

4.68 $$r = 0$$

at the singularity. From equations (4.67) and (4.68) it follows that

4.69 $$\sin^2\theta = \frac{a^2 - \ell^2}{a^2} ,$$

4.70 $$H = \frac{a + \ell^2}{a} ,$$

4.71 $$R = a^2 + \ell^2 = A ,$$

4.72 $$\Delta = a^2 - \ell^2 = B .$$

We/.......

We do not wish to discuss the problems associated with $\sin^2\theta = 0$ and therefore we will not consider the case $a^2 = \ell^2$, which means that $B > 0$ and we may assume $a^2 > 0$. If we substitute from equations (4.69), (4.70), (4.71) and (4.72) into equation (4.24), we find that

4.73 $$0 = (AE - a\Phi)^2 - (\lambda A + K)B \ .$$

The other geodesic equations do not yield any other relations independent of equation (4.73). Since $B > 0$ we must have

4.74 $$\lambda A + K \geqslant 0 \ .$$

It follows that no geodesics with $K \leqslant -A$ can reach $\rho = 0$. Only timelike geodesics with $-A \leqslant K \leqslant 0$, timelike and null geodesics with $0 \leqslant K \leqslant A$ and geodesics of all types with $K \geqslant -A$ can reach $\rho = 0$. We observe that the condition (4.73) can be looked upon as a restriction on the range of K given a, ℓ, B and Φ, or if K is given then it restricts E and Φ.

For instance, if $K = 0$ then no spacelike geodesics can reach the singularity; null geodesics can only reach it if

4.75 $$(a^2 + \ell^2)E = a\Phi$$

and timelike geodesics must have

4.76 $$a^4 - \ell^4 = (AE - a\Phi)^2$$

which provide restrictions on E and Φ given a and ℓ.

CHAPTER IV

EQUATORIAL ORBITS FOR A PARTICLE IN A GRAVITATIONAL FIELD DESCRIBED BY A KINNERSLEY CLASS II a METRIC

§4.1 Introduction.

The motion of a particle in a gravitational field is described by the solution of the geodesic equations associated with the metric. In keeping with the terminology used in Newtonian theory we will call the solution the orbit of the particle. In particular, the equatorial orbits are defined as those for which $\theta = \frac{\pi}{2}$. Thus in the notation of Chapter three,

1.1 $$\cos\theta = 0 \ , \qquad \sin\theta = 1 \ ,$$

1.2 $$H = a \ ,$$

1.3 $$\rho^2 = r^2 + \ell^2 \ ,$$

1.4 $$R = r^2 + \ell^2 + a^2 \ ,$$

1.5 $$\Delta = r^2 - 2mr + a^2 - \ell^2 \ .$$

The geodesic equations (3.4.21), (3.4.22), (3.4.23) and (3.4.24) become;

1.6 $$\rho^2\dot{\phi} = \left(\frac{a^2}{\Delta} - 1\right)\Phi + a\left(1 - \frac{R}{\Delta}\right)E \ ,$$

1.7 $$\rho^2\dot{t} = a\left(\frac{R}{\Delta} - 1\right)\Phi + \left(a^2 - \frac{R^2}{\Delta}\right)E \ ,$$

1.8 $$\rho^2\dot{\theta} = \pm\sqrt{a^2 + K - (aE - \Phi)^2} \ ,$$

1.9/......

1.9 $$\rho^2\dot{r} = \pm\sqrt{(RE - a\Phi)^2 - (R + K)\Delta}\ .$$

For the orbits to be, and to remain, equatorial we require

1.10 $$K = (aE - \Phi)^2 - a^2\ .$$

Some aspects of the equatorial orbits for the Kerr and NUT metrics have been discussed in the literature. The main work on the Kerr metric is contained in papers by Boyer and Lindquist [1] and by Boyer and Price [1]. Some additional work has been done by Carter [2] and [3] and Israel [1], although Israel's contribution has been largely superceded by Carter's. In the case of the NUT metric the main concern has been the extension of the metric and hence indirectly with the geodesics. Some discussion is however given in Newman, Tamburino and Unti [1], Misner [1] and Misner and Taub [1]. The discussion of both these metrics is however far simpler and offers less possibilities than does the Kinnersley class II a metric, since the qualitative behaviour of the geodesics is governed by a quartic, while in the case of the Kerr metric it is governed by a cubic and in the NUT case the discussion of Misner and Taub employs a quadratic.

From equation (1.9)

1.11 $$\rho^4\dot{r}^2 = (E^2 - 1)r^4 + 2mr^3 + (2AE^2 - 2aE\Phi - K - a^2)r^2 + (2mA + 2Km)r + (A^2E^2 - 2aE\Phi A + a^2\Phi^2 - AB - KB)\ ,$$

where/....

where

1.12 $$A = a^2 + \ell^2 ,$$

1.13 $$B = a^2 - \ell^2 .$$

Now we put

1.14 $$\epsilon = E^2 - 1 ,$$

1.15 $$\alpha = 2m ,$$

1.16 $$\beta = E^2(a^2 + 2\ell^2) - \Phi^2 ,$$

1.17 $$\gamma = 2m[\ell^2 + (aE - \Phi)^2] ,$$

1.18 $$\mu = \ell^2[(E^2 + 1)(\ell^2 - a^2) + (2Ea - \Phi)^2] ,$$

in which we note that while ϵ, β and μ can take positive, negative and zero values, α and γ can only take non-negative values unless we are ready to forfeit the interpretation of m as mass and therefore its being positive.

The occurrence of the coefficient ρ^4 of $\dot{r}^2$, in equation (1.11), makes it difficult to introduce a generalised potential and we are therefore prevented from studying the orbits in the usual way (Robertson and Noonan [1] p.242 f). This difficulty can be overcome to some extent as we will now show.

After substitition from equations (1.14), (1.15), (1.16), (1.17) and (1.18) in equation (1.9), we get

1.19 $$\rho^4\dot{r}^2 = \epsilon r^4 + \alpha r^3 + \beta r^2 + \gamma r + \mu ,$$

while equation (1.7) and equation (1.19) together yield:

1.20 (a)/..

1.20 (a) $$\left(\frac{dr}{d\phi}\right)^2 = \frac{\epsilon r^4 + \alpha r^3 + \beta r^2 + \gamma r + \mu}{\left[\left(\frac{a^2}{\Delta} - 1\right)\Phi + a\left(1 - \frac{R}{\Delta}\right)E\right]^2} ,$$

or

1.20 (b) $$\left(\frac{dr}{d\phi}\right)^2 = \frac{\Delta^2(\epsilon r^4 + \alpha r^3 + \beta r^2 + \gamma r + \mu)}{[(a^2 - \Delta)\Phi + a(\Delta - R)E]^2} .$$

For most of the subsequent discussion it is convenient to use a dimensionless form of equation (1.20 (a) and (b)) and therefore we introduce the new variable $x = \frac{\alpha}{r}$, so that

1.21 (a) $$\left(\frac{dx}{d\phi}\right)^2 = \frac{(\epsilon + x + \beta\alpha^{-2}x^2 + \gamma\alpha^{-3}x^3 + \mu\alpha^{-4}x^4)}{\left[\left(\frac{a^2}{\Delta_1} - 1\right)\Phi + a\left(1 - \frac{R_1}{\Delta_1}\right)E\right]^2}$$

or

1.21 (b) $$\left(\frac{dx}{d\phi}\right)^2 = \frac{\Delta_1^2(\epsilon + x + \beta\alpha^{-2}x^2 + \gamma\alpha^{-3}x^3 + \mu\alpha^{-4}x^4)}{[(a^2 - \Delta_1)\Phi + a(\Delta_1 - R_1)E]^2}$$

where the suffix 1 is used to indicate that $\frac{\alpha}{x}$ has been substituted for r, in R and Δ. We can now discuss the behaviour of $\frac{dr}{d\phi}$ (or $\frac{dx}{d\phi}$).

From equation (1.20 (a)) it is apparent that $\frac{dr}{d\phi}$ remains finite for all finite values of r, unless,

1.22 $$\left(\frac{a^2}{\Delta} - 1\right)\Phi + a\left(1 - \frac{R}{\Delta}\right)E = 0 .$$

The behaviour on the Killing horizons $(\Delta = 0)$ will be disregarded here as it is discussed elsewhere. Therefore we take $\Delta \neq 0$ and then the equation (1.22) has roots

1.23 $$r = m\left(1 - \frac{aE}{\Phi}\right) \pm \sqrt{m^2\left(1 - \frac{aE}{\Phi}\right)^2 + \ell^2\left(1 - \frac{2aE}{\Phi}\right)} ,$$

if $\Phi \neq 0$. If $\Phi = 0$, $B \neq 0$ and $a \neq 0$ then

1.24 $$r = -\frac{\ell^2}{m}$$

is the only root of equation (1.22).

On the other hand if $\Phi = 0$, $B = 0$ and/or $a = 0$, then the left hand side of equation (1.22) vanishes identically. However, from equation (1.9), we find that

1.25 $$\rho^2 \dot{r} = \pm\sqrt{-(R + K)\Delta} \;.$$

It is easily shown that in this case $(R + K) > 0$ for all real values of r, ℓ and a. Thus no solution for r is possible unless $\Delta < 0$, i.e. the particle is between the Killing horizons if $\Phi = 0$, $B = 0$ and/or $a = 0$. If $a = 0$, then equation (1.22) gives $\Phi = 0$ and we have the situation just described.

It follows that outside the Killing horizons $\frac{dr}{d\phi}$ remains finite except for isolated values of r. In other words, freely falling particles in the equatorial plane outside the Killing horizon never move directly towards or away from $r = 0$, except perhaps at isolated points.

In equation (1.21 (a)) we note that there will be zeros of $\left(\frac{dx}{d\phi}\right)^2$ where $\Delta = 0$. However, as we have already discussed, these values correspond to the Killing horizons. They require special attention and are therefore discussed in §3.4. All that is left to discuss are the zeros of the quartic:

1.26 $$Q = \epsilon + x + \beta\alpha^{-2}x^2 + \gamma\alpha^{-3}x^3 + \mu\alpha^{-4}x^4 \;.$$

There/....

There are five unspecified parameters in this quartic and therefore a wide range of patterns of behaviour are possible. The coefficients of the quartic cannot all be given physical interpretations although the α, E, Φ and a can be interpreted. In this section we regard the class IIa metric as a generalisation of the Schwarzschild metric and thus $\alpha = 2m$ is the Schwarzschild mass term and ma is the angular momentum of the source (Carter [2]). Equations (3.4.3) and (3.4.4) suggest that we interpret E and Φ, respectively as the total energy and angular momentum per unit rest energy of the orbiting particle in the field (Møller [1] pp.288-294).

In the following discussion of the quartic we will not discuss all the possible cases that can arise. Instead our attention will be focussed on those cases that appear more interesting.

§4.2 A Newtonian Type Solution.

In this section the relevant parameters will be assumed to have values of the same order as those for a body in an approximately circular orbit. In fact the values that we will use will be typical of the planet Mercury. The units are such that $c = 1$ and thus v, the speed of the orbiting particle, is very small. In the case of Mercury v^2 does not exceed 2.56×10^{-8} (McVittie [1] p.88). For simplicity and since actual numerical values are not important, we will use a symbol $\sim$ to mean "of the same order as" and we will denote

terms/....

terms of order v^0, v^1, v^2,..... by O_0, O_1, O_2,..... respectively. On this scale

2.1 $$x \sim O_2 \; ,$$

2.2 $$\frac{h^2 x^2}{\alpha^2} \sim O_2 \; ,$$

where we have replaced Φ by h to conform with normal usage. More precise values of x and $\frac{h^2x^2}{\alpha^2}$ for Mercury and other planets are given in McVittie [1].

The interpretation and estimation of ϵ is a little more difficult. We have followed Møller [1] and called E the energy. However, if we examine equation (1.11) with $a = \ell = 0$ we find that it can be rewritten in the form,

2.3 $$\epsilon = \dot{r}^2 - x + h^2\alpha^{-2}x^2 - h^2\alpha^{-2}x^3 \; ,$$

which led Robertson and Noonan [1] to relate ϵ to the total "Kinetic" and "Potential" energy per unit rest energy of the particle. Irrespective of this interpretation, since all the terms on the right hand side of equation (2.3) are of order O_2 or higher, we have that the magnitude of ϵ,

2.4 $$|\epsilon| \sim O_2 \; .$$

In the following discussion we will use the above mentioned values and make the assumption that a, α and ℓ are all of the same order of magnitude on our scale.

From equations (1.16) and (1.18) it is clear that

2.5 $$\beta < 0$$

and

2.6 $$\mu > 0$$

in (1.26). If we take $\epsilon < 0$, then, by Descertes rule of signs, there are at most three positive roots to the quartic Q. A comparison with the cubic that arises in the Schwarzschild case, shows that at least two of the positive roots will occur.

The cubic that arises when we put $\ell = a = 0$ is

2.7 $$Q_s = \epsilon + x - h^2\alpha^{-2}x^2 + \alpha^{-2}h^2x^3$$

which, as expected, is the cubic that governs the orbits in the Schwarzschild case. Møller [1] showed that Q_s has two roots of O_2. Now for values of $x \sim O_2$ the difference

2.8 $$Q - Q_s \sim O_4 ,$$

so that one would expect Q also to have two roots S_1 and S_2 of order O_2.

Since $\epsilon < 0$ and $\mu > 0$ there must be an odd number of positive roots to the quartic Q and thus, since an odd number of real roots cannot occur, we deduce that if all the roots are real there must be three positive roots and one negative root.

Figure 1/......

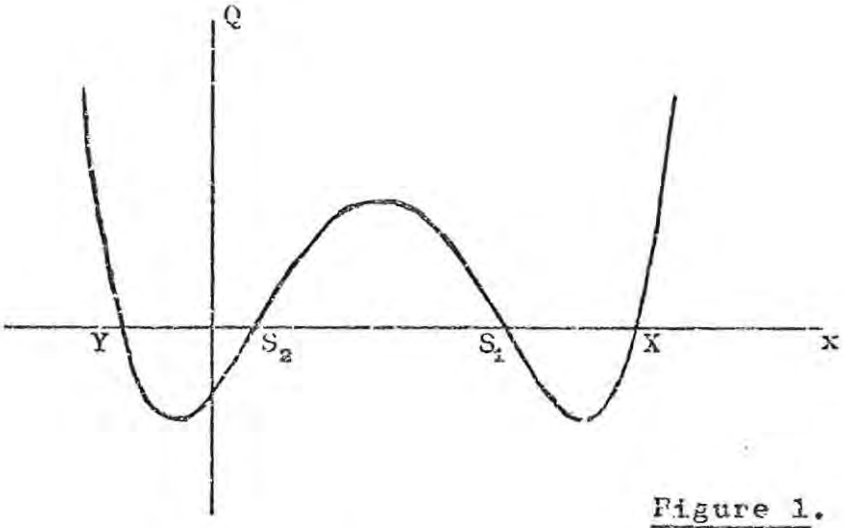

Figure 1.

Since $Q - Q_s \sim O_4$ we might expect X to be large compared with S_1 and S_2. We will not show this here as it is not required, but later, in the case where $a = 0$, we will show that the root corresponding to X is very large compared to S_1 and S_2. The root Y has no analogue in the Schwarzschild case.

According to the graph we obtain real values of $\left(\frac{dr}{d\phi}\right)$ for $S_2 \leqslant x \leqslant S_1$ and for all $x \geqslant X$. Hence, if $\mu \geqslant 0$ and and $\epsilon < 0$, then the particle cannot escape to infinity. In fact it will be captured in an "elliptic" orbit (i.e. confined to $S_1 \leqslant x \leqslant S_2$) or it will fall in to the Killing horizon (1).

The "elliptic" orbits between S_1 and S_2 will be referred to as "Schwarzschild" orbits, since they correspond

to/.......

(1) For the Kerr metric $\mu = 0$, so the condition simplifies to $\epsilon < 0$ (Carter [2]).

to the ones discussed by Møller [1] and others when discussing the perihelion shift. Physically the parameter values correspond, as we have said, to an orbit such as that of Mercury in a Newtonian framework.

A natural question to ask about X is whether it can lie inside the outer Killing horizon. Clearly with the freedom available in the choice of parameters, it can be made to do so. However for the parameter values we are using it will generally lie outside the Killing horizons. This we can illustrate as follows.

Corresponding to the root X of the quartic Q there will be a root ν of the quartic

2.9 $$Q_1 = \epsilon r^4 + \alpha r^3 + \beta r^2 + \gamma r + \mu .$$

The graph of Q_1 will be

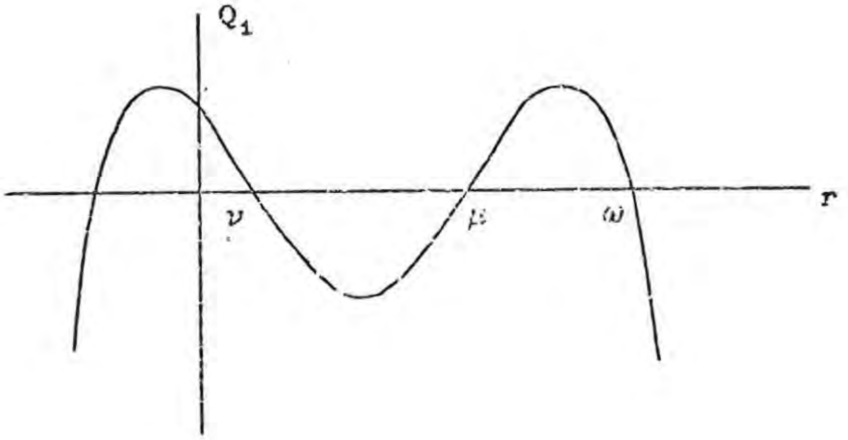

Figure 2.

where μ and ω correspond to the roots S_1 and S_2. If ν is to lie inside the outer Killing horizon then we must have

2.10 $$Q_1 < 0$$

at the outer Killing horizon given by

2.11 $$r = r_+ = m + \sqrt{m^2 - a^2 + \ell^2}$$

(assuming $m^2 - a^2 + \ell^2 \geq 0$). Now if we take $m = \ell = a$ then $r_+ = 2m$ and

2.12 $$Q_1 = \{(6mE - \Phi)^2 + 4m^2\}m^2 > 0 ,$$

for all values of E and Φ, so that ν cannot lie inside the outer Killing horizon. The general case is more difficult to analyse but the behaviour appears to be the same for the values of ℓ, a and α that we are using.

Another interesting feature would be the existence of stable circular orbits, by which we mean stable orbits for which x is constant. Criteria for such orbits are easy to establish.

From equation (1.21 (b))

2.13 $$\left(\frac{dx}{d\phi}\right)^2 = \frac{\Delta_1^2 Q}{D^2} \overset{\text{def}}{=} f(x) ,$$

where

2.14 $$D = (a^2 - \Delta_1)\Phi + a(\Delta_1 - R)E .$$

For a stable circular orbit at b we require that:

2.15 $$f(b) = 0 ,$$

2.16 $$\left.\frac{df}{dx}\right]_b = 0 ,$$

and/......

and

2.17 $$\left.\frac{d^2f}{dx^2}\right]_b < 0 .$$

It follows that $f(b) = 0$ is a local maximum value of $f(x)$. This situation clearly gives rise to a stable orbit since, while $\left.\frac{dx}{d\phi}\right]_b$ is real, $\frac{dx}{d\phi}$ is imaginary for all values of x in a neighbourhood of b except at b. We are not concerned with orbits within or on the Killing horizon and therefore we will look for circular orbits outside the Killing horizons. Thus we assume

2.18 $$\Delta_1 > 0$$

and

2.19 $$0 < b < x_+ ,$$

where x_+ is the smallest value of x for which $\Delta_1 = 0$. The criteria (2.15), (2.16) and (2.17) yield the following results:

2.20 $$f(b) = 0 \Rightarrow Q(b) = 0$$

2.21 $$0 = \left.\frac{df(x)}{dx}\right]_b = \frac{\Delta_2^2}{D^2}\left.\frac{dQ}{dx}\right]_b \Rightarrow \left.\frac{dQ}{dx}\right]_b = 0$$

2.22 $$0 > \left.\frac{d^2f}{dx^2}\right]_b = \frac{\Delta_2^2}{D^2}\left.\frac{d^2Q}{dx^2}\right]_b \Rightarrow \left.\frac{d^2Q}{dx^2}\right]_b < 0 .$$

A detailed analysis of the equations and inequalities (2.20), (2.21) and (2.22) in the general case is not very informative since the freedom in the choice of parameters makes it impossible to draw definite conclusions. Given a, ℓ and m the

conditions/..

conditions (2.20), (2.21) and (2.22) would normally yield restrictions on E and Φ for a stable circular orbit to exist at b. To show that some stable circular orbits do exist we consider the situation described in Figure 1. The three criteria would be satisfied if

2.23 $$S_1 = S_2 = b \ ,$$

and thus it seems worthwhile to consider the possibility of a circular orbit with $x = b \sim O_2$ i.e. r quite large. For simplicity we add the condition that $a = \alpha = \ell$. The conditions (2.20), (2.21) and (2.22) become:

2.24 $$\epsilon + x + \beta\alpha^{-2}x^2 + \gamma\alpha^{-3}x^3 + \mu\alpha^{-4}x^4 = 0 \ ,$$

2.25 $$1 + 2\beta\alpha^{-2}x + 3\gamma\alpha^{-3}x^2 + 4\mu\alpha^{-4}x^3 = 0 \ ,$$

2.26 $$2\beta\alpha^{-2} + 6\gamma\alpha^{-3}x + 12\mu\alpha^{-4}x^2 < 0 \ .$$

We find that

2.27 $$E = 1 - \tfrac{1}{2}b + O_3$$

and

2.28 $$2b\,\frac{\Phi^2}{\alpha^3} = 1 + 3b + O_4$$

satisfy the conditions we imposed. Thus a stable circular orbit exists for particles with parameter values similar to those for a "Schwarzschild" orbit i.e. for the orbit of Mercury.

§4.3/.....

§4.3 The Perihelion Shift for Equatorial Orbits of a Particle in a Gravitational Field Described by a Kinnersley Class II a Metric with $a = 0$.

If a is put equal to zero in the Kinnersley class II a metric then it becomes the NUT metric (Kinnersley [1]), which has been the subject of a number of papers (Newman, Tamburino and Unti [1], Misner [1] and Misner and Taub [1]). The perihelion shift is discussed in Newman, Tamburino and Unti [1] but, as the geodesic equations were not integrated, the explicit form of the shift could not be calculated. As is well known (Bonnor [1], Misner [1]) there are difficulties with the signature of the NUT metric near $\cos\theta = 1$. However we will confine our attention to the equatorial orbits, in the sense that they are equatorial for the Kinnersley class II a metric when $a \neq 0$, for which the difficulty does not arise. One reason for continuing the work on the NUT metric is that the parameter ℓ has no interpretation as yet although it has been labelled as a "magnetic monopole of mass" by Demianski and Newman [1]. Certain properties of ℓ become apparent in the present analysis although none seem encouraging from the physical viewpoint and one is tempted to conjecture that for ℓ to be meaningful it should be ℓ coupled with other parameters.

When $a = 0$ equation (1.10) becomes

3.1 $$K = h^2 ,$$

we will continue to use h instead of Φ to conform with the

usual/....

usual practice in orbit theory. The equation (1.11) becomes

3.2 $$(r^2 + \ell^2)^2 \dot{r}^2 = \epsilon r^4 + 2mr^3 + (2\ell^2 E^2 - h^2)r^2 + 2m(\ell^2 + h^2)r + (\ell^2 E^2 + \ell^2 + h^2)\ell^2$$

where

3.3 $$\epsilon = E^2 - 1 .$$

Equation (1.6) becomes

3.4 $$(r^2 + \ell^2)\dot{\phi} = -h .$$

Again the presence of ℓ^2 on the left hand side of equation (3.2) prevents us from discussing the generalised potential in the usual way. Some of the difficulties can be overcome as before. We introduce a new variable $u = \frac{1}{r}$ and put $\alpha = 2m$ and find from equations (3.3) and (3.4) that

3.5 $$\left(\frac{du}{d\phi}\right)^2 = \frac{1}{h^2}\left[\epsilon + \alpha u + (2\ell^2 E^2 - h^2)u^2 + \alpha(\ell^2 + h^2)u^3 + (E^2\ell^2 + \ell^2 + h^2)\ell^2 u^4\right] .$$

If $\ell = 0$ this reduces to the usual equation in the Schwarzschild case (Robertson and Noonan [1] p.243 where $\alpha = 1$). Now we can make certain

Remarks:

(i) The "potential" due to ℓ is shorter "ranged" than that due to α, in the sense that for large values of r, the effect of ℓ arises from terms in $\frac{1}{r^2}$ while that of α arises from terms in $\frac{1}{r}$.

(ii) In view of the suggestion by Demianski and Newman

that/.....

that ℓ is a magnetic monopole of mass it is surprising that it does not enter the equations in a more symmetric way compared with α. On the contrary, the effect of ℓ appears to be closely allied to the angular momentum of the particle. The net effect seems to be to alter the apparent angular momentum of the particle.

Neither of these remarks enable one to distinguish between the ideas that the NUT metric is due to a magnetic monopole of mass, or a cosmological effect as suggested by Misner [1]. If anything the first remark tends to support the monopole idea in that the effect seems to be related to a source at $r = 0$.

Since we would expect E, ℓ, h and m to be real and m to be positive, more than one positive root of the quartic can only occur if

3.6 $$2\ell^2 E^2 < h^2 .$$

Unless

3.7 $$\varepsilon < 0$$

we get a maximum of two positive roots for x. Since $\mu > 0$ a bound orbit cannot exist between these two roots because $\left(\frac{dx}{d\phi}\right)^2$ is negative. Thus given that $\mu > 0$ (equation (2.6)) condition (3.7) must be satisfied for an elliptic orbit to exist. In view of (3.7) if all the roots are real then at least one must be negative because the quartic is positive for large negative values of u, and negative for $u = 0$. Thus,

in/.......

in this case, we can get bound orbits between the smaller of the two positive roots.

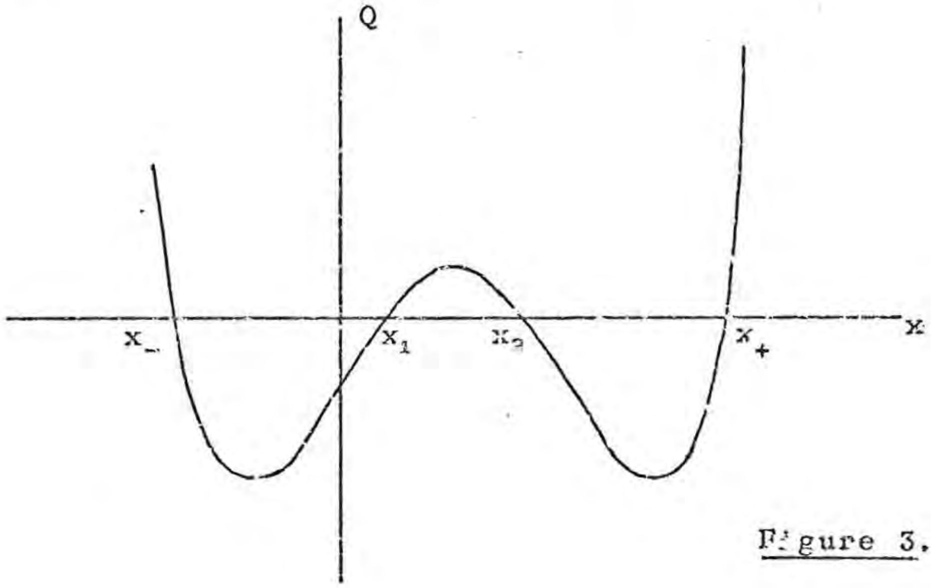

Figure 3.

Motion of the Pericenter.

We have already remarked on the fact that ℓ seems to alter the apparent angular momentum of a particle. Thus it seems resonable to expect that the introduction of ℓ into a Schwarzschild metric should alter the perihelion shift. Of course if ℓ does yield an, in principle, observable shift, then it may be possible to suggest a method of detecting and measuring the magnitude of the magnetic monopole of mass of a given body. Also such a shift may lead to a physical interpretation of ℓ.

As already discussed, if we make appropriate assumptions about the parameters involved, we can expect bound orbits to exist with aphelion and perihelion near those for the Schwarzschild solution. For the purposes of the present analysis it is useful to reduce equation (3.5) to dimensionless form by

putting/..

putting $x = \alpha u$. Then

3.8 $$\left(\frac{dx}{d\phi}\right)^2 = \frac{\epsilon\alpha^2}{h^2} + \frac{\alpha^2 x}{h^2} + \left(\frac{2\ell^2 E^2 - h^2}{h^2}\right)x^2 + \left(\frac{\ell^2 + h^2}{h^2}\right)x^3 + \frac{H^2\ell^2}{\alpha^2 h^2}x^4$$

where

3.9 $$H^2 = \ell^2 E^2 + \ell^2 + h^2 .$$

Also we assume again that for the "Schwarzschild" type of orbit (see page 64)

3.10 $$x \sim O_2$$

3.11 $$\frac{\alpha}{h} \sim O_1$$

3.12 $$\epsilon \sim O_2 .$$

This aspect is discussed in more detail in the previous section.

Now we assume that

3.13 $$\ell = \alpha .$$

Thus

3.14 $$\frac{\epsilon\alpha^2}{h^2} \sim O_4 , \quad \frac{\alpha^2}{h^2} x \sim O_4 , \quad x^2 \sim O_4 ,$$

3.15 $$x^3 \sim O_6 ,$$

3.16 $$\frac{\ell^2 E^2}{h^2} x^2 \sim O_8 , \quad \frac{\ell^2}{h^2} x^3 \sim O_8 , \quad \frac{H^2}{h^2} x^4 \sim O_8 ,$$

in the domain of the elliptic orbit.

The/......

The following analysis is based on that of Møller [1] although the present problem is more complicated. From (3.14) and (3.16) it is apparent that condition (3.6) holds. We will assume that ϵ is negative and that all the roots are real. The two smaller positive roots, the aphelion and perihelion for the orbit, will be denoted by x_1 and x_2. From equation (3.8) it follows that the perihelion advance is described by

3.17 $$\phi = 2\int_{x_1}^{x_2} \frac{dx}{\sqrt{ax^4 + bx^3 + cx^2 + dx + e}}$$

where

3.18 $$a = \frac{H^2}{h^2} = \frac{2\ell^2 + \ell^2\epsilon + h^2}{h^2},$$

where we have used equation (1.14).

3.19 $$b = \frac{\ell^2 + h^2}{h^2},$$

3.20 $$c = \frac{2\ell^2 E^2 - h^2}{h^2},$$

3.21 $$d = \frac{\alpha^2}{h^2},$$

3.22 $$e = \frac{\epsilon\alpha^2}{h^2}.$$

We can write equation (3.17) as

3.23 $$\phi = 2\int_{x_1}^{x_2} \frac{dx}{\sqrt{(x - x_1)(x - x_2)(ax^2 + Bx + D)}}$$

and proceed from here. As we will show explicitly later, D and B are both of order O_0 and thus since ax^2 is of

order/....

order O_4 we can write

$$\phi = \frac{2}{\sqrt{-D}} \int_{x_1}^{x_2} \frac{dx}{\sqrt{(x - x_1)(x_2 - x)}} \left(1 - \frac{B}{2D} x + O_4\right) , \tag{3.24}$$

where $\frac{B}{D} x$ is of order O_2. We now use the following results for elliptic integrals,

$$\int_{x_1}^{x_2} \frac{dx}{\sqrt{(x - x_1)(x_2 - x)}} = \pi , \tag{3.25}$$

$$\int_{x_1}^{x_2} \frac{x\, dx}{\sqrt{(x - x_1)(x_2 - x)}} = \frac{\pi}{2} (x_1 + x_2) , \tag{3.26}$$

and obtain

$$\phi = \frac{2\pi}{\sqrt{-D}} \left(1 - \frac{B}{4D} (x_1 + x_2) + O_4\right) . \tag{3.27}$$

The problem is thus reduced to the estimation of B, D and $(x_1 + x_2)$.

From the quartic in equation (3.23) it is clear that

$$D = a x_- x_+ , \tag{3.28}$$

where x_- is the negative root and x_+ the largest positive root of the quartic. Also we have

$$B = -a(x_+ + x_-) . \tag{3.29}$$

The sum of the roots of the quartic is given by

$$x_1 + x_2 + x_+ + x_- = -\frac{b}{a} , \tag{3.30}$$

where we have reverted to the notation of equation (3.17) in

which/....

which the quartic has the form

3.31 $$Q = ax^4 + bx^3 + cx^2 + dx + e \ .$$

Thus we find that

3.32 $$x_1 + x_2 = -\frac{b}{a} - x_+ - x_- \ .$$

From equation (3.18)

3.33 $$a = \frac{H^2}{h^2} = 1 + \frac{2\alpha^2}{h^2} + O_4 \ ,$$

3.34 $$\frac{b}{a} = 1 - \frac{\alpha^2}{h^2} + O_4 \ ,$$

where we have used equation(3.19) and definition (1.14) to remove the B^2 terms.

We now substitute in equation (3.27) to obtain

3.35 $$\phi = \frac{2\pi}{\sqrt{-x_+ x_-}} \left(1 - \frac{\alpha^2}{h^2}\right)\left\{1 - \frac{x_+ + x_-}{x_+ x_-}\left(1 - \frac{\alpha^2}{h^2} + x_+ + x_-\right)\right\} + O_4 \ ,$$

and the problem has been reduced to the evaluation of x_+ and x_-.

It is easiest to obtain estimates of the roots x_+ and x_- from the quartic in the form given in equation (3.31). From the definitions of a, b, c, d and e given in equations (3.18) to (3.22) it follows that

3.36 $$Q = x^4 + x^3 - x^2 + \frac{\alpha^2}{h^2}\left(2x^4 + x^3 + 2x^2 + x\right) + O_4 \ ,$$

where/....

where the terms of order O_4 are those with coefficients of order O_4. (The E^2 terms were removed using the definition (1.14) of ϵ). The quartic

3.37 $$Q^* = x^4 + x^3 - x^2$$

has four roots, 0, 0, $\frac{1}{2}(-1 - \sqrt{5})$ and $\frac{1}{2}(-1 + \sqrt{5})$. These correspond to the zero order parts of the solution to the quartic Q. This is clear because a Newton iteration procedure based on any of these roots can only give a correction of order O_2, since the terms of order O_0 vanish, that is if the derivative of Q does become of order less than O_2 near any of the roots. However, as we have assumed that all four roots are distinct, it is impossible for $\frac{dQ}{dx}$ to vanish at a root. The convergence will be shown explicitly for the roots that interest us.

The Newton iteration procedure (Turnbull [1] p.92) based on the non zero roots of Q^* yields

3.38 $$x_+ = \tilde{x}_+ + p\,\frac{\alpha^2}{h^2} + O_4 \, ,$$

3.39 $$x_- = \tilde{x}_- + q\,\frac{\alpha^2}{h^2} + O_4 \, ,$$

where

3.40 $$\tilde{x}_+ = -\tfrac{1}{2}(1 - \sqrt{5}) \, ,$$

3.41 $$\tilde{x}_- = -\tfrac{1}{2}(1 + \sqrt{5})$$

3.42 $$p = 3 + \frac{4}{5}\sqrt{5} \, ,$$

3.43 $$q = 5 + 2\sqrt{5} \, .$$

On/.......

On substituting from equations (3.38) and (3.39) in equation (3.35) we obtain

3.44 $$\phi = 2\pi\left\{1 + \frac{\alpha^2}{4h^2}\left[p(2\tilde{x}_- - 1) + q(2\tilde{x}_+ - 1) - 3\right] + O_4\right\}$$

$$= 2\pi\left\{1 + N\frac{m^2}{h^2} + O_4\right\},$$

where

3.45 $$N = p(2\tilde{x}_- - 1) + q(2\tilde{x}_+ - 1) - 3$$

$$= -18.0 \ .$$

Hence one obtains a perihelion shift $\Delta\phi \approx -\frac{36\pi m^2}{h^2}$ which is retrograde and much larger than the shift predicted using the Schwarzschild metric, viz $\Delta\phi = 6\pi\frac{m^2}{h^2}$.

§4.4 Some Special Cases of Interest.

We will now examine some of the possibilities that arise if we allow greater freedom in the choice of the parameters in equation (1.26). The results will be represented diagrammatically; the intention is merely to illustrate the roots and domains in which the quartic Q takes on positive values. Most of the alternatives that arise are not new and consequently are of limited interest, however, a new phenomenon does appear which could be detectable even for small values of l. The following set of cases is by no means exhaustive but covers the more interesting possibilities. In all cases the roots will be assumed to lie outside the values

of/.......

of x on the Killing horizon.

Case 1.

We assume $\epsilon > 0$, $\beta < 0$ and $\mu > 0$, then according to Descartes rule of signs there can be, at most, two positive roots. Assuming that all the roots are real we would have the situation represented in Figure 4.

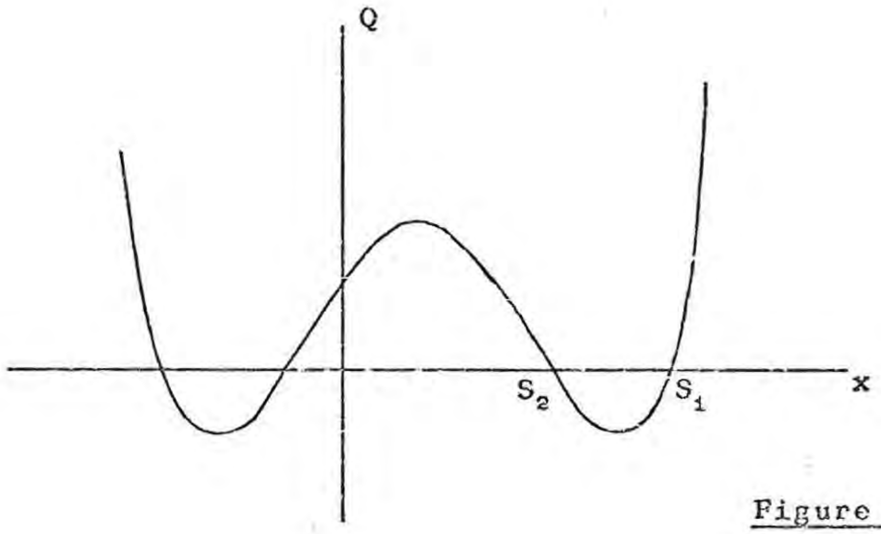

Figure 4

A particle with the above parameter values coming in from infinity is bounded away from the Killing horizon by S_2, at which it will either be captured in an unstable circular orbit if S_1 coincides with S_2 or be reflected back to infinity if $S_1 \neq S_2$. If the particle starts at a value $x > S_1$, and outside the Killing horizons then it will eventually spiral into the Killing horizon.

Case 2.

Assume that $\epsilon > 0$, $\beta < 0$ and $\mu < 0$, which implies that

4.1 $$(E^2 + 1)(a^2 - \ell^2) > (2Ea - \Phi)^2 ,$$

from/.....

from which it follows that we must have $a^2 \geqslant \ell^2$. This means that the condition $\mu < 0$ can only be satisfied for a metric which has a singularity since, as we already know, a singularity in the metric must occur if $a^2 > \ell^2$, at $\rho^2 = 0$. In this case Descartes rule of signs indicates that there can be at most three positive roots. Diagramatically we have:

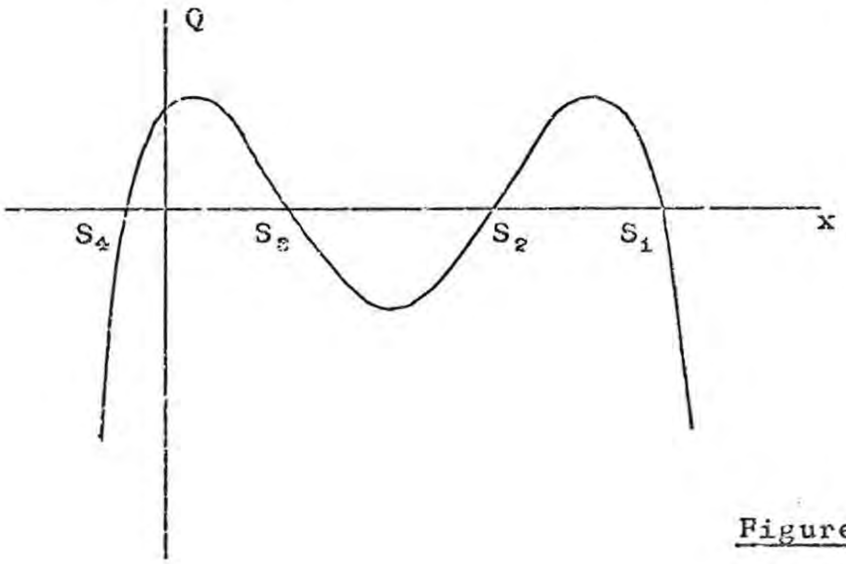

Figure 5.

This situation is entirely new. The particle coming in from infinity is bounded away from the region in which "elliptic" type orbits are possible: it may be trapped at S_3 or return to infinity. A particle with the same energy and momentum between S_2 and S_1 will be trapped in elliptic type orbits and cannot get out to infinity. Thus we have the possibility of either an elliptic or hyperbolic type orbit depending on where the particle is situated. A more interesting possibility arises if we allow $S_2 = S_3 = S$, in which case the particle coming in from infinity on reaching S will enter an unstable circular orbit, since at S

4.2 $$Q = 0 ,$$

$$\frac{dQ}{dx} = 0 , \tag{4.3}$$

and

$$\frac{d^2Q}{dx^2} > 0 , \tag{4.4}$$

which is the condition for an unstable circular orbit. Given the parameters ε, ℓ and α then equations (4.2), (4.3) and (4.4) provide conditions which must be satisfied by Ξ and Φ. Quite clearly, with the freedom available here in the choice of parameters, such orbits are possible. The particle that is captured in the orbit at S may leave this orbit as a result of a perturbation and enter the region between S and S_1 or it may return to infinity. Therefore, in effect, S represents a barrier that may or may not be penetrated.

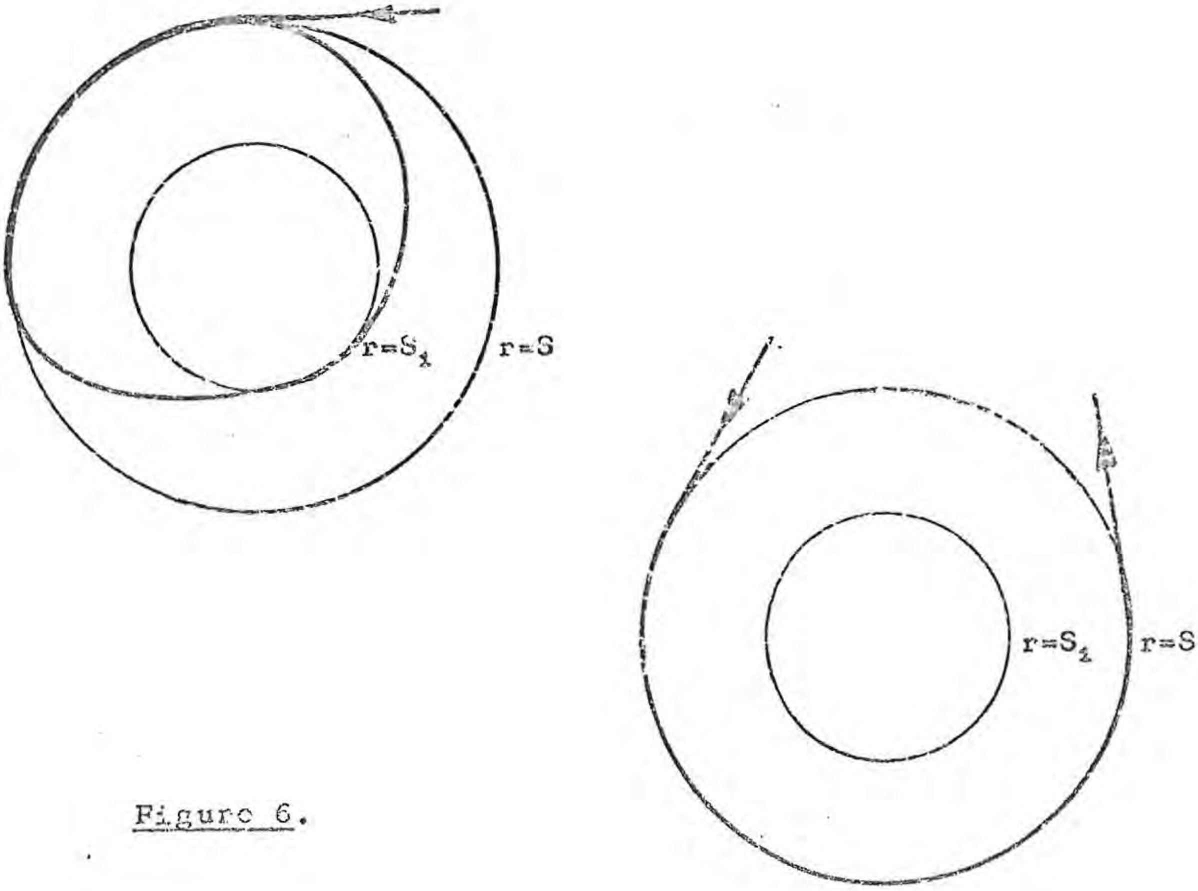

Figure 6.

A similar result holds for the particle if it is in an elliptic orbit between S and S_1 in that on reaching S it enters the same unstable circular orbit.

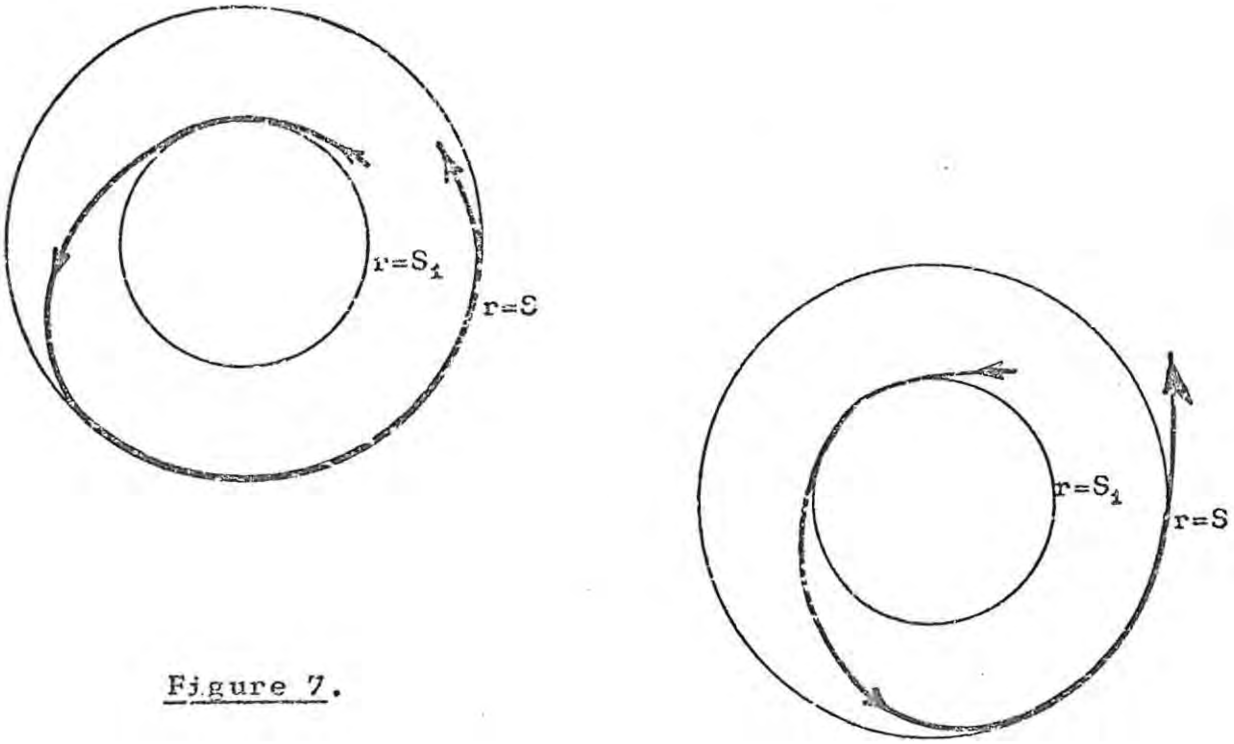

Figure 7.

The next case that we will discuss gives an even more interesting example of the phenomenon that we are considering.

Case 3.

Here we assume that $\varepsilon < 0$, $\beta < 0$ and $\mu < 0$. Again a^2 must be greater than ℓ^2 and a singularity must occur. This in itself makes these cases interesting because one would expect a singularity to occur in a physically interesting metric that is a generalisation of the Schwarzschild metric, and purports to represent a mass source m, together with other parameters. In this case we have from Descartes

rule/.....

rule of signs that all four roots can be positive;

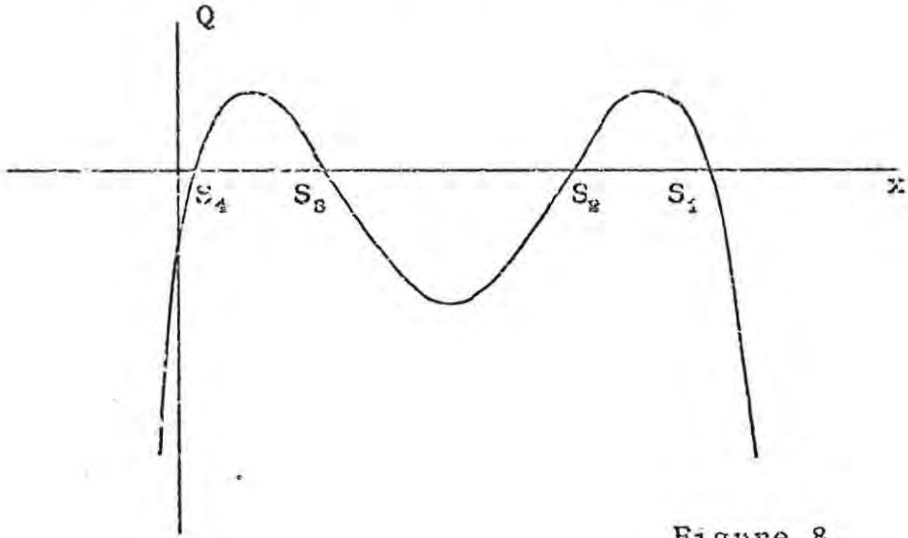

Figure 8.

"Elliptic" type orbits are possible between S_1 and S_2 and between S_3 and S_4 which in itself is interesting because we are allowed orbits at various "levels" of x at the same energy and angular momentum. However, if we allow S_2 and S_3 to coincide, we get a far more interesting phenomenon. Let us call the region between S_1 and S_2, A and that between S_3 and S_4, B. Then if $S_2 = S_3 = S$, a particle orbiting in A or B which reaches S is trapped in an unstable circular orbit (the reason being the same as in Case 2) and then it may escape, as a result of a small perturbation, into either A or B to return later to S and repeat the procedure. Thus we have the possibility of a particle performing an extremely complicated orbit, for example,

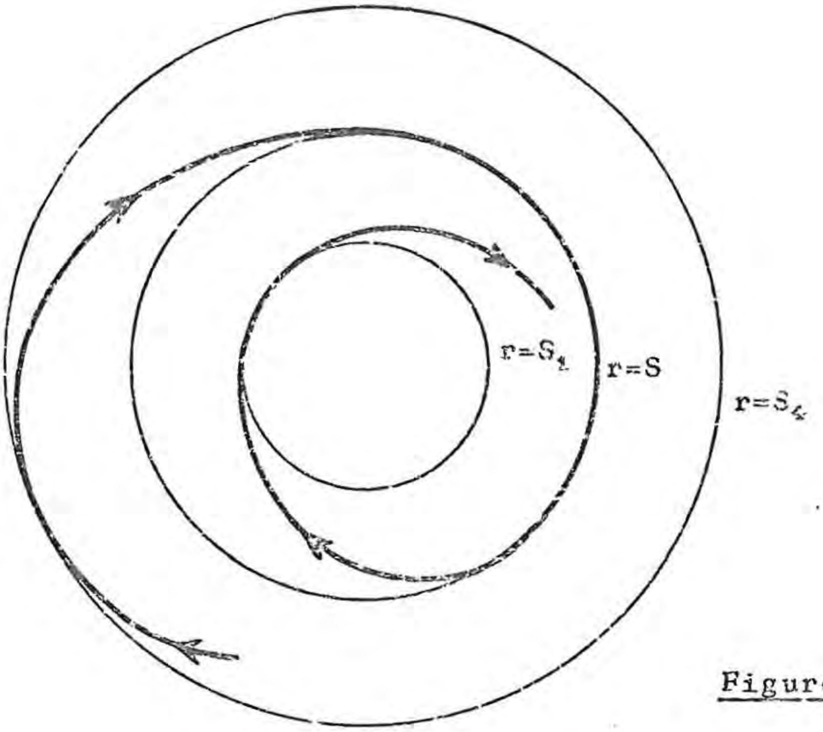

Figure 9.

Case 4.

Here we apply the same conditions to the coefficients of the quartic as in Case 2 but now we consider the case where there are only two real roots; S_4 and S_1.

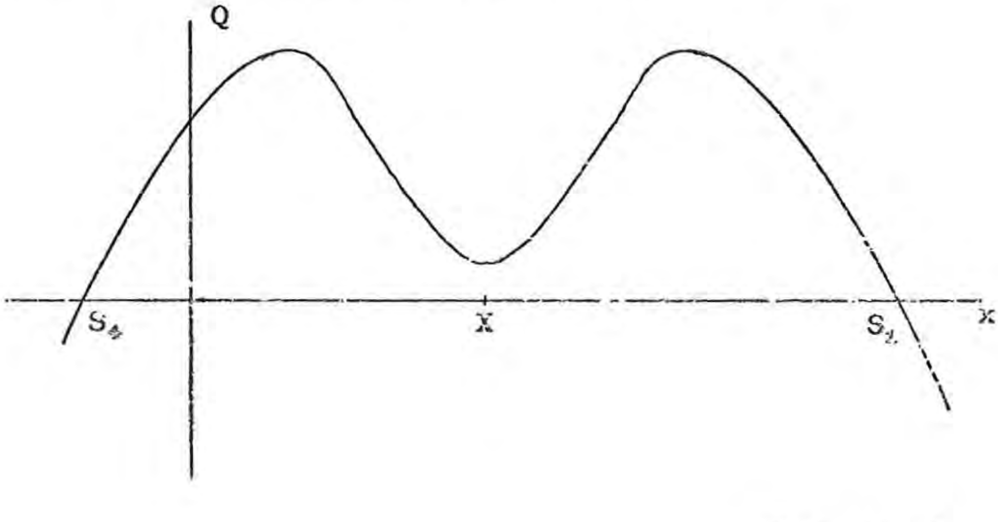

Figure 10.

Here the particle we are considering comes in from infinity and is reflected back to infinity, from S_1. The

orbit/....

orbit, however, differs remarkably from the usual Newtonian type in that it has a kink corresponding to the value $x = X$.

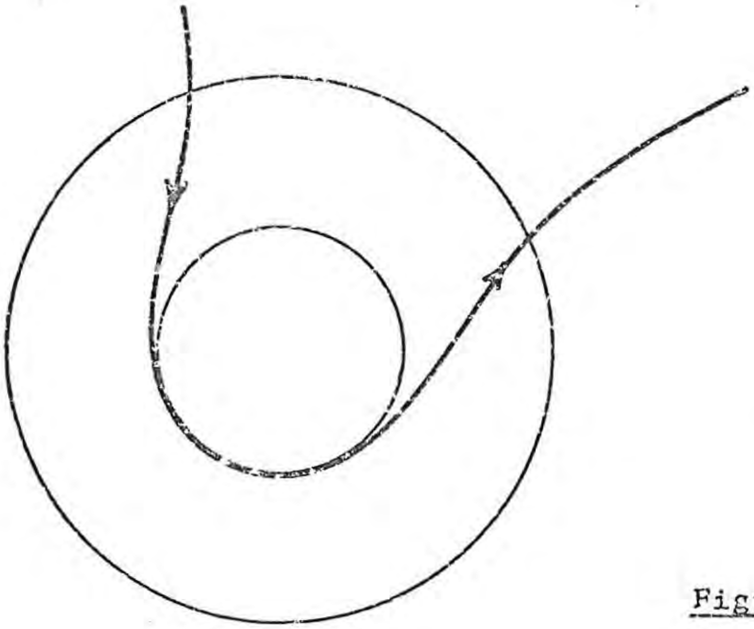

Figure 11.

Case 5.

We now apply the conditions of Case 3 except that we only require two of the roots to be real; S_4 and S_1. Thus Figure 8 becomes

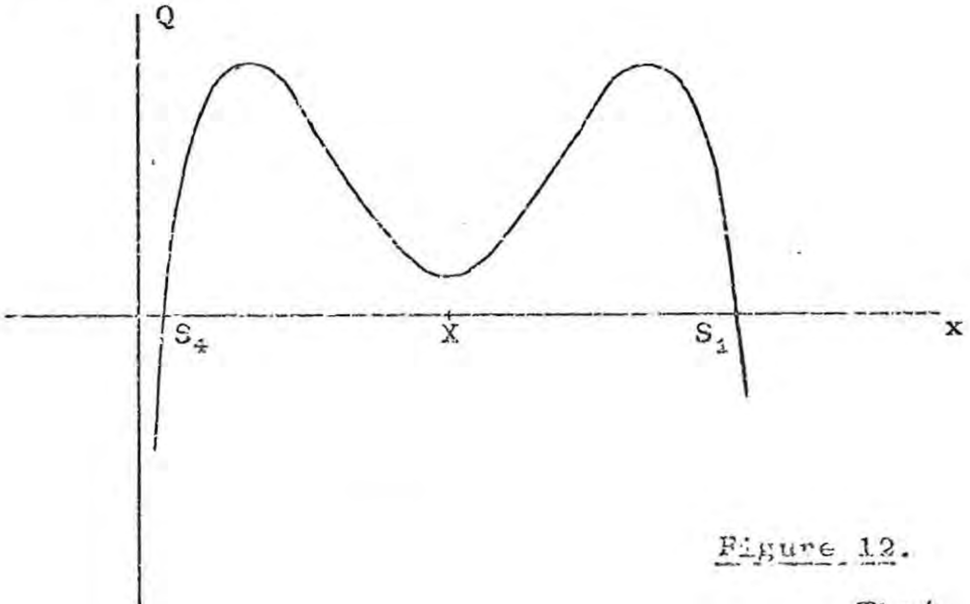

Figure 12.

The/......

The particle is trapped in an "elliptic" type orbit with a kink at $x = X$.

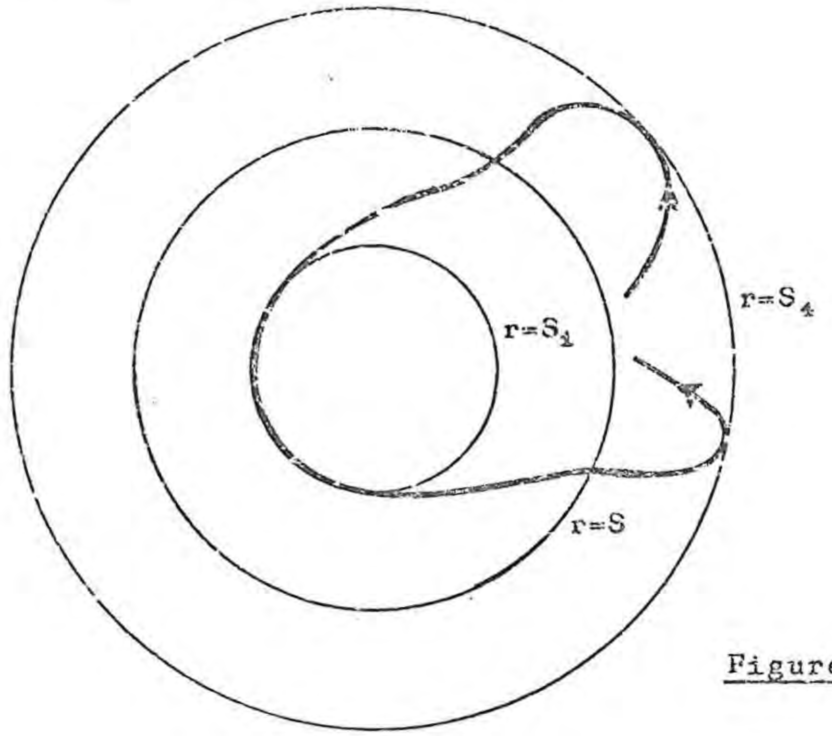

Figure 13.

Remarks.

All the diagrams above are designed to illustrate and clarify the ideas but they are not meant to be in any way quantitative.

The phenomena described in cases 2, 3, 4 and 5 cannot occur with the Schwarzschild or Kerr metrics since the analysis depends on a cubic, and not a quartic. As far as we can ascertain, these phenomena that we have described have not been discussed before.

The main conclusion we can draw from the above analysis is that in the absence of electromagnetic fields the presence of a "magnetic monopole of mass" or a cosmological effect

described/...

described by ℓ is observable, at least in principle, either through the perihelion shift of elliptic orbits or the occurrence of the anomalous orbits that we have described.

CHAPTER V

THE KINNERSLEY CLASS III, AND CLASS IV a and b METRICS

The remaining Kinnersley metrics will not be discussed in the same detail as the class II a metric. The class III metric has already been dealt with, in depth, by Kinnersley and Walker [1]; the only reasons for including it here are firstly to show that the null geodesic equations can be reduced to quadratures (1) and secondly, as an example of a Generalised Carter metric (2). The class IV b metric is the Kundt and Ehlers [1] type B metric and has been known for for some time. Despite this fact there does not appear to be any interpretation available as yet. Kinnersley [1] suggests that the IV a metric is a "rotating" generalisation of the metric IV b. However it suffices here to say that, whereas the IV b metric has two parameters, the IV a metric has three and can therefore be regarded as a generalisation of the class IV b metric. We are able to integrate the geodesic equations, and to discuss certain possibilities for the class IV a metric but no definite interpretation has been obtained.

§5.1/.....

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(1) The methods of Schouten [1] and Penrose and Walker [1] that are used in Kinnersley and Walker [1] only produce the first integrals.

(2) The problem of generating Generalised Carter metrics from Carter metrics or other Generalised Carter metrics is discussed in Appendix B.

§5.1 The Kinnersley Class III Metric.

The Kinnersley class III metric is the Kundt and Ehlers [1] type C metric (Kinnersley [1])(1) and therefore in suitable coordinates it takes the form

1.1 $$ds^2 = \frac{1}{(x + y)^2}\left[Y\,dt^2 - Y^{-1}dy^2 - X^{-1}dx^2 - X\,dz^2\right]$$

where $Y = Y(y)$ is a function of y only and $X = X(x)$ is a function of x only. In fact X and Y are polynomials. The integration of the geodesic equations does not depend on the explicit form of the X and Y and therefore applies equally well, in a formal sense, to the charged C metric of Kinnersley and Walker [1]. All that has to be done is to substitute the correct forms of X and Y. The metric is of the Carter type except for the conformal factor $\frac{1}{(x + y)^2}$, and thus it is an example of what we have called a Generalised Carter metric (§2.2 p.18). Consequently, as explained in Chapter two, the null geodesic equations can be reduced to quadratures, as we will now show.

The geodesic equations are given in terms of a parameter s by $x = x(s)$, $y = y(s)$, $z = z(s)$ and $t = t(s)$ and the function F defined in (2.2.10) is given by;

1.2/......

(1) This metric, first discovered by Levi Civita [1], has been rediscovered a number of times. To the list given in Kinnersley and Walker [1] we add Wang [1].

1.2 $$F = \frac{1}{2} \frac{1}{(x + y)^2} [Y\dot{t}^2 - Y^{-1}\dot{y}^2 - X^{-1}\dot{x}^2 - X\dot{z}^2]$$

where the dots indicate the derivative with respect to s. The parameter s is chosen so that condition (2.2.9) is satisfied. From equation (2.2.12) the "momenta" are

1.3 $$P_t = \frac{1}{(x + y)^2} Y\dot{t}$$

1.4 $$P_y = - \frac{1}{(x + y)^2} Y^{-1}\dot{y}$$

1.5 $$P_x = - \frac{1}{(x + y)^2} X^{-1}\dot{x}$$

1.6 $$P_z = - \frac{1}{(x + y)^2} X\dot{z}$$

where we have replaced the y in (2.2.12) by P to avoid confusion. From the Euler-Lagrange equations (2.2.11) it follows immediately that

1.7 $$P_t = -E ,$$

1.8 $$P_z = \Phi ,$$

where E and Φ are constants. Consequently (2.2.7) implies that

1.9 $$\frac{\partial S}{\partial t} = -E ,$$

1.10 $$\frac{\partial S}{\partial z} = \Phi .$$

Since no cross terms occur in the metric given in equation

(1.2)/.....

(1.2), the contravariant components of the metric tensor can be found by inspection. The Hamilton-Jacobi equation is then easily found to be

$$1.11 \qquad (x + y)^2 \left[Y^{-1}\left(\frac{\partial S}{\partial t}\right)^2 - Y\left(\frac{\partial S}{\partial y}\right)^2 - X\left(\frac{\partial S}{\partial x}\right)^2 - X^{-1}\left(\frac{\partial S}{\partial z}\right)^2 \right] = \lambda ,$$

where $\lambda = 0, +1,$ or -1 depending on the geodesic under consideration. The factor $(x + y)^2$ makes this equation extremely difficult to integrate unless $\lambda = 0$. However, if $\lambda = 0$, then we are dealing with the null geodesics and the equation can be integrated quite easily. If we substitute in equation (1.11) from equations (1.9) and (1.10), the equation, with $\lambda = 0$, becomes

$$1.12 \qquad Y\left(\frac{\partial S}{\partial y}\right)^2 - Y^{-1}E^2 + X\left(\frac{\partial S}{\partial x}\right)^2 + X^{-1}\Phi^2 = 0$$

which clearly has a solution of the form

$$1.13 \qquad S = S_1(y) + S_2(x) + \Phi z - Et .$$

If S, in the form (1.13), is substituted in equation (1.12) it can be separated to yield,

$$1.14 \qquad Y\left(\frac{dS_1}{dy}\right)^2 - Y^{-1}E^2 = -K$$

and

$$1.15 \qquad X\left(\frac{dS_2}{dx}\right)^2 + X^{-1}\Phi^2 = K$$

where K is an integration constant. From equations (1.14), (1.15) and (1.13) we get

1.16 $$S = \int \frac{\pm\sqrt{E^2 - KY}}{Y}\,dy + \int \frac{\pm\sqrt{KX - \Phi^2}}{X}\,dx + \Phi z - Et$$

where the plus or minus sign may be chosen arbitrarily. As was done in Appendix A, we now use the method of Vanstone [1] to obtain the geodesic equations as quadratures. The derivatives of S with respect to E, K and Φ are put equal to constants. This process yields;

1.17 $$t = \int^{y} \frac{E}{Y}\left(\pm\sqrt{E^2 - KY}\right)^{-1} dy\ ,$$

1.18 $$z = \int^{x} \frac{\Phi}{X}\left(\pm\sqrt{KX - \Phi^2}\right) dx\ ,$$

and

1.19 $$\int^{y} \frac{dy}{\pm\sqrt{E^2 - KY}} = \int^{x} \frac{dx}{\pm\sqrt{KX - \Phi^2}}\ ,$$

which completes the reduction of the null geodesic equations. The reduction that has been achieved shows again that, for the Petrov type D metrics, the Hamilton-Jacobi approach has an advantage over the Killing tensor approach of Penrose and Walker [1], since that technique only leads to first integrals of the equations. The advantage of their technique lies in the fact that they do not have to use special coordinates, but there still remain the difficulties of finding the conformal Killing tensor. The first integrals can be obtained from the above equations if required. From the equations (1.3), (1.4), (1.5), (1.6) and (2.2.7) we have;

1.20/.....

1.20 $$\frac{\partial S}{\partial x} = -\frac{1}{(x+y)^2} X^{-1}\dot{x}$$

1.21 $$\frac{\partial S}{\partial y} = -\frac{1}{(x+y)^2} Y^{-1}\dot{y}$$

1.22 $$\frac{\partial S}{\partial z} = -\frac{1}{(x+y)^2} X\dot{z}$$

1.23 $$\frac{\partial S}{\partial t} = +\frac{1}{(x+y)^2} Y\dot{t}$$

where it is assumed that the derivatives on the right hand side of equations (1.20), (1.21), (1.22), (1.23) are to be considered as functions of x and y. Now from equation (1.16) it follows that

1.24 $$\dot{x} = \pm(x+y)^2\sqrt{XK - \Phi^2} ,$$

1.25 $$\dot{y} = \mp(x+y)^2\sqrt{B^2 - KY} ,$$

1.26 $$\dot{z} = -(x+y)^2 X^{-1}\Phi ,$$

1.27 $$\dot{t} = -(x+y)^2 Y^{-1} B .$$

§5.2 <u>The Kinnersley Class IV Metrics.</u>

In this section we do not intend giving a detailed treatment of the Kinnersley class IV metrics(1); rather we will concentrate on some particular aspects. Kundt and

Ehlers/...

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(1) As was done previously the metrics will be referred to as class IV in the subsequent discussion.

Ehlers [1] call the IV b metrics the 'B' metrics, and as yet no physically meaningful interpretation has been suggested. We will show that, under suitable assumptions, all the class IV metrics can be reduced asymptotically to flat space metrics. This does not necessarily imply asymptotic flatness in the sense of Penrose [1]. Also we will reduce the geodesic equations to quadratures and discuss the metric IV a in various coordinate systems that seem useful.

Asymptotic Properties.

The class IV metrics are of Petrov type D and according to Kinnersley [1] the Newman-Penrose invariant Ψ_2 in terms of a principal null tetrad is given by:

2.1 $$\Psi_2 = \frac{m + i\ell}{(x + ia)^3},$$

where m, ℓ, and a are constants and $i = \sqrt{-1}$. From equation (2.1) we see that, as $x \to \infty$, Ψ_2 goes to zero as x^{-3} and thus we are led to suspect that the metric may be asymptotically flat as $x \to \infty$ and we conclude that the possibility of using x as an affine parameter along certain of the geodesics should be investigated.

The asymptotic forms of the metrics are easy to obtain once we have made a simple coordinate transformation. The Kinnersley [1] class IV b metrics with $C \neq 0$ are given by

2.2 $$ds^2 = -2C\frac{r^2}{x^2}du^2 + 2\,du\,dr - 4\frac{r}{x}du\,dx - \frac{1}{2}\left(C + \frac{m}{x}\right)^{-1}dx^2 - 2\left(C + \frac{m}{x}\right)dy^2$$

where/....

where $C = \pm \frac{1}{2}$ and m is a constant. This metric has singularities at $x = 0$ and $x = -\frac{C}{m}$. Although we are interested in the asymptotic form of the metric and therefore not in the singularities, some aspects will have to be discussed. Unlike the Schwarzschild singularity, at which in the usual coordinates, r changes from a space-coordinate to a time-coordinate and t alters in the reverse way, the signature of the class IV b metric goes from $(+---)$ to $(+++-)$ on crossing the singularity at $x = -\frac{C}{m}$. In order to retain the signature $(+---)$ as $x \to \infty$ we will use the metric with $C = \frac{1}{2}$ (1).

We now introduce new coordinates;

2.3 $$v = (\sqrt{2})^{-1}(r + u) ,$$

2.4 $$w = (\sqrt{2})^{-1}(r - u) ,$$

which, when substituted in the metric (2.2), yield;

2.5 $$\begin{aligned} ds^2 = &- \frac{1}{4} \frac{(v + w)^2}{x^2} (dv^2 - 2\,dv\,dw + dw^2) + dv^2 - dw^2 \\ &- \frac{2(v + w)}{x} (dv - dw)\,dx - \left(1 + \frac{2m}{x}\right)^{-1} dx^2 \\ &- \left(1 + \frac{2m}{x}\right) dy^2 , \end{aligned}$$

which/....

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(1) Singularities of the type $\frac{dr^2}{1 - \frac{1}{r}} + \left(1 - \frac{1}{r}\right) d\phi^2$ have been discussed in Kundt and Ehlers [1] p.78, where a method of removing them is given.

which tends to

2.6 $$ds^2 = dv^2 - dw^2 - dx^2 - dy^2 \; ;$$

and thus the metric is asymptotically flat in the spacelike direction defined by $x \to \infty$. If we include only the first order terms in the metric, given in equation (2.5), we get;

2.7 $$ds^2 = dv^2 - dw^2 - \left(1 - \frac{2m}{x}\right) dx^2 - \left(1 + \frac{2m}{x}\right) dy^2 - \frac{2(v + w)}{x} (dv - dw)dx + O(x^{-2}) \; .$$

We will now show that the class IV a metric, taken to the same order, has a similar form.

The class IV a metric in Kinnersley [1] coordinates is

2.8 $$ds^2 = \frac{r^2 \ell a^{-1}}{x^2 + a^2} du^2 + 2\, du\, dr - \frac{4rx}{x^2 + a^2} du\, dx - \frac{a(x^2 + a^2)}{2amx + \ell(a^2 - x^2)} dx^2 - \frac{2amx + \ell(a^2 - x^2)}{a(x^2 + a^2)} dy^2 \; ,$$

in which m, a and ℓ are all constants. There are singularities at the points where $2amx + \ell(a^2 - x^2) = 0$, which again yield changes in the metric from $(- - - +)$ to $(+ + + -)$ on opposite sides of the singularity. In order to maintain the signature $(- - - +)$ which is consistent with the rest of this work we will choose the metric with $\frac{\ell}{a} < 0$. (This choice is not essential but it is convenient for our purposes). The transformation given in equations (2.3) and (2.4) leads to the metric,

2.9/......

$$2.9 \qquad ds^2 = \frac{1}{4}\frac{(v + w)^2 \ell a^{-1}}{x^2 + a^2}(dv^2 - 2\,dv\,dw + dw^2) + dv^2 - dw^2$$

$$- \frac{2(v + w)x}{x^2 + a^2}(dv - dw)dx - \frac{a(x^2 + a^2)}{2amx + \ell(a^2 - x^2)}dx^2$$

$$- \frac{2amx + \ell(a^2 - x^2)}{a(x^2 + a^2)}dy^2 \;.$$

For large values of x,

$$2.10 \qquad \frac{(-\ell x^2 + \ell a^2 + 2amx)}{a(x^2 + a^2)} = -\frac{\ell}{a} + \frac{2m}{x} + O(x^{-2}) \;,$$

and

$$2.11 \qquad \frac{a(x^2 + a^2)}{(-\ell x^2 + \ell a^2 + 2amx)} = -\frac{a}{\ell} - \frac{2a^2 m}{\ell^2 x} + O(x^{-2}) \;.$$

Thus the metric (2.9) becomes:

$$2.12 \qquad ds^2 = dv^2 - dw^2 + \frac{a}{\ell}\left(1 + \frac{2am}{\ell x}\right)dx^2 + \frac{\ell}{a}\left(1 - \frac{2am}{\ell x}\right)dy^2$$

$$- \frac{2(v + w)}{x}(dv - dw)dx + O(x^{-2}) \;,$$

which if we choose $a = -\ell$, reduces to the metric (2.7), so that the IV a metric with $a = -\ell$ is asymptotically equivalent to the IV b metric in the above sense. On the other hand, Kinnersley [1] mentions that if we let $\ell \to 0$ and $a \to 0$ so that $\frac{\ell}{a} \to C$, then the IV a metric tends smoothly to the IV b metric. It is thus interesting to note that even if ℓ and a are non zero then the metrics have similar asymptotic forms if $\ell = -a$. If we consider the metric (2.12) it is again clear that as $x \to \infty$ the metric tends to

$$2.13 \qquad ds^2 = dv^2 - dw^2 - dx'^2 - dy'^2 \;,$$

where $dx'^2 = -\frac{a}{\ell}dx^2$ and $dy'^2 = -\frac{a}{\ell}dy^2$ in which we note that $\frac{a}{\ell} < 0$, and again we conclude that the space-time is asymptotically flat in the spacelike direction given by $x \to \infty$. We will now put the metrics into a form which is analogous to the Boyer-Lindquist form of the class IV a metric and reduce the geodesic equations to quadratures.

Geodesic Equations.

We will now show that the metrics IV a and b are Carter metrics and that the geodesic equations can be reduced to quadratures under certain coordinate conditions.

For convenience we introduce a special notation,

2.14 $$\rho^2 = x^2 + a^2 ,$$

2.15 $$a\Delta = 2amx + \ell(a^2 - x^2) ,$$

2.16 $$k = \frac{\ell}{a} ,$$

in terms of which the metric (2.8) becomes

2.17 $$ds^2 = 2\,du\,dr - 4rx\rho^{-2}\,du\,dx - \Delta^{-1}\rho^2\,dx^2 + r^2k\rho^{-2}du^2 - \Delta\rho^{-2}dy^2 .$$

For values of $r > 0$ we can introduce the transformation

2.18 $$\lambda = \log r - \log \rho^2$$

or

2.19 $$r = \rho^2 e^{\lambda} ,$$

then substitution in equation (2.17) yields

2.20/.....

2.20 $$ds^2 = 2\rho^2 e^{\lambda} du\, d\lambda - \Delta^{-1}\rho^2\, dx^2 + \rho^2 e^{2\lambda} k\, du^2 - \Delta\rho^{-2} dy^2 ,$$

which is already in a form which yields a separable Hamilton-Jacobi equation. It is convenient, however, to go one stage further with the transformation so as to reduce the metric to diagonal form. We put

2.21 $$z = -k^{-1} e^{-\lambda} + u$$

then the metric (2.20) becomes

2.22 $$ds^2 = \rho^2 k e^{2\lambda} dz^2 - \Delta^{-1}\rho^2\, dx^2 - \Delta\rho^{-2} dy^2 - \rho^2 k^{-1} d\lambda^2 ,$$

which is the form we will use. The IV b metrics can be reduced to precisely the same form as equation (2.22) if $C \neq 0$. We put

2.23 $$\bar{\Delta} = 2\left(C + \frac{m}{x}\right) x^2 ,$$

2.24 $$\bar{\rho} = x ,$$

and

2.25 $$\bar{k} = -2C .$$

If we use this notation the metric (2.2) reads

2.26 $$ds^2 = 2\, du\, dr + \bar{k} r^2 (\bar{\rho})^{-2} du^2 - 4rx(\bar{\rho})^{-2} du\, dx - (\bar{\Delta})^{-1}(\bar{\rho})^2 dx^2 - (\bar{\Delta})(\bar{\rho})^{-2} dy^2$$

The transformation

2.27 $$\lambda = \log r - \log (\bar{\rho})^2 ,$$

2.28 $$z = -(\bar{k})^{-1} e^{-\lambda} + u ,$$

takes/....

takes the metric (2.26) into the form

2.29 $$ds^2 = (\bar{\rho})^2 \bar{k} e^{2\lambda} dz^2 - \Delta^{-1}(\bar{\rho})^2 dx^2 - \Delta(\bar{\rho})^{-2} dy^2 - (\bar{\rho})^2 (\bar{k})^{-1} d\lambda^2 ,$$

which is formally identical with the metric (2.22).

If $C = 0$ the metric (2.2) becomes

2.30 $$ds^2 = 2\,du\,dr - \frac{4r}{x}\,du\,dx - \frac{x}{2m}\,dx^2 - \frac{2m}{x}\,dy^2 .$$

Then the transformation of coordinates (t, x, y, z) given by

2.31 $$r = \frac{1}{\sqrt{2}}\,(t - z)x^2 ,$$

2.32 $$u = \frac{1}{\sqrt{2}}\,(t + z)$$

reduces it to the diagonal form

2.33 $$ds^2 = x^2 dt^2 - x^2 dz^2 - \frac{x}{2m}\,dx^2 - \frac{2m}{x}\,dy^2 .$$

Since the metric coefficients depend on only one coordinate, namely x, the geodesic equations can be integrated directly. There are three first integrals from the obvious Killing vectors and then the metric itself, through the associated Lagrangian, provides a fourth equation which yields $\frac{dx}{ds}$ as a function of x.

We can now return to the problem of integrating the geodesic equations for the metrics given in equations (2.29) and (2.22). These two metrics are formally the same so that we need only consider the metric (2.22).

Again/....

Again we assume the geodesics given by $x = x(s)$, $y = y(s)$, $z = z(s)$, $\lambda = \lambda(s)$ as functions of an affine parameter chosen so as to satisfy the conditions in §2.2. Then F defined in (2.2.10) is given by

2.34 $$F = \frac{1}{2}\,(\rho^2 k e^{2\lambda}\dot{z}^2 - \Delta^{-1}\rho^2\dot{x}^2 - \Delta\rho^{-2}\dot{y}^2 - \rho^2 k^{-1}\dot{\lambda}^2) \, ,$$

where a dot indicates the derivative with respect to s. Thus according to (2.2.12)

2.35 $$P_z = \rho^2 k e^{2\lambda}\dot{z} \, ,$$

2.36 $$P_x = -\Delta^{-1}\rho^2\dot{x} \, ,$$

2.37 $$P_y = -\Delta\rho^{-2}\dot{y} \, ,$$

2.38 $$P_\lambda = -\rho^2 k^{-1}\dot{\lambda} \, ,$$

where we have replaced y in (2.2.12) by P to avoid confusion. From the Euler-Lagrange equations (2.2.11) it follows that,

2.39 $$P_z = \rho^2 k e^{2\lambda}\dot{z} = -E \, ,$$

2.40 $$P_y = -\Delta\rho^{-2}\dot{y} = \Phi ,$$

where E and Φ are constants. The contravariant components of the metric tensor follow from (2.22) or (2.29) by inspection and hence the Hamilton-Jacobi equation is given by

2.41 $$\rho^{-2}k^{-1}e^{-2\lambda}\left(\frac{\partial S}{\partial z}\right)^2 - \Delta\rho^{-2}\left(\frac{\partial S}{\partial x}\right)^2 - \Delta^{-1}\rho^2\left(\frac{\partial S}{\partial y}\right)^2 - \rho^{-2}k\left(\frac{\partial S}{\partial \lambda}\right)^2 = \mu \, ,$$

where/....

where $\mu = +1, -1$ or 0. From equations (2.39), (2.40) and (2.2.7) it follows that

2.42 $$\frac{\partial S}{\partial z} = -E \ ;$$

2.43 $$\frac{\partial S}{\partial y} = \Phi \ ,$$

and thus equation (2.37) has a solution of the form

2.44 $$S = \Phi y - Ez + S_1(x) + S_2(\lambda) \ .$$

If we substitute from equation (2.44) in equation (2.41) the Hamilton-Jacobi equation separates and yields the two ordinary differential equations

2.45 $$k^{-1} e^{-2\lambda} E^2 - k \left(\frac{dS_2}{d\lambda} \right)^2 = N \ ,$$

2.46 $$\Delta \left(\frac{dS_1}{dx} \right)^2 + \Delta^{-1} \rho^4 \Phi^2 + \mu \rho^2 = N \ ,$$

where N is an integration constant. From equations (2.45) and (2.46) we get

2.47 $$S_1 = \int \pm\sqrt{\Theta} \ \frac{dx}{\Delta} \ ,$$

2.48 $$S_2 = \int \pm\sqrt{\Lambda} \ \frac{d\lambda}{k} \ ,$$

where

2.49 $$\Theta = N\Delta - \mu\rho^2 - \rho^4 \Phi^2 \ ,$$

2.50 $$\Lambda = E^2 e^{-2\lambda} - Nk \ .$$

The final form of the solution to the Hamilton-Jacobi equation is obtained by substituting from equations (2.47) and (2.48) in equation (2.49). Differentiating this final form of the

solution/...

solution with respect to E, Φ and N yields the geodesic equations as quadratures;

2.51 $$z = \int^{\lambda} \frac{Ek^{-1}e^{-2\lambda}}{\pm\sqrt{\Lambda}}\, d\lambda\ ,$$

2.52 $$y = \int^{x} \frac{\rho^4 \Phi \Delta^{-1}}{\pm\sqrt{\Theta}}\, dx\ ,$$

2.53 $$\int^{x} \frac{dx}{\pm\sqrt{\Theta}} = \int^{\lambda} \frac{d\lambda}{\pm\sqrt{\Lambda}}\ .$$

The solutions to the geodesic equations for the IV b metric are obtained by simply replacing Δ, ρ, k by $\bar{\Delta}$, $\bar{\rho}$, $\bar{k}$ respectively, in equations (2.51), (2.52) and (2.53).

The geodesic equations can also be given in the form of first order ordinary differential equations. We use (2.2.7) and the momenta given in equations (2.35), (2.36), (2.37) and (2.38) and the solution to the Hamilton-Jacobi equation (2.40). The explicit equations are;

2.54 $$\rho^2 \dot{z} = -Ek^{-1}e^{-2\lambda}\ ,$$

2.55 $$\rho^2 \dot{y} = -\Phi\rho^4\Delta^{-1}\ ,$$

2.56 $$\rho^2 \dot{x} = \pm\sqrt{\Theta}\ ,$$

2.57 $$\rho^2 \dot{\lambda} = \pm\sqrt{\Lambda}\ .$$

Again the solutions for the IV b metric can be found by replacing Δ, ρ, k by $\bar{\Delta}$, $\bar{\rho}$ and $\bar{k}$ respectively.

Some/.....

Some Remarks on the Possibility of Interpreting the Class IV a Metric as a Radiation Metric.

The asymptotic flatness of the class IV a metric, together with the fact that, in terms of a principal null tetrad, Ψ_2 is of order x^{-3} tempts one to conclude that, in suitable coordinates and with a corresponding choice of tetrad, Ψ_4 may not vanish. If this is true it should be possible to interpret the class IV a metric as a radiation metric using the methods Kinnersley and Walker [1] have applied to the class III case.

As we have already pointed out we have chosen $\frac{\ell}{a} < 0$ so as to retain the signature $(+ - - -)$. To obviate the nuisance of using negative valued parameters we will replace ℓ by $-b$; the metric (2.22) then reads

$$2.58 \qquad ds^2 = \rho^2 ab^{-1} d\lambda^2 - \rho^2 ba^{-1} e^{2\lambda} dz^2 - \Delta^{-1}\rho^2 dx^2 - \Delta\rho^{-2} dy^2 .$$

The signature of this metric still only remains $(+ - - -)$ if we consider values of x larger than x_+ where x_+ is the larger root of $\Delta = 0$, i.e.

$$2.59 \qquad x_+ = \frac{a}{b}\left[-m + \sqrt{m^2 + b^2}\right] ,$$

which will be greater than zero for all real values of m and b. Now we wish to consider the possibility of treating the metric (2.58), outside x_+ as a radiation metric. This we will do by bringing it to the Robinson and Trautman [1] form and then evaluating the Newman-Penrose [1] invariants. In particular we will attempt to use x as an affine parameter/....

meter, along certain of the geodesics, in view of the form of Ψ_2 given in equation (2.1)

The transformation

2.60 $$du = d\lambda - \frac{b}{a}\Delta^{-\frac{1}{2}}dx$$

brings the metric (2.58) to the form

2.61 $$ds^2 = \rho^2 ab^{-1}du^2 + 2\rho^2\sqrt{\frac{a}{b}}\,\Delta^{-\frac{1}{2}}du\,dx - \rho^2\left(\frac{b}{a}\right)e^{\lambda^*}dz^2 - \Delta\rho^{-2}dy^2$$

where λ^* results from expressing λ as a function of x and u. We now rescale the x coordinate by putting

2.62 $$r = \int_{x_+}^{x}\rho^2\sqrt{\frac{a}{b}}\,\Delta^{-\frac{1}{2}}dt\ .$$

For all values of x that we are considering, r and u exist since the integrals converge. In fact

2.63 $$\int_{x_+}^{x}\rho^2\sqrt{\frac{a}{b}}\,\Delta^{-\frac{1}{2}}dt = \sqrt{\frac{a}{b}}\left\{\left(\frac{t}{2b} - \frac{3am}{4b^3}\right)\Delta^{\frac{1}{2}}\right]_{x_+}^{x} + \frac{3a^2m^2 + 12b^2a^2}{8b^2}\int_{x_+}^{x}\frac{dt}{\Delta^{\frac{1}{2}}}\right\}$$

and

2.64 $$\int_{x_+}^{x}\frac{dt}{\Delta^{\frac{1}{2}}} = \frac{1}{b^{\frac{1}{2}}}\log\left|2(b\Delta)^{\frac{1}{2}} + 2bt + b\right|\Big]_{x_+}^{x}$$

We can rewrite the metric as

2.65 $$ds^2 = h\,du^2 - 2\,du\,dr - f\,dy^2 - m\,dz^2$$

where

2.66 $f = \Delta\rho^{-2}$, considered as a function of r,

2.67 $h = \rho^2 ab^{-1}$, considered as a function of r,

2.68/.....

2.68 $$m = \rho^2 b a^{-1} e^{2(u+g)}, \quad \text{considered as a function of } r \text{ and } u,$$

in which

2.69 $$g = \int_{x_+}^{x} \sqrt{\frac{b}{a}}\, \Delta^{-\frac{1}{2}} dx$$

is again to be considered as a function of r.

The contravariant form of the metric (2.65) is

2.70 $$g^{ij} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -h & 0 & 0 \\ 0 & 0 & -f^{-1} & 0 \\ 0 & 0 & 0 & -m^{-1} \end{bmatrix}$$

where $i, j = 1, 2, 3, 4$ and $x^1 = u$, $x^2 = r$, $x^3 = y$ and $x^4 = z$. This metric is in the required Robinson and Trautman [1] form. The explicit dependence of the components of the metric in (2.70) on r is difficult to obtain. The Newman-Penrose null tetrad corresponding to the metric g^{ij} given in (2.70) is:

2.71 $$\ell^i = \delta^i_2$$

2.72 $$n^i = \delta^i_1 - \tfrac{1}{2} h \delta^i_2$$

2.73 $$m^i = \frac{1}{\sqrt{2}} (f^{-\frac{1}{2}} \delta^i_3 + i m^{-\frac{1}{2}} \delta^i_4) .$$

2.74 $$\bar{m}^i = \frac{1}{\sqrt{2}} (f^{-\frac{1}{2}} \delta^i_3 - i m^{-\frac{1}{2}} \delta^i_4) .$$

This tetrad turns out to be rather special. The vector field, with components ℓ^i, is orthogonal to the null hypersurface

2.75 $$u = \text{constant}$$

since/....

since

2.76 $$\ell_i = u_{,i} .$$

Since the hypersurface (2.75) is null, the vectors ℓ_i are tangent to a family of curves in the surface. In fact these curves are null geodesics since

2.77 $$\ell^i{}_{;j}\ell^j = 0$$

as can be checked directly. Also along these geodesics r is an affine parameter, since

2.78 $$\ell^i = \frac{\partial x^i}{\partial r} .$$

Equations (2.75) to (2.78) characterise the Newman and Unti [1] type (radiation) coordinates and the only remaining freedom lies in the other two coordinates.

It can be shown that the remaining vectors of the tetrad are propagated parallelly along the null geodesics with tangent ℓ^i. Thus among the spin coefficients of Newman and Penrose [1],

2.79 $$\kappa = \epsilon = \pi = 0 ,$$

and the remainder are given by:

2.80 $$\zeta = \nu = 0 ,$$

2.81 $$\rho = -\frac{1}{4} m_2 m^{-1} ,$$

2.82 $$\sigma = \frac{1}{4} m_2 m^{-1} ,$$

2.83/.....

2.83 $$\mu = -\frac{1}{4} f_2 fh - \frac{1}{8} m_2 hm + \frac{1}{4} m_1 m \ ,$$

2.84 $$\lambda = -\frac{1}{4} f_2 fh - \frac{1}{4} m_1 m + \frac{1}{8} m_2 mh \ ,$$

2.85 $$\gamma = \frac{1}{4} h_2 \ ,$$

where the suffix 1 or 2 indicates a partial derivative with respect to u or r respectively. From these spin coefficients and the Newman-Penrose [1] equations we find that

2.86 $$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0 \ ,$$

and

2.87 $$\Psi_2 = \frac{1}{4} h_{22} \ .$$

Thus the tetrad given in (2.71), (2.72), (2.73), (2.74) is principal null.

If one takes as a criterion for radiation that $\Psi_4 = O(r^{-1})$, in terms of a null tetrad adapted to radiation or Newman-Unti coordinates, where r is an affine parameter along a congruence of null geodesics, then the class IV a metric is not a radiation metric.

APPENDIX A

INTEGRATION OF THE GEODESIC EQUATIONS FOR THE KINNERSLEY CLASS II METRICS

The following table is given in the form:

First; the heading indicates which metric, in the Kinnersley classification, is being considered.

Second; the metric is given in Kinnersley or Kerr-Newman coordinates.

Third; the notation that will be used is defined.

Fourth; the transformation which takes the metric into Boyer-Lindquist form is defined.

Fifth; the Boyer-Lindquist form of the metric is given.

The notation has been designed so that all metrics are formally the same in Boyer-Lindquist coordinates. After all the equations have been reduced to this form the integrated geodesic equations are given in a generalised form suitable for application to each metric once the appropriate substitutions have been made.

Class II a.

A.1 $$ds^2 = \frac{r^2 - 2mr - \ell^2 + a^2\cos^2 x}{r^2 + (\ell - a\cos x)^2}\, du^2 + 2\, du\, dr$$

$$+ 4\, \frac{a \sin^2 x(mr + \ell^2) - \ell \cos x(r^2 - 2mr - \ell^2 + a^2)}{r^2 + (\ell - a\cos x)^2}\, du\, dy$$

$$- 2(a\sin^2 x + 2\ell\cos x)\, dr\, dy - [r^2 + (\ell - a\cos x)^2]\, dx^2$$

$$+ \frac{(r^2 - 2mr - \ell^2 + a^2)(a\sin^2 x + 2\ell\cos x)^2 - \sin^2 x(r^2 + \ell^2 + a^2)^2}{r^2 + (\ell - a\cos x)^2}\, dy^2$$

Notation.

A.2 $$H = a\sin^2 x + 2\ell\cos x$$

A.3 $$\Delta = r^2 - 2mr - \ell^2 + a^2$$

A.4 $$\rho^2 = R - aH$$

A.5 $$R = r^2 + \ell^2 + a^2$$

Transformation.

A.6 $$dt = du - R\Delta^{-1} dr$$

A.7 $$d\phi = dy - a\Delta^{-1} dr$$

Boyer-Lindquist form.

A.8 $$ds^2 = -\rho^2 dx^2 - \rho^2\Delta^{-1} dr^2 + \rho^{-2}\Delta(dt - H\, d\phi)^2$$
$$- \rho^{-2}\sin^2 x(a\, dt - R\, d\phi)^2 .$$

Class II b /..

Class II b.

A.9 $$ds^2 = -\frac{r^2 + 2mr - \ell^2 + a^2\cosh^2 x}{r^2 + (\ell - a\cosh x)^2}\, du^2 + 2\, du\, dr$$

$$+ 4\,\frac{\ell\cosh x(r^2 + 2mr - \ell^2 + a^2) - a\sinh x(mr - \ell^2)}{r^2 + (\ell - a\cosh x)^2}\, du\, dy$$

$$+ 2(a\sinh^2 x - 2\ell\cosh x)\, dr\, dy - [r^2 + (\ell - a\cosh x)^2]\, dx^2$$

$$-\frac{(r^2 + 2mr - \ell^2 + a^2)(a\sinh^2 x - 2\ell\cosh x)^2 - \sinh^2 x(r^2 + \ell^2 + a^2)}{r^2 + (\ell - a\cosh x)^2} \times dy^2$$

Notation.

A.10 $$H = -a\sinh^2 x + 2\ell\cosh x$$

A.11 $$\Delta = -(r^2 + 2mr - \ell^2 + a^2)$$

A.12 $$\rho^2 = R - aH$$

A.13 $$R = r^2 + \ell^2 + a^2$$

Transformation.

A.14 $$dt = du + R\Delta^{-1}\, dr$$

A.15 $$d\phi = dy + a\Delta^{-1}\, dr$$

Boyer-Lindquist form.

A.16 $$ds^2 = -\rho^2 dx^2 - \rho^2\Delta^{-1} dr^2 + \rho^{-2}\Delta(dt - H\, d\phi)^2$$
$$- \rho^{-2}\sinh^2 x(a\, dt - R\, d\phi)^2 .$$

Class II c.

A.17 $$ds^2 = -\frac{r^2 + 2mr - \ell^2 + a^2\sinh^2 x}{r^2 + (\ell - a\sinh x)^2}\,du^2 + 2\,du\,dr$$

$$+ 2\,\frac{\ell \sinh x(r^2 + 2mr - \ell^2 - a^2) - a^2\cosh^2 x(mr - \ell^2)}{r^2 + (\ell - a\sinh x)^2}\,du\,dy$$

$$+ 2(a\cosh^2 x - 2\ell\sinh x)\,dr\,dy - [r^2 + (\ell - a\sinh x)^2]\,dx^2$$

$$-\frac{(r^2 + 2mr - \ell^2 - a^2)(-a\cosh^2 x + 2\ell\sinh x)^2 + \cosh^2 x(r^2 + \ell^2 - a^2)^2}{r^2 + (\ell - a\sinh x)^2}\,dy^2$$

Notation.

A.18 $$H = -a\cosh^2 x + 2\ell\sinh x$$

A.19 $$\Delta = -(r^2 + 2mr - \ell^2 - a^2)$$

A.20 $$\rho^2 = R - aH$$

A.21 $$R = r^2 + \ell^2 - a^2$$

Transformation.

A.22 $$dt = du + R\Delta^{-1}\,dr$$

A.23 $$d\phi = dy + a\Delta^{-1}\,dr$$

Boyer-Lindquist form.

A.24 $$ds^2 = -\rho^2 dx^2 - \rho^2\Delta^{-1}dr^2 + \rho^{-2}\Delta(dt - H\,d\phi)^2$$
$$- \rho^{-2}\cosh^2 x(a\,dt - R\,d\phi)^2 .$$

Class II d/...

Class II d.

A.25 $$ds^2 = \frac{r^2 + 2mr - \ell^2 + a^2 e^{2x}}{r^2 + (\ell - ae^x)^2} du^2 + 2 \, du \, dr$$

$$+ 4 \frac{\ell e^x (r^3 + 2mr - \ell^2) - ae^{2x}(mr - \ell^2)}{r^2 + (\ell - ae^x)^2} du \, dy$$

$$+ 2(ae^{2x} - 2\ell e^x) dr \, dy - [r^2 + (\ell - ae^x)^2] dx^2$$

$$- \frac{(r^2 + 2mr - \ell^2)(-ae^{2x} + 2\ell e^x)^2 + (r^2 + \ell^2)^2 e^{2x}}{r^2 + (\ell - ae^x)^2} dy^2$$

Notation.

A.26 $$H = -ae^{2x} + 2\ell e^x$$

A.27 $$\Delta = -(r^2 + 2mr - \ell^2)$$

A.28 $$\rho^2 = R - aH$$

A.29 $$R = r^2 + \ell^2$$

Transformation.

A.30 $$dt = du + R\Delta^{-1} dr$$

A.31 $$d\phi = dy + \Delta^{-1} dr$$

Boyer-Lindquist form.

A.32 $$ds^2 = -\rho^2 dx^2 - \rho^2 \Delta^{-1} dr^2 + \rho^{-2} \Delta (dt - H \, d\phi)^2$$

$$- \rho^{-2} e^{2x} (a \, dt - R \, d\phi)^2 .$$

Class II e/...

Class II e.

A.33 $$ds^2 = -\frac{2mr + 2b + x^2}{r^2 + (b + \frac{1}{2}x^2)^2}\, du^2 + 2\, du\, dr$$

$$+ 2x^2 \frac{r^2 - 2mbr - \frac{1}{2}mx^2 r - b^2 - \frac{1}{4}bx^2}{r^2 + (b + \frac{1}{2}x^2)^2}\, du\, dy$$

$$+ 2(bx^2 + \tfrac{1}{4}x^4)\, dr\, dy - [r^2 + (b + \tfrac{1}{2}x^2)^2] dx^2$$

$$- \frac{(2mr + 2b)(bx^2 + \frac{1}{4}x^4)^2 + x^2(r^2 + b^2)}{r^2 + (b + \frac{1}{2}x^2)^2}\, dy^2$$

Notation.

A.34 $$H = -(bx^2 + \tfrac{1}{4}x^4)$$

A.35 $$R = r^2 + b^2$$

A.36 $$\Delta = -(2mr + 2b)$$

A.37 $$\rho^2 = R - H$$

Transformation.

A.38 $$dt = du + R\Delta^{-1} dr$$

A.39 $$d\phi = dy + \Delta^{-1} dr$$

Boyer-Lindquist form.

A.40 $$ds^2 = -\rho^2 dx^2 - \rho^2\Delta^{-1} dr^2 + \rho^{-2}\Delta(dt - H\, d\phi)^2$$

$$- \rho^{-2}x^2(dt - R\, d\phi)^2 \ .$$

Class II f/...

Class II f.

A.41 $$ds^2 = -\frac{2mr}{r^2 + x^2}du^2 + 2\,du\,dr + 2\frac{r^2 - 2mrx^2 + x^2}{r^2 + x^2}du\,dy$$

$$+ 2x^2 dr\,dy - (r^2 + x^2)dx^2 - \frac{r^4 + 2mx^4 r - x^4}{r^2 + x^2}dy^2$$

Notation.

A.42 $$R = r^2$$

A.43 $$\Delta = 1 - 2mr$$

A.44 $$H = -x^2$$

A.45 $$\rho^2 = R - H$$

Transformation.

A.46 $$dt = du - R\Delta^{-1}dr$$

A.47 $$d\phi = dy - \Delta^{-1}dr$$

Boyer-Lindquist form.

A.48 $$ds^2 = -\rho^2 dx^2 - \rho^2\Delta^{-1}dr^2 + \rho^{-2}\Delta(dt - H\,d\phi)^2$$

$$- \rho^{-2}(dt - R\,d\phi)^2 .$$

The/......

The general class II metric in Boyer-Lindquist coordinates is:

A.49 $$ds^2 = -\rho^2 dx^2 - \rho^2\Delta^{-1}dr^2 + \rho^{-2}\Delta(dt - H\, d\phi)^2 - \rho^{-2}f^2(a\, dt - R\, d\phi)^2 ,$$

where the appropriate expressions for Δ, R, f and ρ^2 are given above. The first integrals of the geodesic equations are given by:

A.50 $$\rho^2\dot{\phi} = \left(\frac{a^2}{\Delta} - \frac{1}{f}\right)\Phi + \left(\frac{H}{f} - \frac{aR}{\Delta}\right)E$$

A.51 $$\rho^2\dot{t} = \left(\frac{aR}{\Delta} - \frac{H}{f}\right)\Phi + \left(\frac{H}{f} - \frac{R^2}{\Delta}\right)E$$

A.52 $$\rho^2\dot{x} = \pm\sqrt{\lambda aH + K - \frac{1}{f}(HE - \Phi)^2}$$

A.53 $$\rho^2\dot{r} = \pm\sqrt{(RE - a\Phi)^2 - (\lambda R + K)\Delta}$$

where B, Φ and K are integration constants and $\lambda = +1$ for timelike geodesics, -1 for spacelike geodesics and 0 for null geodesics. The sign of the roots in (A.52) and (A.53) are chosen independently. The solution S to the Hamilton-Jacobi equation is given by:

A.54 $$S = \int \pm\sqrt{X}\, dx + \int \pm\sqrt{\frac{Q}{\Delta^2}}\, dr + \Phi\phi - Et ,$$

where

A.55 $$X = \lambda aH + K - \frac{1}{f}(HE - \Phi)^2$$

A.56 $$Q = (RE - a\Phi)^2 - (\lambda R + K)\Delta .$$

We/.......

We now use the method suggested in Vanstone [1] to obtain the geodesic in the form of quadratures. All that is required is to solve equations that result from putting $\frac{\partial S}{\partial E}$, $\frac{\partial S}{\partial \Phi}$, $\frac{\partial S}{\partial K}$ equal to constants. The equations that result are;

A.57 $$t = \int^{r} \frac{(RE - a\Phi)}{\pm\sqrt{Q}} \frac{R}{\Delta} \, dr - \int^{x} \frac{(HE - \Phi)}{\pm\sqrt{X}} \frac{H}{f} \, dx \,,$$

A.58 $$\int^{x} \frac{dx}{\pm\sqrt{X}} = \int^{r} \frac{dr}{\pm\sqrt{Q}} \,,$$

A.59 $$\phi = \int^{r} \frac{(RE - a\Phi)}{\pm\sqrt{Q}} \frac{a}{\Delta} \, dr - \int^{x} \frac{(HE - \Phi)}{\pm\sqrt{X}} \frac{1}{f} \, dx \,.$$

APPENDIX B

GENERALISED CARTER METRICS

In Chapter two we defined a Generalised Carter Metric as a metric which satisfies the Carter [1] separability condition except for a conformal factor. This means that the null geodesic equations will be integrable by the Hamilton-Jacobi procedure. As we saw in Chapter five, such vacuum metrics do exist, the example given there being the Kundt and Ehlers [1] C metric. Our intention in this Appendix is to touch on the question of generating further Generalised Carter Metrics from known Carter or Generalised Carter metrics by

conformal/...

conformal transformations [1].

The background to the study of conformally related metrics is well described in Schouten [1] and Petrov [1], both of whom give detailed bibliographies. The basic references for work on conformally related metrics are, however, Brinkmann [1], [2], [3], which contain almost all the results we will require. The initial formulation of the problem here will be slightly more general than that usually used.

Let V_4 be a space-time with metric tensor g_{ij} in a coordinate neighbourhood $\{x^i\}$, and $\bar{V}_4$ be the conformally related space-time which results from the mapping

B.1 $$\bar{g}_{ij} = e^{2\sigma} g_{ij} ,$$

where $\sigma = \sigma(x^i)$ is a class C^2 function of the x^i. It is useful to introduce the notation

B.2 $$\Lambda = \tfrac{1}{4}R + \lambda ,$$

where R is the Ricci tensor and λ is the cosmological constant that appears in the Einstein equations which now take the form:

B.3 $$R_{ij} - \Lambda g_{ij} = L_{ij} .$$

Here the L_{ij} is determined once the metric g_{ij} is given.

In/.......

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(1) An alternative method of generalising these metrics would be to modify the condition iv on p.289 of Carter [1].

In contrast, the approach used by Carter [1] and Kinnersley [1] assumes the L_{ij} given (see footnote on p.3). The corresponding equations for the conformally related space-time are

B.4 $$\bar{R}_{ij} - \bar{\Lambda}\bar{g}_{ij} = \bar{L}_{ij} ,$$

where the transformations for $\bar{R}_{ij}$, $\bar{\Lambda}$ and $\bar{g}_{ij}$ follow from equation (B.1). The tensor $\bar{L}_{ij}$ is again determined by the left hand side of the equation. From equation (B.3) and equation (B.4) it follows that

B.5 $$\bar{R}_{ij} - R_{ij} = (\bar{L}_{ij} - L_{ij}) + \bar{\Lambda}\bar{g}_{ij} - \Lambda g_{ij} .$$

If we follow Petrov [1] and define

B.6 $$w_{ij} = \sigma_{;ij} - \sigma_{;i}\sigma_{;j} + \tfrac{1}{2}g^{k\ell}\sigma_{;k}\sigma_{;\ell}\, g_{ij}$$

and

B.7 $$w = g^{ij}w_{ij} ,$$

then it follows from equations (B.1), (B.5), (B.6) and (B.7) that

B.8 $$\begin{aligned} 2w_{ij} + g_{ij}w &= \bar{R}_{ij} - R_{ij} \\ &= 2(\bar{L}_{ij} - L_{ij}) + 2\bar{\Lambda}\bar{g}_{ij} - 2\Lambda g_{ij} , \end{aligned}$$

and from equations (B.8) and (B.2), that

B.9 $$6w = \bar{R}e^{2\sigma} - R = 2e^{2\sigma}(\bar{\Lambda} - \lambda) - 2(\Lambda - \lambda) ,$$

or

B.10 $$w = \frac{1}{3}\left[e^{2\sigma}(\bar{\Lambda} - \lambda) - (\Lambda - \lambda)\right] .$$

We/.......

We will impose conditions on the L_{ij} and $\bar{L}_{ij}$ in equations (B.8) and (B.10) and consider whether solutions for σ exist. The conditions are

B.11 $$L_{ij} = \bar{L}_{ij} ,$$

B.12 $$\bar{L}_{ij} = 0 , \quad L_{ij} \neq 0 .$$

The other possible condition is to allow $L_{ij} \neq 0$, $\bar{L}_{ij} \neq 0$ and $L_{ij} \neq \bar{L}_{ij}$. This case will not be discussed since to make any solutions physically interesting, ancilliary conditions would have to be applied to the $\bar{L}_{ij}$. These are difficult to include in a formal discussion. For instance one would normally require that it should be possible to relate L_{ij} to the energy momentum tensor of a physical field in a relatively easy way.

Case 1 Metrics which satisfy conditions (B.11)

This case corresponds to going from one Carter metric or Generalised Carter metric to another. Note that the metrics are not necessarily vacuum metrics and therefore this case does not correspond to mapping one Einstein Space (1) into another. Since $L_{ij} = \bar{L}_{ij}$, equation (B.8) and equation (B.10) become respectively

B.13/.....

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(1) Schouten [1] p.148 defines an Einstein Space as one in which "R_{ij} differs from g_{ij} only by a scalar factor".

B.13 $$w_{ij} + \frac{1}{2} g_{ij}w = (\bar{\Lambda}e^{2\sigma} - \Lambda)g_{ij} ,$$

B.14 $$w = \frac{1}{3} [(\bar{\Lambda}e^{2\sigma} - \Lambda) + \lambda(1 - e^{2\sigma})] .$$

We restrict the problem further by requiring that $L^i{}_i = \bar{L}^i{}_i = 0$ as is the case for the Maxwell fields, and $\lambda = 0$. Then we have

B.15 $$R = \bar{R} = 0 ,$$

and thus from equation (B.2)

B.16 $$\Lambda = \bar{\Lambda} = 0 .$$

The equations (B.13) and (B.14) then become:

B.17 $$w_{ij} = 0 ,$$

B.18 $$w = C .$$

The first integrability condition for equation (B.17) is

B.19 $$C_{ijk\ell}\sigma^i = 0 ,$$

where $C_{ijk\ell}$ is the conformal curvature tensor defined in §1.1 and

B.20 $$\sigma^i = g^{ij}\sigma_{,j} .$$

According to Petrov [1] p.265, condition (B.19) implies that either

B.21 $$g^{ij}\sigma_{,i}\sigma_{,j} = 0 ,$$

or/.......

or

B.22 $$C_{ijk\ell} = 0 \; .$$

These are precisely the conditions which arise from mapping Einstein Spaces into Einstein Spaces. Thus they will also apply if $\lambda \neq 0$ but $L_{ij} = \bar{L}_{ij} = 0$.

Equation (B.21) together with equation (B.19) implies that the metric must be of Petrov type N (Pirani [1] p.321). These are more specialised than the Carter metrics that already exist and therefore of limited interest. If equation (B.21) does not hold then equation (B.22) must hold and the space-time must be conformally flat and therefore again of limited interest.

Case II Metrics which satisfy conditions (B.12)

In this case we require $\bar{L}_{ij} = 0$ and thus equations (B.10) and (B.11) become:

B.23 $$w_{ij} + \frac{1}{2} g_{ij} w = - \frac{1}{2} L_{ij} \cdot [3\lambda e^{\sigma} - (\tfrac{1}{2}R + \lambda)] g_{ij}$$

and

B.24 $$w = \frac{1}{3} [2e^{2\sigma}\lambda - (\Lambda - \lambda)] \; .$$

If λ is put equal to zero and $L^{i}_{\;i} = 0$ the equations yield

B.25 $$w_{ij} = -\tfrac{1}{2} L_{ij}$$

and

B.26 $$w = 0 \; ,$$

for which the first integrability condition is

B.27/.....

B.27 $$C^i{}_{jk\ell}\sigma_i = \frac{1}{2}\left(L_{jk;\ell} - L_{j\ell;k}\right) .$$

The further integrability conditions are given in Schouten [1] and Petrov [1] but for these to be of value more information is required about the form of g_{ij}.

Case II offers more scope since one may start with any (Generalised) Carter metric satisfying the above conditions. However, it need not satisfy any particular field equations. The resulting metric $\bar{g}_{ij}$ is then a Generalised Carter vacuum metric. The difficulty lies in suggesting a suitable form for the initial metric g_{ij}, about which all we know is that it must be Carter separable. It would seem therefore that some further guide lines will have to be found before a worthwhile attempt can be made to solve the equations.

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