Aspects of the Symplectic and Metric Geometry of Classical and Quantum Physics

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Abstract

I investigate some algebras and calculi naturally associated with the symplectic and metric Clifford algebras. In particular, I reformulate the well known Lepage decomposition for the symplectic exterior algebra in geometrical form and present some new results relating to the simple subspaces of the decomposition. I then present an analogous decomposition for the symmetric exterior algebra with a metric. Finally, I extend this symmetric exterior algebra into a new calculus for the symmetric differential forms on a pseudo-Riemannian manifold. The importance of this calculus lies in its potential for the description of bosonic systems in Quantum Theory.

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Chapter 1

Introduction

Quantum theory uses principally two types of algebra. One is characteristically antisymmetric and based on the commutator bracket. The other is symmetric and based on the anticommutator bracket. In a geometrisation of the Quantum Theory, one would expect each to arise from some well defined underlying natural geometrical structure and to be associated with some calculus that codifies the content of that structure.

For the commutator bracket, this expectation is not empty. Its associated geometry is the symplectic geometry. Quantum Theory inherits it directly from Classical Mechanics through the classical quantisation procedures of Heisenberg and Dirac. Symplectic geometry is well understood and well developed. It also possesses a substantial body of mathematical knowledge behind it.

The matter is not so clear cut in the case of the anticommutator bracket, but it seems that the associated geometry is almost certainly related to a pseudo Riemannian structure of the hyperbolic type (that is, of zero signature). Unlike its symplectic counterpart, this type of geometry is neither well developed nor well understood. In fact it seems that we do not even yet have a sufficiently well defined mathematical framework against which its distinctive properties may be conveniently discussed.

The object of this thesis is to investigate aspects of the geometrisation of Classical and Quantum Mechanics. In particular, I aim to develop (at least in part) a proper framework against which the geometry underlying the anti-commutator bracket may be understood. But my efforts will not be directed exclusively to that objective. Some interesting aspects of the symplectic geometry are also in need of attention, so a part of this thesis is taken up also with those.

Some of the results that I shall present are already well known. However, I rework them in order to cast them into a form more suited to a thoroughgoing geometrical outlook. The Lepage decomposition discussed in chapter 2 and some of the results on the symplectic geometry in the same chapter are of this type. Other results are, to the best of my knowledge, completely new. Among these are the generalisation of the Lepage decomposition to the symmetric exterior algebra in chapter 3, and the definition of a calculus for symmetric forms in chapter 4.

In the following sections, I shall describe four algebras that form the context for the material presented in this thesis. They are the two algebras associated with the commutator and the anti-commutator, and two other closely related algebras. I shall be concerned principally with the last two. Each of these algebras has an associated geometric calculus which facilitates the study of its underlying geometry and provides a language for the expression of its geometric content. Several of these calculi are well known, but one of them in particular does not seem to have been investigated. It fact, it does not seem even to have been explicitly identified previously. I shall outline below my interest in it.

Symmetric exterior algebra

The symmetric exterior algebra arises naturally in the context of the quantisation of Classical Mechanics. It is associated with the symplectic structure of the phase space.

For a mechanical system with n degrees of freedom, the phase space has position coordinates q^i and momentum coordinates p_i for $i = 1, \ldots, n$ in some local coordinate chart. Indigenous to this space is the symplectic two-form $\omega = dp_i \wedge dq^i$. This nondegenerate form defines an isomorphic map $\flat$ from the vectors to the forms, with inverse $\sharp$. The isomorphism immediately selects a distinguished two-vector λ defined by $\lambda := \sharp\omega$, which allows the symplectic structure of the cotangent spaces to be characterised by the expressions

$$\begin{aligned} \lambda(dq^j, dq^k) &= 0\,, \\ \lambda(dp_j, dp_k) &= 0\,, \\ \lambda(dp_j, dq^k) &= \delta_j^k\,. \end{aligned} \tag{1.1}$$

This structure is more commonly expressed in terms of the Poisson brackets (Fetter and Walecka, 1980 p 198).

The dynamical information is contained in a Hamiltonian function $H(p, q, t)$ on the phase space. The evolution of the system is given by the curve $\gamma(x_0; t)$ passing through the initial point x_0. This curve belongs to a congruence of curves satisfying Hamilton's canonical equations, which may be expressed as

$$\frac{d\gamma}{dt} = \sharp dH\,.$$

This simple framework contains the basic ingredients of Classical Mechanics.

Discussions of similar material may be found in Guillemin and Sternberg (1984), Arnold (1978) and Woodhouse (1980).

One passes to Quantum Mechanics by making the standard replacements

$$\begin{aligned} q^j &\longrightarrow q^j\,, \\ p_j &\longrightarrow \frac{\hbar}{i}\frac{\partial}{\partial q^j}\,. \end{aligned} \tag{1.2}$$

The commutators for these operators are

$$\begin{aligned} \left[q^j\,,\,q^k\right] &= 0\,, \\ \left[\frac{\hbar}{i}\frac{\partial}{\partial q^j}\,,\,\frac{\hbar}{i}\frac{\partial}{\partial q^k}\right] &= 0\,, \\ \left[\frac{\hbar}{i}\frac{\partial}{\partial q^j}\,,\,q^k\right] &= \frac{\hbar}{i}\delta_j^k\,. \end{aligned} \tag{1.3}$$

These expressions are similar, but not identical, to those defining the symplectic structure (1.1). The symplectic two-vector appearing in the classical case is replaced by the commutator bracket in the quantum case. The symplectic structure is still present, however, and for the case of the quantum mechanical harmonic oscillator, is made explicit by the transformations

$$\begin{aligned} b_k &= \sqrt{\frac{m\omega}{2\hbar}}\left(q^k + \frac{i}{m\omega}p_k\right) \\ b_k^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}\left(q^k - \frac{i}{m\omega}p_k\right)\,. \end{aligned}$$

Here p_j and q^k are the operators (1.2). b, $b^\dagger$ are called the *Bose operators* (Haken 1976 p 16) and satisfy the relations

$$b_i b_j^\dagger - b_j^\dagger b_i = \delta_{ij}\,.$$

These relations define the *symplectic Clifford algebra* (Bacry and Boon 1987, Crumeyrolle, 1990). They may be expressed in the basis independent, slightly modified form

$$xy - yx = 2\omega(x,y)\,,$$

where x, y are vectors and xy is the symplectic Clifford product of x and y.

The symplectic Clifford algebra is isomorphic, as a vector space, to the space of symmetric tensors, which we will refer to as the *symmetric exterior algebra.* This infinite dimensional algebra will be one of the principal structures studied in this thesis.

Antisymmetric exterior algebra

The antisymmetric exterior algebra arises naturally in the context of Quantum Mechanics. In Quantum theory, anticommutators are common and occur for several classes of operators in the form

$$xy + yx = 2g(x, y)\,, \tag{1.4}$$

where x, y are again vectors and g is a metric on the vector space. The above expression defines the standard metric Clifford algebra. While for given dimension a symplectic structure ω has only one local canonical form, a symmetric structure g may be of several kinds, depending on its signature, each geometrically distinct. The Pauli spin matrices σ^i, $(i = 1, \ldots, 3)$ satisfy this relation when g is the three dimensional orthogonal metric of the ordinary three dimensional space. The Dirac matrices γ^μ $(\mu = 0, \ldots, 3)$ satisfy this relation when g is the four dimensional spacetime metric. The fermionic operators of Quantum Field Theory (see Haken 1976 p 105) satisfy the anticommutator expression

$$a_i a_j^\dagger + a_j^\dagger a_i = \delta_{ij}\,.$$

This expression also takes the form (1.4) in a $2n$-dimensional vector space with basis $\{a_1, \ldots, a_n; a_1^\dagger, \ldots, a_n^\dagger\}$, where g has the canonical form

$$\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

The recently identified geometry associated with this particular metric has been called the *cosymplectic geometry* (Frescura and Lubczonok, 1990, 1991) due to its similarities with the symplectic geometry.

All these Clifford algebras are isomorphic, as vector spaces, to the exterior algebra $\bigwedge(V^*)$ built on their respective vector spaces. Unlike their symplectic counterparts, the metric Clifford algebras have been studied in some detail (see for instance Benn and Tucker, 1987). The fermionic Clifford algebras however appear to be of a new kind not previously studied and need attention. The attempt by Frescura and Lubczonok to identify them with a cosymplectic structure on the phase space is not very convincing and they now think that they are probably better identified with a cosymplectic fibre bundle on the phase space (Frescura and Lubczonok, private communication).

Four algebras

The two algebras described above are the symmetric exterior algebra $\bigvee(V)$ with a distinguished antisymmetric two-form ω and the antisymmetric exterior algebra $\bigwedge(V)$ with a distinguished symmetric two-form g. In both cases, the distinguished forms do not belong to the algebras themselves. There is reason to believe that some physical systems require the simultaneous existence of both a symplectic and a metric structure for their description (Frescura and Lubczonok, 1991). I will not consider such a coexistence here, but this possibility leads me to investigate two further algebras, closely related to the above ones, namely $\bigvee(V)$ with a distinguished *symmetric* form g, and $\bigwedge(V)$ with a distinguished *antisymmetric* form ω. In these algebras, the distinguished forms have been 'swapped' so that they are elements of the algebras themselves. I denote the two new structures by $\mathcal{M}(V,g)$ and $\mathcal{S}(V,\omega)$ respectively. The four algebras so defined are tabulated in Table (1.1).

Calculus on manifolds

An algebra codifies only the local, pointwise characteristics of a geometrical structure. It is natural to consider the ways in which an algebra on a manifold may be exploited to define a calculus codifying nonlocal information also. This may always be done for differentiable manifolds by means of the exterior differential calculus. This calculus, together with the geometrical information, usually contained in a metric, provides a geometrical language in which the properties of the manifold may be discussed. In physics, this calculus finds one application in the theory of the Dirac equation. In this context, the metric is the spacetime metric and the resulting calculus carries the physical information and properties of the relativistic electron.

The Clifford algebra arising from the Dirac matrices is, as we mentioned, isomorphic to the exterior algebra constructed on the same vector space. By coincidence, or for some more profound reason, the algebra of this physical system has been provided with a naturally defined calculus. One is led to ask whether this scheme works as well for bosonic systems as it does for fermionic ones. Immediately a problem arises in the fact that the bosonic, or symplectic Clifford algebra is isomorphic to the infinite dimensional symmetric tensor algebra, which has no well developed naturally associated calculus. It is these considerations that have motivated us to try to define a calculus for the symmetric tensors, with the eventual aim of using the resulting language as an apparatus for the discussion of bosonic systems in physics.

Outline of thesis

The material in this thesis is organised as follows.

In chapter 2, I investigate the symplectic exterior algebra. The main achievements are a new geometrical interpretation of the Lepage decomposition theorem in terms of a direct sum of subspaces, and various results about the component subspaces. In broad overview, the chapter is arranged as follows. I first review the exterior algebra and introduce a symplectic two-form. In doing so I define and investigate several operations that, while not new, are not yet in common use and have not been systematically discussed in any publications I know. I then use the structures set up to obtain the Lepage decomposition theorem. The proof is adapted from proofs of the same theorem for Kaehler manifolds. I then investigate the component subspaces of the decomposition, giving several completely new results.

In chapter 3, where I investigate the symmetric exterior algebra with a distinguished metric, almost all the results are new. I show that this algebra may be decomposed in a fashion completely analogous to the exterior Lepage decomposition[1]. The development is almost identical to that in chapter 1. After reviewing the algebra of symmetric tensors, a metric is introduced and various structures are defined from it. These are used to prove the decomposition theorem. Again the basic structure of the component subspaces is found.

The purpose of chapter 4 is to present a new, but still tentative calculus for the symmetric tensors on a pseudo-Riemannian manifold. This calculus exhibits many similarities with the standard exterior differential calculus. We have been able to define a symmetric exterior derivative[2], and an associated coderivative. The algebraic structures set up in chapter 3 are closely related to this calculus. I discuss the use of this calculus in the context of symmetric

[1]That this could be done was originally the conjecture of Fabio Frescura.

[2]This definition was proposed by Fabio Frescura.

Killing forms and Harmonic forms, where we put forward definitions for a 'Laplace operator' and a 'Dirac operator'. As mentioned above, the objective is to use this calculus to investigate bosonic systems in Quantum Mechanics, and there are positive indications are that this will be possible.

	antisymmetric exterior algebra $\bigwedge(V)$	symmetric exterior algebra $\bigvee(V)$
metric g	metric Clifford algebra $xy + yx = 2g(x, y)$	$\mathcal{M}(V, g)$
symplectic ω	$\mathcal{S}(V, \omega)$	symplectic Clifford algebra $xy - yx = 2\omega(x, y)$

Table 1.1: The four algebras

Chapter 2

Lepage decomposition of the symplectic exterior algebra

The primary result of this chapter is a new interpretation of the Lepage decomposition theorem for the symplectic exterior algebra. This decomposition theorem is proven on Kaehler spaces in several works, including Hodge (1941), Chern (1956) and Weil (1958). The result for the symplectic case appears in Libermann and Marle (1987), but their presentation, which differs from the above authors, is Incomplete. They attribute the result to Lepage, who seems to have found it in a matrix context.

A secondary objective of this chapter is to set up a framework that may be adopted in the study of the symmetric exterior algebra in the next chapter. Many of the results of that chapter are straightforward generalisations of those in the present one. In those cases, the proofs given here will not be duplicated.

The material in this chapter is distributed as follows.

I first review the exterior algebra built on a vector space, and in particular define generalised interior products. These products, while not new, are not in common use, and are often not discussed in any detail. I then introduce a symplectic structure by means of the two-form ω. Using this structure, I define the isomorphisms $\flat$ and $\sharp$. These correspond to the lowering or raising of all the indices of antisymmetric tensors. The symplectic structure is used to define a volume form Ω, and a Hodge dual operator. A new definition in the context of this algebra, the extended symplectic metric, is introduced. This 'metric' provides a lot of information in the sections that follow. Two operators L and M are defined, and their commutation relations are obtained. These expressions are used to prove the Lepage decomposition theorem. Up to this point the material is based mainly on the first chapter of the book by Libermann and Marle. The remainder of the material is new. The geometrical interpretation of the Lepage decompositon is given in terms of a direct sum of 'simple' subspaces. These subspaces are investigated in detail and expressions for their dimension and signature are found.

2.1 Exterior algebra

The standard interior product has the restrictive feature in that its first argument is limited to only one gradation of the exterior algebra - the vectors, while its second argument may be any exterior form. In this section I define generalisations of the interior products which allow both arguments to be completely general elements. I will attempt to follow this lead throughout, by avoiding definitions that are restricted to subsets of the algebras. I will also discuss a few results on volume forms in this section.

Take a finite dimensional vector space V. Let it have dimension m and let its dual be V^*. Denote the set of exterior forms of degree p by $\bigwedge^p(V^*)$ and the multivectors of degree p by $\bigwedge^p(V)$, according to the usual convention.

2.1.1 Left interior product

The standard interior product $\rfloor$ is defined by as follows.

Definition 2.1

The *interior product* $\rfloor$ is a map

$$\begin{aligned} \rfloor \; : \; & V \times \textstyle\bigwedge^p(V^*) \;\rightarrow\; \textstyle\bigwedge^{p-1}(V^*) \\ : \; & (x, \phi) \;\mapsto\; x\rfloor\phi\,, \end{aligned} \tag{2.1}$$

given by the following. Let x be a vector.

If ϕ is a scalar, define

$$x\rfloor\phi \;:=\; 0.$$

If ϕ is a homogeneous exterior form of degree $p \geq 1$, define

$$(x\rfloor\phi)(u_2, \ldots, u_p) \;:=\; \phi(x, u_2, \ldots, u_p) \tag{2.2}$$

where $u_2, \ldots, u_p$ are all vectors.

$\rfloor$ generalises to non-homogeneous forms by linearity. □

Before generalising $\rfloor$, we state a result, needed later, related to the homogeneity of a form.

Proposition 2.1

Let V be an m-dimensional vector space with any basis $\{e_i\}$. Let $\{e^i\}$ be its dual basis in V^* and let ϕ belong to $\bigwedge^p(V^*)$. Then

$$\sum_{i=1}^{m} e^i \wedge (e_i \rfloor \phi) = p\,\phi \tag{2.3}$$

The result may be found in Sternberg 1964 (p 21), and is analogous to Euler's theorem for homogeneous functions. As it appears here, it is a similar theorem for homogeneous exterior forms.

The definition of the left interior product may be extended so that a k-vector X may be contracted onto a p-form ϕ to yield a $(p-k)$-form $X \rfloor \phi$. The extended definition is in two steps. First define $X \rfloor \phi$ for decomposable X; then extend this to the non-decomposable case. This definition may be found in Yano (1970, p 5).

Definition 2.2

Let ϕ be any p form. Let a be a scalar. Define

$$a \rfloor \phi := a\phi.$$

Let $e_1, \ldots, e_k$ be vectors. Define

$$(e_1 \wedge \ldots \wedge e_k) \rfloor \phi := e_k \rfloor (\ldots \rfloor (e_1 \rfloor \phi)).$$

Let X be a k-vector with coordinate presentation $X^{i_1 \ldots i_k} e_{i_1} \wedge \ldots \wedge e_{i_k}$. If $k \leq p$, define

$$\begin{aligned} X \rfloor \phi &= (X^{i_1 \ldots i_k} e_{i_1} \wedge \ldots \wedge e_{i_k}) \rfloor \phi \\ &:= X^{i_1 \ldots i_k}((e_{i_1} \wedge \ldots \wedge e_{i_k}) \rfloor \phi). \end{aligned}$$

If $k > p$, define

$$X \rfloor \phi := 0.$$

□

This definition is expressed in terms of a coordinate system, but it is coordinate independent, by the extension principle (Crampin and Pirani p. 112). Libermann and Marle (ch I, §15.1) generalise the interior product by means of a different condition, given as follows, but which we will show to be equivalent.

Definition 2.3

Let ϕ be a p-form and let X be a k-vector. If $k > p$ then

$$X \rfloor \phi := 0 \,.$$

If $k \leq p$, then

$$(X \rfloor \phi)(Y) := \phi(X \wedge Y)\,, \tag{2.4}$$

for all $(p-k)$-vectors Y. □

The equivalence of the two definitions may be demonstrated as follows.

Proof

For $k > p$ the two definitions are explicitly equivalent. In the following, consider only $k \leq p$. To show that definition (2.2) may be deduced from definition (2.3), let a be a scalar and let Y be any $(p-k)$-form. Then

$$\begin{aligned}(a \rfloor \phi)(Y) &= \phi(a \wedge Y) \qquad \text{(by definition (2.2))}\\ &= \phi(aY)\\ &= (a\phi)(Y)\,;\end{aligned}$$

$$\begin{aligned}(e_1 \wedge \ldots \wedge e_k \rfloor \phi)(Y) &= \phi(e_1 \wedge \ldots \wedge e_k \wedge Y) \qquad \text{(definition (2.3))}\\ &= (e_1 \rfloor \phi)(e_2 \wedge \ldots \wedge e_k \wedge Y) \qquad \text{(definition (2.3))}\\ &= (e_2 \rfloor e_1 \rfloor \phi)(e_3 \wedge \ldots \wedge e_k \wedge Y) \qquad \text{(definition (2.3))}\end{aligned}$$

$$= \quad (e_k \rfloor \ldots \rfloor e_1 \rfloor \phi)(Y)\,; \qquad \text{(definition (2.3))}$$

$$\begin{aligned}
\{(X^{i_1\ldots i_k} e_{i_1} \wedge \ldots \wedge e_{i_k})\rfloor \phi\}(Y) &= \phi(X^{i_1\ldots i_k} e_{i_1} \wedge \ldots \wedge e_{i_k} \wedge Y) \qquad \text{(definition (2.3))} \\
&= X^{i_1\ldots i_k} \phi(e_{i_1} \wedge \ldots \wedge e_{i_k} \wedge Y) \\
&= X^{i_1\ldots i_k} \{(e_{i_1} \wedge \ldots \wedge e_{i_k} \rfloor \phi)(Y)\} \qquad \text{(definition (2.3))} \\
&= \left\{X^{i_1\ldots i_k}(e_{i_1} \wedge \ldots \wedge e_{i_k} \rfloor \phi)\right\}(Y)\,.
\end{aligned}$$

To show that definition (2.3) may be deduced from (2.2):

$$\begin{aligned}
(X\rfloor\phi)(Y) &= Y\rfloor X\rfloor\phi \qquad \text{(definition (2.2))} \\
&= Y^{i_1\ldots i_{p-k}} e_{i_1} \wedge \ldots \wedge e_{i_{p-k}} \rfloor \left(X^{j_1\ldots j_k} e_{j_1} \wedge \ldots \wedge e_{j_k} \rfloor \phi\right) \\
&= Y^{i_1\ldots i_{p-k}} e_{i_{p-k}} \rfloor \ldots \rfloor e_{i_1} \rfloor \left(X^{j_1\ldots j_k} e_{j_k} \rfloor \ldots \rfloor e_{j_1} \rfloor \phi\right) \qquad \text{(definition (2.2))} \\
&= Y^{i_1\ldots i_{p-k}} X^{j_1\ldots j_k} e_{j_1} \wedge \ldots \wedge e_{j_k} \wedge e_{i_1} \wedge \ldots \wedge e_{i_{p-k}} \rfloor \phi \qquad \text{(definition (2.2))} \\
&= X \wedge Y \rfloor \phi \\
&= \phi(X \wedge Y)\,.
\end{aligned}$$

□

Proposition 2.2

Let X, Y be multivectors and let ϕ, ψ be p-forms. Then

$$(X \wedge Y)\rfloor\phi = Y\rfloor(X\rfloor\phi) \qquad (2.5)$$

$$X\rfloor(\phi + \psi) = X\rfloor\phi + X\rfloor\psi \qquad (2.6)$$

$$X\rfloor(a\phi) = a(X\rfloor\phi) \qquad (2.7)$$

Proof

Let X and Y be of degree r and s respectively. If $r + s > p$, (2.5) is trivial;

if $0 \leq r + s \leq p$, then choose any $(p - r - s)$-form Z, and (2.5) is proven as follows:

$$\begin{aligned} (X \wedge Y)\rfloor\phi(Z) &= \phi(X \wedge Y \wedge Z) && \text{(definition (2.3))} \\ &= (X\rfloor\phi)(Y \wedge Z) && \text{(definition (2.3))} \\ &= \{Y\rfloor(X\rfloor\phi)\}(Z). && \text{(definition (2.3))} \end{aligned}$$

This holds for all Z and so (2.5) is proven.

To demonstrate (2.6), let X be an r-vector. Then for any $(p - r)$-vector Y,

$$\begin{aligned} (X\rfloor(\phi + \psi))(Y) &= (\phi + \psi)(X \wedge Y) && \text{(definition (2.3))} \\ &= \phi(X \wedge Y) + \psi(X \wedge Y) && \text{(definition of +)} \\ &= (X\rfloor\phi)(Y) + (X\rfloor\psi)(Y) && \text{(definition (2.3))} \\ &= (X\rfloor\phi + X\rfloor\psi)(Y) && \text{(definition of +)}. \end{aligned}$$

Since this holds for all Y, (2.6) is proven.

(2.7) may be proven by similar reasoning. □

2.1.2 The right interior product

Because the multivectors are duals of the exterior forms it is possible to define an operation analogous to the left interior product. The only difference between this new operation and the old one is that it allows forms to be contracted onto multivectors, rather than the other way around. Not surprisingly, the definition and properties are essentially the same as those given above. It is a generalisation of the *right interior product* defined by Libermann and Marle (ch I §4.3) and we will use the same name and symbol $\lfloor$ for it.

Definition 2.4

Let X be a multivector of degree k. Let a be a scalar. Define

$$X\lfloor a := aX.$$

Let θ be any one-form. Define

$$(X\lfloor\theta)(e^2,\ldots,e^k) := X(\theta,e^2,\ldots,e^k)$$

where $e^2,\ldots,e^k$ are 1-forms.

Let $e^{i_1},\ldots,e^{i_p}$ be one-forms and let $p \leq k$. Define

$$X\lfloor(e^{i_1}\wedge\ldots\wedge e^{i_p}) := ((X\lfloor e^{i_1})\lfloor\ldots)\lfloor e^{i_p}.$$

Let ϕ be any p-form with a coordinate presentation $\phi_{i_1\ldots i_p}e^{i_1}\wedge\ldots\wedge e^{i_p}$. If $p\leq k$, define

$$\begin{aligned} X\lfloor\phi &= X\lfloor(\phi_{i_1\ldots i_p}e^{i_1}\wedge\ldots\wedge e^{i_p}) \\ &:= \phi_{i_1\ldots i_p}(X\lfloor e^{i_1}\wedge\ldots\wedge e^{i_p}). \end{aligned}$$

If $p>k$, then define

$$X\lfloor\phi := 0\,.$$

□

Again, this definition is expressed in terms of a coordinate system, but it is coordinate independent, by the extension principle (Crampin and Pirani p 112). As for the left interior product, an equivalent definition is possible:

Definition 2.5

Let X be a k-vector and let ϕ be a p-form. If $p > k$ then

$$X \lfloor \phi := 0 .$$

If $p \leq k$, then

$$(X \lfloor \phi)(\psi) := X(\phi \wedge \psi) , \tag{2.8}$$

for all $(k-p)$-forms ψ. □

The following properties may be proven by the methods used above.

Proposition 2.3

Let X, Y be multivectors of equal degree, let ϕ and ψ be forms, and let a be a scalar. Then

$$X \lfloor (\phi \wedge \psi) = (X \lfloor \phi) \lfloor \psi \tag{2.9}$$

$$(X + Y) \lfloor \phi = X \lfloor \phi + Y \lfloor \phi \tag{2.10}$$

$$(aX) \lfloor \phi = a(X \lfloor \phi) . \tag{2.11}$$

Proposition 2.4

Let X be a p-vector, Y any multivector and θ a one-form. Then

$$(X \wedge Y) \lfloor \theta = (X \lfloor \theta) \wedge Y + (-1)^p X \wedge (Y \lfloor \theta) . \tag{2.12}$$

This property is the analogue of the Leibnitz property for the interior product of a 1-vector with an exterior product of forms.

Proposition 2.5

Let X be a multivector, and let ϕ be a form of the same degree. Then

$$X \rfloor \phi = X \lfloor \phi . \tag{2.13}$$

Proof

The proof relies on applying definition (2.3), which requires that $X \rfloor \phi$ have a scalar argument a:

$$\begin{aligned}(X\rfloor\phi)(a) &= \phi(X \wedge a) \qquad \text{(definition (2.3))}\\ &= a\phi(X)\\ &= aX(\phi)\\ &= X(\phi \wedge a)\\ &= (X\lfloor\phi)(a) \qquad \text{(definition (2.5))}\end{aligned}$$

□

2.1.3 The volume form and volume multivector

Further structure may be introduced into the vector space V by assuming it to have a volume m-form Ω. Associated with this form is an m-vector Λ defined by

Definition 2.6

Given the volume form Ω, define the volume vector Λ by

$$\Lambda\rfloor\Omega = 1\,. \tag{2.14}$$

□

There exists a basis $\{e_1, \ldots, e_m\}$ of V with dual basis $\{e^1, \ldots, e^m\}$ in V^* such that

$$\begin{aligned}\Omega &= e^1 \wedge \ldots \wedge e^m\\ \Lambda &= e_1 \wedge \ldots \wedge e_m\,.\end{aligned}$$

We will delay the choice of volume form until after the symplectic structure has been added in the next section. This will enable us to define the volume form in terms of the symplectic two-form. However it is appropriate to consider the volume here because while it may be defined from the symplectic structure, it does not require a symplectic structure to be defined.

Proposition 2.6

Let X be a p-vector of any degree $0 \le p \le m$. Then

$$\Lambda\lfloor(X\rfloor\Omega) = (-1)^{(m-1)p}X \tag{2.15}$$

Proof

It will suffice to show the result for decomposable X since it then extends to all X by linearity. First $X\rfloor\Omega$ will be evaluated, after which $\Lambda\lfloor(X\rfloor\Omega)$ will be found. Choose $X = e_{i_p} \wedge \ldots \wedge e_{i_1}$ where the indexes are arranged, without loss of generality, so that $i_p > \ldots > i_1$.

$$\begin{aligned} X\rfloor\Omega &= e_{i_p} \wedge \ldots \wedge e_{i_1}\rfloor e^1 \wedge \ldots \wedge e^{i_1} \wedge \ldots \wedge e^{i_p} \wedge \ldots \wedge e^m \\ &= (-1)^{i_1-1} e_{i_p} \wedge \ldots \wedge e_{i_1}\rfloor e^{i_1} \wedge e^1 \wedge \ldots \wedge \widehat{e^{i_1}} \wedge \ldots \wedge e^{i_2} \wedge \ldots \wedge e^{i_p} \wedge \ldots \wedge e^m \end{aligned}$$

where the hat indicates a missing factor. The e^i's dual to those in X are each taken to the 'front' of the volume form, as done above for e^{i_1}:

$$\begin{aligned} &= (-1)^{i_1-1}(-1)^{i_2-1} e_{i_p} \wedge \ldots \wedge e_{i_1}\rfloor e^{i_2} \wedge e^{i_1} \wedge e^1 \wedge \ldots \wedge \widehat{e^{i_1}} \wedge \ldots \\ &\qquad \ldots \wedge \widehat{e^{i_2}} \wedge \ldots \wedge e^{i_p} \wedge \ldots \wedge e^m \\ &= (-1)^{i_1-1} \ldots (-1)^{i_p-1} e_{i_p} \wedge \ldots \wedge e_{i_1}\rfloor e^{i_p} \wedge \ldots \wedge e^{i_1} \wedge e^1 \wedge \ldots \\ &\qquad \ldots \wedge \widehat{e^{i_1}} \wedge \ldots \wedge \widehat{e^{i_p}} \wedge \ldots \wedge e^m \\ &= (-1)^{\sum_{\alpha=1}^{p} i_\alpha - p} e^1 \wedge \ldots \wedge \widehat{e^{i_1}} \wedge \ldots \wedge \widehat{e^{i_p}} \wedge \ldots \wedge e^m \end{aligned}$$

Therefore

$$\Lambda\lfloor(e_{i_p} \wedge \ldots \wedge e_{i_1}\rfloor\Omega) = (-1)^{\sum_{\alpha=1}^{p} i_\alpha - p}\Lambda\lfloor e^1 \wedge \ldots \wedge \widehat{e^{i_1}} \wedge \ldots \wedge \widehat{e^{i_p}} \wedge \ldots \wedge e^m$$

$$= (-1)^{\sum i_\alpha - p}\Lambda(e^1, \ldots, e^{i_1-1}, e^{i_1+1}, \ldots, e^{i_p-1}, e^{i_p+1}, \ldots, e^m, \underbrace{-, \ldots, -}_{p \text{ spaces}})$$

We want to move each of the p spaces to the corresponding p positions of the 'missing' e^{i_α}s. First, we move all the p spaces so as to lie between e^{i_p-1} and e^{i_p+1}. This involves moving an argument past $m - i_p$ arguments, p times, and so introduces a factor of $(-1)^{p(m-i_p)}$:

$$= (-1)^{\sum i_\alpha - p}(-1)^{p(m-i_p)}\Lambda(e^1, \ldots, e^{i_1-1}, e^{i_1+1}, \ldots, e^{i_p-1}, \underbrace{-, \ldots, -}_{p \text{ spaces}}, e^{i_p+1}, \ldots, e^m)$$

Now move the leftmost $p-1$ spaces towards the left, past $i_p - 1 - i_{p-1}$ arguments to lie between $e^{i_{p-1}-1}$ and $e^{i_{p-1}+1}$:

$$\begin{aligned} &= (-1)^{\sum i_\alpha - p}(-1)^{p(m-i_p)}(-1)^{(p-1)(i_p-1-i_{p-1})} \\ &\quad \Lambda(e^1, \ldots, e^{i_{p-2}-1}, e^{i_{p-2}+1}, \ldots, e^{i_{p-1}-1}, \underbrace{-, \ldots, -}_{p-1 \text{ spaces}}, e^{i_{p-1}+1}, \ldots, e^{i_p-1}, -, e^{i_p+1}, \ldots, e^m) \end{aligned}$$

Continue until each space is correctly positioned. Then

$$\begin{aligned} &= (-1)^{\sum i_\alpha - p}(-1)^{p(m-i_p)+(p-1)(i_p-1-i_{p-1})+(p-2)(i_{p-1}-1-i_{p-2})+\ldots+2(i_3-1-i_2)+1(i_2-1-i_1)} \\ &\quad \Lambda(e^1, \ldots, e^{i_1-1}, -, e^{i_1+1}, \ldots, e^{i_{p-1}-1}, -, e^{i_{p-1}+1}, \ldots, e^{i_p-1}, -, e^{i_p+1}, \ldots, e^m) \\ &= (-1)^{\sum_{\alpha=1}^{p} i_\alpha - p + mp - \sum_{\alpha=1}^{p} i_\alpha - \sum_{\alpha=1}^{p-1}\alpha} e_{i_1} \wedge \ldots \wedge e_{i_p} \end{aligned}$$

after collecting terms in the exponent. Since $e_{i_p} \wedge \ldots \wedge e_{i_1} = (-1)^{\sum_{\alpha=1}^{p-1}\alpha} e_{i_1} \wedge \ldots \wedge e_{i_p}$, it follows that

$$\begin{aligned} \Lambda\lfloor(X\rfloor\Omega) &= (-1)^{mp-p-\sum_{\alpha=1}^{p-1}\alpha + \sum_{\alpha=1}^{p-1}\alpha} e_{i_p} \wedge \ldots \wedge e_{i_1} \\ &= (-1)^{(m-1)p} e_{i_p} \wedge \ldots \wedge e_{i_1} \\ &= (-1)^{(m-1)p} X. \end{aligned}$$

□

Proposition 2.7

Let ϕ be any homogeneous form of degree p. Then

$$(\Lambda\lfloor\phi)\rfloor\Omega = (-1)^{(m-1)p}\phi. \tag{2.16}$$

Proof

Choose $X = \Lambda\lfloor\phi$. This multivector is of degree $m-p$. Substituting X into (2.15) we have

$$\Lambda\lfloor((\Lambda\lfloor\phi)\rfloor\Omega) = (-1)^{(m-1)(m-p)}\Lambda\lfloor\phi.$$

This is equivalent to

$$\begin{aligned} ((\Lambda\lfloor\phi)\rfloor\Omega) &= (-1)^{(m-1)(m-p)}\phi \\ &= (-1)^{(m-1)p}\phi. \end{aligned}$$

□

2.2 The symplectic exterior algebra

2.2.1 The symplectic two-form

A symplectic structure is now introduced into the vector space V. This is done by selecting a nondegenerate two-form ω and decreeing it to be an invariant. This object generates a wealth of structure on the exterior algebra. In this section I will use it to define an isomorphism between the forms and the multivectors, a volume form, an isomorphism between the p-forms and the $(m-p)$-forms and an 'extended metric'.

The exterior algebra with this dignified form will be called the *symplectic exterior algebra on* V, and will be denoted by $\mathcal{S}(V^*,\omega)$. The homogeneous subspaces of degree p of this exterior algebra will be denoted by $\mathcal{S}^p(V^*,\omega)$, and the duals of these subspaces by $\mathcal{S}^p(V,\omega)$.

Implicit in the requirement that ω be nondegenerate, is the fact that V is of even dimension $m = 2n$ and that there exists a basis $\{e^1,\ldots,e^n;f^1,\ldots,f^n\}$

of V^* such that

$$\omega = \sum_{i=1}^{n} e^i \wedge f^i \,. \tag{2.17}$$

Such a basis is referred to as a *canonical* basis (see Libermann (1987), p. 6). The dual basis $\{e_1, \ldots, e_n; f_1, \ldots, f_n\}$ of V will also be called a canonical basis.

2.2.2 The lowering operator

Libermann and Marle (ch I, §2) describe how a two-form, such as the symplectic two-form ω, sets up a map from the 1-vectors to the 1-forms. This definition may be extended into a map from the p-vectors to the p-forms for $0 \leq p \leq 2n$. We employ the symbol $\flat$ in our notation[1]. This generalised operation and its inverse, the raising operation (see the next section), are not explicitly defined by Libermann and Marle; however they make use of such operations - for example in chapter I (§15.2, p 43).

Definition 2.7
If a is a scalar, then

$$\flat a := a; \tag{2.18}$$

if x is a vector, then

$$\flat x := -x \rfloor \omega; \tag{2.19}$$

if X is any p-vector, then

$$(\flat X)(u_1, \ldots, u_p) := (-1)^p X(\flat u_1, \ldots, \flat u_p) \tag{2.20}$$

where $u_1, \ldots, u_p$ are vectors. □

[1] The symbol presumably originates from the analogy between the lowering of the pitch of musical notes by a semitone, and the 'lowering' of contravariant indices of tensors to produce covariant indices.

Recalling that ω is nondegenerate,

Proposition 2.8

$$\omega \text{ is nondegenerate} \iff \flat : V \to V^* \text{ is a vector space isomorphism.}$$

This result is standard. See for instance Libermann and Marle (1987) ch. I part 1, §2.1.

Proposition 2.9

Let $\{e^i, f^i\}$ be a canonical basis of the symplectic vector space V^*, and let the dual basis for V be $\{e_i, f_i\}$. Then

$$\flat e_i = -f^i \tag{2.21}$$

$$\flat f_i = e^i \tag{2.22}$$

Proof

$$\begin{aligned}
\flat e_i &= -e_i \rfloor \omega \qquad \text{(by definition)} \\
&= -\sum_{j=1}^{n} e_i \rfloor (e^j \wedge f^j) \qquad \text{(the basis is canonical)} \\
&= -\sum_{j=1}^{n} \left\{ (e_i \rfloor e^j) \wedge f^j - e^j \wedge (e_i \rfloor f^j) \right\} \qquad \text{(by the Leibniz rule)} \\
&= -\sum_{j=1}^{n} \left\{ \delta_i^j f^j - 0 \right\} \\
&= -f^i
\end{aligned}$$

Similarly, for $\flat f_i$. □

Proposition 2.10

Let X, Y be multivectors of degree $0, 1, \ldots, 2n$. Then

$$\flat(X+Y) = (\flat X) + (\flat Y) \quad (2.23)$$

$$\flat(aX) = a(\flat X) \quad (\text{ where } a \text{ is a scalar}) \quad (2.24)$$

$$\flat(X \wedge Y) = (\flat X) \wedge (\flat Y) \quad (2.25)$$

Proof

For X, Y of degree zero the results follow trivially. To prove (2.23), let us assume X, Y are p-vectors where p ranges from 1 to $2n$ and let $u_1, \ldots, u_p$ be vectors.

$$\begin{aligned}
\flat(X+Y)(u_1,\ldots,u_p) &= (-1)^p((X+Y)(\flat u_1,\ldots,\flat u_p)) \quad (\text{by definition of } \flat)\\
&= (-1)^p(X(\flat u_1,\ldots,\flat u_p) + Y(\flat u_1,\ldots,\flat u_p))\\
&= (-1)^p(X(\flat u_1,\ldots,\flat u_p) + (-1)^p Y(\flat u_1,\ldots,\flat u_p))\\
&= \flat X(u_1,\ldots,u_p) + \flat Y(u_1,\ldots,u_p) \quad (\text{by definition of } \flat)\\
&= ((\flat X) + (\flat Y))(u_1,\ldots,u_p) \quad (\text{by definition of } +)
\end{aligned}$$

To prove (2.24), let a be a scalar:

$$\begin{aligned}
(\flat(aX))(u_1,\ldots,u_p) &= (-1)^p(aX)(\flat u_1,\ldots,\flat u_p) \quad (\text{by definition of } \flat)\\
&= ((-1)^p a)X(\flat u_1,\ldots,\flat u_p) \quad (\text{by vector properties})\\
&= a((-1)^p X(\flat u_1,\ldots,\flat u_p)) \quad (\text{by vector properties})\\
&= a(\flat X(u_1,\ldots,u_p)) \quad (\text{by definition of } \flat)\\
&= (a(\flat X))(u_1,\ldots,u_p) \quad (\text{scalar multiplication})
\end{aligned}$$

To prove (2.25), let Y be a q-form and let $v_1, \ldots, v_q$ be vectors.

$$(\flat(X \wedge Y))(u_1, \ldots, u_p, v_1, \ldots, v_q)$$

$$= ((-1)^{p+q}(X \wedge Y))(\flat u_1, \ldots, \flat u_p, \flat v_1, \ldots, \flat v_q) \qquad \text{(by definition of } \flat)$$

$$= (-1)^{p+q}\left\{\frac{1}{p!q!}\sum_{\pi} \mathrm{sign}(\pi) X(\pi(\flat u_1), \ldots, \pi(\flat u_p)) Y(\pi(\flat v_1), \ldots, \pi(\flat v_q))\right\}$$

$$\text{(by definition of } \wedge)$$

$$= \frac{1}{p!q!}\sum_{\pi} \mathrm{sign}(\pi)\left\{(-1)^p X(\pi(\flat u_1), \ldots, \pi(\flat u_p)))((-1)^q Y(\pi(\flat v_1), \ldots, \pi(\flat v_q))\right\}$$

$$= \frac{1}{p!q!}\sum_{\pi} \mathrm{sign}(\pi) \flat X(\pi(u_1), \ldots, \pi(u_p)) \flat Y(\pi(v_1), \ldots, \pi(v_q))$$

$$\text{(by definition of } \flat)$$

$$= ((\flat X) \wedge (\flat Y))(u_1, \ldots, u_p, v_1, \ldots, v_q)$$

$$\text{(by definition of } \wedge)$$

□

Proposition 2.11
Let X and Y be k-vectors. Then

$$(\flat X)(Y) = (-1)^k X(\flat Y)\,. \qquad (2.26)$$

2.2.3 The raising operator

By analogy with the definition of the lowering operator, a 'raising' operator will now be defined. It is denoted by $\sharp$ and is the inverse of the $\flat$ operator.

The definition makes use of a two-vector given special status by ω and is an extension of the mapping from 1-forms to 1-vectors defined by Libermann and Marle in chapter I, §4.3.

Definition 2.8

$\lambda \in \mathcal{S}^2(V,\omega)$ is defined by

$$\flat\lambda = \omega. \tag{2.27}$$

□

We call λ the *symplectic two-vector* or the *symplectic bivector* (see Libermann ch 1 §4).

Proposition 2.12

If $\{e_i, f_i\}$ is a canonical basis of a symplectic vector space (V,ω), then

$$\lambda = \sum_{i=1}^{n} e_i \wedge f_i \tag{2.28}$$

Proposition 2.13

$$\lambda \rfloor \omega = n\,. \tag{2.29}$$

Proof

In a canonical basis,

$$\begin{aligned}
\lambda \rfloor \omega &= \sum_{i=1}^{n}\sum_{j=1}^{n} e_i \wedge f_i \rfloor e^j \wedge f^j \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} e_i \wedge f_i(e^j, f^j) - \sum_{i=1}^{n}\sum_{j=1}^{n} e_i \wedge f_i(f^j, e^j) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \delta_i^j \delta_i^j
\end{aligned}$$

$$= \sum_{i=1}^{n} \delta_i^i \delta_i^i$$
$$= n\,.$$

□

Definition 2.9

If a is a scalar, then

$$\sharp a := a; \tag{2.30}$$

if θ is a one-form, then

$$\sharp\theta := \lambda\lfloor\theta; \tag{2.31}$$

if ϕ is any p-form, then

$$(\sharp\phi)(\theta^1, \ldots, \theta^p) := (-1)^p\,\phi(\sharp\theta^1, \ldots, \sharp\theta^p) \tag{2.32}$$

where $\theta^1, \ldots, \theta^p$ are one-forms. □

Proposition 2.14

If $\{e_i, f_i\}$ is a canonical basis of a symplectic vector space (V, ω), and if $\{e^i, f^i\}$ is the dual basis then

$$\sharp e_i = f^i \tag{2.33}$$

$$\sharp f_i = -e^i \tag{2.34}$$

Proposition 2.15

For ϕ, ψ forms of any degree,

$$\sharp(\phi + \psi) = (\sharp\phi) + (\sharp\psi) \tag{2.35}$$

$$\sharp(a\phi) = a(\sharp\phi) \tag{2.36}$$

$$\sharp(\phi \wedge \psi) = (\sharp\phi) \wedge (\sharp\psi) \tag{2.37}$$

The proof of this proposition is essentially identical to the proof of the analogous set of properties for the $\flat$ operator.

Proposition 2.16

Let ϕ and ψ be p-forms. Then

$$(\sharp\phi)(\psi) = (-1)^p \phi(\sharp\psi)\,. \tag{2.38}$$

We have seen that $\flat$ and $\sharp$ define homomorphisms on the exterior algebra, and that $\flat$ is an algebraic isomorphism on V. The next result shows that both operators are algebraic isomorphisms on the entire exterior algebra.

Proposition 2.17

Let ϕ be a form of any degree and let X be a multivector of any degree.

$$\sharp(\flat X) = X \tag{2.39}$$

$$\flat(\sharp\phi) = \phi. \tag{2.40}$$

Proof

For X and ϕ of degree zero the result is trivial. For a vector x, consider the vector $\sharp(\flat x)$. We operate it on an arbitrary form. Because $\flat$ is an isomorphism, we may express this arbitrary form as $\flat u$:

$$\begin{aligned}
(\sharp(\flat x))(\flat u) &= (\lambda\lfloor(\flat x))(\flat u) \quad \text{(by definition of } \sharp) \\
&= \lambda(\flat x, \flat u) \quad \text{(by definition of } \lfloor) \\
&= \flat\lambda(x, u) \quad \text{(by definition of } \flat) \\
&= \omega(x, u) \quad \text{(by definition of } \lambda) \\
&= (-u\rfloor\omega)(x) \quad \text{(by definition of } \rfloor)
\end{aligned}$$

$$= x(\flat u)$$

For a p-vector X with arbitrary one-form arguments $\flat u_1, \ldots, \flat u_p$ we have

$$
\begin{aligned}
(\sharp(\flat X))(\flat u_1, \ldots, \flat u_p) &= (-1)^p (\flat X)(\sharp\flat u_1, \ldots, \sharp\flat u_p) && \text{(by definition of } \sharp) \\
&= (-1)^p (\flat X)(u_1, \ldots, u_p) && \text{(by previous result)} \\
&= (-1)^p (-1)^p X(\flat u_1, \ldots, \flat u_p) && \text{(by definition of } \flat) \\
&= X(\flat u_1, \ldots, \flat u_p)
\end{aligned}
$$

This is sufficient to prove the result since any p-form ϕ (including a one-form) may be written $\flat X$ by the above. Hence $\flat(\sharp\phi) = \flat\{\sharp(\flat X)\} = \flat\{X\}$ which is just ϕ. □

Proposition 2.18

Let X be any r-vector and let ϕ be any homogeneous form. Then

$$\sharp(X \rfloor \phi) = (-1)^r (\sharp\phi) \lfloor (\flat X). \tag{2.41}$$

Proof

Let ϕ be of degree p. If $p < r$ then both sides are zero. In the following consider only $p \geq r$. For any $(p-r)$-form ψ,

$$
\begin{aligned}
\{\sharp(X \rfloor \phi)\}(\psi) &= (-1)^{p-r} (X \rfloor \phi)(\sharp\psi) && \text{(using (2.38))} \\
&= (-1)^{p-r} \phi(X \wedge \sharp\psi) && \text{(definition (2.3))} \\
&= (-1)^{p-r} \phi((\sharp\flat X) \wedge \sharp\psi) \\
&= (-1)^{p-r} (-1)^p (\sharp\phi)((\flat X) \wedge \psi) && \text{(using (2.38))}
\end{aligned}
$$

$$= (-1)^r \{(\sharp\phi)\lfloor(\flat X)\}(\psi) \qquad \text{(definition (2.5))}$$

□

2.2.4 The volume elements of $\mathcal{S}(V^*,\omega)$

It is natural to define a volume m-form in terms of the symplectic two-form ω. This in turn selects a volume m-vector and it will be expressed in terms of the symplectic two-vector λ.

Definition 2.10

The volume form of $\mathcal{S}(V^*,\omega)$, denoted by Ω, is defined by

$$\Omega := \frac{\omega^n}{n!}. \qquad (2.42)$$

□

In a canonical basis, Ω is given by

$$\Omega = (-1)^{\frac{1}{2}n(n-1)} e^1 \wedge \ldots \wedge e^n \wedge f^1 \wedge \ldots \wedge f^n \qquad (2.43)$$

(see Libermann (1987), ch I, §3.4). Woodhouse (1980) (ch. 1, §1.2) refers to Ω as the 'Liouville volume element'.

The volume vector Λ is selected by the condition (2.14) and may be expressed in the canonical basis as

$$\Lambda = (-1)^{\frac{1}{2}n(n-1)} e_1 \wedge \ldots \wedge e_n \wedge f_1 \wedge \ldots \wedge f_n \,. \qquad (2.44)$$

It follows that

$$\Lambda = \frac{\lambda^n}{n!}. \qquad (2.45)$$

Proposition 2.19

$$\begin{aligned} \sharp\Omega &= \Lambda & (2.46) \\ \flat\Lambda &= \Omega & (2.47) \end{aligned}$$

Proof

$$\begin{aligned} \sharp\Omega &= \sharp\left(\frac{\omega^n}{n!}\right) && \text{(by definition)} \\ &= \frac{1}{n!}\sharp(\omega^n) && \text{(by linearity of } \sharp) \\ &= \frac{1}{n!}(\sharp\omega)^n && (\sharp \text{ respects } \wedge) \\ &= \frac{1}{n!}\lambda^n && \text{(definition of } \lambda) \\ &= \Lambda. \end{aligned}$$

To prove (2.47), note that $\flat\Lambda = \flat(\sharp\Omega)$ by (2.46) just proved, and hence $\flat\Lambda = \Omega$. □

2.2.5 Symplectic Hodge dual

The structures defined above may be used to define an operation analogous to the Hodge star operator of metric vector spaces. The definition is due to Libermann and Marle (ch I p 43):

Definition 2.11
The *star* or *adjoint* operator $*$ is given by

$$\begin{aligned} * \ &: \ \mathcal{S}^p(V^*,\omega) \rightarrow \mathcal{S}^{2n-p}(V^*,\omega) \\ &: \ \phi \mapsto *\phi = (\sharp\phi)\rfloor\Omega \end{aligned} \qquad (2.48)$$

where Ω is the volume form defined previously. □

Proposition 2.20

Let ϕ and ψ be homogeneous forms of equal degree, and let a be a scalar. Then

$$*(\phi + \psi) = (*\phi) + (*\psi) \tag{2.49}$$

$$*(a\phi) = a(*\phi). \tag{2.50}$$

For ϕ and ψ not necessarily of equal degree,

$$*(\phi \wedge \psi) = (\sharp\psi)\rfloor(*\phi). \tag{2.51}$$

Proof

To prove (2.49):

$$\begin{aligned} *(\phi + \psi) &= (\sharp(\phi + \psi))\rfloor\Omega \quad \text{(by definition of } *) \\ &= (\sharp\phi + \sharp\psi)\rfloor\Omega \quad \text{(by properties of } \sharp) \\ &= (\sharp\phi)\rfloor\Omega + (\sharp\psi)\rfloor\Omega \quad \text{(by properties of } \rfloor) \\ &= *\phi + *\psi \quad \text{(by definition of } *) \end{aligned}$$

To prove (2.50):

$$\begin{aligned} *(a\phi) &= (\sharp(a\phi))\rfloor\Omega \quad \text{(by definition of } *) \\ &= (a(\sharp\phi))\rfloor\Omega \quad \text{(by linearity of } \sharp) \\ &= a\{(\sharp\phi)\rfloor\Omega\} \quad \text{(by properties of } \rfloor) \\ &= a(*\phi) \quad \text{(by definition of } *) \end{aligned}$$

To prove (2.51):

$$
\begin{aligned}
*(\phi \wedge \psi) &= (\sharp(\phi \wedge \psi)) \rfloor \Omega \qquad \text{(by definition of } *) \\
&= (\sharp\phi \wedge \sharp\psi) \rfloor \Omega \qquad \text{(by properties of } \sharp) \\
&= (\sharp\psi) \rfloor ((\sharp\phi) \rfloor \Omega) \qquad \text{(by properties of } \rfloor) \\
&= (\sharp\psi) \rfloor (*\phi) \qquad \text{(by definition of } *)
\end{aligned}
$$

□

Proposition 2.21

Let ϕ be any homogeneous form. Then

$$
(\phi) = \phi \tag{2.52}
$$

$$
*\Omega = 1. \tag{2.53}
$$

Proof

To prove (2.52), let $\phi \in \mathcal{S}^p(V^*, \omega)$. Then

$$
\begin{aligned}
(\phi) &= *((\sharp\phi) \rfloor \Omega) \qquad \text{(by definition of } *) \\
&= \{\sharp((\sharp\phi) \rfloor \Omega)\} \rfloor \Omega \qquad \text{(by definition of } *) \\
&= (-1)^p ((\sharp\Omega) \lfloor \phi) \rfloor \Omega \qquad \text{(by property of } \sharp \text{ and } \rfloor) \\
&= (-1)^p (\Lambda \lfloor \phi) \rfloor \Omega \qquad \text{(by definition of } \Lambda) \\
&= (-1)^p (-1)^{(2n-1)p} \phi \qquad \text{(by (2.16) with } m = 2n) \\
&= \phi
\end{aligned}
$$

(2.53) is proven as follows.

$$\begin{aligned} *\Omega &= \sharp\Omega\rfloor\Omega \qquad \text{(definition of } *) \\ &= \Lambda\rfloor\Omega \\ &= 1\,. \qquad \text{(definition of } \Lambda) \end{aligned}$$

□

Because of the duality between the forms and the vectors, it is possible to define a similar adjoint operator on the multivectors, using the definition

$$*X := \Lambda\lfloor(\flat X)\,,$$

where X is a multivector. I will not make use of this operation.

2.2.6 The extended symplectic metric

'Metric' here should strictly be 'bilinear form' since we will use the former to refer to antisymmetric as well as symmetric bilinear maps. The symplectic form defines a bilinear map (to the reals) on only one gradation of the algebra $\mathcal{S}(V^*,\omega)$, namely the one-forms. To eliminate this imbalance, I will use ω to induce a bilinear map on the entire exterior algebra $\mathcal{S}(V^*,\omega)$.

The symplectic bivector λ defines a 'metric' on V^* as follows:

$$\begin{aligned} \lambda \;&:\; V^* \times V^* \;\rightarrow\; \Re \\ &:\; \alpha, \beta \;\;\mapsto\; \lambda(\alpha,\beta) \end{aligned} \tag{2.54}$$

By rewriting this expression for the metric, an expression is obtained that generalises to the entire symplectic exterior algebra.

$$\lambda(\alpha,\beta) = (\lambda\lfloor\alpha)\lfloor\beta \qquad \text{(definition of } \lfloor)$$

$$= \sharp\alpha \lfloor \beta \qquad \text{(definition of } \sharp\text{)}$$

$$= \sharp\alpha \rfloor \beta \qquad \text{(property of the interior product)}$$

This expression does not require the arguments α, β to be one-forms, so define

Definition 2.12

Let $\alpha \in \mathcal{S}^p(V^*, \omega)$ and $\beta \in \mathcal{S}^q(V^*, \omega)$. Then define

$$\begin{aligned} \omega(\,,\,) &: \mathcal{S}^p(V^*,\omega) \times \mathcal{S}^q(V^*,\omega) \rightarrow \Re \\ &: \alpha, \beta \qquad\qquad \mapsto \omega(\alpha,\beta) := \delta_{pq}\, \sharp\alpha \rfloor \beta \end{aligned} \tag{2.55}$$

□

The abuse of notation here should not lead to any confusion as the degrees of the arguments will usually be clear from the context.

2.2.7 Properties of the extended symplectic metric

We examine some properties of the extended symplectic metric.

Proposition 2.22

For α, $\phi \in \mathcal{S}^p(V^*, \omega)$,

$$\omega(\alpha, \phi) = (-1)^p \omega(\phi, \alpha). \tag{2.56}$$

Proof

It will suffice to show the result for decomposable α since it extends to the non-decomposable forms by linearity. We assume then that $\alpha = \alpha^1 \wedge \ldots \wedge \alpha^p$.

$$\begin{aligned}
\omega(\alpha, \phi) &= \sharp\alpha \rfloor \phi \quad \text{(definition of metric)} \\
&= \sharp(\alpha^1 \wedge \ldots \wedge \alpha^p) \rfloor \phi \\
&= (\sharp\alpha^1 \wedge \ldots \wedge \sharp\alpha^p) \rfloor \phi \quad \text{(property of } \sharp) \\
&= \phi(\sharp\alpha^1, \ldots, \sharp\alpha^p) \quad \text{(definition of interior product)} \\
&= (-1)^p \sharp\phi(\alpha^1, \ldots, \alpha^p) \quad \text{(definition of } \sharp) \\
&= (-1)^p \sharp\phi \lfloor \alpha \quad \text{(definition of interior product)} \\
&= (-1)^p \sharp\phi \rfloor \alpha \quad \text{(since the degrees are equal)} \\
&= (-1)^p \omega(\phi, \alpha) \quad \text{(definition of metric.)}
\end{aligned}$$

□

This result shows that the extended symplectic metric is symmetric on the homogeneous subspaces of even degree and antisymmetric on the homogeneous subspaces of odd degree.

Proposition 2.23
Let $\alpha,\ \phi \in \mathcal{S}^p(V^*, \omega)$. Then

$$\omega(*\alpha, *\phi) = \omega(\alpha, \phi). \tag{2.57}$$

Proof

$$\omega(*\alpha, *\phi) = \omega(\sharp\alpha \rfloor \Omega, *\phi) \quad \text{(definition of } *)$$

$$
\begin{aligned}
&= \sharp(\sharp\alpha \rfloor \Omega) \rfloor * \phi \qquad \text{(definition of metric)} \\
&= (-1)^p (\Lambda \lfloor \alpha) \rfloor * \phi \qquad \text{(property of } \sharp \text{ and } \rfloor) \\
&= (-1)^p (\Lambda \lfloor \alpha) \lfloor * \phi \qquad \text{(property of } \rfloor) \\
&= (-1)^p (-1)^{p(2n-p)} (\Lambda \lfloor * \phi) \lfloor \alpha \qquad \text{(swapping arguments)} \\
&= (-1)^p (-1)^{2np-p^2} (\Lambda \lfloor (\sharp \phi \rfloor \Omega)) \lfloor \alpha \qquad \text{(definition of } *)
\end{aligned}
$$

Because the dimension of our vector space is $2n$, it follows from (2.15) that $\Lambda \lfloor (\sharp\phi \rfloor \Omega) = (-1)^{(2n-1)p} \sharp\phi$, and this is just equal to $(-1)^p \sharp\phi$. Therefore

$$
\begin{aligned}
\omega(*\alpha, *\phi) &= (-1)^{2np-p^2} \sharp\phi \lfloor \alpha \\
&= (-1)^{p^2} \omega(\phi, \alpha) \qquad \text{(definition of metric)} \\
&= (-1)^{p^2+p} \omega(\alpha, \phi) \qquad \text{(symmetry of metric)} \\
&= \omega(\alpha, \phi) \qquad \text{(since the exponent is always even.)}
\end{aligned}
$$

□

Proposition 2.24

Let ϕ and ψ be forms of equal degree p, for $0 \leq p \leq 2n$. Then

$$\phi \wedge *\psi = \omega(\psi, \phi)\Omega \tag{2.58}$$

Proof

$$*(\phi \wedge *\psi) = \sharp(*\psi) \rfloor * \phi \qquad \text{(using (2.51),)}$$

$$= \omega(*\psi, *\phi) \quad \text{(definition of extended metric)}$$
$$= \omega(\psi, \phi) \quad \text{(by (2.57).)}$$

Since $\omega(\psi, \phi)$ is a scalar, $*\omega(\psi, \phi) = \omega(\psi, \phi)\Omega$. Thus taking $*$ of both sides yields

$$\phi \wedge *\psi = \omega(\psi, \phi)\Omega.$$

□

Definition 2.13

Let $e^1, \ldots, e^n; f^1, \ldots, f^n$ be a canonical basis of V^*. The basis of $\mathcal{S}^p(V^*, \omega)$ generated by exterior products of these 1-forms will be referred to as a *canonical basis of* $\mathcal{S}^p(V^*, \omega)$. □

Proposition 2.25

The extended metric is nondegenerate. That is, for p-forms ϕ, ψ:

$$\omega(\phi, \psi) = 0 \ \ \forall\psi \implies \phi = 0.$$

Proof

$$\begin{aligned} \omega(\phi, \psi) &= 0 \quad \forall\psi \\ \iff \psi \wedge *\phi &= 0 \quad \forall\psi. \qquad \text{(by (2.58))} \end{aligned}$$

If this holds for all ψ, then in particular it holds for the elements of a canonical basis. For instance, if $\psi = e^1 \wedge \ldots \wedge e^k \wedge f^1 \wedge \ldots \wedge f^{p-k}$, then

$$e^1 \wedge \ldots \wedge e^k \wedge f^1 \wedge \ldots \wedge f^{p-k} \wedge *\phi = 0.$$

By a standard result, we know that for a 1-form θ and a p-form ξ, $\theta \wedge \xi = 0 \iff \theta$ is a factor of ξ. Thus, $*\phi$ has factors $e^1, \ldots, e^k, f^1, \ldots, f^{p-k}$. By putting ψ equal to each element of a canonical basis, we find that $*\phi$ has the $2n$ factors $\{e^1, \ldots, e^n, f^1, \ldots, f^n\}$. But $*\phi$ is of degree $2n - p$ only; this contradiction implies that $*\phi = 0$, and consequently that $\phi = 0$, for those cases where $p > 0$.

If $p = 0$, then we have that $*\phi = a\Omega$, with a being a scalar. $\psi = b$ is also a scalar, because the arguments of the metric need to be of equal degree. Then

$$
\begin{aligned}
\psi \wedge *\phi &= 0 \quad \forall \psi \\
\iff \quad & ba\Omega = 0 \quad \forall b \\
\iff \quad & ba = 0 \quad \forall b \\
\iff \quad & a = 0 \\
\iff \quad & \phi = 0.
\end{aligned}
$$

□

Theorem 2.1

The homogeneous subspaces $\mathcal{S}^p(V^*, \omega)$ of the symplectic exterior algebra are symplectic vector spaces for p odd, and metric vector spaces for p even.

Proof

A vector space is symplectic if it has a non-degenerate, antisymmetric, bilinear form on it (see Libermann (1987) ch. I, part 1, §3.1). The extended metric is all of these things, for p odd, by the above results. For p even, the extended metric is nondegenerate, symmetric and bilinear; hence the even degree subspaces are metric vector spaces. □

The matrices representing the metric on the odd degree homogeneous subspaces of $\mathcal{S}(V^*,\omega)$ all have the same symplectic canonical form, by the above result. But the corresponding matrices for the even degree subspaces have different canonical forms, determined by their signatures. In the following results, the signature of $\mathcal{S}^{2r}(V^*,\omega)$, for $0 \le r \le n$, is obtained.

To avoid confusion between different definitions of the signature, we specify that the signature of a symmetric matrix is the number of positives less the number of negatives on the diagonal when expressed in the diagonal basis.

Proposition 2.26

Let ϕ, ψ be elements of a canonical basis of $\mathcal{S}^p(V^*,\omega)$. Then $\omega(\phi,\psi) \neq 0$ if and only if the following two conditions are met:

(i) the set of e-indices of ϕ equals the set of f-indices of ψ,

(ii) the set of f-indices of ϕ equals the set of e-indices of ψ.

Proof

$$\begin{aligned}&\omega(e^{i_1}\wedge\ldots\wedge e^{i_a}\wedge f^{j_1}\wedge\ldots\wedge f^{j_b}, e^{k_1}\wedge\ldots\wedge e^{k_c}\wedge f^{l_1}\wedge\ldots\wedge f^{l_d})\\ &= \sharp(e^{i_1}\wedge\ldots\wedge e^{i_a}\wedge f^{j_1}\wedge\ldots\wedge f^{j_b})\rfloor e^{k_1}\wedge\ldots\wedge e^{k_c}\wedge f^{l_1}\wedge\ldots\wedge f^{l_d} \qquad (2.59)\end{aligned}$$

But from (2.33) and (2.34) we have that $\sharp e^i = f_i$ and $\sharp f^i = -e_i$; hence the above becomes

$$\begin{aligned}&= (-1)^b f_{i_1}\wedge\ldots\wedge f_{i_a}\wedge e_{j_1}\wedge\ldots\wedge e_{j_b}\rfloor e^{k_1}\wedge\ldots\wedge e^{k_c}\wedge f^{l_1}\wedge\ldots\wedge f^{l_d}\\ &= (-1)^b(-1)^{ab} e_{j_1}\wedge\ldots\wedge e_{j_b}\wedge f_{i_1}\wedge\ldots\wedge f_{i_a}\rfloor e^{k_1}\wedge\ldots\wedge e^{k_c}\wedge f^{l_1}\wedge\ldots\wedge f^{l_d}\\ &= (-1)^{b(a+1)}\sum_\pi \mathrm{sign}(\pi) e^{k_1}\wedge\ldots\wedge e^{k_c}\wedge f^{l_1}\wedge\ldots\wedge f^{l_d}(\pi e_{j_1},\ldots,\pi e_{j_b},\pi f_{i_1},\ldots,\pi f_{i_a}),\end{aligned}$$

where each π is a permutation of the index set of $a+b$ elements. From consideration of this expression it is clear that it is non-zero if and only if the following sets are equal:

$$\{k_1, \ldots, k_c\} = \{j_1, \ldots, j_b\}$$
$$\{l_1, \ldots, l_d\} = \{i_1, \ldots, i_a\}.$$

Consequently, $a = d$ and $b = c$, and the result is proven. □

Corollary

Let ϕ be any element of a canonical basis for $\mathcal{S}^p(V^*, \omega)$, where $0 \leq p \leq 2n$. There exists one and only one element ψ of the basis, not necessarily distinct from ϕ, such that $\omega(\phi, \psi) \neq 0$.

Proof

The existence as well as the uniqueness follow quite easily from the construction used in the proposition above. □

Definition 2.14

Given a canonical basis of $\mathcal{S}^p(V^*, \omega)$ for any $0 \leq p \leq 2n$, two distinct elements ϕ, ψ of the basis satisfying $\omega(\phi, \psi) \neq 0$ will be called 'conjugates'; an element ϕ of the basis satisfying $\omega(\phi, \phi) \neq 0$ will be called 'self-conjugate'. □

From the previous corollary, it follows that any canonical basis element is either self-conjugate, or has an unique, distinct, conjugate element. This conjugation relation is symmetric.

Theorem 2.2

For $0 \leq r \leq n$,

$$\text{signature } \mathcal{S}^{2r}(V^*, \omega) = \binom{n}{r}. \tag{2.60}$$

Proof

The result is obtained by considering the matrix representation of the extended metric with respect to the canonical basis.

Because of the symmetry of the extended metric on the spaces we are

considering, any pair of distinct conjugates induces a 2×2 block on the main diagonal, of the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

But both of these blocks are similar to the zero signature block

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The problem of finding the signature of the metric on the whole space is thus reduced to that of finding the signature of the block corresponding to the self-conjugate canonical basis elements only. This block must by definition be zero everywhere except on the main diagonal, where all the elements must be non-zero. Because all self-conjugate elements must be of the form $e^{i_1} \wedge \ldots \wedge e^{i_r} \wedge f^{i_1} \wedge \ldots \wedge f^{i_r}$, we have, by putting $a = b = c = d = r$ into (2.59), that:

$$\begin{aligned}
&\omega(e^{i_1} \wedge \ldots \wedge e^{i_r} \wedge f^{i_1} \wedge \ldots \wedge f^{i_r}, e^{i_1} \wedge \ldots \wedge e^{i_r} \wedge f^{i_1} \wedge \ldots \wedge f^{i_r}) \\
&\quad = (-1)^r(-1)^{r^2} e_{i_1} \wedge \ldots \wedge e_{i_r} \wedge f_{i_1} \wedge \ldots \wedge f_{i_r} \rfloor e^{i_1} \wedge \ldots \wedge e^{i_r} \wedge f^{i_1} \wedge \ldots \wedge f^{i_r} \\
&\quad = (-1)^{r(r+1)} 1 \\
&\quad = 1. \qquad \text{(for } r \text{ even or odd)}
\end{aligned}$$

Thus each self-conjugate element contributes $+1$ to the signature of the metric; the signature $\mathcal{S}^{2r}(V^*, \omega)$ is therefore equal to the number of self- conjugate elements in the canonical basis.

In how many ways may we choose elements $e^{i_1} \wedge \ldots \wedge e^{i_r} \wedge f^{i_1} \wedge \ldots \wedge f^{i_r}$? Clearly, once the e's are selected, the f's are fixed, so we need merely count the number of possible choices of the former. The e's are to be distinct, there must be r of them, and they must be taken from the pool of n available e's. Thus they may be chosen in $\binom{n}{r}$ different ways.

Hence there are $\binom{n}{r}$ self-conjugate elements in any canonical basis of $S^{2r}(V^*, \omega)$, and by the above reasoning, this number equals the signature of the space. This completes the proof. □

2.3 The Lepage decomposition

In this section, I use the existing symplectic structure to introduce two operators L and M. Their definitions are not new - see Libermann and Marle (1987), Hodge (1941), and Chern (1956). The properties of these operators will be investigated and in particular their commutation relations will be found. These relations will be used to prove the Lepage decomposition theorem following arguments similar to those of Chern.

2.3.1 Operators L and M.

Definition 2.15

Let θ be an element of $S^p(V^*, \omega)$. Define

$$\begin{aligned} L_\omega &: S^p(V^*,\omega) \rightarrow S^{p+2}(V^*,\omega) \\ &: \theta \mapsto L_\omega\theta = \omega \wedge \theta \end{aligned} \tag{2.61}$$

Where no ambiguity arises, the subscript will be dropped, and we will write L in place of L_ω. □

Proposition 2.27

Let a, b be reals. Let ϕ, ψ belong to $\mathcal{S}^p(V^*, \omega)$ and ξ to $\mathcal{S}^k(V^*, \omega)$. Then

$$L(a\phi + b\psi) = aL\phi + bL\psi \quad (2.62)$$

$$L(\phi \wedge \xi) = (L\phi) \wedge \xi = \phi \wedge (L\xi). \quad (2.63)$$

Proof

The linearity follows trivially from the bilinearity of the exterior product. To demonstrate the distribution over the exterior product:

$$\begin{aligned} L(\phi \wedge \psi) &= \omega \wedge (\phi \wedge \psi) && \text{(by definition of } L) \\ &= (\omega \wedge \phi) \wedge \psi && \text{(by associativity of } \wedge) \\ &= (L\phi) \wedge \psi ; && \text{(by definition of } L) \end{aligned}$$

furthermore

$$\begin{aligned} L(\phi \wedge \psi) &= \omega \wedge (\phi \wedge \psi) && \text{(by definition of } L) \\ &= (\omega \wedge \phi) \wedge \psi && \text{(by associativity of } \wedge) \\ &= (\phi \wedge \omega) \wedge \psi && \text{(since } \omega \text{ is of even degree)} \\ &= \phi \wedge (\omega \wedge \psi) && \text{(by associativity of } \wedge) \\ &= \phi \wedge (L\psi) . && \text{(by definition of } L) \end{aligned}$$

□

Definition 2.16

Let θ be an element of $\mathcal{S}^p(V^*,\omega)$. Define

$$\begin{aligned} M_\omega \;&:\; \mathcal{S}^p(V^*,\omega) \;\rightarrow\; \mathcal{S}^{p-2}(V^*,\omega) \\ &:\; \theta \;\mapsto\; M_\omega\theta = (\sharp\omega)\rfloor\theta = \lambda\rfloor\theta \end{aligned} \tag{2.64}$$

Where no ambiguity arises, the subscript on M will be dropped and we will write M in place of M_ω. □

Proposition 2.28

M is a linear operator on the exterior algebra. That is, for a, b reals and ϕ, ψ belonging to $\mathcal{S}^p(V^*,\omega)$

$$M(a\phi + b\psi) = a(M\phi) + b(M\psi). \tag{2.65}$$

Furthermore, M is not a derivation with respect to the exterior product.

The above proposition is easily proven; I give no explicit proof. To 'measure' the degree to which M fails to be a derivation, define:

Definition 2.17

Let ϕ, ψ be forms of degree k and r respectively. Define

$$\begin{aligned} P_M \;&:\; \mathcal{S}^k(V^*,\omega)\times\mathcal{S}^r(V^*,\omega) \;\rightarrow\; \mathcal{S}^{k+r-2}(V^*,\omega) \\ &:\; (\phi,\psi) \;\mapsto\; P_M(\phi,\psi) \end{aligned} \tag{2.66}$$

where

$$P_M(\phi,\psi) \doteq M(\phi\wedge\psi) - (M\phi)\wedge\psi - \phi\wedge(M\psi).$$

□

I will not investigate the properties of P_M here.

Proposition 2.29

Let θ be an element of $\mathcal{S}^p(V^*, \omega)$. Then

$$M_\omega \theta = *L_\omega * \theta \tag{2.67}$$

This result appears in Libermann and Marle, but I nevertheless include a proof.

Proof

Let θ be a p-form.

$$\begin{aligned}
L\theta &= *L(\sharp\theta \rfloor \Omega) \quad \text{(by definition of } *) \\
&= *(\omega \wedge (\sharp\theta \rfloor \Omega)) \quad \text{(by definition of } L) \\
&= \sharp(\omega \wedge (\sharp\theta \rfloor \Omega)) \rfloor \Omega \quad \text{(by definition of } *) \\
&= (\lambda \wedge \sharp(\sharp\theta \rfloor \Omega)) \rfloor \Omega \quad \text{(property of } \sharp) \\
&= (-1)^p (\lambda \wedge (\Lambda \lfloor \theta)) \rfloor \Omega \quad \text{(property of } \sharp) \\
&= (-1)^p ((\Lambda \lfloor \theta) \wedge \lambda) \rfloor \Omega \quad (\lambda \text{ is of even degree}) \\
&= (-1)^p \lambda \rfloor ((\Lambda \lfloor \theta) \rfloor \Omega) \quad \text{(property of } \rfloor) \\
&= (-1)^p (-1)^{(2n-1)p} \lambda \rfloor \theta \quad \text{(identity for } \Omega \text{ and } \Lambda) \\
&= M_\omega \theta \quad \text{(by definition of } M)
\end{aligned}$$

□

Proposition 2.30

M and L are adjoint operators with respect to the extended symplectic metric. That is, if degree(α) is two less than degree(β), then

$$\omega(L\alpha, \beta) = \omega(\alpha, M\beta). \tag{2.68}$$

Proof

$$\begin{aligned}
\omega(L\alpha, \beta) &= \omega(\omega \wedge \alpha, \beta) && \text{(definition of } L) \\
&= \sharp(\omega \wedge \alpha) \rfloor \beta && \text{(definition of metric)} \\
&= (\sharp\omega \wedge \sharp\alpha) \rfloor \beta && \text{(property of } \sharp) \\
&= \sharp\alpha \rfloor (\sharp\omega \rfloor \beta) && \text{(property of interior product)} \\
&= \sharp\alpha \rfloor (M\beta) && \text{(definition of } M) \\
&= \omega(\alpha, M\beta) && \text{(definition of metric.)}
\end{aligned}$$

□

Proposition 2.31

Let degree(α) be two less than degree(β) $= p$. Then

$$\omega(\beta, L\alpha) = \omega(M\beta, \alpha). \tag{2.69}$$

Proof

$$\omega(\beta, L\alpha) = (-1)^p \omega(L\alpha, \beta) \qquad \text{(property of metric)}$$

$$= (-1)^p \omega(\alpha, M\beta) \qquad (M \text{ and } L \text{ are adjoints})$$

$$= \omega(M\beta, \alpha) \qquad (\text{property of metric})$$

□

Proposition 2.32

Let $\phi \in \mathcal{S}^p(V^*, \omega)$. Then for any homogeneous form ψ

$$\omega([M^r, L^s]\,\phi, \psi) = \omega(\phi, [M^s, L^r]\,\psi). \tag{2.70}$$

Proof

The result is trivially true if the degree of $[M^r, L^s]\,\phi$ is not equal to the degree of ψ since the extended metric is zero on forms of differing degree. For ψ of degree $p + 2s - 2r$,

$$\begin{aligned}
\omega([M^r, L^s]\,\phi, \psi) &= \omega(M^r L^s \phi - L^s M^r \phi, \psi) \\
&= \omega(M^r L^s \phi, \psi) - \omega(L^s M^r \phi, \psi) \\
&= \omega(L^s \phi, L^r \psi) - \omega(M^r \phi, M^s \psi) \qquad (\text{adjoint property}) \\
&= \omega(\phi, M^s L^r \psi) - \omega(\phi, L^r M^s \psi) \qquad (\text{adjoint property}) \\
&= \omega(\phi, M^s L^r \psi - L^r M^s \psi) \\
&= \omega(\psi, [M^s, L^r]\,\psi).
\end{aligned}$$

□

Proposition 2.33

Let $\phi \in \mathcal{S}^p(V^*, \omega)$.

$$M^r * \phi = *L^r \phi \tag{2.71}$$

$$L^r * \phi = *M^r\phi \tag{2.72}$$

$$[M^r, L^s] * \phi = - * [M^s, L^r]\,\phi \tag{2.73}$$

Proof

It was shown previously that $M\phi = *L * \phi$. From this it follows immediately that

$$M * \psi = *L\psi \tag{2.74}$$

by putting $\phi = *\psi$ and recalling that $* * \psi = \psi$. It also follows that

$$* M\phi = L * \phi \tag{2.75}$$

by acting $*$ on both sides of $M\phi = *L * \phi$. To prove (2.71), we act M on both sides of (2.74); this yields

$$\begin{aligned} M^2 * \psi &= M * (L\psi) \\ &= *LL\psi \qquad \text{(by (2.74))} \\ &= *L^2\psi \end{aligned}$$

The result follows by repeating this process as many times as necessary. To prove (2.72) we act L on both sides of (2.75) :

$$\begin{aligned} L^2 * \phi &= L * (M\phi) \\ &= *M(M\phi) \qquad \text{(by (2.75))} \\ &= *M^2\phi. \end{aligned}$$

The result follows by iterating this procedure. To prove (2.73) :

$$\begin{aligned} [M^r, L^s] * \phi &= M^r L^s * \phi - L^s M^r * \phi \\ &= M^r * M^s\phi - L^s * L^r\phi \qquad \text{(by (2.72) and (2.71))} \\ &= *L^r M^s\phi - *M^s L^r\phi \qquad \text{(by (2.71) and (2.72))} \\ &= - * [M^s, L^r]\,\phi \end{aligned}$$

□

2.3.2 Commutation relations for M and L.

Some of the relations proved here may be found in other references, but a complete list including the most general form of these relations, has not been found in the literature.

Proposition 2.34

Let ϕ belong to $\mathcal{S}^p(V^*,\omega)$, where $0 \leq p \leq 2n$. Then

$$[M,L]\phi = (n-p)\,\phi \tag{2.76}$$

Proof

Let $\{e^i, f^i\}$ be a canonical basis of (V^*,ω) and let $\{e_i, f_i\}$ be the dual basis.

$$\begin{aligned} ML\phi &= \sharp\omega\rfloor(\omega\wedge\phi) \qquad \text{(by definition of } M, L) \\ &= \sum_{i=1}^{n}\{e_i\wedge f_i\rfloor(\omega\wedge\phi)\} \\ &= \sum_{i=1}^{n} f_i\rfloor e_i\rfloor(\omega\wedge\phi) \end{aligned}$$

We apply the product rule for exterior forms twice, to give

$$\begin{aligned} ML\phi &= \sum_{i=1}^{n} f_i\rfloor\{(e_i\rfloor\omega)\wedge\phi + \omega\wedge(e_i\rfloor\phi)\} \\ &= \sum_{i=1}^{n}\{(f_i\rfloor e_i\rfloor\omega)\wedge\phi - (e_i\rfloor\omega)\wedge(f_i\rfloor\phi) + (f_i\rfloor\omega)\wedge(e_i\rfloor\phi) + \omega\wedge(f_i\rfloor e_i\rfloor\phi)\} \\ &= (\sharp\omega\rfloor\omega)\wedge\phi + \omega\wedge(\sharp\omega\rfloor\phi) + \sum_{i=1}^{n}\{-(e_i\rfloor\omega)\wedge(f_i\rfloor\phi) + (f_i\rfloor\omega)\wedge(e_i\rfloor\phi)\} \\ &= n\phi + LM\phi + \sum_{i=1}^{n}\left\{-f^i\wedge(f_i\rfloor\phi) - e^i\wedge(e_i\rfloor\phi)\right\} \end{aligned}$$

The simplification of the first term involves the result (2.29). The summation here is over the entire basis. We invoke (2.3) to yield

$$ML\phi = n\phi + LM\phi - p\phi,$$

which completes the proof. □

Proposition 2.35

Let $\phi \in \mathcal{S}^p(V^*,\omega)$, for $0 \leq p \leq 2n$. Then for $r \geq 0$,

$$[M^r, L]\,\phi = r(n-p+r-1)M^{r-1}\phi. \tag{2.77}$$

Proof

If $r = 0$ or if $r > n$ then (2.77) is just the trivial statement $0 = 0$; the nontrivial range of r values is thus $1 \leq r \leq n$. The result must be proven for these r values. We use a standard identity for operators to express $[M^r, L]\,\phi$ as

$$[M^r, L]\,\phi = \sum_{i=1}^{r} M^{r-i}\,[M,L]\,M^{i-1}\phi.$$

The argument of $[M, L]$ in each term has degree $p-2i+2$. Invoking (2.76) we have

$$\begin{aligned}
[M^r, L]\,\phi &= \sum_{i=1}^{r} M^{r-i}\{n-(p-2i+2)\}M^{i-1}\phi \\
&= \sum_{i=1}^{r}(n-p-2+2i)M^{r-1}\phi \\
&= r(n-p-2)M^{r-1}\phi + 2\left(\sum_{i=1}^{r} i\right)M^{r-1}\phi \\
&= r(n-p-2)M^{r-1}\phi + r(r+1)M^{r-1}\phi \\
&= r(n-p+r-1)M^{r-1}\phi
\end{aligned}$$

□

Proposition 2.36

Let $\phi \in \mathcal{S}^p(V^*,\omega)$, for $0 \leq p \leq 2n$. Then for $r \geq 0$,

$$[M, L^r]\,\phi = r(n-p-r+1)L^{r-1}\phi \tag{2.78}$$

Proof

We note that if $r = 0$ or if $r > n$ then (2.78) is just the trivial statement $0 = 0$; the non-trivial range of r values is thus $1 \leq r \leq n$. For these latter values, let ψ be any $(p + 2r - 2)$–form.

$$
\begin{aligned}
\omega([M, L^r]\,\phi, \psi) &= \omega(\phi, [M^r, L]\,\psi) \\
&= \omega(\phi, r\{n - (p + 2r - 2) + r - 1\}M^{r-1}\psi) \qquad \text{(by (2.77))} \\
&= \omega(\phi, r(n - p - r + 1)M^{r-1}\psi) \\
&= \omega(r(n - p - r + 1)L^{r-1}\phi, \psi)
\end{aligned}
$$

Using the adjoint property and the bilinearity of the extended metric. The result follows since this holds for all such ψ. □

The following four propositions have not been found in the literature.

Proposition 2.37

Let ϕ belong to $\mathcal{S}^p(V^*, \omega)$. Then for $r \geq 1$ and $a \geq 0$,

$$[M^r, L^{r+a}]\,\phi = \sum_{i=1}^{r} \binom{r}{i} \frac{(r+a)!}{(r+a-i)!} \prod_{k=0}^{i-1} (n - p - a - k) L^{r+a-i} M^{r-i} \phi. \qquad (2.79)$$

A proof, using the method of induction, is given in the appendix.

Proposition 2.38

Let ϕ belong to $\mathcal{S}^p(V^*, \omega)$ where $0 \leq p \leq 2n$. Then, for $r \geq 1$ and $a \geq 0$,

$$[M^{r+a}, L^r]\,\phi = \sum_{i=1}^{r} \binom{r}{i} \frac{(r+a)!}{(r+a-i)!} \prod_{k=0}^{i-1} (n - p + a - k) L^{r-i} M^{r+a-i} \phi. \qquad (2.80)$$

Proof

(2.80) is non-trivial for $1 \leq r \leq n$ and $0 \leq a \leq n - r$; outside of these ranges, it is the trivial expression $0 = 0$. For the non-trivial cases, let ψ be of degree $q = p - 2a$. Then

$$\begin{aligned}\omega([M^{r+a}, L^r]\,\phi, \psi) &= \omega(\phi, [M^r, L^{r+a}]\,\psi) \qquad \text{(by (2.70))}\\ &= \sum_{i=1}^{r} \binom{r}{i} \frac{(r+a)!}{(r+a-i)!} \prod_{k=0}^{i-1} (n-p-a-k)\,\omega(\phi, L^{r+a-i} M^{r-i}\psi) \qquad \text{(by (2.79))}\\ &= \sum_{i=1}^{r} \binom{r}{i} \frac{(r+a)!}{(r+a-i)!} \prod_{k=0}^{i-1} (n-p-a-k)\,\omega(L^{r-i} M^{r+a-i}\phi, \psi)\end{aligned}$$

using the adjoint property of M and L. Since $(n-p-a-k) = (n-(q-2a)+a-k) = (n-q+a-k)$,

$$\omega([M^{r+a}, L^r]\,\phi, \psi) = \sum_{i=1}^{r} \binom{r}{i} \frac{(r+a)!}{(r+a-i)!} \prod_{k=0}^{i-1} (n-q+a-k)\,\omega(L^{r-i} M^{r+a-i}\phi, \psi).$$

This holds for all ϕ; hence the result. □

Proposition 2.39

Let ϕ belong to $\mathcal{S}^p(V^*, \omega)$, where $0 \le p \le 2n$. Then, for $r \ge 1$ and $a \ge 0$,

$$[M^{r+a}, L^r]\,\phi = \sum_{i=1}^{r} (-1)^{i+1} \binom{r}{i} \frac{(r+a)!}{(r+a-1)!} \prod_{k=0}^{i-1} (n-p+a+k) M^{r+a-i} L^{r-i}\phi. \tag{2.81}$$

Proof

This identity is non-trivial for $1 \le r \le n$ and $0 \le a \le n-r$; outside of these ranges it is the trivial expression $0 = 0$. We assume that r, a are non-trivial. Since ϕ is of degree p, $*\phi$ is of degree $2n - p$. Then

$$\begin{aligned}[M^{r+a}, L^r]\,\phi &= **\,[M^{r+a}, L^r]\,\phi\\ &= -*\,[M^r, L^{r+a}] * \phi \qquad \text{(by (2.73))}\\ &= -*\sum_{i=1}^{r} \binom{r}{i} \frac{(r+a)!}{(r+a-i)!} \prod_{k=0}^{i-1} (n-(2n-p)-a-k) L^{r+a-i} M^{r-i}\phi\end{aligned}$$

where we have substituted (2.79). Applying (2.72) then (2.71) we have

$$\begin{aligned}[M^{r+a}, L^r]\,\phi &= -\sum_{i=1}^{r}\binom{r}{i}\frac{(r+a)!}{(r+a-i)!}\prod_{k=o}^{i-1}(-n+p-a-k)M^{r+a-i}L^{r-i}\phi \\ &= \sum_{i=1}^{r}(-1)^{i+1}\binom{r}{i}\frac{(r+a)!}{(r+a-i)!}\prod_{k=0}^{i-1}(n-p+a+k)M^{r+a-i}L^{r-i}\phi\end{aligned}$$

and the result follows. □

Proposition 2.40

Let ϕ belong to $\mathcal{S}^p(V^*,\omega)$, where $0 \le p \le 2n$. Then, for $r \ge 1$ and $a \ge 0$,

$$[M^r, L^{r+a}]\,\phi = \sum_{i=1}^{r}(-1)^{i+1}\binom{r}{i}\frac{(r+a)!}{(r+a-i)!}\prod_{k=0}^{i-1}(n-p-a+k)M^{r-i}L^{r+a-i}\phi. \tag{2.82}$$

Proof

This identity is non-trivial for $1 \le r \le n$ and $0 \le a \le n-r$; outside of these ranges it is the trivial expression $0 = 0$. We assume r, a are non-trivial. Since ϕ is of degree p, $*\phi$ is of degree $2n-p$. Then

$$\begin{aligned}[M^r, L^{r+a}]\,\phi &= **\,[M^r, L^{r+a}]\,\phi \\ &= -*\,[M^{r+a}, L^r] * \phi \qquad \text{(by (2.73))}\end{aligned}$$

Invoking (2.80) and then applying (2.71) and (2.72), we obtain

$$\begin{aligned}[M^r, L^{r+a}]\,\phi &= -*\sum_{i=1}^{r}\binom{r}{i}\frac{(r+a)!}{(r+a-i)!}\prod_{k=0}^{i-1}(n-(2n-p)+a-k)L^{r-i}M^{r+a-i} * \phi \\ &= \sum_{i=1}^{r}(-1)^{i+1}\binom{r}{i}\frac{(r+a)!}{(r+a-i)!}\prod_{k=0}^{i-1}(n-p-a+k)M^{r-i}L^{r+a-1}\phi.\end{aligned}$$

□

2.3.3 The Lepage decomposition for $p \leq n$

The Lepage decomposition theorem states that any p-form in $\mathcal{S}^p(V^*,\omega)$ may be uniquely decomposed in terms of simple forms, to be defined later. In this section, the proof is given for forms of degree less than n. First, I will define the effective forms (see Libermann and Marle p 64):

Definition 2.18

Let $\phi \in \mathcal{S}(V^*,\omega)$. Then ϕ is called *effective* or *primitive* if

$$M\,\phi = 0\,.$$

□

Proposition 2.41

Let ϕ be an effective p-form. Then for $r \geq 1$

$$M^r L^r \phi = r! \prod_{k=0}^{r-1} (n-p-k)\phi. \tag{2.83}$$

(See Lemma C of Hodge, 1941, section 42.2.)

Proof

The result holds trivially for $r > n$. We consider the non-trivial values, where $1 \leq r \leq n$. We have from (2.79) that

$$[M^r, L^{r+a}]\,\phi = \sum_{i=1}^{r} \binom{r}{i} \frac{(r+a)!}{(r+a-i)!} \prod_{k=0}^{i-1} (n-p-a-k) L^{r+a-i} M^{r-i} \phi.$$

Since ϕ is effective, all the terms except the $i = r$ term are zero. Including also the requirement $a = 0$, yields

$$[M^r, L^r]\,\phi = \binom{r}{r} \frac{r!}{0!} \prod_{k=0}^{r-1} (n-p-k)\phi.$$

Hence

$$M^r L^r \phi = r! \prod_{k=0}^{r-1} (n-p-k)\phi.$$

□

Theorem 2.1 (The Lepage decomposition for $p \le n$)

Every form ϕ of degree $p \le n$ may be uniquely decomposed into a sum of simple forms

$$\phi = \sum_{i=0}^{q} L^i \, \psi_{p-2i} \tag{2.84}$$

where each ψ_{p-2i} is an effective form of degree $(p-2i)$ for $0 \le i \le q$ and $q \le \left[\frac{p}{2}\right]$. (See section 42.3 of Hodge)

Proof

If $\phi = 0$, then it is already effective and the result holds trivially.

We define q by the requirements

$$M^q \phi \neq 0 \tag{2.85}$$

$$M^{q+1}\phi = 0\,. \tag{2.86}$$

Since (2.85), it follows that $p - 2q \ge 0$ and hence that $q \le \left[\frac{p}{2}\right]$.

We next induce a series of p-forms from ϕ. They are defined in the following iterative way, where $a_0, \ldots, a_q$ are constants to be selected later.

$$\begin{aligned}
\phi_{(0)} &:= \phi \\
\phi_{(1)} &:= \phi_{(0)} - a_0 L^q M^q \phi_{(0)} \\
&\vdots \qquad \vdots \\
\phi_{(q-k+1)} &:= \phi_{(q-k)} - a_{q-k} L^k M^k \phi_{(q-k)} \qquad (2.87)\\
&\vdots \qquad \vdots
\end{aligned}$$

$$\begin{aligned}\phi_{(q)} &:= \phi_{(q-1)} - a_{q-1}LM\phi_{(q-1)} \\ \phi_{(q+1)} &:= \phi_{(q)} - a_q\phi_{(q)}.\end{aligned}$$

If we can select $a_0, \ldots, a_q$ so that

$$\begin{aligned}M^{q+1}\phi_{(0)} &= M^q\phi_{(1)} = \ldots = M^k\phi_{(q-k+1)} \\ \ldots &= M^2\phi_{(q-1)} = M\phi_{(q)} = 0\end{aligned} \tag{2.88}$$

then we may define a series of effective forms as follows:

$$\begin{aligned}\psi_{p-2q} &:= a_0M^q\phi_{(0)} \\ \psi_{p-2(q-1)} &:= a_1M^{q-1}\phi_{(1)} \\ \vdots & \quad \vdots \\ \psi_{p-2k} &:= a_{q-k}M^k\phi_{(q-k)} \\ \vdots & \quad \vdots \\ \psi_{p-2} &:= a_{q-1}M\phi_{(q-1)} \\ \psi_p &:= a_q\phi_{(q)}\end{aligned} \tag{2.89}$$

Using (2.89) we may write (2.87) as

$$\begin{aligned}\phi_{(0)} &= \phi \\ \phi_{(1)} &= \phi_{(0)} - L^q\psi_{p-2q} \\ \phi_{(2)} &= \phi_{(1)} - L^{q-1}\psi_{p-2(q-1)} \\ \vdots & \quad \vdots \\ \phi_{(q-k+1)} &= \phi_{(q-k)} - L^k\psi_{p-2k} \\ \vdots & \quad \vdots \\ \phi_{(q)} &= \phi_{(q-1)} - L\psi_{p-2} \\ \phi_{(q+1)} &= \phi_{(q)} - \psi_p\end{aligned} \tag{2.90}$$

By substituting each expression of (2.90) into the one below it, we obtain

$$\phi_{(q+1)} = \phi - \sum_{i=0}^{q} L^i \psi_{p-2i}. \tag{2.91}$$

The above relies on the assumption that (2.88) holds true. I prove this to be the case in the following lemma.

Lemma

If $\phi_{(0)}, \ldots, \phi_{(q+1)}$ are defined as in (2.87), then there exist numbers $a_0, \ldots, a_q$ such that

$$M^{q-i}\phi_{(i)} \qquad \text{is effective} \tag{2.92}$$

for $0 \leq i \leq q$, and they are given by

$$a_i = \frac{(n-p+q-i)!}{(q-i)!(n-p+2q-2i)!}. \tag{2.93}$$

Proof

By induction on i.

For $i = 0$: $M^q\phi_{(0)}$ is already effective by (2.86).

We assume that for some r satisfying $0 \leq r < q$,

$$M^{q-r}\phi_{(r)} \qquad \text{is effective.} \tag{2.94}$$

For $M^{q-(r+1)}\phi_{(r+1)}$ to be effective, we require

$$\begin{aligned} 0 &= M^{q-r}\phi_{(r+1)} \\ &= M^{q-r}\left\{\phi_{(r)} - a_r L^{q-r} M^{q-r}\phi_{(r)}\right\} \qquad \text{(by (2.87))} \\ &= M^{q-r}\phi_{(r)} - a_r M^{q-r} L^{q-r}\left(M^{q-r}\phi_{(r)}\right). \end{aligned}$$

By the inductive hypothesis (2.94), $M^{q-r}\phi_{(r)}$ is effective and we may make use of the result (2.83), to yield:

$$\begin{aligned} 0 &= M^{q-r}\phi_{(r)} - a_r(q-r)!\prod_{k=0}^{q-r-1}(n-(p-2q+2r)-k)M^{q-r}\phi_{(r)} \\ &= \left\{1 - a_r(q-r)!\prod_{k=0}^{q-r-1}(n-p+2q-2r-k)\right\}M^{q-r}\phi_{(r)}. \qquad (2.95) \end{aligned}$$

Consider

$$\prod_{k=0}^{q-r-1}(n-p+2q-2r-k) =$$

$$(n-p+2q-2r)(n-p+2q-2r-1)\ldots(n-p+q-r+1) \qquad (2.96)$$

The smallest term is the last one, $(n-p+q-r+1)$. Since $p \leq n$ and $r < q$, it follows that this term and thus the whole expression (2.96) is strictly positive. It is therefore possible to select a_r so that the coefficient of (2.95) vanishes; this is done by the choice

$$\begin{aligned} a_r &= \frac{1}{(q-r)!\prod_{k=0}^{q-r-1}(n-p+2q-2r-k)} \\ &= \frac{(n-p+q-r)!}{(q-r)!(n-p+2q-2r)!} \quad \text{for } 0 \leq r \leq q, \end{aligned}$$

which completes the lemma. □

Examining (2.91) it is clear that the result will be proven if we can show that $\phi_{(q+1)} = 0$. From (2.87) we have

$$\begin{aligned} \phi_{(q+1)} &= \phi_{(q)} - a_q\phi_{(q)} \\ &= \phi_{(q)} - \frac{(n-p)!}{0!(n-p)!}\phi(q) \qquad \text{(using (2.93).)} \\ &= 0. \end{aligned}$$

Thus (2.91) becomes

$$\phi = \sum_{i=0}^{q} L^i \psi_{p-2i} \,.$$

Because q and the ψ_{p-2i}'s were explicitly constructed, they must be unique. This completes the proof. □

2.3.4 The Lepage decomposition for $p > n$

The Lepage decomposition is more difficult to prove for $p > n$ than for $p \leq n$. It is necessary to prove several preliminary results first.

I use the following notation:

$$(p-n)^{+} \; = \; \max(0, p-n)\,.$$

I will also use $\left[\frac{p}{2}\right]$ to indicate the integral part of $\frac{p}{2}$.

Proposition 2.42

Let ψ be any effective form of degree p. For all $a \geq 1$,

$$M^{r+a} L^{r} \psi = 0 \tag{2.97}$$

Proof

Since ψ is effective,

$$\begin{aligned} M^{r+a} L^{r} \psi &= \left[M^{r+a}, L^{r}\right] \psi \\ &= \sum_{i=1}^{r} \binom{r}{i} \frac{(r+a)!}{(r+a-i)!} \prod_{k=0}^{i-1} (n-p+a-k) L^{r-i} M^{r+a-i} \psi, \end{aligned}$$

using (2.80). Each term of the expansion contains $M\psi$ because $a \geq 1$; therefore the summation is zero and the result follows. □

Proposition 2.43

Let ψ_{p-2i} be an effective form of degree $p-2i$, for any value of p from 0 to $2n$. If i lies in the range $(p-n)^{+} \leq i \leq \left[\frac{p}{2}\right]$, then

$$M^{i} L^{i} \psi_{p-2i} = 0 \implies \psi_{p-2i} = 0. \tag{2.98}$$

Proof

From our previous result (2.83), it follows that

$$M^iL^i\psi_{p-2i} = i!\prod_{k=0}^{i-1}(n-(p-2i)-k)\psi_{p-2i}$$
$$= i!(n-p+2i)(n-p+2i-1)\dots(n-p+i+1)\psi_{p-2i}$$

Therefore

$$\begin{aligned} 0 &= M^iL^i\psi_{p-2i} \\ \Longrightarrow 0 &= (n-p+2i)(n-p+2i-1)\dots(n-p+i+1)\ \psi_{p-2i} \end{aligned} \quad (2.99)$$

To deduce the result, it will suffice to show that the coefficient of (2.99) is non-zero. Consider the smallest term in the coefficient, $(n-p+i+1)$. If $p \leq n$, then $(p-n)^+ = 0$ so that $0 \leq i \leq \left[\frac{p}{2}\right]$. The minimum value this term assumes is

where $i = 0$, being $(n-p+0+1)$. This is strictly positive and so for this case the coefficient of (2.99) is non-zero.

If $p > n$, then $(p-n)^+ = p-n$ so that $p-n \leq i \leq \left[\frac{p}{2}\right]$. The term $(n-p+i+1)$ has minimum value when $i = p-n$, being $(n-p+(p-n)+1) = 1$. Thus the coefficient of (2.99) is non-zero for all values of p. □

Before proving the Lepage decomposition for $p > n$, it will be necessary to prove the uniqueness for these degrees. The following theorem does this but also does so for degrees $p \leq n$.

Proposition 2.44

Let ϕ be any p-form. The decomposition

$$\phi = \sum_{i=(p-n)^+}^{q} L^i\,\psi_{p-2i} \quad (2.100)$$

where the ψ_{p-2i} are effective, and q satisfies $M^q\phi \neq 0$ but $M(M^q\phi) = 0$, is unique if it exists. That is to say, q is unique and so is each ψ_{p-2i}.

Proof

It will suffice to prove that if

$$0 = \sum_{i=(p-n)^+}^{q} L^i \psi_{p-2i} \tag{2.101}$$

then

$$\psi_{p-2i} = 0 \text{ for } (p-n)^+ \leq i \leq q.$$

I will prove this latter assertion by induction on i, starting with $i = q$ and reducing it by one each time.

$i = q$: Apply M^q to both sides of (2.101) to obtain

$$\begin{aligned} 0 &= \sum_{i=(p-n)^+}^{q} M^q L^i \psi_{p-2i} \\ &= M^q L^q \psi_{p-2q} \end{aligned}$$

by (2.97) since in all the other terms $i < q$. Because $M^q\phi \neq 0$, it follows that $p - 2q \geq 0$ and hence that $q \leq \left[\frac{p}{2}\right]$. Therefore we may invoke (2.98) to deduce

$$\psi_{p-2q} = 0.$$

Now make the inductive hypothesis that the result holds for all values of i from q down to $r+1$, where $(p-n)^+ \leq r < q \leq \left[\frac{p}{2}\right]$. Then

$$\psi_{p-2q} = \psi_{p-2(q-1)} = \ldots = \psi_{p-2(r+2)} = \psi_{p-2(r+1)} = 0. \tag{2.102}$$

$i = r$: With (2.102) substituted into (2.101) we obtain

$$0 = \sum_{i=(p-n)^+}^{r} L^i \psi_{p-2i}\,;$$

apply M^r to both sides, to give

$$0 = \sum_{i=(p-n)^+}^{r} M^r L^i \psi_{p-2i}.$$

By (2.97) only one term survives, since all the terms with $i < r$

are zero; therefore

$$0 = M^r L^r \psi_{p-2r}.$$

Since $(p-n)^+ \leq r < \left[\frac{p}{2}\right]$ we invoke (2.98) to give

$$\psi_{p-2r} = 0. \tag{2.103}$$

This completes the induction, and we conclude that every ψ is zero. □

Proposition 2.45

Let ϕ be any form of degree $p > n$. Then

$$M^{p-n}\phi = 0 \implies \phi = 0. \tag{2.104}$$

Proof

Consider $*\phi$ of degree $2n - p = r$, say. Then $r < n$ and we may decompose $*\phi$ using the Lepage decomposition (2.84) for degree less than n. Hence

$$* \phi = \sum_{i=0}^{s} L^i \psi_{r-2i} \tag{2.105}$$

where $s \leq \left[\frac{r}{2}\right]$ and satisfies $M^s * \phi \neq 0,\ M(M^s * \phi) = 0$. Then

$$\begin{aligned}
0 &= M^{p-n}\phi \\
&= *L^{p-n} * \phi \qquad (\text{since } M = *L*) \\
&= *L^{p-n} \sum_{i=0}^{s} L^i \psi_{r-2i} \\
&= * \sum_{i=0}^{s} L^{p-n+i} \psi_{r-2i}.
\end{aligned}$$

Taking $*$ of both sides, we have

$$0 = \sum_{i=0}^{s} L^{p-n+i}\psi_{r-2i} \tag{2.106}$$

Re-express (2.106) by redefining the indices. Put $j = p - n + i$. Then $i = j - p + n$, and $r - 2i = 2n - p - 2(j - p + n) = p - 2j$. The summation begins at $j = p - n$, where $i = 0$ and ends at $j = p - n + s$, where $i = s$.

Defining $q := p - n + s$, we will show that $q \leq \left[\frac{p}{2}\right]$. By the definition of s, $s \leq \left[\frac{r}{2}\right] = \left[\frac{2n-p}{2}\right]$. For p even, we have

$$\begin{aligned} q &\leq p - n + \left[\frac{2n-p}{2}\right] \\ &= p - n + \frac{2n-p}{2} \\ &= \frac{p}{2} \\ &= \left[\frac{p}{2}\right]; \end{aligned}$$

for p odd,

$$\begin{aligned} q &\leq p - n + \left[\frac{2n-p}{2}\right] \\ &= p - n + \frac{2n-p-1}{2} \\ &= \frac{p-1}{2} \\ &= \left[\frac{p}{2}\right]. \end{aligned}$$

This demonstrates that $q \leq \left[\frac{p}{2}\right]$. So we may write (2.106) as

$$0 = \sum_{j=p-n}^{q} L^j \psi_{p-2j}.$$

This is the Lepage decomposition of the zero p-form, with $p > n$. By the uniqueness of the decomposition (2.100), all the ψ_{p-2j} are zero. Therefore in (2.105) we have

$$\begin{aligned} *\phi &= 0 \\ \iff \phi &= 0. \end{aligned}$$

□

Corollary

Let ϕ be any form of degree $p > n$. Then for $r \leq p - n$

$$M^r \phi = 0 \implies \phi = 0. \tag{2.107}$$

(See corollary 15.15 of Libermann (1987) ch. I, part 3.)

Proof

If $r = p-n$, then the result follows from the proposition above. For $r < p-n$, we have $p - n - r > 0$, so that

$$\begin{aligned} M^r \phi = 0 &\implies M^{p-n-r} M^r \phi = 0 \\ &\iff M^{p-n} \phi = 0 \\ &\implies \phi = 0 \quad \text{(by the above proposition.)} \end{aligned}$$

□

Theorem 2.2 (The Lepage decomposition for $p > n$)

Let ϕ be any p-form, with $p > n$. Then ϕ may be decomposed as

$$\phi = \sum_{i=p-n}^{q} L^i \psi_{p-2i} \tag{2.108}$$

where the ψ_{p-2i} are effective of degree $p - 2i$, and $q \leq \left[\frac{p}{2}\right]$.

Proof

If $\phi = 0$, then it is already effective and the result holds trivially.

If $\phi \neq 0$, we define q by the requirements

$$M^q\phi \neq 0 \tag{2.109}$$

$$M(M^q)\phi = 0. \tag{2.110}$$

The range of allowed q values is found as follows. (2.109) implies that $p-2q \geq 0$; hence $q \leq \left[\frac{p}{2}\right]$. If $q+1 \leq p-n$, then from (2.110) and (2.107) it follows that $\phi = 0$. Therefore to eliminate this contradiction, $q+1 > p-n$, or $q \geq p-n$. Hence the range of values for q is

$$p-n \leq q \leq \left[\frac{p}{2}\right] \tag{2.111}$$

I now follow a series of definitions and manipulations very similar to those carried out in the proof of the Lepage decomposition for $p \leq n$. Define a series of p-forms

$$\begin{aligned}
\phi_{(0)} &:= \phi \\
\phi_{(1)} &:= \phi_{(0)} - a_0 L^q M^q \phi_{(0)} \\
\phi_{(2)} &:= \phi_{(1)} - a_1 L^{q-1} M^{q-1} \phi_{(1)} \\
&\vdots \\
\phi_{(q-k+1)} &:= \phi_{(q-k)} - a_{q-k} L^k M^k \phi_{(q-k)} \qquad (2.112) \\
&\vdots \\
\phi_{(q-(p-n))} &:= \phi_{(q-(p-n)-1)} - a_{q-(p-n)-1} L^{p-n+1} M^{p-n+1} \phi_{(q-(p-n)-1)} \\
\phi_{(q-(p-n)+1)} &:= \phi_{(q-(p-n))} - a_{q-(p-n)} L^{p-n} M^{p-n} \phi_{(q-(p-n))},
\end{aligned}$$

where the a's are scalars. If we can select $a_0, \ldots, a_{q-(p-n)}$ such that

$$\begin{aligned}
M^{q+1}\phi_{(0)} &= M^q\phi_{(1)} = \ldots = M^k\phi_{(q-k+1)} = \ldots \\
\ldots &= M^{p-n+1}\phi_{(q-(p-n))} = M^{p-n}\phi_{(q-(p-n)+1)} = 0 \qquad (2.113)
\end{aligned}$$

then we may define a series of effective forms as follows:

$$\psi_{p-2q} := a_o M^q \phi_{(0)}$$

$$\begin{aligned}
\psi_{p-2(q-1)} &:= a_1 M^{q-1}\phi_{(1)} \\
&\vdots \quad \vdots \\
\psi_{p-2k} &:= a_{q-k} M^k \phi_{(q-k)} \\
&\vdots \quad \vdots \\
\psi_{p-2(p-n)} &:= a_{q-(p-n)} M^{p-n}\phi_{(q-(p-n))}\,.
\end{aligned} \tag{2.114}$$

Using (2.114) we may write (2.112) as

$$\begin{aligned}
\phi_{(0)} &= \phi \\
\phi_{(1)} &= \phi_{(0)} - L^q \psi_{p-2q} \\
\phi_{(2)} &= \phi_{(1)} - L^{q-1}\psi_{p-2(q-1)} \\
&\vdots \quad \vdots \\
\phi_{(q-k+1)} &= \phi_{(q-k)} - L^k \psi_{p-2k} \\
&\vdots \quad \vdots \\
\phi_{(q-(p-n))} &= \phi_{(q-(p-n)-1)} - L^{p-n+1}\psi_{p-2(p-n)-2} \\
\phi_{(q-(p-n)+1)} &= \phi_{(q-(p-n))} - L^{p-n}\psi_{p-2(p-n)}.
\end{aligned} \tag{2.115}$$

By substituting each expression of (2.115) into the one below it, we obtain

$$\phi_{(q-(p-n)+1)} = \phi - \sum_{i=p-n}^{q} L^i \psi_{p-2i}. \tag{2.116}$$

Assuming (2.113) to be correct, we have that

$$M^{p-n}\phi_{(q-(p-n)+1)} = 0,$$

where $\phi_{(q-(p-n)+1)}$ is a p-form. Recalling that $p > n$, it follows from (2.104) that

$$\phi_{(q-(p-n)+1)} = 0.$$

Thus (2.116) becomes

$$\phi = \sum_{i=p-n}^{q} L^i \psi_{p-2i}.$$

To complete the proof, it will suffice to show that the coefficients $a_0, \ldots, a_{q-(p-n)}$, satisfying (2.113), do indeed exist. This I show in the following lemma.

Lemma

If $\phi_{(0)}, \ldots, \phi_{(q-(p-n)+1)}$ are defined as in (2.112), then there exist numbers $a_0, \ldots, a_{q-(p-n)}$ such that

$$M^{q-i}\phi_{(i)} \qquad \text{is effective} \tag{2.117}$$

for $i = 0, \ldots, q-(p-n)+1$.

Proof

By induction on i.

For $i = 0$: $M^q\phi_{(0)}$ is already effective by (2.110).

Assume that for some r satisfying $0 \leq r < q-(p-n)+1$,

$$M^{q-r}\phi_{(r)} \qquad \text{is effective.} \tag{2.118}$$

For $M^{q-(r+1)}\phi_{(r+1)}$ to be effective, we require

$$\begin{aligned} 0 &= M^{q-r}\phi_{(r+1)} \\ &= M^{q-r}\left\{\phi_{(r)} - a_r L^{q-r} M^{q-r}\phi_{(r)}\right\} \qquad \text{(by (2.112))} \\ &= M^{q-r}\phi_{(r)} - a_r M^{q-r} L^{q-r}\left(M^{q-r}\phi_{(r)}\right). \end{aligned}$$

By the inductive hypothesis (2.118), $M^{q-r}\phi_{(r)}$ is effective and we may make use of our earlier result (2.83), to yield:

$$\begin{aligned} 0 &= M^{q-r}\phi_{(r)} - a_r(q-r)! \prod_{k=0}^{q-r-1}(n-(p-2q+2r)-k)M^{q-r}\phi_{(r)} \\ &= \left\{1 - a_r(q-r)! \prod_{k=0}^{q-r-1}(n-p+2q-2r-k)\right\} M^{q-r}\phi_{(r)} \end{aligned} \tag{2.119}$$

Consider

$$\prod_{k=0}^{q-r-1}(n-p+2q-2r-k) = (n-p+2q-2r)(n-p+2q-2r-1)\dots(n-p+q-r+1) \quad (2.120)$$

The smallest term is the last one, $(n-p+q-r+1)$. The largest value that r may take on determines the minimum value of this smallest term. Since r has maximum value $q-(p-n)$, it follows that $(n-p+q-r+1) \geq 1$. Thus, the product in (2.119) never vanishes, and is always positive. The Lemma is proven if we select

$$\begin{aligned} a_r &= \frac{1}{(q-r)!\prod_{k=0}^{q-r-1}(n-p+2q-2r-k)} \\ &= \frac{(n-p+q-r)!}{(q-r)!(n-p+2q-2r)!} \quad \text{for } 0 \leq r \leq q-(p-n) \end{aligned}$$

This completes the Lemma, □

and hence the entire proof of the decomposition. □

2.3.5 Simple subspaces

I will show that the Lepage decomposition may be interpreted as a direct sum decomposition of $\mathcal{S}(V^*, \omega)$ in terms of vector subspaces to be defined below. The decomposition states that any p-form ϕ may be expressed, uniquely, as

$$\phi = L^{(p-n)^+}\psi_{p-2(p-n)^+} + \cdots + L^q\psi_{p-2q}$$

where the ψ's are effective. It is natural then to consider in more detail forms that are expressible as $\phi = L^r\psi_{p-2r}$ where ψ_{p-2r} is effective.

Define

Definition 2.19

Let $\phi \in \mathcal{S}(V^*, \omega)$. ϕ is called *simple* if there exists an effective form ψ and an integer $r \in \{0, \ldots, n\}$ such that

$$\phi = L^r \psi$$

(Libermann and Marle, 1987 p 64). We will refer to r as the *height* of ϕ, and to ψ as the *effective part* of ϕ. □

It follows as an immediate consequence of the Lepage decomposition that the height of any simple form is unique, as is its effective part. We also note that a non-zero simple form is effective if and only if it has height is zero. The standard result that a non-zero effective form must be of degree no greater than the half dimension n (Libermann and Marle ch I, proposition 15.7, 15.10) may also be deduced from the decomposition.

I next demonstrate that the set of simple forms of given degree and height constitute a vector subspace.

Proposition 2.46

The effective forms of fixed degree constitute a vector subspace of $\mathcal{S}(V^*, \omega)$.

Proof

Consider the effective p-forms. Noting that the zero p-form is effective, it remains to show the closure of this set. Let ϕ, ψ be effective p-forms. For scalars a, b we have

$$\begin{aligned} M(a\phi + b\psi) &= aM\phi + bM\psi \quad \text{(by linearity of } M) \\ &= a.0 + b.0 \quad \text{(since } \phi, \psi \text{ effective)} \\ &= 0. \end{aligned}$$

Linear combinations of effective p-forms are thus also effective, and the result is proven. □

Proposition 2.47

The simple forms of fixed height and degree constitute a vector subspace of $\mathcal{S}(V^*, \omega)$.

Proof

We consider the set of simple p-forms of height k. It will suffice to show that this set contains the zero p-form and that it is closed. The zero p-form is indeed simple for it may be written $0_p = L^k 0_{p-2k}$, and 0_{p-2k} is effective by the previous proposition. Let ϕ, ψ be simple p-forms of height k. Then there exist effective forms ξ and η of degree $p - 2k$ such that

$$\begin{aligned} a\phi + b\psi &= aL^k\xi + bL^k\eta \\ &= L^k(a\xi + b\eta) \qquad \text{(by linearity of } L\text{)}. \end{aligned}$$

This expression is simple because the argument of L^k is effective by (2.46). This completes the proposition. □

Therefore make the

Definition 2.20

The subspace of simple exterior forms of degree p and height h will be denoted by S_p^h. We will refer to these subspaces as *simple* subspaces. □

The Lepage decomposition may now be interpreted as giving a direct sum decomposition of the homogeneous subspace $\mathcal{S}^p(V^*,\omega)$ of the symplectic exterior algebra into simple subspaces of various heights. For $0 \le p \le 2n$

$$\mathcal{S}^p(V^*,\omega) = S_p^{(p-n)^+} \oplus S_p^{(p-n)^++1} \oplus \cdots \oplus S_p^q$$

where $q = \left[\frac{p}{2}\right]$.

To clarify this consider the case of $n = 5$. The decompositions of each of the ten homogeneous subspaces of $\mathcal{S}(V^*,\omega)$ are:

$$\begin{array}{rcl}
\mathcal{S}^0(V^*,\omega) & = & S_0^0 \\
\mathcal{S}^1(V^*,\omega) & = & S_1^0 \\
\mathcal{S}^2(V^*,\omega) & = & S_2^0 \oplus S_2^1 \\
\mathcal{S}^3(V^*,\omega) & = & S_3^0 \oplus S_3^1 \\
\mathcal{S}^4(V^*,\omega) & = & S_4^0 \oplus S_4^1 \oplus S_4^2 \\
\mathcal{S}^5(V^*,\omega) & = & S_5^0 \oplus S_5^1 \oplus S_5^2 \\
\mathcal{S}^6(V^*,\omega) & = & S_6^1 \oplus S_6^2 \oplus S_6^3 \\
\mathcal{S}^7(V^*,\omega) & = & S_7^2 \oplus S_7^3 \\
\mathcal{S}^8(V^*,\omega) & = & S_8^3 \oplus S_8^4 \\
\mathcal{S}^9(V^*,\omega) & = & S_9^4 \\
\mathcal{S}^{10}(V^*,\omega) & = & S_{10}^5
\end{array}$$

The geometrical interpretation given here has not been explicitly stated in the literature; it is felt that these comments provide a better understanding of what the Lepage decomposition is.

2.3.6 Dimensions of the simple subspaces

To investigate the structure of the simple subspaces, it is natural to try and find their dimensions. This is accomplished in this section where a series of results not present in the literature are given.

A few preliminary results are necessary.

Proposition 2.48

Let $0 \leq r \leq n$. Then

$$\prod_{k=0}^{r-1}(n-p-k) = \begin{cases} \frac{(n-p)!}{(n-p-r)!} \neq 0, & \text{for } 0 \leq p \leq n-r \\ 0, & \text{for } n-r+1 \leq p \leq n \\ (-1)^r \frac{(p-n+r-1)!}{(p-n-1)!} \neq 0, & \text{for } n+1 \leq p \leq 2n \end{cases} \quad (2.121)$$

Proof

$$\prod_{k=0}^{r-1}(n-p-k) = (n-p)(n-p-1)(n-p-2),\ldots,(n-p-r+1) \quad (2.122)$$

By inspection, it is clear that the expression is zero for $n-r+1 \leq p \leq n$.

For $0 \leq p \leq n-r$, consider the smallest factor, $(n-p-r+1)$. Since $p \leq n-r$, it follows that $(n-p-r+1) \geq 1$. Therefore the whole expression is strictly positive, and the factorial expression is justified.

For $n+1 \leq p \leq 2n$, consider the largest factor, $(n-p)$. Since $p \geq n+1$, it follows that $(n-p) \leq -1$, so the whole expression is strictly negative. Taking out -1 from each factor, we obtain

$$\begin{aligned} \prod_{k=0}^{r-1}(n-p-k) &= (-1)^r(p-n)(p-n+1)(p-n+2),\ldots,(p-n+r-1) \\ &= (-1)^r \frac{(p-n+r-1)!}{(p-n-1)!} \end{aligned}$$

□

We saw previously that for an effective form ϕ_p,

$$[M^r, L^r]\,\phi_p = r! \prod_{k=0}^{r-1}(n-p-k)\phi_p$$

By the proposition (2.121), it follows that for $0 \leq r \leq n$ and $0 \leq p \leq n-r$,

$$M^r L^r \phi_p = \frac{r!(n-p)!}{(n-p-r)!}\phi_p. \qquad (2.123)$$

From this expression it is evident that an isomorphism may be set up as follows.

Definition 2.21

For $0 \leq r \leq n$ and $0 \leq p \leq n-r$,

$$\begin{aligned} f_p^r \ &: \ S_p^0 \ \rightarrow \ S_{p+2r}^r \\ &: \ \phi_p \ \mapsto \ f_p^r(\phi_p) := \sqrt{\tfrac{(n-p-r)!}{r!(n-p)!}} L^r \phi_p \,. \end{aligned} \qquad (2.124)$$

□

This linear map has inverse given by

$$\begin{aligned} \left(f_p^r\right)^{-1} \ &: \ S_{p+2r}^r \ \rightarrow \ S_p^0 \\ &: \ L^r\phi_p \ \mapsto \ \left(f_p^r\right)^{-1}(L^r\phi_p) := \sqrt{\tfrac{(n-p-r)!}{r!(n-p)!}} M^r \left(L^r\phi_p\right) , \end{aligned} \qquad (2.125)$$

as can be seen from (2.123).

Consequently, we may assert that

Proposition 2.49

For $0 \leq r \leq n$ and $0 \leq p \leq n-r$,

$$S_p^0 \ \cong \ S_{p+2r}^r \qquad (2.126)$$

where $\cong$ denotes a vector space isomorphism.

Corollary

Let $0 \leq r \leq n$, and let $2r \leq p \leq n+r$. Then

$$S_{p-2r}^0 \ \cong \ S_p^r. \qquad (2.127)$$

Proof

From (2.126) $S_q^0 \cong S_{q+2r}^r$ only if $0 \leq q \leq n-r$. This range is equivalent to the range $2r \leq q+2r \leq n+r$. Putting $p = q+2r$ yields

$$2r \leq p \leq n+r \implies S_{p-2r}^0 \cong S_p^r.$$

□

Proposition 2.50

Let r be any given height in the range $1 \leq r \leq n$. Then

$$2r \leq p \leq n+r \implies S_p^r \cong S_{p-2}^{r-1}. \tag{2.128}$$

Proof

We assume that $2r \leq p \leq n+r$. From (2.127),

$$S_p^r \cong S_{p-2r}^0. \tag{2.129}$$

But

$$\begin{aligned} 2r \leq p \leq n+r &\implies 2r-2 \leq p-2 \leq n+r-2 \\ &\implies 2(r-1) \leq p-2 \leq n+(r-1)-1 \\ &\implies 2(r-1) \leq p-2 < n+(r-1), \end{aligned}$$

and so applying (2.127) again,

$$S_{p-2}^{r-1} \cong S_{p-2-2(r-1)}^0 = S_{p-2r}^0. \tag{2.130}$$

The result follows by comparison of (2.129) and (2.130). □

Theorem 2.3 (The dimension of the effective subspaces)
For $0 \leq p \leq n$:

$$\dim(S_p^0) = \binom{2n}{p} - \binom{2n}{p-2} \,. \tag{2.131}$$

Proof

We use the convention that defines $\binom{n}{r} := 0$ for $r < 0$ (see Riordan (1958) ch. 1, §3.1, p. 5). Then for $p = 0, 1$ the result holds, since all scalars and 1-forms are effective; for these p values,

$$\dim(S_p^0) = \dim\left(\mathcal{S}^p(V^*, \omega)\right) = \binom{2n}{p} \,.$$

For the remaining p values, $2 \leq p \leq n$, the Lepage decomposition yields

$$\mathcal{S}^p(V^*, \omega) = S_p^0 \oplus S_p^1 \oplus .... \oplus S_p^q$$

where $q = \left[\frac{p}{2}\right]$. Therefore

$$\dim(S_p^0) = \dim(\mathcal{S}^p(V^*, \omega)) - \dim\left(S_p^1 \oplus \cdots \oplus S_p^q\right).$$

Consider the subspaces S_p^i for $1 \leq i \leq q$. Since $q = \left[\frac{p}{2}\right]$, it follows that $2q \leq p$. Consequently, $2i \leq p$ for each i. In addition, since $p \leq n$, we have that $p < n + i$, so for $1 \leq i \leq q$, p satisfies $2i \leq n + i$. So, (2.128) may be invoked to yield

$$\dim(S_p^0) = \dim(\mathcal{S}^p(V^*, \omega)) - \dim(S_{p-2}^0 \oplus \cdots \oplus S_{p-2}^{q-1}).$$

It is easily demonstrated that $\left[\frac{p-2}{2}\right] = q - 1$, since
for p even

$$\left[\frac{p-2}{2}\right] = \frac{p-2}{2} = \frac{p}{2} - 1 = \left[\frac{p}{2}\right] - 1 = q - 1;$$

and for p odd

$$\left[\frac{p-2}{2}\right] = \frac{p-3}{2} = \frac{p-1}{2} - 1 = \left[\frac{p}{2}\right] - 1 = q - 1.$$

Therefore, by the Lepage decomposition (2.84):

$$\begin{aligned}\dim(S^0_p) &= \dim(\mathcal{S}^p(V^*,\omega)) - \dim(\mathcal{S}^{p-2}(V^*,\omega)) \\ &= \binom{2n}{p} - \binom{2n}{p-2}.\end{aligned}$$

□

Corollary

Given $0 \le r \le n$ and $2r \le p \le n + r$,

$$\dim(S^r_p) = \binom{2n}{p-2r} - \binom{2n}{p-2r-2}. \tag{2.132}$$

Proof

By (2.127),

$$\begin{aligned}\dim\,(S^r_p) &= \dim\,(S^0_{p-2r}) && (2.133)\\ &= \binom{2n}{p-2r} - \binom{2n}{p-2r-2} \quad \text{(by (2.131))} && (2.134)\end{aligned}$$

for these values of r and p. □

2.3.7 Metric structure of the simple subspaces

We will demonstrate that the simple subspaces are mutually orthogonal and that they are nondegenerate. We calculate the signatures of the even degree simple subspaces, which are metric spaces.

The following result demonstrates that the simple subspaces are pairwise orthogonal.

Proposition 2.51

Let the extended metric act on two simple forms of degree $p+2r$ and heights $r+a$ and r respectively. They may be uniquely expressed as $L^{r+a}\phi_{p-2a}$ and $L^r\psi_p$ where ϕ_{p-2a} and ψ_p are effective. We have $0 \le r \le n$, $0 \le p \le n-r$, and $0 \le a \le \left[\frac{p}{2}\right]$. Then

$$\omega(L^{r+a}\phi_{p-2a}, L^r\psi_p) = \begin{cases} 0, & \text{for } a > 0 \\ \frac{r!(n-p)!}{(n-p-r)!}\,\omega(\phi_p, \psi_p), & \text{for } a = 0. \end{cases} \tag{2.135}$$

Proof

$$\begin{aligned}
\omega(L^{r+a}\phi_{p-2a}, L^r\psi_p) &= \sharp(\omega^{r+a} \wedge \phi_{p-2a}) \rfloor L^r\psi_p \qquad \text{(by definition)} \\
&= \sharp\phi_{p-2a} \rfloor \sharp\omega^{r+a} \rfloor L^r\psi_p \\
&= \sharp\phi_{p-2a} \rfloor M^{r+a} L^r\psi_p \\
&= \sharp\phi_{p-2a} \rfloor [M^{r+a}, L^r]\psi_p \qquad \text{(since } \psi_p \text{ is effective)} \\
&= \sum_{i=1}^{r} \binom{r}{i} \frac{(r+a)!}{(r+a-i)!} \prod_{k=0}^{i-1}(n-p+a-k)\sharp\phi_{p-2a} \rfloor L^{r-i} M^{r+a-i}\psi_p\,,
\end{aligned}$$

using (2.80). If $a > 0$, every term contains $M\psi_p$ so that the expression vanishes due to the effectiveness of ψ_p. If $a = 0$, one term, the term with $i = r$, survives. Thus

$$\begin{aligned}
\omega(L^r\phi_p, L^r\psi_p) &= r! \prod_{k=0}^{r-1}(n-p-k)\sharp\phi_p \rfloor \psi_p \\
&= \frac{r!(n-p)!}{(n-p-r)!}\omega(\phi_p, \psi_p). \qquad \text{(from (2.121), \& definition of } \omega(\ ,\ )\text{)}
\end{aligned}$$

□

Proposition 2.52

The simple subspaces are nondegenerate.

Proof

We use a contradiction argument. Assume the simple subspace S_p of degree p is degenerate. Then

$$\exists x \in S_p,\ x \neq 0, \text{ such that } \omega(x,u)=0 \quad \forall u \in S_p .$$

By the Lepage decomposition, $\mathcal{S}^p(V^*,\omega) = S_p \oplus R_p$, where R_p is the direct sum of the other simple subspaces of degree p. Consider any $v \in \mathcal{S}^p(V^*,\omega)$:

$$\begin{aligned} \omega(x,v) &= \omega(x, v_S + v_R) \qquad \text{(unique components)} \\ &= \omega(x,v_S) + \omega(x,v_R) \\ &= \omega(x,v_S) \qquad (S_p \text{ and } R_p \text{ are orthogonal}) \\ &= 0 . \qquad \text{(by the assumption)} \end{aligned}$$

This holds for all v in $\mathcal{S}^p(V^*,\omega)$ and and implies that $\mathcal{S}^p(V^*,\omega)$ is degenerate. To resolve the contradiction, we must conclude that x cannot be nonzero, and hence that S_p is nondegenerate. □

We recall that the extended metric was symmetric or antisymmetric on the homogeneous subspaces, depending on whether the degree of the space was even or odd, respectively. Since the simple subspaces are also subspaces of the homogeneous subspaces, they too are either symmetric or antisymmetric. Hence

Proposition 2.53

Let $0 \leq r \leq n$ and $2r \leq p \leq n+r$.

$$S_p^r \text{ is } \begin{cases} \text{metric,} & \text{if } p \text{ even;} \\ \text{symplectic,} & \text{if } p \text{ odd.} \end{cases} \tag{2.136}$$

Symplectic forms have only one canonical form, for given dimension. There is thus no further structure to be found in the odd degree simple subspaces. The even degree subspaces are metric, and it is natural to find their signatures. This is done in the following material.

Proposition 2.54

For $0 \le r \le n$ and $0 \le 2m \le 2\left[\frac{n-r}{2}\right]$,

$$\text{signature}(S^r_{2(m+r)}) = \text{signature}(S^0_{2m}). \qquad (2.137)$$

Proof

Let $0 \le r \le n$ and $0 \le 2m \le 2\left[\frac{n-r}{2}\right]$. Consider the isomorphism defined above:

$$f^r_{2m} : S^0_{2m} \longrightarrow S^r_{2m+2r}.$$

For ϕ_{2m} and ψ_{2m} effective,

$$\begin{aligned}
\omega(f^r_{2m}\,\phi_{2m}, f^r_{2m}\,\psi_{2m}) &= \frac{(n-2m-r)!}{r!(n-2m)!}\omega(L^r\phi_{2m}, L^r\psi_{2m}) \qquad \text{(by definition of } f^r_{2m}\text{)} \\
&= \frac{(n-2m-r)!}{r!(n-2m)!}\frac{r!(n-2m)!}{(n-2m-r)!}\omega(\phi_{2m},\psi_{2m}) \qquad \text{(by (2.135))} \\
&= \omega(\phi_{2m},\psi_{2m}).
\end{aligned}$$

Because the isomorphism preserves the extended metric, it must preserve the signature of the metric as well. Hence the signatures of S^0_{2m} and $S^r_{2(m+r)}$ are equal. □

Corollary

For $0 \le r \le n$ and $2r \le 2q \le 2\left[\frac{n+r}{2}\right]$,

$$\text{signature}(S^r_{2q}) = \text{signature}(S^0_{2(q-r)}). \qquad (2.138)$$

Proof

We had in (2.137) that for $0 \leq r \leq n$ and $0 \leq 2m \leq 2\left[\frac{n-r}{2}\right]$,

$$\text{signature } (S^r_{2(m+r)}) = \text{signature } (S^0_{2m}).$$

Put $q = m + r$ and eliminate m in favour of q. Then

$$\begin{aligned} 0 \leq 2m \leq 2\left[\frac{n-r}{2}\right] &\iff 0 \leq 2(q-r) \leq 2\left[\frac{n-r}{2}\right] \\ &\iff 2r \leq 2q \leq 2\left[\frac{n-r}{2}\right] + 2r. \end{aligned}$$

Consider $2\left[\frac{n-r}{2}\right] + 2r$.
For $n - r$ even,

$$2\left[\frac{n-r}{2}\right] + 2r = n - r + 2r = n + r = 2\left[\frac{n+r}{2}\right];$$

and for $n - r$ odd

$$2\left[\frac{n-r}{2}\right] + 2r = n - r - 1 + 2r = n + r - 1 = 2\left[\frac{n+r}{2}\right].$$

So the previous proposition (2.137) is translated into the statement

$$\text{signature}(S^r_{2q}) = \text{signature}(S^0_{2(q-r)}),$$

for $0 \leq r \leq n$ and $2r \leq 2q \leq 2\left[\frac{n+r}{2}\right]$. □

Corollary

For $1 \leq r \leq n$ and $2r \leq 2q \leq 2\left[\frac{n+r}{2}\right]$,

$$\text{signature}(S^r_{2q}) = \text{signature} S^{r-1}_{2q-2} \tag{2.139}$$

Proof

Assume that $1 \leq r \leq n$ and that $2r \leq 2q \leq 2\left[\frac{n+r}{2}\right]$. Then by (2.138),

$$\text{signature}(S^r_{2q}) = \text{signature}(S^0_{2(q-r)}). \tag{2.140}$$

But $2r \leq 2q \leq 2\left[\frac{n+r}{2}\right]$ from which it follows that $2(r-1) \leq 2q-2 \leq 2\left[\frac{n+r}{2}\right]-2$. Consider this maximum value for $2(q-1)$, namely $2\left[\frac{n+r}{2}\right]-2$: for $n+r$ even,

$$2\left[\frac{n+r}{2}\right]-2 = n+r-2 = 2\left[\frac{n+(r-1)}{2}\right];$$

and for $n+r$ odd

$$2\left[\frac{n+r}{2}\right]-2 = n+r-3 = 2\left[\frac{n+(r-1)}{2}\right]-2 < 2\left[\frac{n+(r-1)}{2}\right].$$

So we may rewrite the above ranges for r and for $2(q-1)$ as $1 \leq r \leq n$ and $2(r-1) \leq 2(q-1) < 2\left[\frac{n+(r-1)}{2}\right]$.

This allows us to invoke the previous corollary (2.138) to yield

$$\begin{aligned} \text{signature}(S^{r-1}_{2(q-1)}) &= \text{signature}(S^0_{2(q-1-(r-1))}) \\ &= \text{signature}(S^0_{2(q-r)}). \end{aligned} \tag{2.141}$$

Comparison of (2.140) and (2.141) yields the result. □

Theorem 2.4 (The signature of an even degree effective subspace)

For $0 \leq 2q \leq 2\left[\frac{n}{2}\right]$,

$$\text{signature}(S^0_{2q}) = \binom{n}{q} - \binom{n}{q-1}. \tag{2.142}$$

Proof

If $q = 0$, then the result holds trivially: S^0_{2q} is just S^0_0, which is the set of scalars, or $\mathcal{S}^0(V^*,\omega)$. Putting any non-zero number, say a, into the arguments of the extended metric produces a positive number, since $\omega(a,a) = \sharp a \rfloor a = a^2$. Following the convention where $\binom{n}{q} = 0$ for $q < 0$, the result holds in this case.

For $0 < 2q \leq 2\left[\frac{n}{2}\right]$, we have by the Lepage decomposition (2.84) that

$$\mathcal{S}^{2q}(V^*,\omega) = S^0_{2q} \oplus S^1_{2q} \oplus \cdots \oplus S^q_{2q}.$$

Because the simple subspaces are mutually orthogonal by (2.135), it follows that

$$\text{signature}(S^0_{2q}) = \text{signature}(\mathcal{S}^{2q}(V^*,\omega)) - \text{signature}(S^1_{2q} \oplus \cdots \oplus S^q_{2q}).$$

Consider the subspaces S^i_{2q} for $1 \leq i \leq q$. Since $2q \leq n$, it is certainly true that $q < n$. Thus we have $1 \leq i < n$. For the range of

values of $2q$, note that $i \leq q$ from which $2i \leq 2q$; in addition, since $2q \leq 2\left[\frac{n}{2}\right]$, it is clear that $2q \leq 2\left[\frac{n+i}{2}\right]$. Thus the ranges of values for i and $2q$ are $1 \leq i \leq n$ and $2i \leq 2q \leq 2\left[\frac{n+i}{2}\right]$.

This allows us to invoke corollary (2.139) to yield

$$\begin{aligned} \text{signature}(S^0_{2q}) &= \text{signature}(\mathcal{S}^{2q}(V^*,\omega)) - \text{signature}(S^0_{2q-2} \oplus \cdots \oplus S^{q-1}_{2q-2}) \\ &= \text{signature}(\mathcal{S}^{2q}(V^*,\omega)) - \text{signature}(\mathcal{S}^{2q-2}(V^*,\omega)) \end{aligned}$$

by the Lepage decomposition (2.84). Thus, using our earlier result (2.60),

$$\text{signature}(S^0_{2q}) = \binom{n}{q} - \binom{n}{q-1}.$$

□

Corollary

For $1 \leq r \leq n$ and $2r \leq 2q \leq 2\left[\frac{n+r}{2}\right]$,

$$\text{signature}(S^r_{2q}) = \binom{n}{q-r} - \binom{n}{q-r-1}. \qquad (2.143)$$

Proof

For $0 \leq r \leq n$ and $2r \leq 2q \leq 2\left[\frac{n+r}{2}\right]$ we had in (2.138) that

$$\text{signature}(S^r_{2q}) = \text{signature}(S^0_{2(q-r)}).$$

We saw in the proof of (2.140) that for $0 \leq r \leq n$, the range $2r \leq 2q \leq 2\left[\frac{n+r}{2}\right]$ is equivalent to the range $0 \leq 2(q-r) \leq 2\left[\frac{n-r}{2}\right]$. So clearly $0 \leq 2(q-r) \leq 2\left[\frac{n}{2}\right]$, which allows us to invoke (2.142), yielding

$$\text{signature}(S^r_{2q}) = \binom{n}{q-r} - \binom{n}{q-r-1}.$$

□

A natural development of the material of this chapter is to extend the results onto a manifold. This line is not followed here. Accounts of symplectic manifolds may be found in Libermann and Marle (1987), or Guillemin and Sternberg (1977,1984).

Chapter 3

Lepage decomposition for the symmetric tensor algebra

The main result of this chapter is the proof of a Lepage decomposition theorem for the symmetric exterior algebra endowed with a metric. As in the symplectic case, this decomposition has been interpreted in terms of a direct sum of simple spaces. As far as I know, these results are new.

My strategy here has been to imitate the development of the material in the previous chapter. This turned out to be a most successful scheme, and with only a few exceptions, the structures of the symplectic exterior algebra carried over into the present case. This chapter is consequently shorter than chapter 2 as many proofs are omitted, being identical to their symplectic counterparts.

The material is distributed as follows.

There are three sections. In the first section, I review the symmetric exterior algebra, and define generalised interior products. In the second section, I introduce a metric and use it to define the isomorphisms $\flat$ and $\sharp$. These structures closely follow those defined in the symplectic case. However, there is no volume form, so a Hodge dual operator cannot be defined. The next structure set up is an extended metric. In the final section, I define operators L and M, obtain their commutation relations, and use the results to prove a Lepage decomposition theorem in this exterior algebra. I interpret this theorem in terms of a direct sum of simple subspaces and investigate the properties of these spaces.

3.1 Algebra of symmetric tensors

I review some definitions and properties of the algebra of symmetric tensors in its two guises, with upper and lower indices. Similar material can be found in Shaw (1983) vol. II, ch. 10, and Crumeyrolle (1990).

3.1.1 Symmetric forms

Given an m dimensional vector space V, a symmetric p-form ϕ is a multilinear map from p cartesian products of V into the reals which is totally symmetric. In other words, it is a totally symmetric tensor with p covariant indices. Addition of symmetric forms is defined as for tensors or exterior forms, by requiring that the valuation of a sum is the sum of the valuations. This does not affect the symmetry, and the set of all symmetric p-forms is a vector space, denoted by $\bigvee^p(V^*)$. I will refer to this vector space as the 'homogeneous symmetric p-forms'. The dimensions of the homogeneous spaces of degree p are given by (Crumeyrolle p 32, Shaw p 370)

$$\dim \overset{p}{\bigvee}(V^*) = \binom{m+p-1}{p} . \tag{3.1}$$

As in the exterior algebra, the zero-forms are identified with the reals; furthermore, the 1-forms coincide with the antisymmetric 1-forms. Unlike the exterior algebra which terminates after the antisymmetric forms of degree equal to the dimension of V, the symmetric algebra, consisting of the direct sum of symmetric forms of all degrees, never terminates; it is infinite dimensional. In this chapter, 'form' will refer to 'symmetric forms', unless otherwise stated.

The symmetric product, denoted by $\vee$, produces a $(p+q)$-form from a p-form and a q-form. It is essentially a symmetrised tensor product, defined by

$$(\phi\vee\psi)(X_1,\ldots,X_{(p+q)}) := \frac{1}{p!q!}\sum_{\pi}\phi(X_{\pi(1)},\ldots,X_{\pi(p)})\,\psi(X_{\pi(p+1)},\ldots,X_{\pi(p+q)})\,. \tag{3.2}$$

The symmetric product is bilinear and associative.

The interior product of a p-form and a vector produces a $(p-1)$-form, defined by

$$(X\rfloor\phi)(Y_1,\ldots,Y_{p-1}) := \phi(X,Y_1,\ldots,Y_{p-1}) \tag{3.3}$$

where $X, Y_1, \ldots, Y_{p-1}$ are arbitrary vectors. This operation is also referred to as the 'contraction of X onto ϕ'. The interior product is bilinear, and acts as a derivation in distributing over the symmetric product:

$$X\rfloor(\phi\vee\psi) = (X\rfloor\phi)\vee\psi + \phi\vee(X\rfloor\psi)\,. \tag{3.4}$$

The symmetric algebra on the m dimensional vector space V is isomorphic to the polynomials in m indeterminates (Shaw p 390). The following result which I shall use later is essentially Euler's theorem for homogeneous functions in m unknowns.

Proposition 3.1

Let V be an m-dimensional vector space with any basis $\{e_i\}$. Let $\{e^i\}$ be a dual basis of V^* and let ϕ belong to $\bigvee^r(V^*)$. Then

$$\sum_{i=1}^{m} e^i \vee (e_i \rfloor \phi) = r\,\phi \tag{3.5}$$

Proof

It suffices to show the result for decomposable ϕ only, as it then extends to all ϕ by the linearity of the interior and symmetric products. Since

$$e_i \rfloor e^{j_1} \vee \ldots \vee e^{j_r} = \begin{cases} e^{j_1} \vee \ldots \vee \widehat{e^i} \vee \ldots \vee e^{j_r} & i \in \{j_1, \ldots, j_r\} \\ 0 & i \notin \{j_1, \ldots, j_r\} \end{cases}$$

it follows that

$$\underbrace{e^i \vee (e_i}_{\text{no sum}} \rfloor e^{j_1} \vee \ldots \vee e^{j_r}) = \begin{cases} e^{j_1} \vee \ldots \vee e^{j_r} & i \in \{j_1, \ldots, j_r\} \\ 0 & i \notin \{j_1, \ldots, j_r\}. \end{cases}$$

If we sum over all values of i then clearly

$$\sum_{i=1}^{m} e^i \vee (e_i \rfloor e^{j_1} \vee \ldots \vee e^{j_r}) = r e^{j_1} \vee \ldots \vee e^{j_r}.$$

□

A corresponding result, (2.3), was stated for the exterior algebra.

I now generalise the definition of the interior product so that any multivector may be contracted onto a form. As in the case of the exterior algebra, the definition is in two steps. First, $X \rfloor \phi$ is defined for a decomposable X; in the second step this is extended to the non-decomposable case.

Definition 3.1

Let ϕ be any (symmetric) p-form and let $e_1, \ldots, e_k$ be vectors, with $k \leq p$. Then define

$$(e_1 \vee \ldots \vee e_k) \rfloor \phi := e_k \rfloor (\ldots \rfloor (e_1 \rfloor \phi)).$$

Let a k-form X have a coordinate presentation $X^{i_1 \ldots i_k} e_{i_1} \vee \ldots \vee e_{i_k}$. Then define

$$\begin{aligned} X \rfloor \phi &= (X^{i_1 \ldots i_k} e_{i_1} \vee \ldots \vee e_{i_k}) \rfloor \phi \\ &:= X^{i_1 \ldots i_k}((e_{i_1} \vee \ldots \vee e_{i_k}) \rfloor \phi). \end{aligned}$$

To complete the definition, we define, for a scalar a:

$$a \rfloor \phi := a\phi ,$$

and we define $X \rfloor \phi := 0$ if X is of degree greater than that of ϕ. □

It is easy to demonstrate the following properties:

Proposition 3.2

Let X, Y be multivectors and let ϕ, ψ be p-forms. Then

$$X \rfloor (Y \rfloor \phi) = Y \rfloor (X \rfloor \phi) \tag{3.6}$$

$$(X \vee Y) \rfloor \phi = X \rfloor (Y \rfloor \phi) \tag{3.7}$$

$$X \rfloor (\phi + \psi) = X \rfloor \phi + X \rfloor \psi \tag{3.8}$$

$$X \rfloor (a\phi) = a(X \rfloor \phi) \tag{3.9}$$

3.1.2 Symmetric multivectors

The symmetric forms are tensors with lower indices that are symmetric; an isomorphic structure may be set up for the upper index symmetric tensors. This is the algebra of symmetric multivectors.

Let V^* be the dual of the m dimensional vector space V. Then a symmetric p-vector X is a multilinear map from p cartesian products of V^* into the reals, which is totally symmetric. The set of all symmetric p-vectors is a vector space under the operations of scalar multiplication and addition defined as for the exterior forms. Denote this space by $\bigvee^p(V)$ and refer to its elements as 'homogeneous multivectors of degree p'. The dimensions of these spaces are given by the same expression as above:

$$\dim \overset{p}{\bigvee}(V) \;=\; \binom{m+p-1}{p}\,. \tag{3.10}$$

The multivectors of degree zero are identified with the reals, and those of degree one make up the vector space V. The algebra consisting of the direct sum of the symmetric multivectors of all degrees is also infinite dimensional. It will be denoted by $\bigvee(V)$. In this chapter, 'p-vector' will mean 'symmetric p-vector', unless otherwise stated.

A symmetric product of multivectors is defined along the same lines as for the symmetric forms. It is denoted by the same symbol $\vee$, and produces a $(p+q)$-vector from a p-vector and a q-vector. It is defined by

$$(X \vee Y)(\theta^1, \ldots, \theta^{(p+q)}) \;:=\; \frac{1}{p!q!}\sum_{\pi} X(\theta^{\pi(1)}, \ldots, \theta^{\pi(p)})\, Y(\theta^{\pi(p+1)}, \ldots, \theta^{\pi(p+q)}) \tag{3.11}$$

where X , Y are multivectors of degree p and q respectively and $\theta^1, \ldots, \theta^{(p+q)}$ are arbitrary 1-forms. This product is again bilinear and associative.

The interior product of a p-vector and a 1-form produces a $(p-1)$-vector, defined by

$$(X\lfloor\theta)(\eta^1, \ldots, \eta^{p-1}) := X(\theta, \eta^1, \ldots, \eta^{p-1}) \tag{3.12}$$

where $\theta, \eta^1, \ldots, \eta^{p-1}$ are arbitrary 1-forms. Again, the interior product is bilinear, and acts as a (right) derivation in distributing over the symmetric product of multivectors:

$$(X \vee Y)\lfloor\theta = (X\lfloor\theta) \vee Y + X \vee (Y\lfloor\theta)\,. \tag{3.13}$$

As in the case of the left interior product, we may generalise the definition of this interior product to allow any form to be contracted onto a multivector.

Definition 3.2

Let X be a multivector of degree p and let $\theta^{i_1}, \ldots, \theta^{i_k}$ be one-forms. If $k \leq p$, define

$$X\lfloor(\theta^{i_1} \vee \ldots \vee \theta^{i_k}) := ((X\lfloor\theta^{i_1})\lfloor\ldots)\lfloor\theta^{i_k}.$$

Let ϕ be any k-form with a coordinate presentation $\phi_{i_1 \ldots i_k}\theta^{i_1} \vee \ldots \vee \theta^{i_k}$. If $k \leq p$, define

$$\begin{aligned} X\lfloor\phi &= X\lfloor(\phi_{i_1 \ldots i_k}\theta^{i_1} \vee \ldots \vee \theta^{i_k}) \\ &:= \phi_{i_1 \ldots i_k}(X\lfloor\theta^{i_1} \vee \ldots \vee \theta^{i_k}). \end{aligned}$$

For any scalar a,

$$X\lfloor a := aX\,.$$

If ϕ has degree $k > p$, define

$$X\lfloor\phi := 0\,.$$

□

We have the following properties:

Proposition 3.3

Let X, Y be multivectors, ϕ a form, and let a be a scalar. Then

$$(X\lfloor\phi)\lfloor\psi = (X\lfloor\psi)\lfloor\phi \tag{3.14}$$

$$X\lfloor(\phi \vee \psi) = (X\lfloor\phi)\lfloor\psi \tag{3.15}$$

$$(X+Y)\lfloor\phi = X\lfloor\phi + Y\lfloor\phi \tag{3.16}$$

$$(aX)\lfloor\phi = a(X\lfloor\phi)\,. \tag{3.17}$$

We may also prove that

Proposition 3.4

Let X be a homogeneous multivector, and let ϕ be a form of the same degree. Then

$$X \rfloor \phi = X \lfloor \phi . \tag{3.18}$$

3.2 Symmetric algebra with a metric

3.2.1 The metric

Recalling that our scheme in this chapter is to set up a framework parallel to the framework established for the symplectic exterior algebra, it is necessary to define a structure analogous to the symplectic two form, for the symmetric exterior algebra. The object has to be of degree two and must belong to the algebra itself. It is therefore a symmetric two-form, or metric. By analogy with the symplectic case it must also be nondegenerate.

From this point on, we may assume our vector space V has such a distinguished nondegenerate metric g belonging to $\bigvee^2(V^*)$. We know from the standard theory (Göckeler and Schücker, 1987 pp 33-4) that there always exist orthonormal bases of V in which g, considered as a matrix, takes on a diagonal form having only $+1$'s or -1's on the diagonal. The number of positive and negative entries is independent of the orthonormal basis chosen, and is thus a property of g. Denote the number of positives and negatives by p and q respectively. The nondegeneracy of g then requires that $p + q = m$.

Thus there will always exist a basis $\{e_1, \ldots, e_p, f_1, \ldots, f_q\}$ of V such that $g(e_i, e_i) = +1$ and $g(f_j, f_j) = -1$ for i, j running from 1 to p, q respectively.

g is then given in terms of the dual basis $\{e^1, \ldots, e^p, f^1, \ldots, f^q\}$ as

$$g = \frac{1}{2}\left(e^1 \vee e^1 + \cdots + e^p \vee e^p - f^1 \vee f^1 - \cdots - f^q \vee f^q\right). \tag{3.19}$$

3.2.2 The lowering operator

We may use the metric g to define an isomorphism from $\bigvee^r(V)$ to $\bigvee^r(V^*)$ for all r. The development here is aimed at imitating that followed for the exterior algebra with a distinguished symplectic form.

Definition 3.3

If a is any scalar, then

$$\flat a \;:=\; a\,;$$

if x is a vector, then

$$\flat x \;:=\; x \rfloor g\,;$$

and if X is any symmetric r-vector, then

$$(\flat X)(x_1, ..., x_r) \;:=\; X(\flat x_1, ..., \flat x_r)$$

where $x_1, ..., x_r$ are arbitrary vectors. □

It is easily verified that the above operation expressed in index form corresponds to the standard method of lowering tensor indices using a metric.

Proposition 3.5

Let $\{e_1, \ldots, e_p, f_1, \ldots, f_q\}$ be a canonical basis of V and let $\{e^1, \ldots, e^p, f^1, \ldots, f^q\}$ be its dual basis. Then

$$\begin{aligned} \flat e_i &= e^i \\ \flat f_j &= -f^j. \end{aligned} \tag{3.20}$$

Proof

$$\begin{aligned}
\flat f_j &= f_j \rfloor g \qquad \text{(by definition)} \\
&= \frac{1}{2} f_j \rfloor \left(e^1 \vee e^1 + \cdots + e^p \vee e^p - f^1 \vee f^1 - \cdots - f^q \vee f^q\right) \\
&= -\frac{1}{2} f_j \rfloor f^j \vee f^j \qquad \text{(no summation over } j\text{)}
\end{aligned}$$

But for any vector x,

$$\begin{aligned}
\left(f_j \rfloor f^j \vee f^j\right)(x) &= f^j \vee f^j\,(f_j, x) \qquad \text{(no summation)} \\
&= f^j(f_j) f^j(x) + f^j(x) f^j(f_j) \qquad \text{(no summation)} \\
&= 2 f^j(x)\,,
\end{aligned}$$

and substituting $f_j \rfloor f^j \vee f^j = 2f^j$ into the above expression yields the result that $\flat f_j = -f^j$ for $j = 1$ to q. Similarly, we have $\flat e_i = e^i$ for $i = 1$ to p. □

Proposition 3.6

Let a be a scalar, and let X, Y be symmetric r-vectors, and let Z be an s-vector, with s not necessarily equal to r. Then

$$\begin{aligned}
\flat(aX) &= a(\flat X) \\
\flat(X + Y) &= \flat X + \flat Y \\
\flat(X \vee Z) &= (\flat X) \vee (\flat Z)\,.
\end{aligned}$$

The proof of this proposition closely follows the analogous proof in the symplectic case.

3.2.3 Raising operator

I will define a distinguished 2-vector and use it to express the inverse of the lowering map.

Definition 3.4
G is a symmetric 2-vector satisfying

$$\flat G \;:=\; g\,.$$

□

It is easily shown that in a canonical basis $\{e_1, \ldots, e_p, f_1, \ldots, f_q\}$, G is expressed as

$$G = \frac{1}{2}(e_1 \vee e_1 + \cdots + e_p \vee e_p - f_1 \vee f_1 - \cdots - f_q \vee f_q)\,. \tag{3.21}$$

Definition 3.5
If a is a scalar, then

$$\sharp a \;:=\; a\,;$$

if θ is a 1-form, then

$$\sharp\theta \;:=\; G\lfloor\theta\,;$$

and if ϕ is any symmetric r-form, then

$$(\sharp\phi)(\theta^1, \ldots, \theta^r) \;:=\; \phi(\sharp\theta^1, \ldots, \sharp\theta^r) \tag{3.22}$$

where $\theta^1, \ldots, \theta^r$ are arbitrary 1-forms. □

Proposition 3.7

Let $\{e_1, \ldots, e_p, f_1, \ldots, f_q\}$ be a canonical basis of V with dual basis $\{e^1, \ldots, e^p, f^1, \ldots, f^q\}$. Then

$$\begin{aligned} \sharp e^i &= e_i \\ \sharp f^j &= -f_j \end{aligned}$$

The properties of $\sharp$ are no different from those of $\flat$:

Proposition 3.8

Let a be any scalar, let ϕ, ψ be symmetric r-forms, and let ξ be any s-form. Then

$$\begin{aligned} \sharp(a\phi) &= a(\sharp\phi) \\ \sharp(\phi + \psi) &= \sharp\phi + \sharp\psi \\ \sharp(\phi \vee \xi) &= (\sharp\phi) \vee (\sharp\xi) \end{aligned}$$

We also have

Proposition 3.9

For any symmetric form ϕ and any symmetric vector X,

$$\begin{aligned} \flat\sharp\phi &= \phi \\ \sharp\flat X &= X \\ \sharp(X \rfloor \phi) &= (\sharp\phi) \lfloor (\flat X) . \end{aligned}$$

3.2.4 Extended metric

The vector space V^* has metric G. I will extend this metric to act on symmetric forms of any degree. The homogeneous subspaces will be mutually orthogonal by this definition. I will show that the extended metric on the homogeneous subspaces is symmetric in all cases. The signature of the extended metric on these subspaces will be found.

Definition 3.6

Define a metric that maps the cartesian product of $\bigvee^r(V^*)$ and $\bigvee^s(V^*)$ into the reals. Referring to it as the *extended metric* and denoting it by $g(\ ,\ )$, we define

$$g(\phi,\psi) \quad := \quad \delta_{rs}\,\sharp\phi\rfloor\psi\,.$$

where $\phi\,,\psi$ belong to $\bigvee^r(V^*)$ and $\bigvee^s(V^*)$ respectively, □

Proposition 3.10

The extended metric $g(\ ,\ )$ is bilinear and symmetric in its arguments, for all degrees.

Proof

The bilinearity is evident from the fact that $\sharp$ and $\rfloor$ act linearly. If the degrees of the arguments are unequal, the outcome is zero and symmetry is trivial; to prove the symmetry for arguments of equal degree, it will suffice to consider only decomposable $\phi = \phi^1 \vee \ldots \vee \phi^r$ as the result follows by linearity.

$$\begin{aligned}
g(\phi,\psi) &= \sharp\phi\rfloor\psi \qquad \text{(by definition)} \\
&= \sharp(\phi^1 \vee \ldots \vee \phi^r)\rfloor\psi \\
&= \psi(\sharp\phi^1,\ldots,\sharp\phi^r) \\
&= \sharp\psi(\phi^1,\ldots,\phi^r) \qquad \text{(by definition of } \sharp) \\
&= (\sharp\psi)\lfloor\phi^1 \vee \ldots \vee \phi^r \\
&= \sharp\psi\lfloor\phi
\end{aligned}$$

$$= \sharp\psi \rfloor \phi \qquad \text{(since arguments are of equal degree)}$$
$$= g(\psi, \phi)$$

□

From here on, $\mathcal{M}^k(V^*, g)$ will denote the homogeneous subspace $\bigvee^k(V^*)$ with the above extended metric on it. The complete algebra formed from the direct sum of the homogeneous subspaces will be denoted by $\mathcal{M}(V^*, g)$. The two dual spaces will be denoted by $\mathcal{M}^k(V, g)$ (for the homogeneous k-forms) and $\mathcal{M}(V, g)$ (for the direct sum of the homogeneous subspaces of k-forms).

Proposition 3.11

The extended metric expressed in the basis of $\mathcal{M}^r(V^*, g)$ generated from any diagonal basis of V^* is diagonal. If the general element of this basis is given by $e^{i_1} \vee \ldots \vee e^{i_{r-a}} \vee f^{j_1} \vee \ldots \vee f^{j_a}$ for $0 \le a \le r$, then the sign of the corresponding diagonal entry is $(-1)^a$.

Proof

Consider the extended metric operating on an arbitrary pair of basis r-forms in $\mathcal{M}^r(V^*, g)$. Suppose in the following that $a \le b$. Then

$$g\left(e^{i_1} \vee \ldots \vee e^{i_{r-a}} \vee f^{j_1} \vee \ldots \vee f^{j_a}, e^{k_1} \vee \ldots \vee e^{k_{r-b}} \vee f^{l_1} \vee \ldots \vee f^{l_b}\right)$$
$$= \sharp(e^{i_1} \vee \ldots \vee e^{i_{r-a}} \vee f^{j_1} \vee \ldots \vee f^{j_a}) \rfloor e^{k_1} \vee \ldots \vee e^{k_{r-b}} \vee f^{l_1} \vee \ldots \vee f^{l_b}.$$

But we have that $\sharp e^i = e_i$ and $\sharp f^i = -f_i$; hence the above becomes

$$= (-1)^a e_{i_1} \vee \ldots \vee e_{i_{r-a}} \vee f_{j_1} \vee \ldots \vee f_{j_a} \rfloor e^{k_1} \vee \ldots \vee e^{k_{r-b}} \vee f^{l_1} \vee \ldots \vee f^{l_b}$$
$$= (-1)^a e^{k_1} \vee \ldots \vee e^{k_{r-b}} \vee f^{l_1} \vee \ldots \vee f^{l_b}(e_{i_1}, \ldots, e_{i_{r-a}}, f_{j_1}, \ldots, f_{j_a})$$
$$= (-1)^a \sum_{\pi} e^{k_1}(\pi e_{i_1}) \ldots e^{k_{r-b}}(\pi e_{i_{r-b}}) f^{l_1}(\pi e_{i_{r-b+1}}) \ldots$$
$$\ldots f^{l_{b-a}}(\pi e_{i_{r-a}}) f^{l_{b-a+1}}(\pi f_{j_1}) \ldots f^{l_b}(\pi f_{j_a})$$

where each π is a permutation of the index set of r elements. As long as $a \neq b$ the above expression will be zero because there are more e_i's than f^j's. Thus a non-zero result is possible only when $a = b$. In these cases, the expression becomes

$$\begin{aligned} & g\left(e^{i_1} \vee \ldots \vee e^{i_{r-a}} \vee f^{j_1} \vee \ldots \vee f^{j_a}, e^{k_1} \vee \ldots \vee e^{k_{r-a}} \vee f^{l_1} \vee \ldots \vee f^{l_a}\right) \\ = \; & (-1)^a \sum_{\pi} e^{k_1}(\pi e_{i_1}) \ldots e^{k_{r-a}}(\pi e_{i_{r-a}}) f^{l_1}(\pi f_{j_1}) \ldots f^{l_a}(\pi f_{j_a}) . \end{aligned}$$

Excluding the coefficient $(-1)^a$, the summation has at least one non-zero term, where the permutation π takes the index set $\{i_1, \ldots, i_{r-a}, j_1, \ldots, j_a\}$ to
$\{k_1, \ldots, k_{r-a}, l_1, \ldots, l_a\}$, but there may be more due to the possibility of repeated factors. However the summation itself has positive sign, and so the sign of the expression is given by $(-1)^a$. □

Definition 3.7

Any basis for $\mathcal{M}^k(V^*, g)$ generated from a diagonal basis for $\mathcal{M}^1(V^*, g)$ will be called a *diagonal* basis. □

Corollary

The extended metric $g(\; , \;)$ is nondegenerate on each homogeneous subspace.

Proof

By the arguments in proposition (3.11) there can never be a zero on the diagonal of the matrix representing the extended metric in a diagonal basis. This guarantees the nondegeneracy. □

The signature of the metric g on the vector space is determined by the pair of numbers p and q, where $p+q=m$. As in the previous chapter, we will express the signature as the difference $p-q$. Then the signature of the vector space is $p-q=p-(m-p)=2p-m$.

Proposition 3.12

If $\mathcal{M}^1(V^*,g)$ has signature $2p-m$ where p is the number of positive entries on the diagonal of any orthonormal basis, then the signature of $\mathcal{M}^r(V^*,g)$ is given by

$$\text{signature}\,\mathcal{M}^r(V^*,g) \;=\; \sum_{i=0}^{r}(-1)^i\binom{p+r-i-1}{p-1}\binom{m-p+i-1}{m-p-1}\,. \quad (3.23)$$

Proof

As in the previous theorem, we work in the basis induced from a diagonal basis

$\{e^1,\ldots,e^p,f^1,\ldots,f^q\}$ of V^*. The proof is a simple counting procedure. We write out the basis of $\binom{m+r-1}{r}$ symmetric r-forms gathering together first those elements with no f^j's, then those with one f^j, those

with two, and so on, until finally we get to those with r f^j's.

Letting i denote the number of f's in a particular category, we claim that the number of basis elements in this category having i f^j's and $r-i$ e^j's is

$$\binom{p+r-i-1}{p-1}\binom{m-p+i-1}{m-p-1}\,. \quad (3.24)$$

To justify this, note that the $r-i$ e^j's are selected from a pool of p, with unlimited repetition. This may be done in

$$\binom{p+r-i-1}{p-1}$$

different ways. In the case of the f^j's, there are i of them, to be selected from a stock of $m-p$, also with unrestricted repetition. This may be done in

$$\binom{m-p+i-1}{m-p-1}$$

different ways. The product of these two combination formulae therefore yields the number of basis elements in this category.

By the previous theorem, the negatives in the signature are contributed by the basis elements with i odd, and the positives by those with i even. To obtain the signature, we must find the sum of all products like (3.24) with i even, and subtract all those with i odd. This is precisely what is expressed in the given formula (3.23). □

3.3 The Lepage decomposition

3.3.1 Operators L and M

By analogy with the symplectic exterior algebra, define two operators which arise naturally from the structure of $\mathcal{M}^r(V^*, g)$.

Definition 3.8

Let θ be an element of $\mathcal{M}^r(V^*, g)$. Then define

$$\begin{aligned} L_g \ &: \ \mathcal{M}^r(V^*, g) \ \rightarrow \ \mathcal{M}^{r+2}(V^*, g) \\ &: \ \theta \ \mapsto \ L_g\theta = g \vee \theta \end{aligned} \tag{3.25}$$

Where no ambiguity arises, we will write L in place of L_g. □

L has the following properties:

Proposition 3.13

Let a, b be reals. Let ϕ, ψ belong to $\mathcal{M}^r(V^*, g)$ and let ξ belong to $\mathcal{M}^k(V^*, g)$. Then

$$\begin{aligned} L(a\phi + b\psi) &= aL\phi + bL\psi && (3.26)\\ L(\phi \vee \xi) &= (L\phi) \vee \xi = \phi \vee (L\xi). && (3.27) \end{aligned}$$

Definition 3.9

Let θ be an element of $\mathcal{M}^r(V^*, g)$. Then define

$$\begin{aligned} M_g &: \mathcal{M}^r(V^*, g) \rightarrow \mathcal{M}^{r-2}(V^*, g)\\ &: \theta \mapsto M_g\theta = (\sharp g)\rfloor\theta = G\rfloor\theta \end{aligned} \qquad (3.28)$$

Where no ambiguity arises, we will write M in place of M_g. □

Proposition 3.14

M is a linear operator on the exterior algebra. That is, for a, b reals and ϕ, ψ belonging to $\mathcal{M}^r(V^*, g)$

$$M(a\phi + b\psi) = a(M\phi) + b(M\psi). \qquad (3.29)$$

Furthermore, M is not a derivation with respect to the exterior product.

Because M is not a derivation, it seems natural to define

Definition 3.10

Let ϕ, ψ be forms of degree k and r respectively. Define

$$\begin{aligned} P_M &: \mathcal{M}^k(V^*, g) \times \mathcal{M}^r(V^*, g) \rightarrow \mathcal{M}^{k+r-2}(V^*, g)\\ &: (\phi, \psi) \mapsto P_M(\phi, \psi) \end{aligned} \qquad (3.30)$$

where

$$P_M(\phi, \psi) \quad := \quad M(\phi \vee \psi) - (M\phi) \vee \psi - \phi \vee (M\psi).$$

□

P_M measures the degree of departure of M from being a derivation. I do not investigate P_M any further here.

Proposition 3.15

M and L are adjoint operators with respect to the extended metric. That is, if the degree of ϕ is two less than that of ψ, then

$$\begin{aligned} g(L\phi, \psi) &= g(\phi, M\psi)\,; \\ g(\psi, L\phi) &= g(M\psi, \phi)\,. \end{aligned}$$

Proof

$$\begin{aligned} g(L\phi, \psi) &= g(g \vee \phi, \psi) && \text{(definition of } L\text{)} \\ &= \sharp(g \vee \phi) \rfloor \psi && \text{(definition of metric)} \\ &= (\sharp g \vee \sharp\phi) \rfloor \psi && \text{(property of } \sharp\text{)} \\ &= \sharp\phi \rfloor (\sharp g \rfloor \psi) && \text{(property of interior product)} \\ &= \sharp\phi \rfloor (M\psi) && \text{(definition of } M\text{)} \\ &= g(\phi, M\psi) && \text{(definition of metric.)} \end{aligned}$$

The second statement follows trivially from the symmetry of the extended metric. □

Proposition 3.16

Let $\phi \in \mathcal{M}^k(V^*, g)$. Then for any homogeneous form ψ

$$g([M^r, L^s]\,\phi, \psi) = g(\phi, [M^s, L^r]\,\psi)\,. \tag{3.31}$$

Proof

The result is trivially true if the degree of ψ is not equal to the degree of $[M^r, L^s]\,\phi$ since the extended metric is zero on forms of differing degree. For ψ of degree $k + 2s - 2r$,

$$\begin{aligned}
g([M^r, L^s]\,\phi, \psi) &= g(M^r L^s \phi - L^s M^r \phi, \psi) \\
&= g(M^r L^s \phi, \psi) - g(L^s M^r \phi, \psi) \qquad \text{(since } g \text{ is bilinear)} \\
&= g(L^s \phi, L^r \psi) - g(M^r \phi, M^s \psi) \qquad \text{(adjoint property)} \\
&= g(\phi, M^s L^r \psi) - g(\phi, L^r M^s \psi) \qquad \text{(adjoint property)} \\
&= g(\phi, M^s L^r \psi - L^r M^s \psi) \qquad \text{(bilinearity of } g\text{)} \\
&= g(\phi, [M^s, L^r]\,\psi)\,.
\end{aligned}$$

□

Proposition 3.17

$$\sharp g \rfloor g = \frac{1}{2} m\,. \tag{3.32}$$

Proof

$$\begin{aligned}
\sharp g \rfloor g &= G \rfloor g \\
&= \frac{1}{4}\left\{\sum_{i=1}^{p} e_i \vee e_i - \sum_{j=1}^{q} f_j \vee f_j\right\} \rfloor \left\{\sum_{h=1}^{p} e^h \vee e^h - \sum_{k=1}^{q} f^k \vee f^k\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\left\{\sum_{i=1}^{p}\sum_{h=1}^{p}(e_i \vee e_i)\rfloor(e^h \vee e^h) + \sum_{j=1}^{q}\sum_{k=1}^{q}(f_j \vee f_j)\rfloor(f^k \vee f^k)\right\} \\
&= \frac{1}{4}\{2p+2q\} \\
&= \frac{1}{2}m\,.
\end{aligned}
$$

□

For convenience, I will make the following definition.

Definition 3.11

$$
n \;:=\; \frac{1}{2}m\,. \tag{3.33}
$$

□

Note that n is not necessarily an integer as m, the dimension of V, may be an odd number.

3.3.2 The commutators of L and M

Proposition 3.18

Let ϕ belong to $\mathcal{M}^k(V^*, g)$. Then

$$
[M, L]\,\phi = (n+k)\,\phi\,. \tag{3.34}
$$

Proof

We work in a diagonal basis $\{e^1, \ldots, e^p, f^1, \ldots, f^q\}$ of (V^*, g).

$$
ML\phi \;=\; \sharp g\rfloor(g \vee \phi) \qquad \text{(by definition of } M, L\text{)}
$$

$$\begin{aligned}
&= \frac{1}{2}\left\{\sum_{i=1}^{p} e_i \vee e_i - \sum_{j=1}^{q} f_j \vee f_j\right\}\rfloor(g \vee \phi) \\
&= \frac{1}{2}\sum_{i=1}^{p} e_i\rfloor e_i\rfloor(g \vee \phi) - \frac{1}{2}\sum_{j=1}^{q} f_j\rfloor f_j\rfloor(g \vee \phi)
\end{aligned}$$

The Leibnitz product rule is applied twice, giving

$$\begin{aligned}
ML\phi &= \frac{1}{2}\sum_{i=1}^{p} e_i\rfloor\{(e_i\rfloor g) \vee \phi + g \vee (e_i\rfloor\phi)\} - \frac{1}{2}\sum_{j=1}^{q} f_j\rfloor\{(f_j\rfloor g) \vee \phi + g \vee (f_j\rfloor\phi)\} \\
&= \frac{1}{2}\sum_{i=1}^{p}\{(e_i\rfloor e_i\rfloor g) \vee \phi + (e_i\rfloor g) \vee (e_i\rfloor\phi) + (e_i\rfloor g) \vee (e_i\rfloor\phi) + g \vee (e_i\rfloor e_i\rfloor\phi)\} \\
&- \frac{1}{2}\sum_{j=1}^{q}\{(f_j\rfloor f_j\rfloor g) \vee \phi + (f_j\rfloor g) \vee (f_j\rfloor\phi) \\
&\qquad + (f_j\rfloor g) \vee (f_j\rfloor\phi) + g \vee (f_j\rfloor f_j\rfloor\phi)\}
\end{aligned}$$

Recalling that $e_i\rfloor g = \flat e_i = e^i$, that $f_j\rfloor g = \flat f_j = -f^j$, and gathering like terms, we have

$$\begin{aligned}
ML\phi &= \left\{\frac{1}{2}\left(\sum_{i=1}^{p} e_i \vee e_i - \sum_{j=1}^{q} f_j \vee f_j\right)\rfloor g\right\} \vee \phi \\
&+ g \vee \left\{\frac{1}{2}\left(\sum_{i=1}^{p} e_i \vee e_i - \sum_{j=1}^{q} f_j \vee f_j\right)\rfloor\phi\right\} \\
&+ \sum_{i=1}^{p} e^i \vee (e_i\rfloor\phi) + \sum_{j=1}^{q} f^j \vee (f_j\rfloor\phi)
\end{aligned}$$

Using (3.5) the last two summations may be replaced by $k\,\phi$ since ϕ is of degree k. Thus

$$\begin{aligned}
ML\,\phi &= (\sharp g\rfloor g) \vee \phi + g \vee (\sharp g\rfloor\phi) + k\,\phi \\
&= n\,\phi + LM\phi + k\,\phi \qquad \text{(using (3.32),)}
\end{aligned}$$

which completes the proof. □

Proposition 3.19

Let $\phi \in \mathcal{M}^k(V^*, g)$. Then for $r \geq 1$,

$$[M^r, L]\,\phi = r(n + k - r + 1)M^{r-1}\phi. \tag{3.35}$$

Proof

The method of proof is identical to that used in the symplectic case: use a standard identity for operators to express $[M^r, L]\,\phi$ as

$$[M^r, L]\,\phi = \sum_{i=1}^{r} M^{r-i}[M, L]\, M^{i-1}\phi.$$

The argument of $[M, L]$ in each term has degree $k - 2i + 2$. Invoking (3.34) we have

$$\begin{aligned}
[M^r, L]\,\phi &= \sum_{i=1}^{r} M^{r-i}\{n + (k - 2i + 2)\}M^{i-1}\phi \\
&= \sum_{i=1}^{r}(n + k + 2 - 2i)M^{r-1}\phi \\
&= r(n + k + 2)M^{r-1}\phi - 2\left(\sum_{i=1}^{r} i\right) M^{r-1}\phi \\
&= r(n + k + 2)M^{r-1}\phi - r(r + 1)M^{r-1}\phi \\
&= r(n + k - r + 1)M^{r-1}\phi\,.
\end{aligned}$$

□

Proposition 3.20

Let $\phi \in \mathcal{M}^k(V^*, g)$. Then for $r \geq 1$,

$$[M, L^r]\,\phi = r(n + k + r - 1)L^{r-1}\phi\,. \tag{3.36}$$

Proof

Let ψ be any symmetric $(k + 2r - 2)$–form.

$$g([M, L^r]\,\phi, \psi) \;=\; g(\phi, [M^r, L]\,\psi) \qquad \text{(by (3.31))}$$

$$= g(\phi, r\{n + (k + 2r - 2) - r + 1\}M^{r-1}\psi) \qquad \text{(by (3.35))}$$
$$= g(\phi, r(n + k + r - 1)M^{r-1}\psi)$$
$$= g(r(n + k + r - 1)L^{r-1}\phi, \psi) \qquad \text{(adjoint property)}$$

The result follows since this holds for all such ψ. □

Proposition 3.21

Let ϕ belong to $\mathcal{M}^k(V^*, g)$. Then for $r \geq 1$ and $a \geq 0$,

$$[M^r, L^{r+a}]\,\phi = \sum_{i=1}^{r} \binom{r}{i} \frac{(r+a)!}{(r+a-i)!} \prod_{j=0}^{i-1} (n + k + a + j) L^{r+a-i} M^{r-i} \phi. \quad (3.37)$$

This result may be proved by induction, and is almost identical to the proof for the symplectic case (which is given in the appendix).

Proposition 3.22

Let ϕ belong to $\mathcal{M}^k(V^*, g)$. Then, for $r \geq 1$ and $a \geq 0$,

$$[M^{r+a}, L^r]\,\phi = \sum_{i=1}^{r} \binom{r}{i} \frac{(r+a)!}{(r+a-i)!} \prod_{j=0}^{i-1} (n + k - a + j) L^{r-i} M^{r+a-i} \phi. \quad (3.38)$$

Proof

Let ψ be of degree $k - 2a$. Then

$$g([M^{r+a}, L^r]\,\phi, \psi) = g(\phi, [M^r, L^{r+a}]\,\psi) \qquad \text{(using (3.31))}$$
$$= \sum_{i=1}^{r} \binom{r}{i} \frac{(r+a)!}{(r+a-i)!} \prod_{j=0}^{i-1} (n + (k - 2a) + a + j) g(\phi, L^{r+a-i} M^{r-i} \psi) \qquad \text{(from (3.37))}$$
$$= \sum_{i=1}^{r} \binom{r}{i} \frac{(r+a)!}{(r+a-i)!} \prod_{k=0}^{i-1} (n + k - a + j) g(L^{r-i} M^{r+a-i} \phi, \psi)$$

using the adjoint property of M and L. This holds for all ψ ; hence the result.

□

Recall that two other expressions were found for these commutators in the symplectic case. Their derivations made use of the Hodge star and since there is no such operation in the symmetric algebra, there are no further commutator expressions analogous to the ones for the symplectic case.

3.3.3 The Lepage decomposition

A decomposition very similar to the Lepage decomposition for the exterior algebra is possible for the symmetric algebra.

Definition 3.12

Any symmetric form ϕ such that $M\,\phi = 0$ will be called *effective.* □

Proposition 3.23

Let ψ be any effective symmetric form of degree k. Then

$$M^{r+a}L^r\psi = 0 \quad (\text{for } a \geq 1) \tag{3.39}$$

$$M^r L^r\psi = r!\frac{(n+k+r-1)!}{(n+k-1)!}\psi \tag{3.40}$$

$$M^r L^r\psi = 0 \implies \psi = 0\,. \tag{3.41}$$

Proof

To prove the first result we invoke (3.38).

$$\begin{aligned} M^{r+a}L^r\psi &= [M^{r+a}, L^r]\,\psi \qquad (\text{since } \psi \text{ effective}) \\ &= \sum_{i=1}^{r}\binom{r}{i}\frac{(r+a)!}{(r+a-i)!}\prod_{j=0}^{i-1}(n+k-a+j)L^{r-i}M^{r+a-i}\psi \end{aligned} \tag{3.42}$$

Since every term contains $M\psi$ for $a > 0$, the whole expression is zero.

For the second result (3.40), put $a = 0$ into (3.42) to obtain

$$[M^r, L^r]\psi = \sum_{i=1}^{r} \binom{r}{i} \frac{r!}{(r-i)!} \prod_{j=0}^{i-1} (n+k+j) L^{r-i} M^{r-i} \psi ;$$

since ψ is effective, all the terms except the $i = r$ term are zero. Thus

$$[M^r, L^r]\psi = r! \prod_{j=0}^{r-1} (n+k+j)\psi ,$$

from which follows

$$M^r L^r \psi = r! \frac{(n+k+r-1)!}{(n+k-1)!} \psi .$$

To prove (3.41) we need only note that both r and k are non-negative. Because of this, the coefficient of (3.40) cannot vanish. Therefore, the requirement that $M^r L^r \psi = 0$ implies that ψ must be zero. □

Theorem 3.1 (The Lepage decomposition)

Every form ϕ of degree k may be decomposed as

$$\phi = \sum_{i=0}^{q} L^i \, \psi_{k-2i} \tag{3.43}$$

where each ψ_{k-2i} is an effective form of degree $(k - 2i)$ and $q \leq \left[\frac{k}{2}\right]$.

Proof

If $\phi = 0$, then it is already effective and the result holds trivially.

We define q by the requirements

$$M^q \phi \neq 0 \tag{3.44}$$

$$M^{q+1} \phi = 0 . \tag{3.45}$$

Due to (3.44), it follows that $k - 2q \geq 0$ and hence that $q \leq \left[\frac{k}{2}\right]$.

We next induce a series of k-forms from ϕ. They are defined in the following iterative way, where $a_0, \ldots, a_q$ are constants to be selected later.

$$\begin{aligned}
\phi_{(0)} &:= \phi \\
\phi_{(1)} &:= \phi_{(0)} - a_0 L^q M^q \phi_{(0)} \\
&\vdots \\
\phi_{(q-i+1)} &:= \phi_{(q-i)} - a_{q-i} L^i M^i \phi_{(q-i)} \qquad (3.46) \\
&\vdots \\
\phi_{(q)} &:= \phi_{(q-1)} - a_{q-1} L M \phi_{(q-1)} \\
\phi_{(q+1)} &:= \phi_{(q)} - a_q \phi_{(q)}.
\end{aligned}$$

If we can select $a_0, \ldots, a_q$ so that

$$\begin{aligned}
M^{q+1}\phi_{(0)} &= M^q \phi_{(1)} = \ldots = M^i \phi_{(q-i+1)} \\
\ldots &= M^2 \phi_{(q-1)} = M\phi_{(q)} = 0 \qquad (3.47)
\end{aligned}$$

then we may define a series of effective forms as follows:

$$\begin{aligned}
\psi_{k-2q} &:= a_0 M^q \phi_{(0)} \\
\psi_{k-2(q-1)} &:= a_1 M^{q-1} \phi_{(1)} \\
&\vdots \\
\psi_{k-2i} &:= a_{q-i} M^i \phi_{(q-i)} \qquad (3.48) \\
&\vdots \\
\psi_{k-2} &:= a_{q-1} M \phi_{(q-1)} \\
\psi_k &:= a_q \phi_{(q)}
\end{aligned}$$

Using (3.48) we may write (3.46) as

$$\phi_{(0)} = \phi$$

$$\begin{aligned}
\phi_{(1)} &= \phi_{(0)} - L^q \psi_{k-2q} \\
\phi_{(2)} &= \phi_{(1)} - L^{q-1} \psi_{k-2(q-1)} \\
&\vdots \\
\phi_{(q-i+1)} &= \phi_{(q-i)} - L^i \psi_{k-2i} \\
&\vdots \\
\phi_{(q)} &= \phi_{(q-1)} - L \psi_{k-2} \\
\phi_{(q+1)} &= \phi_{(q)} - \psi_k \,.
\end{aligned} \tag{3.49}$$

By substituting each expression of (3.49) into the one below it, we obtain

$$\phi_{(q+1)} = \phi - \sum_{i=0}^{q} L^i \psi_{k-2i}. \tag{3.50}$$

This relies on the assumption that (3.47) holds true. I prove this to be the case in the following lemma.

Lemma

If $\phi_{(0)}, \ldots, \phi_{(q+1)}$ are defined as in (3.46), then there exist numbers $a_0, \ldots, a_q$ such that

$$M^{q-i} \phi_{(i)} \qquad \text{is effective} \tag{3.51}$$

for $0 \leq i \leq q$, and they are given by

$$a_i = \frac{(n+k-2q+2i-1)!}{(q-i)!(n+k-q+i-1)!}. \tag{3.52}$$

Proof

By induction on i.

For $i = 0$: $M^q \phi_{(0)}$ is already effective by (3.45).

We assume that for some i satisfying $0 \leq i < q$,

$$M^{q-i} \phi_{(i)} \qquad \text{is effective.} \tag{3.53}$$

Now we show that the assertion holds also if we increment i by one unit. For $M^{q-(i+1)}\phi_{(i+1)}$ to be effective, we require

$$\begin{aligned} 0 &= M^{q-i}\phi_{(i+1)} \\ &= M^{q-i}\left\{\phi_{(i)} - a_i L^{q-i} M^{q-i}\phi_{(i)}\right\} \qquad \text{(by (3.46))} \\ &= M^{q-i}\phi_{(i)} - a_i M^{q-i} L^{q-i}\left(M^{q-i}\phi_{(i)}\right). \end{aligned}$$

By the inductive hypothesis (3.53), $M^{q-i}\phi_{(i)}$ is effective of degree $k-2q+2i$ and we may make use of the result (3.40), to yield:

$$\begin{aligned} 0 &= M^{q-i}\phi_{(i)} - a_i(q-i)!\frac{(n+(k-2q+2i)+(q-i)-1)!}{(n+(k-2q+2i)-1)!}M^{q-i}\phi_{(i)} \\ &= \left\{1 - a_i(q-i)!\frac{(n+k-q+i-1)!}{(n+k-2q+2i-1)!}\right\}M^{q-i}\phi_{(i)}. \end{aligned} \tag{3.54}$$

Noting that the degree of $M^{q-i}\phi_{(i)}$ is greater than zero, it follows

that $(k-2q+2i) \geq 0$. Thus both the numerator and the denominator of the fraction in (3.54) are positive. It is therefore possible to solve for a_i to yield

$$a_i = \frac{(n+k-2q+2i-1)!}{(q-i)!(n+k-q+i-1)!}.$$

We have shown by induction that this holds for $0 \leq i \leq q$, which completes the lemma. □

Examining (3.50) it is clear that the result will be proven if we can show that $\phi_{(q+1)} = 0$. From (3.46) we have

$$\begin{aligned} \phi_{(q+1)} &= \phi_{(q)} - a_q\phi_{(q)} \\ &= \phi_{(q)} - \frac{(n+k-1)!}{0!(n+k-1)!}\phi(q) \qquad \text{(using (3.52).)} \\ &= 0. \end{aligned}$$

Thus (3.50) becomes

$$\phi = \sum_{i=0}^{q} L^i \psi_{k-2i},$$

and the proof is complete. □

Because the above theorem constructs the forms $\psi_k - 2i$ explicitly, it also proves the uniqueness of the decomposition. However I choose to demonstrate the uniqueness by a second method also.

Proposition 3.24

The Lepage decomposition stated above is unique. That is, q is unique and so too are the ψ_{k-2i}'s.

Proof

q is unique by the way it is constructed. To show the uniqueness of the ψ's, it suffices to prove that if

$$0 = \sum_{i=0}^{q} L^i \psi_{k-2i} \tag{3.55}$$

then each ψ_{k-2i} is zero (for $0 \leq i \leq q$). Applying M^q to (3.55), we obtain

$$\begin{aligned} 0 &= \sum_{i=0}^{q} M^q L^i \psi_{k-2i} \\ &= M^q L^q \psi_{k-2q} \qquad \text{(by (3.39).)} \end{aligned}$$

From (3.41), it follows that $\psi_{k-2q} = 0$.

So (3.55) becomes

$$0 = \sum_{i=0}^{q-1} L^i \psi_{k-2i} \,. \tag{3.56}$$

By applying M^{q-1} to this expression, the same reasoning allows us to deduce that $\psi_{k-2q+2} = 0$. Following this procedure iteratively, we can show that all the ψ_{k-2i}'s are zero. □

3.3.4 The simple subspaces

Following the same development as for the symplectic case, we note that the Lepage decomposition uniquely expresses any k-form ϕ as

$$\phi = \psi_k + L^1\psi_{k-2} + \cdots + L^q\psi_{k-2q}$$

where the ψ's are effective. It is natural then to consider in more detail forms that are expressible as $\phi = L^r\psi_{k-2r}$ where ψ_{k-2r} is effective. Define:

Definition 3.13

Let $\phi \in \mathcal{M}(V^*, g)$. ϕ is called *simple* if there exists an effective form ψ and a non-negative integer r such that

$$\phi = L^r\,\psi\,.$$

I will refer to r as the *height* of ϕ, and to ψ as the *effective part* of ϕ. □

It follows as an immediate consequence of the Lepage decomposition that the height of any simple form is unique, as is its effective part. Also note that a non-zero simple form is effective if and only if its height is zero.

Proposition 3.25

The effective forms of fixed degree constitute a vector subspace of $\mathcal{M}(V^*, g)$.

The proof is no different from the analogous proof in the symplectic case.

Proposition 3.26

The simple forms of fixed height and degree constitute a vector subspace of $\mathcal{M}(V^*, g)$.

Again I omit the proof as it is no different from the symplectic case.

Definition 3.14

The subspace of simple symmetric forms of degree k and height h will be denoted by S_k^h. I will refer to these subspaces as *simple* subspaces. □

The Lepage decomposition may now be interpreted as giving a direct sum decomposition of the homogeneous subspace $\mathcal{M}^k(V^*, g)$ of the symmetric algebra into simple subspaces of various heights.

$$\mathcal{M}^k(V^*, g) = S_k^0 \oplus S_k^1 \oplus \cdots \oplus S_k^q$$

where $q = \left[\frac{k}{2}\right]$.

Writing out a few cases more explicitly gives the following.

$$\begin{array}{rcl} \mathcal{M}^0(V^*,g) & = & S_0^0 \\ \mathcal{M}^1(V^*,g) & = & S_1^0 \\ \mathcal{M}^2(V^*,g) & = & S_2^0 \oplus S_2^1 \\ \mathcal{M}^3(V^*,g) & = & S_3^0 \oplus S_3^1 \\ \mathcal{M}^4(V^*,g) & = & S_4^0 \oplus S_4^1 \oplus S_4^2 \\ \mathcal{M}^5(V^*,g) & = & S_5^0 \oplus S_5^1 \oplus S_5^2 \\ \mathcal{M}^6(V^*,g) & = & S_6^0 \oplus S_6^1 \oplus S_6^2 \oplus S_6^3 \\ \vdots & & \vdots \quad\; \vdots \quad\; \vdots \quad\; \vdots \end{array}$$

3.3.5 Dimensions of the simple subspaces

Definition 3.15

For $r\,, k \geq 0$,

$$\begin{aligned} \Upsilon_k^r \;&:\; S_k^0 \;\rightarrow\; S_{k+2r}^r \\ &:\; \phi_k \;\mapsto\; \Upsilon_k^r(\phi_k) \;:=\; \sqrt{\tfrac{(n+k-1)!}{r!(n+k+r-1)!}}\, L^r \phi_k\,. \end{aligned} \tag{3.57}$$

□

This map, which is linear, has inverse given by

$$\begin{aligned} (\Upsilon_k^r)^{-1} \;&:\; S_{k+2r}^r \;\rightarrow\; S_k^0 \\ &:\; L^r\phi_k \;\mapsto\; (\Upsilon_k^r)^{-1}(L^r\phi_k) \;:=\; \sqrt{\tfrac{(n+k-1)!}{r!(n+k+r-1)!}}\, M^r\,(L^r\phi_k)\;, \end{aligned} \tag{3.58}$$

where ϕ_k is effective, as can be verified from (3.40).

Because this map is invertible, it follows immediately that

Proposition 3.27

$$S_k^0 \;\cong\; S_{k+2r}^r \tag{3.59}$$

where $\cong$ denotes a vector space isomorphism.

Corollary

For $k \geq 2r$,

$$S_k^r \;\cong\; S_{k-2r}^0\,. \tag{3.60}$$

Proof

Put $t = k - 2r$ and note that this implies $t \geq 0$. Invoking (3.59), we have

$$S_t^0 \;\cong\; S_{t+2r}^r\,,$$

and so we have $S_{k-2r}^0 \;\cong\; S_k^r$. □

Proposition 3.28

Let S_k^r be a simple subspace with $r \geq 1$ and $k \geq 2r$. Then

$$S_k^r \;\cong\; S_{k-2}^{r-1}\,. \tag{3.61}$$

Proof

From (3.59) we have $S_t^0 \;\cong\; S_{t+2r}^r$ for $t \geq 0$. Similarly, for $r \geq 1$, we have $S_t^0 \;\cong\; S_{t+2(r-1)}^{r-1}$, so that for $r \geq 1$,

$$S_{t+2r}^r \;\cong\; S_{t+2(r-1)}^{r-1}\,.$$

Putting $k = t + 2r$, this becomes

$$S_k^r \;\cong\; S_{k-2}^{r-1}\,,$$

and $k - 2r \geq 0$ since $t \geq 0$; hence $k \geq 2r$. □

Theorem 3.2 (Dimensions of the effective subspaces)

$$\dim(S_k^0) = \binom{n+k-1}{n-1} - \binom{n+k-3}{n-1}\,. \tag{3.62}$$

Proof

The Lepage decomposition states

$$\mathcal{M}^k(V^*, g) \;=\; S_k^0 \oplus S_k^1 \oplus \cdots \oplus S_k^q$$

where $q = \left[\frac{k}{2}\right]$. Therefore

$$\dim(S_k^0) = \dim(\mathcal{M}^k(V^*, g)) - \dim\left\{S_k^1 \oplus \cdots \oplus S_k^q\right\}.$$

In each of the subspaces S_k^i for $1 \leq i \leq q$, we have that $k \geq 2i$ since $q = \left[\frac{k}{2}\right]$. Therefore, (3.61) may be invoked to yield

$$\dim(S_k^0) = \dim(\mathcal{M}^k(V^*, g)) - \dim\left\{S_{k-2}^0 \oplus \cdots \oplus S_{k-2}^{q-1}\right\}.$$

Since $\left[\frac{k-2}{2}\right] = q - 1$, it follows that the direct sum in the second term is the Lepage decomposition of $\mathcal{M}^{k-2}(V^*, g)$; thus

$$\begin{aligned} \dim(S_k^0) &= \dim(\mathcal{M}^k(V^*, g)) - \dim(\mathcal{M}^{k-2}(V^*, g)) \\ &= \binom{n+k-1}{n-1} - \binom{n+k-3}{n-1} \qquad \text{(using (3.1).)} \end{aligned}$$

□

Corollary

For $r \geq 1$ and $k \geq 2r$,

$$\dim(S_k^r) = \binom{n+k-2r-1}{n-1} - \binom{n+k-2r-3}{n-1}. \qquad (3.63)$$

Proof

Because $k \geq 2r$, (3.60) may be invoked to yield

$$\begin{aligned} \dim\,(S_k^r) &= \dim\,(S_{k-2r}^0) \\ &= \binom{n+k-2r-1}{n-1} - \binom{n+k-2r-3}{n-1}. \end{aligned}$$

□

3.3.6 Signatures of the simple subspaces

I will show that the simple subspaces are mutually orthogonal and degenerate. They are all metric spaces and we will calculate their signatures.

The following proposition demonstrates the orthogonality.

Proposition 3.29
Let $r \geq 0$, and let $0 \leq a \leq \left[\frac{k}{2}\right]$. Then for ϕ_{k-2a} and ψ_k effective of degree $k-2a$ and k respectively,

$$g(L^{r+a}\phi_{k-2a}, L^r\psi_k) = \begin{cases} 0, & \text{for } a > 0 \\ \frac{r!(n+k+r-1)!}{(n+k-1)!} g(\phi_k, \psi_k), & \text{for } a = 0. \end{cases} \tag{3.64}$$

Proof

$$\begin{aligned} g(L^{r+a}\phi_{k-2a}, L^r\psi_k) &= \sharp(g^{r+a} \vee \phi_{k-2a}) \rfloor L^r\psi_k \qquad \text{(by definition)} \\ &= \sharp\phi_{k-2a} \rfloor \sharp g^{r+a} \rfloor L^r\psi_k \\ &= \sharp\phi_{k-2a} \rfloor M^{r+a} L^r\psi_k \end{aligned}$$

If $a > 0$ then from (3.39) the expression is zero. For $a = 0$,

$$\begin{aligned} g(L^r\phi_k, L^r\psi_k) &= \sharp\phi_k \rfloor M^r L^r\psi_k \\ &= \frac{r!(n+k+r-1)!}{(n+k-1)!} g(\phi_k, \psi_k). \qquad \text{(from (3.40))} \end{aligned}$$

□

Proposition 3.30
The simple subspaces are nondegenerate.

I omit the proof, which follows by the same methods as for the analogous theorem in the symplectic case.

We saw that all the homogeneous subspaces in $\mathcal{M}(V^*, g)$ are symmetric, and so it follows that the simple subspaces are all symmetric too. It is natural then to look for their signatures.

Proposition 3.31

$$\text{signature}(S_k^0) = \text{signature}(S_{k+2r}^r). \tag{3.65}$$

Proof

It suffices to show that the extended metric is preserved under our isomorphism

$$\Upsilon_k^r : S_k^0 \longrightarrow S_{k+2r}^r.$$

Let ϕ_k, ψ_k be effective k-forms. Then

$$\begin{aligned} g(\Upsilon_k^r \phi_k, \Upsilon_k^r \psi_k) &= \frac{(n+k-1)!}{r!(n+k+r-1)!} g(L^r \phi_k, L^r \psi_k) \qquad \text{(by definition of } \Upsilon_k^r) \\ &= \frac{(n+k-1)!}{r!(n+k+r-1)!} \frac{r!(n+k+r-1)!}{(n+k-1)!} g(\phi_k, \psi_k) \qquad \text{(by (3.64),)} \\ &= g(\phi_k, \psi_k). \end{aligned}$$

□

Corollary

For $k \geq 2r$,

$$\text{signature}(S_k^r) = \text{signature}(S_{k-2r}^0). \tag{3.66}$$

Proof

Put $t = k - 2r$ so that $t \geq 0$. Invoking (3.65), we have

$$\text{signature}(S_t^0) \;=\; \text{signature}(S_{t+2r}^r)\,,$$

which is

$$\text{signature}(S_{k-2r}^0) \;=\; \text{signature}(S_k^r)\,.$$

□

Corollary

Let $k \geq 2r$. Then

$$\text{signature}(S_k^r) \;=\; \text{signature}(S_{k-2}^{r-1}) \tag{3.67}$$

Proof

For $t \geq 0$ we had in (3.65) that

$$\text{signature}(S_t^0) \;=\; \text{signature}(S_{t+2r}^r).$$

So too, for $r \geq 1$, we have that

$$\text{signature}(S_t^0) \;=\; \text{signature}(S_{t+2(r-1)}^{r-1}).$$

By comparison,

$$\text{signature}(S_{t+2r}^r) \;=\; \text{signature}(S_{t+2(r-1)}^{r-1}),$$

and putting $k = t + 2r$, this becomes

$$\text{signature}(S_k^r) \;=\; \text{signature}(S_{k-2}^{r-1})\,,$$

and $k - 2r \geq 0$ since $t \geq 0$. Thus $k \geq 2r$. □

Theorem 3.3 (Signature of an effective subspace)

$$\text{signature}(S_k^0) \;=\; \text{signature}(\mathcal{M}^k(V^*,g)) - \text{signature}(\mathcal{M}^{k-2}(V^*,g)) . \quad (3.68)$$

Proof

The Lepage decomposition states that

$$\mathcal{M}^k(V^*,g) \;=\; S_k^0 \oplus S_k^1 \oplus \cdots \oplus S_k^q$$

where $q = \left[\frac{k}{2}\right]$. Because the components of the sum are orthogonal,

$$\text{signature}(S_k^0) = \text{signature}(\mathcal{M}^k(V^*,g)) - \text{signature}\left\{S_k^1 \oplus \cdots \oplus S_k^q\right\} .$$

In each of the subspaces S_k^i for $1 \le i \le q$, we have that $k \ge 2i$ since $q = \left[\frac{k}{2}\right]$. Therefore, (3.67) may be invoked to yield

$$\text{signature}(S_k^0) = \text{signature}(\mathcal{M}^k(V^*,g)) - \text{signature}\left\{S_{k-2}^0 \oplus \cdots \oplus S_{k-2}^{q-1}\right\} .$$

Since $\left[\frac{k-2}{2}\right] = q - 1$, it follows that the direct sum in the second term is the Lepage decomposition of $\mathcal{M}^{k-2}(V^*,g)$; thus

$$\text{signature}(S_k^0) = \text{signature}(\mathcal{M}^k(V^*,g)) - \text{signature}(\mathcal{M}^{k-2}(V^*,g)) .$$

□

To evaluate the right hand side of this expression, we substitute the earlier result (3.23).

Corollary

For $k \ge 2r$,

$$\text{signature}(S_k^r) = \text{signature}(\mathcal{M}^{k-2r}(V^*,g)) - \text{signature}(\mathcal{M}^{k-2r-2}(V^*,g)) . \quad (3.69)$$

Proof

Because $k \geq 2r$, (3.66) may be invoked to yield

$$\begin{aligned} \text{signature}(S_k^r) &= \text{signature}\ (S_{k-2r}^0) \\ &= \text{signature}(\mathcal{M}^{k-2r}(V^*, g)) - \text{signature}(\mathcal{M}^{k-2r-2}(V^*, g)), \end{aligned}$$

using (3.68). □

The Lepage decomposition, found in this chapter for the symmetric exterior algebra endowed with a metric, exists also for the symplectic exterior algebra and Kaehler spaces. This suggests that it might be useful to find a general Lepage decomposition theorem on an abstract algebra. The conditions sufficient to set up such a decomposition and the ways in which they may be relaxed are possible areas for further research.

Note also that the geometrical interpretation of the decomposition theorem found for the symplectic and metric spaces must presumably also apply to the Kaehler case and any other spaces in which the theorem has been found.

Chapter 4

A calculus for symmetric tensors

4.1 Introduction

In this chapter I will define a calculus for the symmetric differential forms on a manifold. As outlined in the introduction, such a calculus is of interest in the study of bosonic systems in Quantum Mechanics which, unlike fermionic systems, do not seem to have a natural mathematical environment for their description. My object is to find a structure that parallels the calculus associated with the Dirac equation. As it turns out, this is possible if the manifold is equipped with a pseudo-Riemannian metric. As far as I know, this calculus is new.

In the case of the Dirac equation, the Dirac matrices γ^μ form a metric Clifford algebra. This algebra extends in a completely natural way onto a manifold. The way this works is as follows. The extension involves setting up a fibre bundle on the manifold where each fibre is a copy of the Clifford algebra. But all differentiable manifolds admit another, naturally defined fibre bundle. In this bundle, each fibre is the exterior algebra constructed from the cotangent space at the base point. This fibre bundle over a manifold P, usually denoted by $\bigwedge(T^*P)$, has a natural calculus associated with it, namely the exterior differential calculus. The natural extension of the Dirac equation to a manifold can now be seen to arise from the fact that the fibres in the above two bundles are isomorphic and can therefore be identified with each other. The exterior differential calculus is thus immediately available in the context of the Dirac equation on a manifold.

My strategy is to imitate these fermionic structures as far as possible to define a mathematical context appropriate for bosonic systems. Bosonic systems in Quantum Mechanics are described by the symplectic Clifford algebra, as discussed in the introduction. This algebra is infinite dimensional and, considered as a vector space, is isomorphic to the *symmetric* exterior algebra. To follow the analogy of the Dirac equation, we must consider two fibre bundles. In the first, each fibre is a copy of the symplectic Clifford algebra above the base point. In the second, each fibre is the symmetric exterior algebra constructed from the cotangent space above the base point. The latter fibre bundle on the manifold P is denoted by $\bigvee(T^*P)$. At this point the analogy with the Dirac case breaks down, because the symmetric exterior forms have no naturally associated calculus. Continuation of the analogy using only the structures defined so far, seems impossible. However, further progress can be made by introducing a metric g into the manifold. We will define a calculus for the symmetric forms using a metric, and in particular the Levì Cività connection derived from it.

Clearly, the manifold of interest will have two coexisting structures. It will have a symplectic structure, because each fibre is a symplectic Clifford algebra, and it will need a metric structure by the above arguments. The way the two structures relate to each other will not be investigated here. A preliminary discussion of this problem may be found in Frescura and Lubczonok (1991).

Because the calculus we will consider uses only the metric structure for its definition, it will be convenient to overlook the symplectic structure and to consider a manifold with metric structure only. The symplectic structure can be re-introduced at a later stage.

Consider then a differentiable manifold P with metric g. The tangent and cotangent spaces of P generate symmetric tensor algebras at each point of P. The algebraic structures defined in chapter 3 are all carried over into these fibres and their extensions to the manifold are defined by the usual point by point method. These structures include the algebra of symmetric tensors (addition, scalar multiplication, symmetric exterior products, and generalised interior products), lowering and raising operations $\flat$ and $\sharp$, the extended metric, the operators L_g, M_g and the Lepage decomposition. For some of our considerations, we will assume P has an invariant volume form Ω on it, and that it has no boundary.

The material of this chapter is distributed as follows.

I first review some properties of the covariant derivative of symmetric tensors. I then define a symmetric exterior derivative D, analogous to the standard exterior derivative d. A symmetric coderivative ∂, analogous to the exterior coderivative δ, is also defined. The properties of both these operators are discussed. I obtain some expressions involving these derivatives and the algebraic operations M_g and L_g from chapter 3. I then relate this material to a Lie algebra of symmetric forms defined by N.M.J. Woodhouse, and also to symmetric Killing forms. I also investigate a possible Harmonic theory of symmetric forms. A global inner product is defined and it is demonstrated that D and ∂ are adjoints with respect to it. Self adjoint 'Laplacian' and a 'Dirac' operators are defined by analogy with their standard definitions.

I will use the following notation. Denote the covariant symmetric tensors of degree r by $\mathcal{M}^r(T^*P, g)$, and refer to them as *symmetric differential forms* or just *symmetric forms.* The direct sum of all the homogeneous subspaces of symmetric forms is denoted by $\mathcal{M}(T^*P, g)$. The contravariant symmetric tensors of degree k are denoted by $\mathcal{M}^k(TP, g)$ and are called *symmetric multivectors*; the direct sum of the homogeneous subspaces of symmetric multivectors is denoted by $\mathcal{M}(TP, g)$. I use the symbol ∇ to denote the Levì Cività connection defined by g.

4.2 The covariant derivative of symmetric forms

Although the covariant derivative of tensors is a well known operation, the expressions obtained for the covariant derivatives of symmetric tensors can sometimes be simplified by exploiting their symmetry. For this reason and also because the properties of the covariant derivative with respect to the non-standard operations like the generalised interior products are not generally familiar, I will derive a few expressions needed later.

Proposition 4.1
For a vector field X and a symmetric form ϕ,

$$\nabla_Y(X\rfloor\phi) = (\nabla_Y X)\rfloor\phi + X\rfloor(\nabla_Y\phi). \tag{4.1}$$

Proof

$X \rfloor \phi$ is an interior product of tensors, and the result follows immediately from a result quoted in most books on tensor analysis (for example see Spain p. 35). However we give a coordinate independent proof. Assuming ϕ to be of degree r and letting $U_1, \ldots, U_{r-1}$ be arbitrary vector fields,

$$\begin{aligned}
\{\nabla_Y(X\rfloor\phi)\}(U_1,\ldots,U_{r-1}) &= \nabla_Y\{(X\rfloor\phi)(U_1,\ldots,U_{r-1})\} - (X\rfloor\phi)(\nabla_Y U_1,\ldots,U_{r-1}) \\
&\quad - \cdots - (X\rfloor\phi)(U_1,\ldots,\nabla_Y U_{r-1}) \qquad \text{(definition of } \nabla \text{ on any tensor)} \\
&= \nabla_Y\{\phi(X,U_1,\ldots,U_{r-1})\} - \phi(X,\nabla_Y U_1,\ldots,U_{r-1}) - \cdots \\
&\qquad -\phi(X,U_1,\ldots,\nabla_Y U_{r-1}) \qquad \text{(definition of } \rfloor\text{)} \\
&= (\nabla_Y\phi)(X,U_1,\ldots,U_{r-1}) + \phi(\nabla_Y X,U_1,\ldots,U_{r-1}) \\
&\quad +\phi(X,\nabla_Y U_1,\ldots,U_{r-1}) + \cdots + \phi(X,U_1,\ldots,\nabla_Y U_{r-1}) \\
&\quad -\phi(X,\nabla_Y U_1,\ldots,U_{r-1}) - \cdots - \phi(X,U_1,\ldots,\nabla_Y U_{r-1}) \\
&\qquad \text{(properties of } \nabla \text{ on tensors)} \\
&= (X\rfloor\nabla_Y\phi)(U_1,\ldots,U_{r-1}) + \{(\nabla_Y X)\rfloor\phi\}(U_1,\ldots,U_{r-1}).
\end{aligned}$$

□

Proposition 4.2

For a symmetric multivector X and a symmetric form ϕ,

$$\nabla_Y(X\rfloor\phi) = (\nabla_Y X)\rfloor\phi + X\rfloor(\nabla_Y\phi). \tag{4.2}$$

Proof

Again, we expect the result to hold, because $X\rfloor\phi$ is an interior product. First consider $X = X_1 \vee \ldots \vee X_r$:

$$\begin{aligned}
\nabla_Y(X\rfloor\phi) &= \nabla_Y(X_1\rfloor(\ldots\rfloor(X_r\rfloor\phi))) \\
&= (\nabla_Y X_1)\rfloor(X_2\rfloor\ldots\rfloor X_r\rfloor\phi) + X_1\rfloor\nabla_Y(X_2\rfloor\ldots\rfloor X_r\rfloor\phi) \qquad \text{(from (4.1))}
\end{aligned}$$

$$
\begin{aligned}
&= (\nabla_Y X_1)\rfloor(X_2\rfloor\ldots\rfloor X_r\rfloor\phi) + X_1\rfloor(\nabla_Y X_2)\rfloor(X_3\rfloor\ldots\rfloor X_r\rfloor\phi) \\
&\qquad\qquad\qquad + X_1\rfloor X_2\rfloor\nabla_Y(X_3\rfloor\ldots\rfloor X_r\rfloor\phi) \\
&\qquad\qquad\qquad\vdots \\
&= (\nabla_Y X_1)\rfloor(X_2\rfloor\ldots\rfloor X_r\rfloor\phi) + X_1\rfloor(\nabla_Y X_2)\rfloor(X_3\rfloor\ldots\rfloor X_r\rfloor\phi) + \cdots \\
&\qquad\qquad + X_1\rfloor\ldots\rfloor X_{r-1}(\nabla_Y X_r)\rfloor\phi + X_1\rfloor\ldots\rfloor X_r\rfloor(\nabla_Y\phi) \\
&= \{(\nabla_Y X_1)\vee\ldots\vee X_r + \cdots + X_1\vee\ldots\vee(\nabla_Y X_r)\}\rfloor\phi + X\rfloor\nabla_Y\phi \\
&= (\nabla_Y X)\rfloor\phi + X\rfloor(\nabla_Y\phi)\,.
\end{aligned}
$$

The result extends to arbitrary symmetric r-vectors X by linearity. □

Proposition 4.3

For any two symmetric forms ϕ and ψ,

$$\nabla_X(\phi\vee\psi) \;=\; (\nabla_X\phi)\vee\psi + \phi\vee(\nabla_X\psi)\,. \tag{4.3}$$

The proof follows without difficulty from the definition of the symmetric exterior product and the properties of the covariant derivative.

Proposition 4.4

For vectors X, Y

$$\flat(\nabla_X Y) \;=\; \nabla_X(\flat Y)\,. \tag{4.4}$$

Proof

$$
\begin{aligned}
\nabla_X(\flat Y) &= \nabla_X(Y\rfloor g) \\
&= (\nabla_X Y)\rfloor g + Y\rfloor\nabla_X g \qquad \text{(by (4.1))} \\
&= (\nabla_X Y)\rfloor g \qquad (\text{since } \nabla_X g = 0) \\
&= \flat(\nabla_X Y)\,.
\end{aligned}
$$

□

Corollary

For a vector field X and a one-form field θ,

$$\sharp(\nabla_X \theta) = \nabla_X(\sharp\theta). \tag{4.5}$$

Proof

Put $Y = \sharp\theta$ in (4.4):

$$\flat\nabla_X\sharp\theta = \nabla_X(\flat\sharp\theta)$$

and sharpening both sides yields the result. □

Proposition 4.5

For any vector field X, any symmetric r-form ϕ, and any symmetric multi-vector Y,

$$\sharp(\nabla_X \phi) = \nabla_X(\sharp\phi) \tag{4.6}$$

$$\flat(\nabla_X Y) = \nabla_X(\flat Y). \tag{4.7}$$

Proof

Consider decomposable $\phi = \phi_1 \vee \ldots \vee \phi_r$.

$$\begin{aligned}
\sharp\nabla_X(\phi_1 \vee \ldots \vee \phi_r) &= \sharp\{(\nabla_X\phi_1) \vee \ldots \vee \phi_r + \phi_1 \vee (\nabla_X\phi_2) \vee \ldots \vee \phi_r + \cdots \\
&\quad + \phi_1 \vee \ldots \vee \phi_{r-1} \vee (\nabla_X\phi_r)\} \qquad \text{(properties of } \nabla) \\
&= \sharp(\nabla_X\phi_1) \vee \ldots \vee \sharp\phi_r + \sharp\phi_1 \vee (\sharp\nabla_X\phi_2) \vee \ldots \vee \sharp\phi_r + \cdots \\
&\quad + \sharp\phi_1 \vee \ldots \vee \sharp\phi_{r-1} \vee \sharp(\nabla_X\phi_r) \qquad \text{(properties of } \sharp)
\end{aligned}$$

$$= \nabla_X \{\sharp\phi_1 \vee \ldots \vee \sharp\phi_r\}$$
$$= \nabla_X(\sharp\phi).$$

The result extends to non-decomposable forms by linearity. The proof of (4.7) follows similar lines. □

Many of these results will perhaps be more familiar in tensor notation. For example, (4.6) expressed this way takes the form:

$$g^{ri}\, g^{si}\, (X^k\, \phi_{ij\,;k}) \;=\; X^k\, (g^{ri}\, g^{si}\, \phi_{ij})_{;\,k}\,.$$

4.3 Symmetric exterior derivative

In this section I will define a derivative analogous to the exterior derivative of the exterior calculus.[1]

To motivate and arrive at our definition, consider the exterior derivative of an antisymmetric r-form ϕ,

$$d\,\phi \;=\; (-1)^r(\nabla_i\phi) \wedge dx^i\,,$$

where ∇ is a connection. This may be proven by evaluating $(\nabla_i\phi) \wedge dx^i$ in a coordinate system and utilising the symmetry of the Christoffel symbols together with the antisymmetry of the exterior product. This suggests the following definition.

Definition 4.1

Given a differentiable manifold P with a metric g and Levì-Cività connection

[1]The definition of this derivative is due to F A M Frescura (private communication).

∇, we define the *symmetric exterior derivative* D by

$$\begin{array}{rcl} D & : & \mathcal{M}^r(T^*P, g) \;\rightarrow\; \mathcal{M}^{r+1}(T^*P, g) \\ & : & \phi \qquad\qquad\quad \mapsto \; D\phi \;:=\; (\nabla_i\phi) \vee dx^i \, . \end{array} \tag{4.8}$$

□

Since the covariant derivative and the symmetric product both preserve the symmetry, $\mathcal{M}(T^*P)$ is closed under the operation D.

The next result guarantees the coordinate independence of D by expressing it in coordinate free terms. It therefore provides an alternative definition of D.

Proposition 4.6

For any symmetric p-form ϕ,

$$(D\phi)(U_1, \ldots, U_{p+1}) \;=\; \sum_{i=1}^{p+1} (\nabla_{U_i}\phi)\,(U_1, \ldots, U_{i-1}, U_{i+1}, \ldots, U_{p+1})\,, \tag{4.9}$$

where $U_1, \ldots, U_{p+1}$ are arbitrary vector fields.

Proof

We will prove the result only for a two-form ϕ; the same method may be adopted to prove the general case. Let U, V, W be arbitrary vector fields. Then

$$\begin{aligned} (D\phi)(U, V, W) &= \left\{(\nabla_i\phi) \vee dx^i\right\}(U, V, W) \qquad \text{(by definition)} \\ &= \frac{1}{2!\,1!}\{(\nabla_i\phi)(U, V)\, dx^i(W) + (\nabla_i\phi)(V, U)\, dx^i(W) \\ &\quad +(\nabla_i\phi)(U, W)\, dx^i(V) + (\nabla_i\phi)(W, U)\, dx^i(V) \\ &\quad +(\nabla_i\phi)(W, V)\, dx^i(U) + (\nabla_i\phi)(V, W)\, dx^i(U)\} \\ &\qquad \text{(by definition of } \vee\text{)} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2!\,1!}2!\{(\nabla_i\phi)(U,V)\,dx^i(W) + (\nabla_i\phi)(U,W)\,dx^i(V) \\
&\quad +(\nabla_i\phi)(V,W)\,dx^i(U)\} \qquad \text{(by symmetry of } \nabla_i\phi) \\
&= W^i(\nabla_i\phi)(U,V) + V^i(\nabla_i\phi)(U,W) + U^i(\nabla_i\phi)(V,W) \\
&= (\nabla_W\phi)(U,V) + (\nabla_V\phi)(U,W) + (\nabla_U\phi)(V,W)\,,
\end{aligned}$$

which is the result asserted. □

An analogous result may be proved for the exterior derivative (de Rham p 128, §26, Crampin and Pirani p 124 Ex 12). For an antisymmetric p-form ϕ,

$$(d\phi)(U_1,\ldots,U_{p+1}) \;=\; \sum_{i=1}^{p+1}(-1)^{i-1}\,(\nabla_{U_i}\phi)\,(U_1,\ldots,U_{i-1},U_{i+1},\ldots,U_{p+1})\,,$$

or, in coordinates

$$(d\phi)_{i_1\ldots i_{p+1}} \;=\; \sum_{a=1}^{p+1}(-1)^{a-1}\nabla_{i_a}\phi_{i_1\ldots\widehat{i_a}\ldots i_{p+1}}\,.$$

Proposition 4.7

Let ϕ and ψ be symmetric forms of equal degree. Then

$$D(\phi+\psi) \;=\; D\phi + D\psi\,. \tag{4.10}$$

For any two symmetric forms ϕ and ψ,

$$D(\phi\vee\psi) \;=\; (D\phi)\vee\psi + \phi\vee(D\psi)\,. \tag{4.11}$$

On the coordinate basis one-forms dx^i,

$$D(dx^i) \;=\; -\Gamma^i{}_{jk}dx^j\vee dx^k\,. \tag{4.12}$$

For a function f,

$$Df \;=\; df\,. \tag{4.13}$$

A special case of (4.11) is

$$D(f\phi) \;=\; (df)\vee\phi + f(D\phi)\,. \tag{4.14}$$

Proof

To prove (4.10):

$$\begin{aligned} D(\phi + \psi) &= \{\nabla_i(\phi + \psi)\} \vee dx^i \qquad \text{(by definition)} \\ &= (\nabla_i\phi + \nabla_i\psi) \vee dx^i \qquad \text{(by linearity of } \nabla) \\ &= (\nabla_i\phi) \vee dx^i + (\nabla_i\psi) \vee dx^i \qquad \text{(property of } \vee) \\ &= D\phi + D\psi \end{aligned}$$

To prove (4.11):

$$\begin{aligned} D(\phi \vee \psi) &= \{\nabla_i(\phi \vee \psi)\} \vee dx^i \qquad \text{(by definition)} \\ &= \{(\nabla_i\phi) \vee \psi + \phi \vee (\nabla_i\psi)\} \vee dx^i \qquad \text{(property of } \nabla) \\ &= \left\{(\nabla_i\phi) \vee dx^i\right\} \vee \psi + \phi \vee \left\{(\nabla_i\psi) \vee dx^i\right\} \qquad \text{(property of } \vee) \\ &= (D\phi) \vee \psi + \phi \vee (D\psi) \end{aligned}$$

To prove (4.12):

$$\begin{aligned} Ddx^i &= (\nabla_j dx^i) \vee dx^j \qquad \text{(by definition)} \\ &= -\Gamma^i{}_{kj} dx^k \vee dx^j \end{aligned}$$

To prove (4.13):

$$\begin{aligned} Df &= (\nabla_i f) \vee dx^i \qquad \text{(by definition)} \\ &= \frac{\partial f}{\partial x^i} dx^i \\ &= df \end{aligned}$$

To prove (4.14):

$$\begin{aligned} D(f\phi) &= D(f \vee \phi) \\ &= (Df) \vee \phi + f \vee (D\phi) \qquad \text{(invoking (4.11))} \\ &= (df) \vee \phi + f(D\phi)\,. \qquad \text{(invoking (4.13))} \end{aligned}$$

□

Another coordinate free definition of D is obtained from (4.13), (4.9) with $p = 1$, (4.10), and (4.11):

Definition 4.2

Let f be a differentiable function on P. Define

$$Df := df\,.$$

Let θ be a differential one-form. Define

$$(D\theta)(X,Y) := \langle X, \nabla_Y\theta\rangle + \langle Y, \nabla_X\theta\rangle\ ,$$

for arbitrary vector fields X and Y.

Let ϕ and ψ be symmetric differential forms. Define

$$\begin{aligned} D(\phi+\psi) &= D\phi + D\psi\,, \\ D(\phi\vee\psi) &= (D\phi)\vee\psi + \phi\vee(D\psi)\,. \end{aligned}$$

□

Proposition 4.8

In a coordinate basis, the general expression for the symmetric derivative of a p-form ϕ is

$$D\phi = \phi_{i_1\ldots i_p;i}\,dx^i\vee dx^{i_1}\vee\ldots\vee dx^{i_p}\,; \tag{4.15}$$

and in terms of the Christoffel symbols,

$$D\phi = \left\{\frac{\partial\phi_{i_1\ldots i_p}}{\partial x^j} - p\,\phi_{i_1\ldots i_{p-1}s}\,\Gamma^s{}_{i_pj}\right\}dx^j\vee dx^{i_1}\vee\ldots\vee dx^{i_p}\,. \tag{4.16}$$

Proof

(4.15) is just a trivial rewriting of the definition. To prove (4.16),

$$\begin{aligned} D\phi &= \phi_{i_1\ldots i_p;j} dx^j \vee dx^{i_1} \vee \ldots \vee dx^{i_p} \\ &= \left\{ \frac{\partial \phi_{i_1\ldots i_p}}{\partial x^j} - \phi_{s i_2 \ldots i_p} \Gamma^s{}_{j i_1} - \cdots - \phi_{i_1 \ldots i_{p-1} s} \Gamma^s{}_{j i_p} \right\} dx^j \vee dx^{i_1} \vee \ldots \vee dx^{i_p} . \end{aligned}$$

Due to the symmetry in the indices $i_1, \ldots, i_p$, the last p terms are equal; hence

$$D\phi = \left\{ \frac{\partial \phi_{i_1\ldots i_p}}{\partial x^j} - p\, \phi_{i_1\ldots i_{p-1} s}\, \Gamma^s{}_{i_p j} \right\} dx^j \vee dx^{i_1} \vee \ldots \vee dx^{i_p} .$$

□

We will need the following result:

Proposition 4.9

$$Dg = 0 . \tag{4.17}$$

The proof is trivial since the Levì-Cività connection ∇ satisfies the compatibility condition $\nabla g = 0$.

4.4 Symmetric coderivative

I now propose a definition for a coderivative for the symmetric differential forms. It is the analogue of the coderivative δ of the exterior algebra. It produces a symmetric differential form of degree one unit less than that of its argument.

To motivate the definition consider, as a first line of attack, the coderivative for the exterior calculus. Ideally, one would like to adopt without alteration the defining expression used there. Except possibly for a sign, it is defined as

$$\delta\phi := (-1)^r * d * \phi\,,$$

where $*$ is the Hodge dual operator (Choquet-Bruhat et al (1982) p 296, Göckeler and Schücker (1987) p 40). Unfortunately, this expression has no direct analogue in the symmetric calculus because this calculus has no Hodge star operator.

The following approach, however, does work. Consider the expression for the coderivative in the exterior calculus, given in Choquet et al on page 317. Up to a coefficient, it is

$$\delta\phi \;=\; \nabla^j\, \phi_{jk_1\ldots k_{p-1}} dx^{k_1} \wedge \ldots \wedge dx^{k_{p-1}}\,.$$

Some authors give this as the definition of the coderivative (Yano (1970) p 64, Guggenheimer (1963) p 329, Ex 36). In older texts, this operation is referred to as the 'divergence' (Gerretsen (1962) p 161; Misner, Thorne and Wheeler (1970), p 261 Ex 10.11). A more modern way of denoting it is

$$\delta\phi \;=\; \sharp dx^j \rfloor \nabla_j \phi\,.$$

This expression *is* suitable for use in the context of the symmetric calculus and motivates our definition of the symmetric coderivative.

Definition 4.3

Let P be a differentiable manifold with a metric g. Define the coderivative of a symmetric form , denoted by the symbol ∂, by

$$\begin{aligned} \partial \;:\; & \mathcal{M}^r(T^*P) \;\to\; \mathcal{M}^{r-1}(T^*P) \\ :\; & \phi \qquad\quad \mapsto\; \partial\phi \;:=\; -\sharp dx^i \rfloor \nabla_i \phi\,. \end{aligned} \tag{4.18}$$

□

The negative sign is introduced for later convenience.

The divergence of a tensor on a manifold with a metric is a well known operation. However ∂ as defined here is specifically a differential operator on the symmetric algebra and so strictly constitutes a new object.

The coordinate invariance of ∂ is inherited from the coordinate invariance of the divergence on the general tensors. The symmetric coderivative preserves the symmetry of its argument, and so $\mathcal{M}(T^*P, g)$ is closed under the operation.

Proposition 4.10

Let ϕ and ψ be symmetric forms of equal degree. Then

$$\partial(\phi + \psi) \;=\; \partial\phi + \partial\psi\,. \tag{4.19}$$

For any two symmetric forms ϕ and ψ,

$$\partial(\phi \vee \psi) \;=\; (\partial\phi)\vee\psi + \phi\vee(\partial\psi) - (\nabla_i\phi)\vee(\sharp dx^i \rfloor \psi) - (\nabla_i\psi)\vee(\sharp dx^i \rfloor \phi)\,. \tag{4.20}$$

On the coordinate basis one-forms dx^i,

$$\partial(dx^i) \;=\; g^{jk}\Gamma^i{}_{kj}\,. \tag{4.21}$$

For a function f,

$$\partial f \;=\; 0\,. \tag{4.22}$$

A special case of (4.20) is

$$\partial(f\phi) \;=\; f(\partial\phi) - (\sharp df)\rfloor\phi\,. \tag{4.23}$$

Proof

To prove (4.19):

$$\partial(\phi + \psi) \;=\; -\sharp dx^i \rfloor \nabla_i(\phi + \psi) \qquad \text{(by definition)}$$

$$
\begin{aligned}
&= -\sharp dx^i \rfloor (\nabla_i \phi + \nabla_i \psi) \qquad \text{(properties of } \nabla) \\
&= -\sharp dx^i \rfloor \nabla_i \phi - \sharp dx^i \rfloor \nabla_i \psi \qquad \text{(properties of } \rfloor) \\
&= \partial \phi + \partial \psi \,. \qquad \text{(by definition)}
\end{aligned}
$$

To prove (4.20):

$$
\begin{aligned}
\partial(\phi \vee \psi) &= -\sharp dx^i \rfloor \nabla_i (\phi \vee \psi) \qquad \text{(by definition)} \\
&= -\sharp dx^i \rfloor \{(\nabla_i \phi) \vee \psi + \phi \vee (\nabla_i \psi)\} \qquad \text{(properties of } \nabla) \\
&= -(\sharp dx^i \rfloor \nabla_i \phi) \vee \psi - (\nabla_i \phi) \vee (\sharp dx^i \rfloor \psi) - (\sharp dx^i \rfloor \phi) \vee (\nabla_i \psi) \\
&\qquad - \phi \vee (\sharp dx^i \rfloor \nabla_i \psi) \qquad \text{(properties of } \rfloor) \\
&= (\partial \phi) \vee \psi + \phi \vee (\partial \psi) - (\nabla_i \phi) \vee (\sharp dx^i \rfloor \psi) - (\nabla_i \psi) \vee (\sharp dx^i \rfloor \phi) \,.
\end{aligned}
$$

To prove (4.21):

$$
\begin{aligned}
\partial(dx^i) &= -\sharp dx^j \rfloor \nabla_j dx^i \qquad \text{(by definition)} \\
&= \sharp dx^j \rfloor \Gamma^i{}_{kj} dx^k \\
&= g^{jk} \Gamma^i{}_{kj} \,. \qquad (\text{since } \sharp dx^j \rfloor dx^k \;=\; g^{jk})
\end{aligned}
$$

To prove (4.22):

$$
\begin{aligned}
\partial f &= -\sharp dx^i \rfloor \nabla_i f \qquad \text{(by definition)} \\
&= -\sharp dx^i \rfloor \frac{\partial f}{\partial x^i} \\
&= 0 \,. \quad (\text{since } \frac{\partial f}{\partial x^i} \text{ is of lower degree than } \sharp dx^i)
\end{aligned}
$$

To prove (4.23):

$$
\begin{aligned}
\partial(f\phi) &= \partial(f \vee \phi) \\
&= (\partial f) \vee \phi + f \vee (\partial \phi) - (\nabla_i f) \vee (\sharp dx^i \rfloor \phi) - (\nabla_i \phi) \vee (\sharp dx^i \rfloor f) \\
&\qquad \text{(from result (4.20))}
\end{aligned}
$$

$$
\begin{aligned}
&= f \vee (\partial\phi) - (\nabla_i f) \vee (\sharp dx^i \rfloor \phi) \qquad \text{(the first and last terms are zero)} \\
&= f(\partial\phi) - \frac{\partial f}{\partial x^i} (\sharp dx^i \rfloor \phi) \\
&= f(\partial\phi) - (\sharp \frac{\partial f}{\partial x^i} dx^i) \rfloor \phi \\
&= f(\partial\phi) - (\sharp df) \rfloor \phi .
\end{aligned}
$$

□

The first four properties of this proposition may be adopted as an alternative definition of ∂.

Proposition 4.11

In a coordinate basis, the general expression for the symmetric coderivative of a p-form ϕ is

$$\partial \phi = p\, g^{ij} \phi_{ji_2 \ldots i_p\, ;i}\, dx^{i_2} \vee \ldots \vee dx^{i_p} , \tag{4.24}$$

and in terms of the Christoffel symbols,

$$\partial\phi = p\, g^{ij} \left\{ \frac{\partial \phi_{ji_2 \ldots i_p}}{\partial x^i} - (p-1)\phi_{ji_2 \ldots i_{p-1}s} \Gamma^s{}_{i_p i} - \phi_{i_2 \ldots i_p s} \Gamma^s{}_{ji} \right\} dx^{i_2} \vee \ldots \vee dx^{i_p} . \tag{4.25}$$

Proof

I will illustrate the proof only in the case of a symmetric three-form. The same method may be employed for the general case.

$$
\begin{aligned}
\partial\phi &= -\sharp dx^i \rfloor \nabla_i \phi \qquad \text{(by definition)} \\
&= -\sharp dx^i \rfloor \left(\phi_{rst\,;i}\, dx^r \vee dx^s \vee dx^t \right) \\
&= -\phi_{rst\,;i} \left\{ (\sharp dx^i \rfloor dx^r) dx^s \vee dx^t + (\sharp dx^i \rfloor dx^s) dx^r \vee dx^t + (\sharp dx^i \rfloor dx^t) dx^r \vee dx^s \right\} \\
&\qquad\qquad \text{(Leibnitz property of } \rfloor \text{)} \\
&= -\phi_{rst\,;i} \left\{ g^{ir} dx^s \vee dx^t + g^{is} dx^r \vee dx^t + g^{it} dx^r \vee dx^s \right\} .
\end{aligned}
$$

ϕ is symmetric in the indices r, s, t so a change of dummy indices allows us to collect the three terms, giving (4.24):

$$\partial\phi \;=\; -3\,\phi_{rst\,;\,i}\,g^{ir}\,dx^s \vee dx^t\,.$$

To obtain (4.25), we substitute the full expression for $\phi_{rst\,;\,i}$ in terms of the Christoffel symbols:

$$\partial\phi \;=\; -3\left\{\frac{\partial\phi_{rst}}{\partial x^i} - \phi_{kst}\Gamma^k{}_{ir} - \phi_{rkt}\Gamma^k{}_{is} - \phi_{rsk}\Gamma^k{}_{it}\right\} g^{ir}dx^s \vee dx^t\,;$$

due to the symmetries in the various indices, the third and fourth terms are equal. Thus

$$\partial\phi \;=\; -3\left\{\frac{\partial\phi_{rst}}{\partial x^i} - \phi_{kst}\Gamma^k{}_{ir} - 2\phi_{rkt}\Gamma^k{}_{is}\right\} g^{ir}dx^s \vee dx^t\,.$$

□

A result I will use is:

Proposition 4.12

$$\partial g \;=\; 0\,. \tag{4.26}$$

The proof follows trivially from the fact that $\nabla g = 0$.

The standard divergence of a vector field X is related to the symmetric coderivative by

Proposition 4.13

$$\mathrm{div}X \;=\; -\partial(\flat X)\,. \tag{4.27}$$

Proof

div$X$ is defined by the requirement

$$\mathcal{L}_X \Omega \;=\; (\text{div}X)\Omega$$

(Burke p 369, Matsushima p 292, Crampin and Pirani p 183). Burke (1985) states

$$\text{div}X = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}(\sqrt{|g|}X^i)\,,$$

and in Adler et al (1965) pp 72-3 it is shown how

$$\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}(\sqrt{|g|}X^i) \;=\; X^i{}_{;i}\,.$$

This in turn is just $-\partial\,(\flat X)$. □

4.5 Results involving symmetric derivatives

In this section I calculate the commutators of the differential operators D and ∂ with the algebraic operators M and L. I also calculate the commutator $[\partial, D]$. These results are new, but are analogous to the expressions found in Sorani (1969) p 191 for the exterior calculus.

Proposition 4.14

Let ϕ be a symmetric form. Then

$$[\partial, M]\,\phi \;=\; 0 \tag{4.28}$$

$$[D, L]\,\phi \;=\; 0 \tag{4.29}$$

$$[\partial, L]\,\phi \;=\; -D\,\phi \tag{4.30}$$

$$[\partial, L^r]\,\phi \;=\; -r\,L^{r-1}\,D\,\phi \tag{4.31}$$

$$[D, M]\,\phi \;=\; 2\,\partial\,\phi \tag{4.32}$$

$$[D, M^r]\,\phi \;=\; 2^r\, M^{r-1}\,\partial\,\phi\,. \tag{4.33}$$

Proof

To prove (4.28):

$$\begin{aligned}
\partial(M\phi) &= \partial(\sharp g \rfloor \phi) \qquad \text{(by definition of } M\text{)} \\
&= -\sharp dx^i \rfloor \nabla_i(\sharp g \rfloor \phi) \qquad \text{(by definition of } \partial\text{)} \\
&= -\sharp dx^i \rfloor \{(\nabla_i \sharp g)\rfloor \phi + \sharp g \rfloor (\nabla_i \phi)\} \qquad \text{(property of } \nabla\text{)}
\end{aligned}$$

The first term here is zero since $\nabla_i \sharp g = \sharp \nabla_i g = 0$, and the connection is metric compatible. Hence

$$\begin{aligned}
\partial(M\phi) &= -\sharp dx^i \rfloor (\sharp g \rfloor \nabla_i \phi) \\
&= -\sharp g \rfloor (\sharp dx^i \rfloor \nabla_i \phi) \\
&= M(\partial\phi)\,.
\end{aligned}$$

To prove (4.29):

$$\begin{aligned}
D(L\phi) &= D(g \vee \phi) \qquad \text{(by definition of } L\text{)} \\
&= (Dg) \vee \phi + g \vee (D\phi) \qquad \text{(properties of } D\text{)} \\
&= g \vee (D\phi) \qquad \text{(since } Dg = 0\text{)} \\
&= L(D\phi)\,.
\end{aligned}$$

To prove (4.30):

$$\begin{aligned}
\partial(L\phi) &= \partial(g \vee \phi) \qquad \text{(definition of } L\text{)} \\
&= (\partial g) \vee \phi + g \vee (\partial\phi) - (\nabla_i g) \vee (\sharp dx^i \rfloor \phi) - (\sharp dx^i \rfloor g) \vee (\nabla_i \phi) \\
&\qquad\qquad \text{(by properties of } \partial\text{)} \\
&= 0 + g \vee (\partial\phi) - 0 - (\flat\sharp dx^i) \vee (\nabla_i \phi) \qquad \text{(since } \partial g = 0 \text{ and } \nabla_i g = 0\text{)}
\end{aligned}$$

$$= L(\partial\phi) - (\nabla_i\phi) \vee dx^i$$
$$= L(\partial\phi) - D\,\phi$$

(4.31) is proved by induction as follows. It holds for $r = 1$ as shown in (4.30). We assume that for some larger integer r,

$$[\partial, L^{r-1}]\phi = -(r-1)\,L^{r-2}\,D\,\phi\,. \tag{4.34}$$

Now

$$\begin{aligned}
\partial(L^r\phi) &= \partial L(L^{r-1}\phi) \\
&= L\partial(L^{r-1}\phi) - D(L^{r-1}\phi) \qquad \text{(invoking (4.30))} \\
&= L(\partial L^{r-1}\phi) - L^{r-1}D\phi \qquad (L \text{ and } D \text{ commute by (4.29))} \\
&= L\left\{(L^{r-1}\partial\phi) - (r-1)L^{r-2}D\phi\right\} - L^{r-1}D\phi \qquad \text{(by hypothesis (4.34))} \\
&= L^r\partial\phi - (r-1)L^{r-1}D\phi - L^{r-1}D\phi \\
&= L^r\partial\phi - rL^{r-1}D\phi\,.
\end{aligned}$$

To prove (4.32):

$$\begin{aligned}
DM\phi &= \{\nabla_i(\sharp g\rfloor\phi)\} \vee dx^i \qquad \text{(by definition of } D \text{ and } M) \\
&= \{(\nabla_i\sharp g)\rfloor\phi + \sharp g\rfloor\nabla_i\phi\} \vee dx^i \qquad \text{(properties of } \nabla) \\
&= (\sharp g\rfloor\nabla_i\phi) \vee dx^i \qquad \text{(since } \nabla \text{ is metric compatible)} \\
&= (g^{jk}\partial_j\rfloor(\partial_k\rfloor\nabla_i\phi)) \vee dx^i \\
&= g^{jk}\partial_j\rfloor\{(\partial_k\rfloor\nabla_i\phi) \vee dx^i\} - g^{jk}(\partial_k\rfloor\nabla_i\phi) \vee (\partial_j\rfloor dx^i) \\
&\qquad\qquad \text{(Leibnitz property of } \rfloor) \\
&= g^{jk}\partial_j\rfloor\{(\partial_k\rfloor\nabla_i\phi) \vee dx^i\} - g^{ik}(\partial_k\rfloor\nabla_i\phi) \\
&= g^{jk}\partial_j\rfloor\partial_k\rfloor\{\nabla_i\phi \vee dx^i\} - g^{jk}\partial_j\rfloor\{(\nabla_i\phi) \vee (\partial_k\rfloor dx^i)\} + \partial\phi \\
&\qquad\qquad \text{(Leibnitz property of } \rfloor) \\
&= MD\phi - g^{ji}\partial_j\rfloor(\nabla_i\phi) + \partial\phi \\
&= MD\phi + 2\,\partial\phi
\end{aligned}$$

We prove (4.33) by induction. The result holds for $r = 1$ as demonstrated in (4.32). For larger values of r we will assume

$$[D, M^{r-1}]\phi = 2^{r-1}M^{r-2}\partial\phi. \qquad (4.35)$$

Now

$$\begin{aligned}
DM^r\phi &= DM(M^{r-1}\phi) \\
&= MD(M^{r-1}\phi) + 2\partial(M^{r-1}\phi) \qquad \text{(from (4.32))} \\
&= M(DM^{r-1}\phi) + 2M^{r-1}\partial\phi \qquad \text{(from (4.28))} \\
&= M\left\{M^{r-1}D\phi + 2^{r-1}M^{r-2}\partial\phi\right\} + 2M^{r-1}\partial\phi \qquad \text{(by the hypothesis (4.35))} \\
&= M^r D\phi + 2^{r-1}M^{r-1}\partial\phi + 2M^{r-1}\partial\phi \\
&= M^r D\phi + 2^r M^{r-1}\partial\phi.
\end{aligned}$$

□

Proposition 4.15

$$[\partial, D]\phi = \{\sharp dx^i \rfloor (\nabla_j\nabla_i\phi - \nabla_i\nabla_j\phi)\} \vee dx^j + g^{jk}\Gamma^i{}_{jk}\nabla_i\phi - g^{ij}\nabla_i\nabla_j\phi \quad (4.36)$$

Proof

$$\begin{aligned}
\partial D\phi - D\partial\phi &= \partial\{(\nabla_i\phi) \vee dx^i\} + D\{\sharp dx^i \rfloor \nabla_i\phi\} \\
&= (\partial\nabla_i\phi) \vee dx^i + (\nabla_i\phi) \vee (\partial dx^i) - (\nabla_j\nabla_i\phi) \vee (\sharp dx^j \rfloor dx^i) \\
&\quad -(\sharp dx^j \rfloor \nabla_i\phi) \vee (\nabla_j dx^i) + \nabla_j(\sharp dx^i \rfloor \nabla_i\phi) \vee dx^j \qquad \text{('Leibnitz' property)} \\
&= -(\sharp dx^j \rfloor \nabla_j\nabla_i\phi) \vee dx^i - (\nabla_i\phi)(\sharp dx^j \rfloor \nabla_j dx^i) - g^{ji}(\nabla_j\nabla_i\phi) \\
&\quad +\Gamma^i{}_{kj}(\sharp dx^j \rfloor \nabla_i\phi) \vee dx^k + (\sharp\nabla_j dx^i \rfloor \nabla_i\phi) \vee dx^j + (\sharp dx^i \rfloor \nabla_j\nabla_i\phi) \vee dx^j
\end{aligned}$$

$$
\begin{aligned}
&\quad\text{(definition of } \partial \text{ and properties of } \nabla_i) \\
&= \left\{\sharp dx^i \rfloor (\nabla_j \nabla_i \phi - \nabla_i \nabla_j \phi)\right\} \vee dx^j + \Gamma^i{}_{kj} g^{jk} (\nabla_i \phi) - g^{ij} (\nabla_i \nabla_j \phi) \\
&\qquad -(\sharp \nabla_k dx^i \rfloor \nabla_i \phi) \vee dx^k + (\sharp \nabla_j dx^i \rfloor \nabla_i \phi) \vee dx^j \qquad \text{(since } \nabla_i \text{ and } \sharp \text{ commute)} \\
&= \left\{\sharp dx^i \rfloor (\nabla_j \nabla_i \phi - \nabla_i \nabla_j \phi)\right\} \vee dx^j + \Gamma^i{}_{kj} g^{jk} (\nabla_i \phi) - g^{ij} (\nabla_i \nabla_j \phi),
\end{aligned}
$$

since the final two terms cancel out. □

Further developments could include the calculation of the commutators $[\partial^r, L^s]$, $[\partial^r, M^s]$, $[D^r, L^s]$, $[D^r, M^s]$, and $[\partial^r, D^s]$. It might also be interesting to calculate the analogous *anti*-commutators.

4.6 Lie algebra of symmetric forms

In a paper entitled 'Killing Tensors and the Separation of the Hamilton-Jacobi Equation' (1975), Woodhouse defines a Lie bracket on the space of contravariant symmetric tensors (p 16). In this section, I imitate his constructions for the covariant symmetric tensors. These results are directly the analogues of those given by Woodhouse, and are included here for completeness.

Based on the coordinate expression of equation (3.10) of the Woodhouse paper, we define an operation, denoted by { , } :

Definition 4.4

$$\begin{aligned} \{\ ,\ \} \ &:\ \mathcal{M}^r(T^*P,g)\times\mathcal{M}^s(T^*P,g) \ \rightarrow\ \mathcal{M}^{r+s-1}(T^*P,g) \\ &:\ (\phi,\psi) \ \mapsto\ \{\phi,\psi\}, \end{aligned} \tag{4.37}$$

where

$$\{\phi,\psi\} := (\nabla_i\phi)\vee(\sharp dx^i\rfloor\psi) - (\sharp dx^i\rfloor\phi)\vee(\nabla_i\psi).$$

□

Note that this expression bears a close resemblance to the last two terms obtained in the product rule[2] for the coderivative of a symmetric product (4.20):

$$\partial(\phi\vee\psi) = (\partial\phi)\vee\psi + \phi\vee(\partial\psi) - (\nabla_i\phi)\vee(\sharp dx^i\rfloor\psi) - (\nabla_i\psi)\vee(\sharp dx^i\rfloor\phi).$$

Proposition 4.16

The space of symmetric tensors $\mathcal{M}(T^*P,g)$ with the above operation is a Lie

[2]This product rule is the analogue of the Leibnitz rule and might be called a "pseudo Leibnitz product rule.

algebra. That is,

- $\{\ ,\ \}$ is bilinear (4.38)
- $\{\phi,\psi\} = -\{\psi,\phi\}$ (4.39)
- $\{\phi,\{\psi,\mu\}\}+\{\psi,\{\mu,\phi\}\}+\{\mu,\{\phi,\psi\}\} = 0\,.$ (4.40)

Proof

Proof of the bilinearity is trivial, because of the linearity properties of $\vee$, $\rfloor$ and ∇ over the reals. (4.39) may be proven as follows:

$$\begin{aligned}\{\phi,\psi\} &= (\nabla_i\phi)\vee(\sharp dx^i\rfloor\psi)-(\sharp dx^i\rfloor\phi)\vee(\nabla_i\psi)\\ &= -\left[(\nabla_i\psi)\vee(\sharp dx^i\rfloor\phi)-(\sharp dx^i\rfloor\psi)\vee(\nabla_i\phi)\right]\\ &= -\{\psi,\phi\}\,.\end{aligned}$$

The proof of the Jacobi identity (4.40) is tedious and I omit it. It requires applications of the definition of the $\{\ ,\ \}$-operation and the derivation properties of ∇ with respect to $\rfloor$. □

Proposition 4.17

The Lie bracket $\{,\}$ satisfies the Leibnitz rule:

$$\{\phi,\psi\vee\mu\} = \{\phi,\psi\}\vee\mu+\psi\vee\{\phi,\mu\} \tag{4.41}$$

$$\{\phi\vee\mu,\psi\} = \{\phi,\psi\}\vee\mu+\phi\vee\{\mu,\psi\}\,. \tag{4.42}$$

Proof

$$\begin{aligned}\{\phi,\psi\vee\mu\} &= (\nabla_i\phi)\vee(\sharp dx^i\rfloor(\psi\vee\mu))-(\sharp dx^i\rfloor\phi)\vee\nabla_i(\psi\vee\mu) \qquad \text{(by definition)}\\ &= (\nabla_i\phi)\vee(\sharp dx^i\rfloor\psi)\vee\mu+(\nabla_i\phi)\vee\psi\vee(\sharp dx^i\rfloor\mu)-(\sharp dx^i\rfloor\phi)\vee(\nabla_i\psi)\vee\mu\end{aligned}$$

$$
\begin{aligned}
& -(\sharp dx^i \rfloor \phi) \vee \psi \vee (\nabla_i \mu) \qquad \text{(by Leibnitz rules)} \\
= \;& (\nabla_i \phi) \vee (\sharp dx^i \rfloor \psi) \vee \mu - (\sharp dx^i \rfloor \phi) \vee (\nabla_i \psi) \vee \mu + \psi \vee (\nabla_i \phi) \vee (\sharp dx^i \rfloor \mu) \\
& -\psi \vee (\sharp dx^i \rfloor \phi) \vee (\nabla_i \mu) \qquad \text{(rearranging)} \\
= \;& \{\phi, \psi\} \vee \mu + \psi \vee \{\phi, \mu\} \,. \qquad \text{(by definition)}
\end{aligned}
$$

The second result follows from the antisymmetry of the bracket { , } . □

Note also that the set of symmetric one-forms is closed under the bracket, so that $\mathcal{M}^1(T^*P, g)$ is a Lie subalgebra of the Lie algebra of symmetric forms.

4.7 Killing vector fields

A vector field X in a manifold P with metric g is called a 'Killing field' if

$$\mathcal{L}_X g = 0 \tag{4.43}$$

(Crampin and Pirani p 194, Burke p 142). The symbol $\mathcal{L}$ denotes the Lie derivative here. I will show that this condition may be expressed in terms of the symmetric derivative defined previously.

Proposition 4.18
Assuming the manifold is defined using the Levì-Cività connection ∇,

$$\mathcal{L}_X g = 0 \quad \Leftrightarrow \quad D(\flat X) = 0 \,. \tag{4.44}$$

Proof

$$
\begin{aligned}
\mathcal{L}_X g = 0 \;&\Leftrightarrow\; (\mathcal{L}_X g)(Y, Z) = 0 \qquad (\forall\, X, Y \text{ vector fields}) \\
&\Leftrightarrow\; \mathcal{L}_X[g(Y, Z)] - g(\mathcal{L}_X Y, Z) - g(Y, \mathcal{L}_X Z) = 0 \qquad (\text{definition of } \mathcal{L}_X g) \\
&\Leftrightarrow\; \nabla_X[g(Y, Z)] - g(\nabla_X Y - \nabla_Y X, Z) - g(Y, \nabla_X Z - \nabla_Z X) = 0
\end{aligned}
$$

since the space is torsionless and thus $\mathcal{L}_X Y = [X, Y] = \nabla_X Y - \nabla_Y X$. This may be rearranged to give

$$\begin{aligned} \nabla_X[g(Y,Z)] - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) + g(\nabla_Y X, Z) + g(Y, \nabla_Z X) &= 0 \\ \Leftrightarrow \quad (\nabla_X g)(Y,Z) + g(\nabla_Y X, Z) + g(Y, \nabla_Z X) &= 0 \,, \end{aligned}$$

by the definition of the covariant derivative of a two-form. Since the connection is metric compatible,

$$\begin{aligned} & g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0 \\ \Leftrightarrow \quad & (\flat\nabla_Y X)(Z) + (\flat\nabla_Z X)(Y) = 0 \\ \Leftrightarrow \quad & (\nabla_Y \flat X)(Z) + (\nabla_Z \flat X)(Y) = 0 \qquad \text{(using (4.7))} \\ \Leftrightarrow \quad & (D\flat X)(Y,Z) = 0 \qquad (\forall \text{ vector fields } Y, Z) \\ \Leftrightarrow \quad & D\flat X = 0 \,. \end{aligned}$$

□

4.8 Killing forms

Motivated by the previous proposition, it is natural to define a Killing one-form as one having a zero symmetric exterior derivative. This in turn leads to a natural definition for a symmetric Killing form of any degree, which may be found in Kramer et al (1979). They do not define a derivative like D, but their definition is equivalent.

Definition 4.5

Let ϕ be any symmetric differential form in $\mathcal{M}(T^*P, g)$. ϕ is called a *symmetric Killing form* if

$$D\phi = 0 \,.$$

□

The following three propositions may all be found in Woodhouse (p 17). I list them and include their proofs.

Proposition 4.19

$$\phi \text{ is a Killing form} \;\Leftrightarrow\; \{\phi, g\} = 0\,.$$

Proof

$$\begin{aligned}\{\phi, g\} = 0 \;&\Leftrightarrow\; (\nabla_i\phi) \vee (\sharp dx^i \rfloor g) - (\sharp dx^i \rfloor \phi) \vee (\nabla_i g) \;=\; 0 \qquad \text{(by definition)}\\ &\Leftrightarrow\; (\nabla_i\phi) \vee dx^i - 0 \;= 0 \qquad \text{(since } \nabla \text{ is metric compatible)}\\ &\Leftrightarrow\; D\,\phi \;= 0\,. \qquad \text{(by definition)}\end{aligned}$$

□

Proposition 4.20
The set of Killing forms is a Lie subalgebra of the Lie algebra of symmetric forms.

Proof
The set of Killing forms is a vector space since, trivially, any linear combination of Killing forms will be a Killing form. It will suffice then, to demonstrate closure of this vector space, as all the Lie algebra properties are inherited from the larger space. Let ϕ and ψ be Killing forms. By the Jacobi identity,

$$\{\{\phi, \psi\}, g\} + \{\{\psi, g\}, \phi\} + \{\{g, \phi\}, \psi\} \;=\; 0\,.$$

The second and third terms are zero since ϕ and ψ are Killing forms. Therefore

$$\{\{\phi,\psi\},g\} = 0,$$

and $\{\phi,\psi\}$ is a Killing form. □

Proposition 4.21

The Lie subalgebra of Killing forms is closed under the symmetric product.

Proof

Let ϕ and ψ be Killing forms.

$$\begin{aligned} D(\phi \vee \psi) &= (D\phi)\vee\psi + \phi\vee(D\psi) \qquad \text{(Leibnitz property)} \\ &= 0. \qquad \text{(since } \phi \text{ and } \psi \text{ are Killing)} \end{aligned}$$

□

Other results may be found in Woodhouse and Kramer et al with application to differential equations.

4.9 Harmonic theory of symmetric forms

It is common to define an inner product of exterior (antisymmetric) differential forms on manifolds having a Hodge star operation by an expression of the form

$$(\phi,\psi) = \int \phi \wedge *\psi,$$

where the arguments are of the same degree (De Rham p 122, Warner p 221, Choquet-Bruhat et al p 297, Flanders p 137).

Direct imitation of this definition for the symmetric forms is prevented by the lack of a Hodge operator. Taking a cue from the result (2.58) proved for the symplectic exterior algebra, namely

$$\phi \wedge *\psi \;=\; (\sharp\phi \rfloor \psi)\,\Omega\,,$$

we find an expression that can be carried over into the symmetric algebra provided we have a volume form (antisymmetric) on P.

From here on we will assume P to have a volume form Ω. This allows us, then, to define:

Definition 4.6

The *global inner product* or *inner product* on the symmetric forms is defined by:

$$\begin{aligned} G(\,,\,) \;&:\; \mathcal{M}^r(T^*P,g) \times \mathcal{M}^s(T^*P,g) \;\to\; \Re \\ &:\; (\phi,\psi) \qquad\qquad\qquad \mapsto\; G(\phi,\psi) \;:=\; \delta_{rs} \textstyle\int_P (\sharp\phi \rfloor \psi)\,\Omega\,, \end{aligned} \tag{4.45}$$

where Ω is the invariant volume on the manifold P. $G(\,,\,)$ is extended to nonhomogeneous forms by requiring it to be linear over sums of terms of differing degrees. □

By the properties obtained for the extended metric on the symmetric algebra in the previous chapter, it follows that $G(\;,\;)$ is bilinear over the reals, and symmetric.

Strictly speaking, $G(\;,\;)$ is not an inner product because inner products are positive definite, by definition (Shephard p 135). Positive definiteness is not guaranteed for $G(\;,\;)$ unless g is itself positive-definite.

The standard developments of Harmonic theory proceed from here to show that the exterior derivative and the coderivative are adjoints with respect to the inner product (see above references). It is possible to arrive at an analogous result in the symmetric exterior calculus if we introduce an additional assumption about P: that it has no boundary. From here on, this property will be assumed to hold.

This allows us to prove Green's theorem (Yano 1970 p 11):

Proposition 4.22

Let P be a manifold with metric g, volume form Ω and no boundary. For any one-form θ,

$$\int_P (\partial\theta)\,\Omega \;=\; 0\,. \tag{4.46}$$

Proof

$$\begin{aligned}
\int_P (\partial\theta)\Omega &= -\int_P \operatorname{div}(\sharp\theta)\Omega \qquad \text{(invoking (4.27))} \\
&= -\int_P \mathcal{L}_{\sharp\theta}\Omega \qquad \text{(definition of coderivative of a vector field)} \\
&= -\int_P \sharp\theta \rfloor d\Omega - \int_P d(\sharp\theta\rfloor\Omega) \qquad \text{(Cartan identity)} \\
&= 0 - \int_{\partial P} (\sharp\theta\rfloor\Omega) \qquad \text{(since } d\Omega = 0 \text{ and by Stokes' theorem)} \\
&= 0\,. \qquad (P \text{ has no boundary, so } \partial P = 0)
\end{aligned}$$

□

Proposition 4.23

D and ∂ are adjoint with respect to the global inner product. In other words, for any symmetric r-form ϕ and any symmetric $(r+1)$-form ψ,

$$G(D\phi, \psi) \;=\; G(\phi, \partial\psi)\,. \tag{4.47}$$

Proof

Consider the argument of the integral $G(D\phi, \psi) = \int \sharp(D\phi)\rfloor\psi\ \Omega$:

$$\begin{aligned}
\sharp D\,\phi\rfloor\psi &= \sharp\left[(\nabla_i\phi)\vee dx^i\right]\rfloor\psi \qquad \text{(by definition of } D) \\
&= \sharp dx^i\rfloor((\sharp\nabla_i\phi)\rfloor\psi) \qquad \text{(properties of } \sharp \text{ and } \rfloor) \\
&= \sharp dx^i\rfloor((\nabla_i\sharp\phi)\rfloor\psi) \qquad \text{(by (4.6))} \\
&= \sharp dx^i\rfloor\left[\nabla_i(\sharp\phi\rfloor\psi) - \sharp\phi\rfloor\nabla_i\psi\right] \qquad \text{(Leibnitz property of } \nabla) \\
&= -\partial(\sharp\phi\rfloor\psi) - \sharp\phi\rfloor\sharp dx^i\rfloor\nabla_i\psi \qquad \text{(by definition of } \partial) \\
&= -\partial(\sharp\phi\rfloor\psi) + \sharp\phi\rfloor\partial\psi \qquad \text{(by definition of } \partial)
\end{aligned}$$

Therefore

$$\begin{aligned}
G(D\,\phi, \psi) &= \int_P(\sharp D\,\phi\rfloor\psi)\,\Omega \\
&= -\int_P \partial(\sharp\phi\rfloor\psi)\,\Omega + \int_P(\sharp\phi\rfloor\partial\psi)\,\Omega \\
&= 0 + \int_P(\sharp\phi\rfloor\partial\psi)\,\Omega \qquad \text{(by Green's theorem)} \\
&= G(\phi\,, \partial\psi)\,. \qquad \text{(by definition of the inner product)}
\end{aligned}$$

□

In the calculus of exterior differential forms, a Laplacian operator is defined in terms of the exterior derivative and the coderivative by :

$$\Delta := \delta d + d\delta$$

(Warner p 220, Flanders p 137, De Rham p 125, Choquet-Bruhat et al p 318, Burke p 200). It is also referred to as the 'De Rham' operator, the 'Laplace-de Rham' operator or the 'harmonic' operator. We may therefore make an analogous definition:

Definition 4.7

The *Laplacian* operator Δ is defined by

$$\begin{aligned} \Delta \ &: \ \mathcal{M}^r(T^*P,g) \ \rightarrow \ \mathcal{M}^r(T^*P,g) \\ &: \ \phi \ \mapsto \ \Delta\phi \ := \ D\partial\phi \ + \ \partial D\phi \,. \end{aligned} \tag{4.48}$$

□

Proposition 4.24

The Laplacian Δ is self adjoint. That is, for ϕ and ψ of the same degree,

$$G(\Delta\phi, \psi) \ = \ G(\phi, \Delta\psi) \,. \tag{4.49}$$

Proof

The result follows immediately from the adjoint property of D and ∂.

$$\begin{aligned} G(\Delta\phi,\psi) \ &= \ G(D\partial\phi,\psi) \ + \ G(\partial D\phi,\psi) \quad \text{(by definition)} \\ &= \ G(\partial\phi,\partial\psi) \ + \ G(D\phi, D\psi) \quad \text{(by adjoint property)} \\ &= \ G(\phi, D\partial\psi) \ + \ G(\phi,\partial D\psi) \quad \text{(by adjoint property)} \\ &= \ G(\phi,\Delta\psi) \,. \quad \text{(by definition)} \end{aligned}$$

□

The Laplacian operator is usually used to define harmonic forms (Warner p 223, Choquet-Bruhat et al p 318, Flanders p138):

Definition 4.8

Any $\phi \ \in \ \mathcal{M}^r(T^*P,g)$ is called *harmonic* if

$$\Delta\,\phi \ = \ 0 \,. \tag{4.50}$$

□

If the global inner product is positive definite, it may be shown that an equivalent requirement is the two conditions $D\phi = 0$ and $\partial\phi = 0$.

In the exterior case, the Laplacian Δ commutes with d and δ as can be seen from

$$\begin{aligned} \Delta d - d\Delta &= d\delta d + \delta dd - dd\delta - d\delta d \\ &= d\delta d + 0 - 0 - d\delta d \qquad \text{(since } dd = 0 \text{ and } \delta\delta = 0\text{)} \\ &= 0\,, \end{aligned}$$

and similarly for δ (Choquet-Bruhat p 319, De Rham p 125). However, for the symmetric calculus, $DD \neq 0$ and $\partial\partial \neq 0$; consequently

$$\Delta D - D\Delta = \partial DD - DD\partial\,,$$

and

$$\Delta\partial - \partial\Delta = D\partial\partial - \partial\partial D\,.$$

Another self-adjoint operator defined in the standard theory of harmonic forms is the 'Dirac' operator, $d + \delta$. It can also be defined for calculus:

Definition 4.9

The *Dirac* operator is the sum of the two derivatives D and ∂, that is,

$$D + \partial\,. \tag{4.51}$$

□

Proposition 4.25

The Dirac operator in the symmetric calculus is self-adjoint. Let ϕ and ψ be symmetric differential forms in $\mathcal{M}(T^*P, g)$. Then:

$$G([D+\partial]\phi, \psi) = G(\phi, [D+\partial]\psi). \tag{4.52}$$

Proof

The self-adjointness of $D + \partial$ is almost trivial:

$$\begin{aligned} G([D+\partial]\phi, \psi) &= G(D\phi, \psi) + G(\partial\phi, \psi) \\ &= G(\phi, \partial\psi) + G(\phi, D\psi) \qquad \text{(adjoint properties)} \\ &= G(\phi, [\partial + D]\psi) \\ &= G(\phi, [D+\partial]\psi). \end{aligned}$$

□

In the exterior case, two iterations of the Dirac operator yield the Laplacian. This does not happen in the symmetric calculus, because $DD \neq 0$ and $\partial\partial \neq 0$:

$$\begin{aligned} (D+\partial)^2 &= DD + D\partial + \partial D + \partial\partial \\ &= \Delta + DD + \partial\partial. \end{aligned}$$

Discussion and Prospects

The above does not provide a complete theory of harmonic symmetric forms. It only provides a starting point for further investigations. Two directions for further study immediately suggest themselves.

The Hodge decomposition theorem (Warner p 223) says that any exterior differential form may be uniquely decomposed into the sum of a closed form, an exact form and an harmonic form. One is naturally led to ask whether a similar result holds for the symmetric calculus. Can a symmetric form ϕ be uniquely decomposed into a sum

$$\phi = D\alpha + \partial\beta + \gamma\,,$$

where γ is harmonic?

Any vector space with an inner product may be decomposed by making use of a self-adjoint operator (Shephard p 160, 'Third Reduction Theorem'). If we let the metric g on our space be positive-definite, then our space would appear to have this structure, together with two possible self-adjoint operators, Δ and $D + \partial$. To what decompositions do these operators lead? The relationship between these decompositions and the known structures, such as the homogeneous subspaces and the Lepage decomposition would also be of interest.

Chapter 5

Conclusion

In this thesis I have investigated some mathematical structures that arise in a natural way in the geometrisation of Classical and Quantum Mechanics. I have investigated in particular the role and properties of four geometric algebra and of their associated calculi. The emphasis has been on two of these, the symplectic exterior algebra (i.e. the classical exterior algebra endowed with a symplectic structure) and the symmetric exterior algebra endowed with a metric structure, with a view to applying the same techniques to the remaining two, the symplectic and metric Clifford algebras. This has yielded new results in areas that have already been investigated and has also led to the introduction of what appear to be new concepts and definitions, some of the consequences of which I have investigated. I shall now briefly list what I consider to be the more significant contributions of the work reported in this thesis, together with suggestions for further development.

In chapter 2, I examined the structure of the symplectic exterior algebra. The principal advance here was the geometrical interpretation of the Lepage decomposition and the consequent results on the simple subspaces. The new findings hinged critically on the use of the extended symplectic metric. I showed that the even degree homogeneous subspaces were metric spaces, while those of odd degree were symplectic. The simple subspaces had similar metric structures, determined by their degree. The signatures for all the spaces with metric structure were obtained. I also calculated the dimensions of the simple subspaces, and showed them to be orthogonal to each other. The commutation relations of the operators L_ω and M_ω were calculated for use in later material.

In chapter 3, I considered the symmetric exterior algebra with a distinguished metric. Almost all the results of this section were new. The primary finding was that a Lepage-like decomposition theorem holds for this algebra also. Following the methods of the previous chapter, an extended metric was defined. All the homogeneous subspaces and all the simple subspaces were found to be metric spaces. The signatures for these spaces were obtained as well as the dimensions of the simple ones. The simple subspaces were again found to be orthogonal to each other. As in the previous chapter, commutation relations were calculated for the operators L_g and M_g for use in the sequel. One aspect of the symplectic exterior algebra, the Hodge dual operator, could not be duplicated in the symmetric exterior algebra.

The main achievement in chapter 4 was the definition of a calculus for the symmetric tensors on a pseudo-Riemannian manifold. As far as I know, this tentative calculus is new. A symmetric exterior derivative D was defined which reflected many of the properties of the regular antisymmetric exterior derivative. A coderivative ∂ was defined which, in similar style, reproduced many of the properties of the coderivative defined for the antisymmetric exterior forms in the presence of a metric. Various commutation relations involving these derivatives and the operators M_g and L_g were calculated. The symmetric derivative D provided a natural means to express the condition that a symmetric form be a Killing form. We were also able to mimic the standard theory of Harmonic forms to an extent. We defined a global inner product, and demonstrated that the two symmetric derivatives were adjoints with respect to it. We defined possible 'Laplacian' and 'Dirac' operators, by analogy with their standard definitions.

There are many questions related to the symmetric calculus and the two exterior algebras studied here that warrant further investigation. I shall now outline a few suggestions in this regard.

The metric structures developed in chapters 3 and 4 were developed with the eventual objective of applying them to the symplectic Clifford algebra. To succeed in this, it will be necessary to introduce a symplectic structure concurrently with the metric structure. I have not pursued this problem here, but the work of Frescura and Lubczonok (1991) on associated symplectic and cosymplectic structures, or some similar synthesis of metric and symplectic geometry, may be relevant.

If a symplectic two-form can be re-introduced into the context of the symmetric algebra $\mathcal{M}(V^*, g)$, a number of new questions will arise. Due to the dual metric and symplectic structures, the symplectic Clifford algebra would admit a metric Lepage decomposition. It would then be natural to consider the relationship between the two structures. One question would be whether the Clifford product respected the structure of the orthogonal simple subspaces. The simple subspaces and the operators M_g, L_g might also be of use in explicitly expressing the Clifford product of arbitrary symmetric forms. The results in Oziewicz and Sitarczyk (1992) might be useful here.

A parallel consideration would apply to the metric Clifford algebra. Is it perhaps also possible to incorporate a metric g into the context of the symplectic exterior algebra? The result would be a metric Clifford algebra with a symplectic Lepage decomposition. Again the relationship between the Clifford product and the simple subspaces would be of interest. One could also consider ways of expressing the Clifford product explicitly with the aid of the operators M_ω and L_ω.

The material given in chapter 4 on a symmetric calculus consisted largely of definitions and simple properties of derivatives on the symmetric differential forms. But a theory requires more than this. To make it into more than merely a formalism and to give it real content, non-trivial theorems of wide application must be found. The symmetric calculus, while exhibiting many promising features, requires further research to put it on a firm foundation, and success in this respect would immediately remove its tentative nature. There is every indication that this new symmetric calculus could contribute significantly towards the establishment of a framework against which the geometry associated with the symplectic Clifford algebra may be better understood.

Appendix A

Commutation relation proof

I give a proof of the commutation relation (2.79).

Proposition

Let ϕ belong to $\mathcal{S}^p(V^*, \omega)$. Then for $r \geq 1$ and $a \geq 0$,

$$[M^r, L^{r+a}]\,\phi = \sum_{i=1}^{r} \binom{r}{i} \frac{(r+a)!}{(r+a-i)!} \prod_{k=0}^{i-1} (n-p-a-k) L^{r+a-i} M^{r-i} \phi.$$

Proof

It is easily seen that this expression is just the trivial statement $0 = 0$ for $r > n$ or $a > n - r$. The result must be demonstrated for $1 \leq r \leq n$ and $0 \leq a \leq n - r$ and will be proven by induction. We choose to work with (2.79) expressed in a slightly different form:

$$[M^r, L^s]\,\phi = \sum_{i=1}^{r} \binom{r}{i} \frac{(s)!}{(s-i)!} \prod_{j=r+1-i}^{r} (n-s-p+j) L^{s-i} M^{r-i} \phi \qquad \text{(A.1)}$$

where $1 \leq r \leq n$ and $0 \leq a \leq n - r$.

Induction on r.

Treating s as a constant, the result holds for $r = 1$, since we have already seen

$$[M, L^s]\,\phi = s(n-s-p+1)L^{s-1}\phi. \tag{A.2}$$

We hypothesize that for $r < s$

$$[M^{r-1}, L^s]\,\phi = \sum_{j=1}^{r-1} \binom{r-1}{j} \frac{(s)!}{(s-j)!} \prod_{k=r-j}^{r-1} (n-s-p+k) L^{s-j} M^{r-1-j}\phi. \tag{A.3}$$

For $r \leq s$,

$$\begin{aligned} [M^r, L^s]\,\phi &= M^r L^s \phi - L^s M^r \phi \\ &= M M^{r-1} L^s \phi - L^s M(M^{r-1}\phi). \end{aligned}$$

Substituting $[M, L^s]\,\phi = ML^s\phi - L^s M\phi$ into the second term yields

$$\begin{aligned} [M^r, L^s]\,\phi &= M M^{r-1} L^s \phi + [M, L^s]\,(M^{r-1}\phi) - ML^s(M^{r-1}\phi) \\ &= M[M^{r-1}, L^s]\,\phi + [M, L^s]\,(M^{r-1}\phi) \end{aligned}$$

We substitute (A.3) and (A.2) into the two terms, noting that the arguments are of degree p and $p - 2r + 2$ respectively.

$$\begin{aligned} [M^r, L^s]\,\phi &= M \left\{ \sum_{j=1}^{r-1} \binom{r-1}{j} \frac{(s)!}{(s-j)!} \prod_{k=r-j}^{r-1} (n-s-p+k) L^{s-j} M^{r-1-j}\phi \right\} \\ &\quad + s(n-s-(p-2r+2)+1)L^{s-1}M^{r-1}\phi. \end{aligned}$$

Thus,

$$\begin{aligned} [M^r, L^s]\,\phi &= \sum_{j=1}^{r-1} \binom{r-1}{j} \frac{(s)!}{(s-j)!} \prod_{k=r-j}^{r-1} (n-s-p+k) M L^{s-j} M^{r-1-j}\phi \\ &\quad + s(n-s-p+2r-1)L^{s-1}M^{r-1}\phi. \end{aligned} \tag{A.4}$$

We next simplify the expression $ML^{s-j}M^{r-1-j}\phi$ which appears in the first term here. Noting that $M^{r-1-j}\phi$ is of degree $p-2r+2+2j$, we have from (A.2) that

$$\begin{aligned}[M, L^{s-j}]\, M^{r-1-j}\phi &= (s-j)(n-(s-j)-(p-2r+2+2j)+1)L^{s-j-1}M^{r-j-1}\phi \\ &= ML^{s-j}M^{r-1-j}\phi - L^{s-j}M^{r-j}\phi;\end{aligned}$$

hence

$$ML^{s-j}M^{r-1-j}\phi = (s-j)(n-s-p+2r-j-1)L^{s-j-1}M^{r-j-1}\phi + L^{s-j}M^{r-j}\phi. \tag{A.5}$$

Substituting (A.5) into (A.4) yields

$$\begin{aligned}&[M^r, L^s]\,\phi = \\ &\quad \sum_{j=1}^{r-1}\binom{r-1}{j}\frac{s!}{(s-j)!}\prod_{k=r-j}^{r-1}(n-s-p+k)L^{s-j}M^{r-j}\phi \\ &\quad +\sum_{j=1}^{r-1}\binom{r-1}{j}\frac{s!}{(s-j)!}\prod_{k=r-j}^{r-1}(n-s-p+k)(s-j)(n-s-p+2r-j-1)\times \\ &\qquad\qquad L^{s-j-1}M^{r-j-1}\phi + s(n-s-p+2r-1)L^{s-1}M^{r-1}\phi\end{aligned} \tag{A.6}$$

By putting $i=j+1$ in the second summation and replacing the index j in the first summation with i, (A.6) may be written as

$$\begin{aligned}&[M^r, L^s]\,\phi = \\ &\quad \left\{\binom{r-1}{1}\frac{s!}{(s-1)!}\prod_{k=r-1}^{r-1}(n-s-p+k) + s(n-s-p+2r-1)\right\}L^{s-1}M^{r-1}\phi \\ &\quad +\sum_{i=2}^{r-1}\frac{s!}{(s-i)!}\left\{\binom{r-1}{i}\prod_{k=r-i}^{r-1}(n-s-p+k)\right. \\ &\qquad\qquad \left. +\binom{r-1}{i-1}\prod_{k=r-i+1}^{r-1}(n-s-p+k)(n-s-p+2r-i)\right\}L^{s-i}M^{r-i}\phi \\ &\quad +\binom{r-1}{r-1}\frac{s!}{(s-r)!}\prod_{k=1}^{r-1}(n-s-p+k)(n-s-p+2r-r)\,L^{s-r}\phi\end{aligned} \tag{A.7}$$

It is a simple algebraic exercise to verify that the coefficients in (A.7) are equal to those in (A.1). This completes the induction on r.

Induction on s.

We treat r as a constant and show that (A.1) holds for all values of

s where $r < s$. We need not consider $r = s$, since it was treated above. The lowest possible value of s is 2; in this instance (A.1) becomes

$$[M, L^2]\,\phi = 2(n - p - 1)L\phi,$$

which agrees with (A.2). We hypothesize that for $s - 1 \geq r$

$$[M^r, L^{s-1}]\,\phi = \sum_{j=1}^{r} \binom{r}{j} \frac{(s-1)!}{(s-j-1)!} \prod_{k=r+1-j}^{r} (n - s + 1 - p + k) L^{s-j-1} M^{r-j} \phi \tag{A.8}$$

Then, for $s > r$ we have

$$\begin{aligned} [M^r, L^s]\,\phi &= M^r L^s \phi - L^s M^r \phi \\ &= M^r L^{s-1} L\phi - L^{s-1}(LM^r \phi). \end{aligned}$$

Because $[M^r, L]\,\phi = M^r L\phi - LM^r\phi$, it follows that

$$\begin{aligned} [M^r, L^s]\,\phi &= M^r L^{s-1}(L\phi) + L^{s-1}[M^r, L]\,\phi - L^{s-1}M^r(L\phi) \\ &= [M^r, L^{s-1}]\,(L\phi) + L^{s-1}[M^r, L]\,\phi \end{aligned} \tag{A.9}$$

In the first term we substitute the hypothesis (A.8), noting that the

degree of the argument is $p+2$; in the second term we substitute the previously proven result

$$[M^r, L]\,\phi = r(n - p + r - 1)M^{r-1}\phi \tag{A.10}$$

This yields

$$\begin{aligned} [M^r, L^s]\,\phi &= \sum_{j=1}^{r} \binom{r}{j} \frac{(s-1)!}{(s-j-1)!} \prod_{k=r+1-j}^{r} (n - s + 1 - (p+2) + k) L^{s-j-1} M^{r-j}(L\phi) \\ &\quad + r(n - p + r - 1)L^{s-1}M^{r-1}\phi. \end{aligned} \tag{A.11}$$

We separate the term where $j = r$ from the summation to yield

$$\begin{aligned}[M^r, L^s]\,\phi &= r(n-p+r-1)L^{s-1}M^{r-1}\phi \\ &+\sum_{j=1}^{r-1}\binom{r}{j}\frac{(s-1)!}{(s-j-1)!}\prod_{k=r+1-j}^{r}(n-s-p-1+k)L^{s-j-1}M^{r-j}L\phi \\ &+\frac{(s-1)!}{(s-r-1)!}\prod_{k=1}^{r}(n-s-p-1+k)L^{s-r}\phi \qquad (A.12)\end{aligned}$$

To simplify the summation in this expression, we make use of (A.10); since $[M^{r-j}, L]\,\phi = M^{r-j}L\phi - LM^{r-j}\phi$, it follows that

$$\begin{aligned}M^{r-j}L\phi &= [M^{r-j}, L]\,\phi + LM^{r-j}\phi \\ &= (r-j)(n-p+r-j-1)M^{r-j-1}\phi + LM^{r-j}\phi \quad (A.13)\end{aligned}$$

and substitution of this into (A.12) yields

$$\begin{aligned}[M^r, L^s]\,\phi &= r(n-p+r-1)L^{s-1}M^{r-1}\phi \\ &+\sum_{j=1}^{r-1}\binom{r}{j}\frac{(s-1)!}{(s-j-1)!}\prod_{k=r+1-j}^{r}(n-s-p-1+k)(r-j)(n-p+r-j-1)\times \\ &\qquad L^{s-j-1}M^{r-j-1}\phi \\ &+\sum_{j=1}^{r-1}\binom{r}{j}\frac{(s-1)!}{(s-j-1)!}\prod_{k=r+1-j}^{r}(n-s-p-1+k)L^{s-j}M^{r-j}\phi \\ &+\frac{(s-1)!}{(s-r-1)!}\prod_{k=1}^{r}(n-s-p-1+k)L^{s-r}\phi. \qquad (A.14)\end{aligned}$$

In the first summation of (A.14), we put $i = j + 1$ and we change the index j in the second summation to i. This allows us to give the entire expression as

$$\begin{aligned}[M^r, L^s]\,\phi &= \left\{r(n-p+r-1) + \binom{r}{1}\frac{(s-1)!}{(s-2)!}(n-s-p-1+r)\right\}L^{s-1}M^{r-1}\phi \\ &+\sum_{i=2}^{r-1}\left\{\binom{r}{i}\frac{(s-1)!}{(s-i-1)!}\prod_{k=r+1-i}^{r}(n-s-p-1+k)\right. \\ &\left.+\binom{r}{i-1}\frac{(s-1)!}{(s-i)!}\prod_{k=r+2-i}^{r}(n-s-p-1+k)(r-i+1)(n-p+r-i)\right\}L^{s-i}M^{r-i}\phi \\ &+\left\{\frac{(s-1)!}{(s-r-1)!}\prod_{k=1}^{r}(n-s-p-1+k)\right.\end{aligned}$$

$$+\binom{r}{r-1}\frac{(s-1)!}{(s-r)!}\prod_{k=2}^{r}(n-s-p-1+k)(n-p)\Bigg\}L^{s-r}\phi. \quad \text{(A.15)}$$

It is again not difficult to demonstrate that (A.15) is equivalent to (A.1). This completes the induction on s, and the theorem is proven. □

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