# CONTRIBUTIONS TO THE STUDY OF A CLASS OF OPTIMAL CONTROL PROBLEMS ON THE ORTHOGONAL GROUPS $\mathsf{SO}(3)$ AND $\mathsf{SO}(4)$

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# Abstract

In this thesis we investigate a class of invariant optimal control problems, and their associated quadratic Hamilton-Poisson systems, on the orthogonal groups $\mathsf{SO}(3)$ and $\mathsf{SO}(4)$. Specifically, we are concerned with the class of left-invariant control affine systems. We begin by classifying all cost-extended systems on $\mathsf{SO}(3)$ under cost equivalence. (Cost-extended systems are closely related to optimal control problems.) A classification of all quadratic Hamilton-Poisson systems on the (minus) Lie-Poisson space $\mathfrak{so}(3)^*_-$, under affine equivalence, is also obtained. For the normal forms obtained in our classification (of Hamilton-Poisson systems) we investigate the (Lyapunov) stability nature of the equilibria using spectral and energy-Casimir methods. For a subclass of these systems, we obtain analytic expressions for the integral curves of the associated Hamiltonian vector fields in terms of (basic) Jacobi elliptic functions. The explicit relationship between the classification of cost-extended systems on $\mathsf{SO}(3)$ and the classification of quadratic Hamilton-Poisson systems on $\mathfrak{so}(3)^*_-$ is provided. On $\mathsf{SO}(4)$, a classification of all left-invariant control affine systems under $\mathfrak{L}$-equivalence is obtained. We then determine which of these representatives are controllable, thus obtaining a classification under detached feedback equivalence. We also obtain a partial classification of quadratic Hamilton-Poisson systems on the Lie-Poisson space $\mathfrak{so}(4)^*_-$. An investigation of the stability nature of the equilibria for a subclass of these systems is also done. Several illustrative examples of optimal control problems on the orthogonal group $\mathsf{SO}(3)$ are provided. More specifically, we consider an optimal control problem corresponding to a representative of our classification (of cost-extended system) for each possible number of control inputs. For each of these problems, we obtain explicit expressions for the extremal trajectories on the homogeneous space $\mathbb{S}^2$ by projecting the extremal trajectories on the group $\mathsf{SO}(3)$. The examples provided show how our classifications of cost-extended systems and Hamilton-Poisson systems can be used to obtain the optimal controls and the extremal trajectories corresponding to a large class of optimal control problems on $\mathsf{SO}(3)$. An example of a four-input optimal control problem on $\mathsf{SO}(4)$ is also provided. This example is provided to show how the solutions of certain problems on $\mathsf{SO}(4)$ can be related to the solutions of certain optimal control problems on $\mathsf{SO}(3)$.

**Keywords and phrases.** Affine equivalence, cost-extended system, cost equivalence, detached feedback equivalence, left-invariant control affine system, Jacobi elliptic function, Lyapunov stability.

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# Chapter 1

# Introduction

This chapter contains a brief introduction of the use and developments of geometric control theory over the past several decades. We also introduce the main aims of the thesis as well as the general results used in the analysis of the classes of control systems considered. A summary of the contents of each chapter is also provided. Lastly, we give a list of the original contributions made in this thesis.

## History

Mathematical control theory is the area of application-oriented mathematics that treats the basic mathematical principles, theory and problems underlying the analysis and design of control systems. Control theory has applications in many areas including engineering, robotics, physics and biology. Invariant geometric control theory is the study of invariant control systems evolving on Lie groups.

Geometric control theory started in the early 1970s with the realisation of the significance of the Lie bracket in control problems by of R. Brockett, H. Hermes and C. Lobry. Geometric control theory approaches the problems of nonlinear control theory using the methods and tools of differential geometry and Lie theory. The development of geometric control theory has also been heavily influenced by practical problems in mechanics, physics, and engineering by researchers such as Jurdjevic and Sussman. The following quote motivates well the study of Lie groups and their role in applications: "due to their unique combination of geometric and algebraic properties, Lie groups arise naturally as the models for the configuration space of mechanical systems which provide [one of] the major practical motivations for their study", [58]. It should also be noted that one of the major contributions that lead to the field of geometric control theory was the discovery of the Maximum Principle by L.S. Pontryagin in the late 1950s, "a far reaching generalization of Weirstrass's necessary conditions for strong minima, which provides geometric conditions for a (strong) minimum of an integral criterion, called the *cost*, over the trajectories of a differential control system" [34]. The Maximum Principle, in its original form, suffers from some serious limitations; geometric control theory forms a theoretical foundation for extensions of the Maximum Principle to optimal control problems on arbitrary differential manifolds. "This theoretical foundation comprises important results concerning the topological and differential properties of the reachable sets and is an essential complement to modern optimal control theory" [34]. Geometric control theory is an active area of research. Some notable researchers of this topic include: R. Brockett, V. Jurdjevic, A.A. Agrachev, Y.L. Sachkov, H.J. Sussmann and A. Krener.

## Invariant control systems

Mathematically, a control system is a family of smooth dynamical systems together with a class of admissible controls. In particular, a left-invariant control system on a Lie group $\mathsf{G}$ consists of a family of left-invariant vector fields $\Xi = (\Xi_u)_{u \in U}$ on $\mathsf{G}$, parametrized smoothly by controls. An admissible control is a $U$-valued map, defined on some interval of $\mathbb{R}$, which is (Lebesgue) measurable or piecewise constant, or of some regularity type between these two possibilities. The input set $U$ is usually equipped with a separable metric space structure. A *trajectory* of such a system is an integral curve of the (non-autonomous) vector field $\Xi_{u(t)}$, where $u(\cdot)$ is an "admissible control". Thus a control system can be viewed as a family of ordinary differential equations parametrized by control parameters, which can be used to influence the behaviour of the system.

In this thesis, a (left-invariant) control system is a pair $\Sigma = (\mathsf{G}, \Xi)$, where the *state space* $\mathsf{G}$ is a (matrix) Lie group and the dynamics $\Xi : \mathsf{G} \times U \to T\mathsf{G}$ are left-invariant. In classical notation, such a control system $\Sigma$ is written as

$$\Sigma\,:\,\dot{g} = \Xi(g, u) = g\,\Xi(\mathbf{1}, u), \quad g \in \mathsf{G},\, u \in U.$$

For our purposes we shall assume that $U = \mathbb{R}^\ell$.

One of the first natural questions to ask in the study of control systems is the following: given any initial state of the system, does there exist an admissible control taking the system to any final state? This is known as the *controllability problem.* Using the tools of differential geometry and Lie theory, a number of powerful results have been developed to answer this question (see, e.g., [54], [37]). The next natural question is then, given a controllable control system, can we find an admissible control which transforms the system to a given final state in some optimal way? This is known in the literature as the *optimal control problem.* Such problems involve considering some (practical) cost function associated to a control system which we want minimize, given some appropriate boundary data. More specifically, in this thesis, we will be considering *invariant control problems* of the form

$$\begin{aligned} &\dot{g}(t) = g(t)\,\Xi(\mathbf{1}, u(t)), \quad g(\cdot) : [0,T] \to \mathsf{G},\; u(\cdot) : [0,T] \to \mathbb{R}^\ell \\ &g(0) = g_0, \quad g(T) = g_1, \quad g_0, g_1 \in \mathsf{G}, \quad T > 0 \text{ fixed} \\ &\mathcal{J} = \int_0^T (u(t) - \mu)^\top Q (u(t) - \mu)dt \to \min, \quad \mu \in \mathbb{R}^\ell. \end{aligned}$$

on the matrix Lie groups $\mathsf{G} = \mathsf{SO}(3)$ and $\mathsf{G} = \mathsf{SO}(4)$. Here $Q$ is a positive definite $\ell \times \ell$ matrix. In addition, we will assume that the underlying control system is affine in controls, i.e., we will be concerned with left-invariant control affine systems. There has been much research devoted in the last several decades to such types of optimal control problems, particularly on lower-dimensional Lie groups; see e.g., [33, 53]. Numerous applications can be modelled in this fashion and include problems such as Euler's elastic problem [33] [53], the motion of a rigid body [19], and the sub-Riemannian length-minimization problem [45] [54].

The introduction of appropriate equivalence relations can drastically simplify the investigation of invariant control systems on Lie groups (see, e.g., [39]). In this thesis we present a classification of left-invariant control affine system on the Lie groups $\mathsf{SO}(3)$ and $\mathsf{SO}(4)$ under *detached feedback equivalence* (or the closely related concept of *$\mathfrak{L}$-equivalence)*. Roughly, two systems are detached feedback equivalent if there exists a diffeomorphism mapping trajectories to trajectories, and an affine isomorphism mapping the corresponding controls to each other. Detached feedback equivalence turns out to be the most general feedback equivalence for which the state component (i.e., the diffeomorphism between state spaces) preserves left-invariant vector fields [20]. This motivates it as the right choice of equivalence relation as we are interested in left-invariant control systems. This approach has been used, for example, in the investigation of control systems on the Euclidean group $\mathsf{SE}(2)$ (see, [3], [7]). A full classification, under detached feedback equivalence, of all (real) three-dimensional Lie groups has been completed in [23], [24], [25].

After obtaining a classification of control systems, another natural question arises: can we find an appropriate equivalence relation under which to classify the associated optimal control problems? That is, can we find an appropriate way to decide when two optimal control problems are essentially the same and given this relation, can we practically classify classes of these problems? The answer to this question is yes, at least for low-dimensional Lie groups. The notion of *cost-extended equivalence* was introduced by Biggs and Remsing in [26] to answer just this question. It turns out that cost equivalence is also closely related to the detached feedback equivalence of control systems. In this thesis, we classify all left-invariant control affine problems on the Lie group $\mathsf{SO}(3)$ under cost equivalence.

Invariant optimal control problems on Lie groups can be lifted to the cotangent bundle, via the Pontryagin Maximum Principle, and then reduced to Hamilton-Poisson systems on the dual spaces of Lie algebras, equipped with the Lie-Poisson structure. The problem of determining the optimal controls for a large class of such optimal control problems on Lie groups has been shown to reduce to the study of the integral curves of Hamilton-Poisson systems, see, [21], [26]. Hamilton-Poisson systems have been considered by several authors (see, e.g., [5, 16, 33, 38, 43, 60, 59]), most notably in the context of invariant optimal control and geometric mechanics. The study of such systems has received increased attention in recent years. For instance, (Lyapunov) stability as well as (numerical and analytical) integration for systems on $\mathfrak{se}(1,1)^*$, $\mathfrak{se}(2)^*$, and $\mathfrak{so}(3)^*$ were treated in [13], [5], [6], [11]. A natural approach is again to try and classify these classes of Hamilton-Poisson systems under some appropriate equivalence relation. In order to achieve this, we use the notion of *affine equivalence*. Roughly, two Hamilton-Poisson systems are affinely equivalent if there exists an affine isomorphism such that their associated Hamiltonian vector fields are compatible. The notion of affine equivalence has also been shown to be closely related with that of cost equivalence (see [26]). In this thesis we are interested in determining the (Lyapunov) stability nature of the equilibria of Hamilton-Poisson systems. Energy-Casimir methods (see appendix C) are used to prove stability. On the other hand, instability usually follows from spectral instability; however, a direct approach is required in some cases. We are also interested in obtaining (analytic) expressions for the integral curves of the associated Hamiltonian vector fields. In general, this is done as follows. The equations of motion are reduced (using the constants of motion) to a single separable differential equation, which is then transformed into a standard form (see appendix D.2). An appropriate elliptic integral is then used to obtain (after some manipulation) an explicit expression for the integral curve in terms of (basic) Jacobi elliptic functions.

Given a left-invariant control affine problem the optimal controls are affinely related to the integral curves of the associated quadratic Hamilton-Poisson system. Thus, obtaining the integral curves for the Hamilton-Poisson systems allows us to determine the optimal controls for a given optimal control problem. What remains is to find the explicit expressions of the trajectories on the group and its associated homogeneous space, given the optimal controls obtained. There seems to be no standard technique for achieving this on an arbitrary Lie group, and different approaches have been employed for various Lie groups. In this thesis we are concerned with optimal control problems on the orthogonal groups $\mathsf{SO}(3)$ and $\mathsf{SO}(4)$. Techniques for obtaining the trajectories on the groups (and their homogeneous spaces) have been considered in [15, 33, 34, 36].

In recent years, advances in computer algebra software have provided tools for solving problems that historically were too computationally difficult. Throughout this thesis, MATHEMATICA is used extensively to facilitate calculations as well as for plotting figures. In appropriate sections and/or theorems, propositions, etc., we will include the path and file name of the associated MATHEMATICA file. For example

Thesis Mathematica\SO(3)\Affine equivalence\AequivVerification.nb

These files will be included with the thesis separately. Remarks concerning the use of Mathematica will appear as follows:

MATHEMATICA. *Throughout the thesis, when we use phrases such as "simple to show"/ "straightforward"/"easy to see (verify)" etc. the result or claim will either be obvious or will have been verified using Mathematica. In the latter case, we will make it explicitly clear which file the calculations are contained in.*

In summary, the aim of this thesis is to contribute to the study of a class of optimal control problems on the orthogonal groups $\mathsf{SO}(3)$ and $\mathsf{SO}(4)$. In particular, we are concerned with the class of invariant control problems. For each such problem, the underlying control system is a left-invariant control affine system and the cost functional to be minimized is the integral of an affine quadratic function of the controls. Our primary contribution to this class of problems is the use of various equivalence relations to classify cost-extended systems (optimal control problems) and to classify quadratic Hamilton-Poisson systems on the dual space of the associated Lie algebra. The approach of first classifying classes of objects, under appropriate equivalence relations, allows us to drastically simplify the work required in the investigation of such optimal control problems. Our various classifications also allow us to obtain a greater overall understanding of the different possible qualitative behaviours of the systems under investigation. In chapter 6 we provide several illustrative examples of optimal control problems on $\mathsf{SO}(3)$ and $\mathsf{SO}(4)$. These examples show how the classifications we have obtained in this thesis can be used in the study of such optimal control problems. In particular, the examples we provide show how our classification of quadratic Hamilton-Poisson systems can be used to find the optimal controls associated to a large class of optimal control problems. We also provide an example to show how our classification of cost-extended systems can be used to obtain the extremal curves on the group $\mathsf{SO}(3)$ (associated to some optimal control problem) from the extremal curves on $\mathsf{SO}(3)$ of an equivalent optimal control problem.

## Overview

We now give an overview of each of the chapters in this thesis.

In chapter 2 we define and recall the necessary results that will be used in this thesis. In particular, we include the necessary theory of invariant control systems, optimal control problems, and Hamilton-Poisson systems. We also provide the definitions and results used for the various types of equivalences used in the paper, i.e., detached feedback, cost, and affine equivalence.

Chapter 3 is devoted to the various topics considered on the Lie group $\mathsf{SO}(3)$. More specifically, we provide a classification of all left-invariant optimal control affine problems on $\mathsf{SO}(3)$ under cost equivalence. (This is based on a classification of left-invariant control affine systems under detached feedback equivalence.) A classification of all quadratic Hamilton-Poisson systems on $\mathfrak{so}(3)^*_-$ under affine equivalence is done, and a list of normal forms obtained. The normal forms are divided into two types. Type I systems correspond to those whose equilibria are a union of lines or planes. On the other hand, systems of type II are those whose equilibria are not just unions of lines or planes. For type I systems, an investigation of the (Lyapunov) stability and (analytic) integration of each of the normal forms is done (in terms of Jacobi elliptic functions). (The well-known results for the homogeneous systems are included for the sake of completeness.) For type II systems, an analysis of the stability nature of the equilibria is included. (We were unable to obtain explicit expressions for the integral curves for the systems of this type due to computational complexities.) We then investigate the relationship between cost equivalent optimal control problems on $\mathsf{SO}(3)$ and quadratic Hamilton-Poisson systems on $\mathfrak{so}(3)^*$. An approach for how extremal trajectories on the group $\mathsf{SO}(3)$, and the homogeneous space $\mathbb{S}^2$, is provided.

Chapter 4 is dedicated to our investigation of the Lie group $\mathsf{SO}(4)$. In particular, we investigate the relationship between the Lie algebra $\mathfrak{so}(4)$ and the Lie algebra $\mathfrak{so}(3) \times \mathfrak{so}(3)$, as well as the corresponding group of Lie algebra automorphisms. A classification of all left-invariant control affine systems on $\mathsf{SO}(4)$, under $\mathfrak{L}$-equivalence, is obtained. A list of which of these system are full rank is also provided (and thus a classification under detached feedback equivalence). We then attempt to classify homogeneous quadratic Hamilton-Poisson systems on the Lie-Poisson space $\mathfrak{so}(4)^*_-$. We only manage to obtain a partial classification of these systems (due to certain computational difficulties). In particular, we determine the class of Hamilton-Poisson systems that are 'decomposable' as the product of two systems on $\mathfrak{so}(3)^*_-$.

In chapter 5 we investigate several classes of quadratic Hamilton-Poisson systems on $\mathfrak{so}(4)^*_-$ obtained in chapter 4. Specifically, we show how the integral curves of a Hamilton-Poisson system $\mathfrak{so}(4)^*_-$, which decomposes as the product of two systems on $\mathfrak{so}(3)^*_-$, can be found from the integral curves of each subsystem on $\mathfrak{so}(3)^*_-$. For such systems on $\mathfrak{so}(4)^*_-$, we also show how the stability nature of the equilibria is related to that of each subsystem on $\mathfrak{so}(3)^*_-$. We also investigate the simplest class of systems which are not directly decomposable as a product of two systems on $\mathfrak{so}(3)^*_-$. For these systems we attempt to investigate the stability nature of the equilibrium states and for one case obtain the integral curves of the associated Hamiltonian vector field.

Finally, in chapter 6 provide several illustrative examples of optimal control problems on the group $\mathsf{SO}(3)$. Our main goal in this chapter is to obtain explicit expressions for the trajectories on the group and on the associated homogeneous space $\mathbb{S}^2$, corresponding to each of these optimal control problems. More specifically, we consider some selected cases obtained from our classification of cost-extended systems in chapter 3. We also consider an optimal control problem that is associated to a cost-extended system which is not a representative of our classification. For this system, we solve for the trajectories on the group independently, as well as by transforming the results obtained from an equivalent system (which is a representative of our classification). We then discuss the difference between these two approaches. The specific examples considered are chosen to show how our various classifications can be used in the investigation of optimal control problems on $\mathsf{SO}(3)$. In addition, we investigate a four-input optimal control problem on the group $\mathsf{SO}(4)$. The quadratic Hamilton-Poisson system associated to this optimal control problem is equivalent to a Hamilton-Poisson system that is decomposable as the product of two systems on $\mathfrak{so}(3)^*_-$. The investigation of this optimal control problem shows how we can use the solutions of optimal control problems on $\mathsf{SO}(3)$ to obtain the solutions on $\mathsf{SO}(4)$.

## Original Contributions

To the best of our knowledge, the following contributions in this thesis are original.

**Chapter 3.** The classification (under cost equivalence) of all full-rank cost-extended systems on $\mathsf{SO}(3)$; propositions 3.2.2, 3.2.4, 3.2.6, and 3.2.7. The classification (under affine equivalence) of all quadratic Hamilton-Poisson systems on $\mathfrak{so}(3)^*_-$; theorem 3.3.1. The investigation of the stability nature of the equilibria for the inhomogeneous quadratic Hamilton-Poisson systems on $\mathfrak{so}(3)^*_-$; theorems 3.3.8, 3.3.10, 3.3.21, 3.3.31, 3.3.32, 3.3.34, and 3.3.37. The integration of the inhomogeneous quadratic Hamilton-Poisson systems on $\mathfrak{so}(3)^*_-$ of type I; theorems 3.3.9, 3.3.12 through 3.3.20, and 3.3.23 through 3.3.29. The explicit relations of the cost-extended systems on $\mathsf{SO}(3)$ to the quadratic Hamilton-Poisson systems obtained in 3.3.1; propositions 3.4.1 through 3.4.6.

**Chapter 4.** The classification (under $\mathfrak{L}$-equivalence) of all control affine systems on $\mathsf{SO}(4)$; theorems 4.2.2, 4.2.7, 4.2.12, 4.2.15, 4.2.17 , 4.2.19, 4.2.21, and 4.2.23, and corollaries 4.2.9 and 4.2.4. The controllability of each of these representatives, and thus a classification under detached-feedback equivalence of all control affine systems on $\mathsf{SO}(4)$. The partial classification of homogeneous quadratic Hamilton-Poisson systems on the Lie-Poisson space $\mathfrak{so}(4)^*_-$; propositions 4.3.2, 4.3.3, and 4.3.4.

**Chapter 5.** A useful result on the stability nature of equilibria of Hamilton-Poisson systems on $\mathfrak{so}(4)^*_-$ which are decomposable as Hamilton-Poisson systems on $\mathfrak{so}(3)^*_-$; theorem 5.1.3. The investigation of the stability nature of the equilibria for some subclasses of indecomposable Hamilton-Poisson systems on $\mathfrak{so}(4)^*_-$; theorems 5.2.2, 5.2.3, and 5.2.5. The integration of a particular indecomposable Hamilton-Poisson system on $\mathfrak{so}(4)^*_-$; theorem 5.2.4.

**Chapter 6.** The explicit expressions for the trajectories on the groups $\mathsf{SO}(3)$ and $\mathsf{SO}(4)$, as well as the trajectories on the base space $\mathbb{S}^2$. Each sub-section of this chapter contains a specific example of an optimal control problem and the style of this chapter is quite different to the rest of the thesis. In particular, the results in this chapter are not stated as propositions or theorems. Therefore, we make more explicit our original contributions in this chapter itself. We also discuss similar problems investigated by other authors in the conclusion.

## Notation

We outline the notational conventions used in this thesis. Lie groups are denoted using uppercase Sans Serif letters (e.g., $\mathsf{G}$). Lie algebras are denoted using lowercase Fraktur letters (e.g., $\mathfrak{g}$). We shall also use the following notation:

| | |
|---|---|
| $\mathbf{1}$ | identity element of a Lie group. |
| $\rtimes$ | semidirect product of Lie groups (normal subgroup on the left). |
| $C^\infty(\mathsf{M})$ | the set of (smooth) real-valued functions on a smooth manifold $\mathsf{M}$. |
| $\mathrm{Vec}(\mathsf{M})$ | the set of (smooth) vector fields on a smooth manifold $\mathsf{M}$. |
| $\mathsf{GL}(\mathsf{V})$ | group of invertible linear transformations of a vector space $\mathsf{V}$. |
| $\mathfrak{gl}(\mathsf{V})$ | Lie algebra of $\mathsf{GL}(\mathsf{V})$. |
| $\langle S \rangle$ | linear span of a subset $S \subseteq \mathfrak{g}$ or of elements $B_1, \ldots, B_\ell \in \mathfrak{g}$. |
| $\mathrm{Lie}\,(S)$ | Lie algebra generated by $S \subseteq \mathfrak{g}$, *i.e.*, the smallest Lie subalgebra containing $S$. |
| $\langle \cdot, \cdot \rangle$ | natural pairing $\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$, $(p, X) \mapsto p(X)$ between $\mathfrak{g}^*$ and $\mathfrak{g}$. |
| $\mathbf{d}F$ | linearisation of $F \in C^\infty(\mathsf{M})$; the linearisation at $x$ is denoted $\mathbf{d}F(x)$. |
| $T\phi$ | tangent map (differential) of a smooth map $\phi$ between manifolds; the tangent map at $x$ is denoted $T_x\phi$. |
| $X[F]$ | directional derivative of $F \in C^\infty(\mathsf{M})$ in the direction of $X$. |
| $\mathsf{Aut}(\mathsf{G})$ | group of Lie group automorphisms of $\mathsf{G}$. |
| $\mathsf{Aut}(\mathfrak{g})$ | group of Lie algebra automorphisms of $\mathfrak{g}$. |
| $\vec{H}$ | Hamiltonian vector field associated to a smooth function $H$. |
| $\mathsf{Aff}\,(\mathbb{R}^\ell)$ | group of affine isomorphisms of $\mathbb{R}^\ell$. |
| $L_g$ | left translation by $g$, i.e., $L_g : \mathsf{G} \to \mathsf{G}$, $L_g : h \mapsto gh$. |
| $\mathrm{sgn}(x)$ | sign of $x \in \mathbb{R}$. |

# Chapter 2

# Invariant control systems

## 2.1 Left-invariant control affine systems

Let $\mathsf{G}$ be a (real, finite-dimensional) matrix Lie group. A left-invariant control system $\Sigma = (\mathsf{G}, \Xi)$ is a (smooth) control system evolving on $\mathsf{G}$, whose dynamics $\Xi : \mathsf{G} \times U \to T\mathsf{G}$ are invariant under left translations. For the purposes of this work we shall assume $U = \mathbb{R}^\ell$. In particular, we are interested in those systems which are affine in controls, i.e., the parametrization map $\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}$ is an affine embedding. Thus we have the following definition.

2.1.1 Definition. A **left-invariant control affine system** is a pair $\Sigma = (\mathsf{G}, \Xi)$ such that:

- The *state space* $\mathsf{G}$ is a (real, finite dimensional) connected matrix Lie group with Lie algebra $\mathfrak{g}$.
- The *dynamics* $\Xi : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}$ are left-invariant, i.e., for any $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$,

$$(g, u) \mapsto \Xi(g, u) = g\,\Xi(\mathbf{1}, u).$$

- The *parametrization map* is an affine embedding, i.e.,

$$\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}, \quad u \mapsto A + u_1 B_1 + \cdots + u_\ell B_\ell$$

  where the set $\{B_i\}_{i=\overline{1,\ell}}$ is linearly independent.

The "product" $g\,\Xi(\mathbf{1}, u)$ denotes the left translation $T_\mathbf{1} L_g \cdot \Xi(\mathbf{1}, u)$ of $\Xi(\mathbf{1}, u) \in \mathfrak{g}$ by $g$. The *trace* $\Gamma = \operatorname{im} \Xi(\mathbf{1}, \cdot) = A + \langle B_1, \ldots, B_\ell \rangle = A + \Gamma^0$ is affine subspace of the Lie algebra $\mathfrak{g}$. A system $\Sigma$ is called *homogeneous* if $A \in \Gamma^0$, and *inhomogeneous* otherwise. $\Sigma$ is said to have *full rank* if the Lie algebra generated by its trace coincides with $\mathfrak{g}$, i.e., $\operatorname{Lie}(\Gamma) = \mathfrak{g}$. ($\operatorname{Lie}(\Gamma)$ is the smallest Lie subalgebra containing $\Gamma$.)

2.1.2 Remark. Henceforth when we refer to a *control system* we will implicitly assume we are talking about a left-invariant control affine system, unless explicitly specified otherwise.

*Admissible controls* are piecewise continuous maps $u(\cdot) : [0, T] \to \mathbb{R}^\ell$, $0 < T < \infty$. A *trajectory*, corresponding to an admissible control $u(\cdot)$, is an absolutely continuous curve $g(\cdot) : [0, T] \to \mathsf{G}$

which satisfies the differential equation $\dot{g}(t) = g(t)\,\Xi(\mathbf{1}, u(t))$ for almost every $t \in [0, T]$. The pair $(g(\cdot), u(\cdot))$ will be referred to as a *trajectory-control pair*. A system $\Sigma$ is said to be *controllable* if for any pair of points $g_0, g_1 \in \mathsf{G}$, there exists a $T \geq 0$ and a trajectory $g(\cdot) : [0, T] \to \mathsf{G}$ such that $g(0) = g_0$ and $g(T) = g_1$. Two necessary conditions for controllability are that the group $\mathsf{G}$ be connected and that the system $\Sigma$ have full rank (see e.g., [52]). If $\Sigma$ is homogeneous or $\mathsf{G}$ is compact, then these conditions are also sufficient for controllability (see, e.g., [52]).

2.1.3 REMARK. The affine group is given by

$$\mathsf{Aff}(\mathbb{R}^\ell) = \left\{ \begin{bmatrix} 1 & 0 \\ v & N \end{bmatrix} : N \in \mathsf{GL}(\ell, \mathbb{R}), \quad v \in \mathbb{R}^\ell \right\}.$$

This group is isomorphic to the group of affine transformations of $\mathbb{R}^\ell$ given by

$$u \mapsto Nu + v, \quad \det(N) \neq 0, \quad v \in \mathbb{R}^\ell.$$

Let $\Sigma = (\mathsf{G}, \Xi)$, with $\dim(\mathsf{G}) = n$, be a control system where $\Xi(\mathbf{1}, u) = A + u_1 B_1 + \cdots + u_\ell B_\ell$, $1 \leq \ell \leq n$. Given a basis $E_1, \ldots, E_n$ of the Lie algebra $\mathfrak{g}$ such a control system is specified by $\Xi(\mathbf{1}, u) = \sum_{i=1}^n a^i E_i + u_1 \sum_{i=1}^n b_1^i E_i + \cdots + u_\ell \sum_{i=1}^n b_\ell^i E_i$. When convenient such a system which will be represented in matrix form as

$$\Sigma : \begin{bmatrix} a^1 & b_1^1 & \cdots & b_\ell^1 \\ \vdots & \vdots & & \vdots \\ a^n & b_1^n & \cdots & b_\ell^n \end{bmatrix}.$$

For a system written in matrix form $\psi \cdot \Xi(\mathbf{1}, u)$ (for a linear map $\psi$) is then just matrix multiplication.

Applying an affine transformation $\varphi : u \mapsto Nu + v$ is then equivalent to multiplying the matrix form on the right by the corresponding element $K \in \mathsf{Aff}(\mathbb{R}^\ell)$ , i.e.,

$$\Xi(\mathbf{1}, \varphi(u)) = \begin{bmatrix} a^1 & b_1^1 & \cdots & b_\ell^1 \\ \vdots & \vdots & & \vdots \\ a^n & b_1^n & \cdots & b_\ell^n \end{bmatrix} K, \qquad K = \begin{bmatrix} 1 & 0 \\ v & N \end{bmatrix}.$$

Here $K$ corresponds to a *reparametrization* $\Xi(\mathbf{1}, Nu + v)$ of the system $\Sigma$. For *drift free systems*, i.e., where $\Xi(\mathbf{1}, 0) = \mathbf{0}$, a reparametrization of $\Sigma$ is given by

$$\begin{bmatrix} b_1^1 & \cdots & b_\ell^1 \\ \vdots & & \vdots \\ b_1^n & \cdots & b_\ell^n \end{bmatrix} K, \quad K \in \mathsf{GL}(\ell, \mathbb{R}).$$

Here $K$ corresponds to a reparametrization $\Xi(\mathbf{1}, Ku)$ of the system $\Sigma$.

### The optimal control problem

An (invariant) optimal control problem is defined by the specification of (i) a left-invariant control affine system $\Sigma$, (ii) a cost function (or Lagrangian) $\chi : \mathbb{R}^\ell \to \mathbb{R}$, and (iii) boundary data, consisting of an initial state $g_0 \in \mathsf{G}$, a terminal state $g_1 \in \mathsf{G}$ and a (fixed) terminal time $T > 0$. The problem is then to minimize the cost functional $\mathcal{J}(u(\cdot)) = \int_0^T \chi(u(t))dt$ over all admissible trajectory-control pairs $(g(\cdot), u(\cdot))$ such that $g(0) = g_0$ and $g(T) = g_1$.

2.1.4 Definition. Let $\Sigma = (\mathsf{G}, \Xi)$ be a left-invariant control affine system. A **left-invariant control affine problem** (or just an **optimal control problem**) is given by

$$\begin{aligned} &\dot{g}(t) = g(t)\,\Xi(\mathbf{1}, u(t)), \quad g(\cdot) : [0,T] \to \mathsf{G},\ u(\cdot) : [0,T] \to \mathbb{R}^\ell \\ &g(0) = g_0, \quad g(T) = g_1, \quad g_0, g_1 \in \mathsf{G}, \quad T > 0 \text{ fixed} \\ &\mathcal{J}(u(\cdot)) = \int_0^T \chi(u(t))dt = \int_0^T (u(t) - \mu)^\top Q(u(t) - \mu)dt \to \min, \quad \mu \in \mathbb{R}^\ell. \end{aligned} \tag{2.1}$$

Here $Q$ is a positive definite $\ell \times \ell$ matrix.

To each such problem we associate a **cost-extended system** $(\Sigma, \chi)$. Here $\Sigma$ is the control system and the cost function $\chi : \mathbb{R}^\ell \to \mathbb{R}$ has the form given in (2.1). Each cost-extended system corresponds to a family of invariant optimal control problems; by specification of the boundary data $(g_0, g_1, T)$, the associated problem is uniquely defined.

### The Pontryagin Maximum Principle

The Pontryagin Maximum Principle provides necessary conditions for trajectory-control pairs to be optimal and is most naturally expressed in the language of the cotangent bundle $T^*\mathsf{G}$ of $\mathsf{G}$. The cotangent bundle can be trivialized (from the left) such that $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$. To an optimal control problem (2.1), we associate a family of Hamiltonian functions on $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$:

$$\begin{aligned} H_u^\lambda(\xi) &= \lambda\chi(u) + \xi(\Xi_u(g)) \\ &= \lambda\chi(u) + p(\Xi_u(\mathbf{1})), \quad \xi = (g, p) \in T^*\mathsf{G}. \end{aligned}$$

Maximum Principle ([9]) *Suppose the trajectory-control pair* $(\bar{g}(\cdot), \bar{u}(\cdot))$, *defined over the interval* $[0, T]$, *is a solution for the optimal control problem* (2.1). *Then, there exists a curve* $\xi(\cdot) : [0, T] \to T^*\mathsf{G}$ *with* $\xi(t) \in T^*_{\bar{g}(t)}\mathsf{G}$, $t \in [0, T]$, *and a real number* $\lambda \leq 0$ *such that the following conditions hold for almost every* $t \in [0, T]$*:*

$$(\lambda, \xi(t)) \not\equiv (0, 0) \tag{2.2}$$

$$\dot{\xi}(t) = \vec{H}^\lambda_{\bar{u}(t)}(\xi(t)) \tag{2.3}$$

$$H^\lambda_{\bar{u}(t)}(\xi(t)) = \max_u H^\lambda_u(\xi(t)) = \textit{constant.} \tag{2.4}$$

An *optimal trajectory* $\bar{g}(\cdot) : [0, T] \to \mathsf{G}$ is the projection of an integral curve $\xi(\cdot)$ of the (time-varying) Hamiltonian vector field $\vec{H}^\lambda_{\bar{u}(t)}$ for all $t \in [0, T]$. A pair $(\xi(\cdot), u(\cdot))$ defined on $[0, T]$ is said to be an *extremal pair* if $\xi(\cdot)$ satisfies the conditions (2.2),(2.3), and (2.4). The projection $\xi(\cdot)$ of an extremal pair is called an *extremal*. An extremal curve is called *normal* if $\lambda < 0$ and *abnormal* if $\lambda = 0$. (In this work, we shall only be concerned with normal extremals.)

For an optimal control problem suppose the maximality condition (2.4) eliminates the parameter $u$ from the family of Hamiltonians $(H_u)_{u\in\mathbb{R}^\ell}$. The result is a $\mathsf{G}$-invariant Hamiltonian function $H$ defined on $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$. The fact that the cotangent bundle can be left-trivialized in this way allows for a reduction of the Poisson structure on $T^*\mathsf{G}$ to a Poisson structure (the (minus) Lie-Poisson structure) on the dual space $\mathfrak{g}^*$ (see, [40] for details). Accordingly, we obtain a Hamiltonian function $H$ on $\mathfrak{g}^*$. The extremal controls are affinely related to the integral curves of $\vec{H}$ (see theorem 2.1.5 below). As such, the investigation of the extremal controls is essentially reduced to the study of the Hamilton-Poisson system $(\mathfrak{g}^*_-, H)$.

Given a control system $\Sigma = (\mathsf{G}, \Xi)$, let $\mathbf{B}$ be the $n \times \ell$ matrix formed by taking the coordinate vector of $B_i$ (with respect to some given basis) for the $i^{th}$ column of $\mathbf{B}$; thus, in coordinates $\Xi(\mathbf{1}, u) = A + \mathbf{B}u$.

2.1.5 Theorem. ([26]) *Any trajectory-control pair* $(g(\cdot), u(\cdot))$ *of* $(\Sigma, \chi)$ *is given by* $\dot{g}(t) = \Xi(g(t), u(t))$, $u(t) = Q^{-1}\boldsymbol{B}^\top p(t)^\top + \mu$. *Here* $p(\cdot) : [0, T] \to \mathfrak{g}^*$ *is an integral curve of the Hamilton-Poisson system on* $\mathfrak{g}^*_-$ *specified by*

$$H(p) = p(A + \boldsymbol{B}\mu) + \frac{1}{2} p\boldsymbol{B}Q^{-1}\boldsymbol{B}^\top p^\top. \tag{2.5}$$

*Here* $p$ *is written as a row vector (in terms of the dual basis of* $\mathfrak{g}^*$*).*

The Hamiltonian function (2.5) on $\mathfrak{g}^*_-$ is called the *reduced Hamiltonian*. Since $Q$ is positive definite and $\boldsymbol{B}$ does not have full rank in general, it follows that $\boldsymbol{B}Q^{-1}\boldsymbol{B}^\top$ is positive semi-definite. Consequently, $H$ is of the form

$$H(p) = H_{A,\mathcal{Q}}(p) = p(A) + \mathcal{Q}(p)$$

where $A \in \mathfrak{g}$ and $\mathcal{Q}$ is a positive semidefinite quadratic form on $\mathfrak{g}^*$. The system $(\mathfrak{g}^*_-, H_{A,\mathcal{Q}})$ will be called **homogeneous** if $A = 0$, and **inhomogeneous** otherwise. Let $L_A(p) = p(A)$. Then, we have that $\vec{H}_{\mathcal{Q}}(p) = \operatorname{ad}^*_{\mathbf{d}H_{\mathcal{Q}}(p)}(p)$ and $\vec{L}_A(p) = \operatorname{ad}^*_{\mathbf{d}L_A(p)}(p) = \operatorname{ad}^*_A(p)$. As $X \mapsto \operatorname{ad}^*_X(p)$ is a linear map on $\mathfrak{g}$ for any $p \in \mathfrak{g}^*$, it follows that

$$\vec{H}_{A,\mathcal{Q}}(p) = \vec{L}_A(p) + \vec{H}_{\mathcal{Q}}(p).$$

That is, $\vec{H}_{A,\mathcal{Q}}$ decomposes as the sum $\vec{L}_A + \vec{H}_{\mathcal{Q}}$.

## 2.2 Equivalences

The idea of classification (in order to study the properties of large classes of objects simultaneously) is a central theme in mathematics. In this section we introduce the various equivalence relations we use in classifying control systems, cost-extended systems, and their associated Hamilton-Poisson systems. We also explain how each of these equivalences are related to one another.

### 2.2.1 State-space equivalence

In order to understand the (local) geometry of (nonlinear) control systems, it is very useful to introduce natural equivalence relations. The most natural equivalence relation for (smooth, nonlinear) control systems is equivalence up to coordinate change in the state space. Although we do not explicitly classify any control systems in this thesis under state space equivalence, the notion helps to make clear how detached feedback equivalence of control systems (defined in the next subsection) fits into the picture of our analysis of control systems. Two control systems are state space equivalent if they are related by a diffeomorphism, in which case their trajectories, corresponding to the same controls, are also related by that diffeomorphism. State space equivalence is well understood (cf. [20, 51]).

Let $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}', \Xi')$ be two (left-invariant) control systems.

2.2.1 Definition. $\Sigma$ and $\Sigma'$ are called (locally) **state space equivalent** at $g \in \mathsf{G}$ and $g' \in \mathsf{G}'$ if there exist open neighbourhoods $N$ and $N'$ of $g$ and $g'$, respectively, and a (local) diffeomorphism $\phi : N \to N'$, with $\phi(g) = g'$, such that

$$T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g),u) \quad \text{for all} \quad g \in N, \quad u \in \mathbb{R}^\ell.$$

$\Sigma$ and $\Sigma'$ are called *globally state space equivalent* if this happens globally (i.e., $N = \mathsf{G}$ and $N' = \mathsf{G}'$).

This equivalence can also be viewed as the commutativity of the diagram

$$\begin{tikzcd}
N \times \mathbb{R}^\ell \arrow[r, "\phi \times \mathrm{id}_{\mathbb{R}^\ell}"] \arrow[d, "\Xi"'] & N' \times \mathbb{R}^\ell \arrow[d, "\Xi'"] \\
TN \arrow[r, "T\phi"'] & TN'
\end{tikzcd}$$

This equivalence relation is very strong. Consequently, there are so many equivalence classes that any general classification appears to be very difficult, if not impossible. However, there is a chance of some reasonable classification in low dimensions (see, e.g., [2]).

2.2.2 Proposition. ([20]) *$\Sigma$ and $\Sigma'$ are state space equivalent if and only if there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, u)$ for all $u \in \mathbb{R}^\ell$.*

### 2.2.2 Detached feedback equivalence

Feedback equivalence is such that we now allow a transformation of the controls (the feedback component) as well as the state space (the state component). (State space equivalence is a specialisation of feedback equivalence.) Here the transformation of the controls is done in a way that is dependent on the state: thus feeding the system back into itself (see, e.g., [51]). This level of equivalence, however, turns out to be too general when considering the class of left-invariant control systems as we want our transformations to be compatible with the Lie group structure (i.e., left-invariance). More precisely, the push-forward of a left-invariant vector field (by a diffeomorphism) is left-invariant if and only if the feedback component is independent of the state (see [20]). In this case the feedback component is said to be $\mathsf{G}$-invariant.

Thus, for our purposes, we wish to be able to transform the controls in a way that is independent of the state. This type of equivalence is called detached feedback equivalence.

2.2.3 Definition. $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}', \Xi')$ are called **detached feedback equivalent** (or *DF*-**equivalent**) if there exist diffeomorphisms $\phi : \mathsf{G} \to \mathsf{G}'$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $T_g\phi \cdot \Xi\,(g,u) = \Xi'\,(\phi(g), \varphi(u))$ for every $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$.

2.2.4 Remark. From this definition one can see that two control systems $\Sigma$ and $\Sigma'$ are detached feedback equivalent if they are state space equivalent under some reparametrization of the controls.

2.2.5 Proposition. *DF-equivalence is an equivalence relation.*

PROOF. Let $\Sigma = (\mathsf{G}, \Xi)$, $\Sigma' = (\mathsf{G}', \Xi')$, and $\Sigma'' = (\mathsf{G}'', \Xi'')$ be left-invariant control affine systems.

Let $\phi = \mathrm{id}_{\mathsf{G}}$ and $\varphi = \mathrm{id}_{\mathbb{R}^\ell}$. Then $T_g\phi = \mathrm{id}_{T_g\mathsf{G}}$ for every $g \in \mathsf{G}$. Hence $T_g\phi \cdot \Xi(g,u) = \Xi(g,u) = \Xi(\phi(g), \varphi(u))$. Thus, $\Sigma$ is $DF$-equivalent to itself, i.e., $DF$-equivalence is reflexive.

Suppose $\Sigma$ is $DF$-equivalent to $\Sigma'$. Then there exist diffeomorphisms $\phi : \mathsf{G} \to \mathsf{G}'$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g), \varphi(u))$ for every $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$. Now, as $(T_g\phi)^{-1} = T_{\phi(g)}\phi^{-1}$, we have that $T_{\phi(g)}\phi^{-1} \cdot \Xi'(\phi(g), \varphi(u)) = \Xi(\phi^{-1}(\phi(g)), \varphi^{-1}(\varphi(u))) = \Xi(g,u)$ for every $\phi(g) \in \mathsf{G}$ and $\varphi(u) \in \mathbb{R}^\ell$. That is, $\Sigma'$ is $DF$-equivalent to $\Sigma$, i.e., $DF$-equivalence is symmetric.

Finally, suppose $\Sigma$ is $DF$-equivalent to $\Sigma'$ and $\Sigma'$ is $DF$-equivalent to $\Sigma''$. That is, there exist diffeomorphisms $\phi_1 : \mathsf{G} \to \mathsf{G}'$, $\phi_2 : \mathsf{G}' \to \mathsf{G}''$, and $\varphi_1, \varphi_2 : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that

$$T_g\phi_1 \cdot \Xi(g,u) = \Xi'(\phi_1(g), \varphi_1(u)) \quad \text{and} \quad T_h\phi_2 \cdot \Xi'(h,u) = \Xi''(\phi_2(h), \varphi_2(u))$$

for every $g \in \mathsf{G}$, $h \in \mathsf{G}'$, and $u \in \mathbb{R}^\ell$. Let $\phi = \phi_2 \circ \phi_1$ and $\varphi = \varphi_2 \circ \varphi_1$. Then, for every $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$, we have

$$\begin{aligned} T_g\phi \cdot \Xi(g,u) &= T_{\phi_1(g)}\phi_2 \cdot T_g\phi_1 \cdot \Xi(g,u) \\ &= T_{\phi_1(g)}\phi_2 \cdot \Xi'(\phi_1(g), \varphi_1(u)) \\ &= \Xi''(\phi_2(\phi_1(g)), \varphi_2(\varphi_1(u))) \\ &= \Xi''(\phi(g), \varphi(u)). \end{aligned}$$

Thus, $\Sigma$ is $DF$-equivalent to $\Sigma''$, i.e., $DF$-equivalence is transitive. □

The following two results motivate why $DF$-equivalence is the natural choice for the equivalence relation to be used in the investigation of left-invariant control systems.

2.2.6 PROPOSITION. *If $\Sigma$ is $DF$-equivalent to $\Sigma'$, then the trajectory-control pairs of $\Sigma$ and $\Sigma'$ are in a one-to-one correspondence.*

PROOF. Let $(g(\cdot), u(\cdot))$ be a trajectory-control pair of $\Sigma$. Since $\Sigma$ and $\Sigma'$ are $DF$-equivalent there exist diffeomorphisms $\phi : \mathsf{G} \to \mathsf{G}'$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g), \varphi(u))$ for every $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$. We now show that $(\phi(g(\cdot)), \varphi(u(\cdot)))$ is the unique trajectory-control pair of $\Sigma'$ corresponding to $(g(\cdot), u(\cdot))$. Indeed, for almost every $t$ it follows that

$$\begin{aligned} \frac{d}{dt}\phi(g(t)) &= T_{g(t)}\phi \cdot \dot{g}(t) \\ &= T_{g(t)}\phi \cdot \Xi(g(t), u(t)) \\ &= \Xi'(\phi(g(t)), \varphi(u(t))). \end{aligned}$$

Thus, $(\phi(g(\cdot)), \varphi(u(\cdot)))$ is a trajectory-control pair of $\Sigma'$. Now, suppose $\phi(g_1(\cdot)) = \phi(g_2(\cdot))$ and $\varphi(u_1(\cdot)) = \varphi(u_2(\cdot))$, where $(g_1(\cdot), u_1(\cdot))$ and $(g_2(\cdot), u_2(\cdot))$ are trajectory control pairs of $\Sigma$. Applying $\phi^{-1}$ and $\varphi^{-1}$, we have $g_1(\cdot) = g_2(\cdot)$ and $u_1(\cdot) = u_2(\cdot)$. Hence, trajectory-control pairs are mapped injectively from $\Sigma$ to $\Sigma'$. Next let $(g'(\cdot), u'(\cdot))$ be a trajectory control pair of $\Sigma'$. Then $(\phi^{-1}(g'(\cdot)), \varphi^{-1}(u'(\cdot)))$ is the trajectory-control pair of $\Sigma$ that is mapped to $(g'(\cdot), u'(\cdot))$ by $\phi \times \varphi$. Thus, trajectory-control pairs are mapped surjectively. Hence, the trajectory-control pairs of $\Sigma$ and $\Sigma'$ are in a one-to-one correspondence.

□

2.2.7 Proposition. ([20]) *Suppose $\Sigma$ and $\Sigma'$ are $DF$-equivalent. $\Sigma$ is controllable if and only if $\Sigma'$ is controllable.*

The above result simply relies on the fact that attainable sets are preserved by $DF$-equivalence. The following results provide a useful way of characterising the concept of detached feedback equivalence at the level of the Lie algebra. Throughout the remainder of this section we shall assume that all systems are of full-rank.

2.2.8 Lemma. *$\Sigma$ and $\Sigma'$ are $DF$-equivalent if and only if there exist diffeomorphisms $\phi : \mathsf{G} \to \mathsf{G}'$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$, satisfying definition 2.2.3, such that (the feedback component) $\varphi$ is $\mathsf{G}$-invariant.*

This just follows from the definition of $DF$-equivalence.

2.2.9 Proposition. ([20]) *Let $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}', \Xi')$ be two systems and let $\phi : \mathsf{G} \to \mathsf{G}'$ be diffeomorphism such that $\phi(\mathbf{1}) = \mathbf{1}'$. There exist unique diffeomorphisms $\phi : \mathsf{G} \to \mathsf{G}'$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$, satisfying definition 2.2.3, such that (the feedback component) $\varphi$ is $\mathsf{G}$-invariant if and only if $\phi$ is a Lie group isomorphism such that $T_\mathbf{1}\phi \cdot \Gamma = \Gamma'$.*

2.2.10 Corollary. *Two systems $\Sigma$ and $\Sigma'$ are $DF$-equivalent if and only if there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $T_\mathbf{1}\phi \cdot \Gamma = \Gamma'$.*

Proof. Suppose $\Sigma$ and $\Sigma'$ are $DF$-equivalent. Thus there exist smooth maps $\phi : \mathsf{G} \to \mathsf{G}'$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g), \varphi(u))$ for every $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$. We may assume $\phi(\mathbf{1}) = \mathbf{1}'$. Indeed, suppose $\phi(\mathbf{1}) = h$. Then $(L_{h^{-1}} \circ \phi)(\mathbf{1}) = \mathbf{1}'$ and

$$\begin{aligned} T_g (L_{h^{-1}} \circ \phi) \cdot \Xi(g,u) &= T_{\phi(g)} L_{h^{-1}} \cdot T_g\phi \cdot \Xi(g,u) \\ &= T_{\phi(g)} L_{h^{-1}} \cdot \Xi'(\phi(g), \varphi(u)) \\ &= \Xi'(h^{-1}\phi(g), \varphi(u)) \\ &= \Xi'((L_{h^{-1}} \circ \phi)(g), \varphi(u)) \end{aligned}$$

That is, $\Sigma$ and $\Sigma'$ are also $DF$-equivalent with respect to $(L_{h^{-1}} \circ \phi)$ and $\varphi$. Hence, by replacing $\phi$ with $(L_{h^{-1}} \circ \phi)$ we can always arrange for $\phi$ to preserve the identity. Thus, by proposition 2.2.9 it follows that $\phi$ is a Lie group isomorphism such that $T_\mathbf{1}\phi \cdot \Gamma = \Gamma'$.

Conversely, suppose we have a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $T_\mathbf{1}\phi \cdot \Gamma = \Gamma'$. Then, again by proposition 2.2.9, it follows that the systems $\Sigma$ and $\Sigma'$ are $DF$-equivalent.

□

Assume the map $\mathbf{d} : \mathsf{Aut}(\mathsf{G}) \to \mathsf{Aut}(\mathfrak{g})$, $\phi \mapsto T_\mathbf{1}\phi$, is a bijection. (This holds, for example, when $\mathsf{G}$ is simply connected.) Then the following result follows immediately.

2.2.11 Corollary. *Two systems $\Sigma$ and $\Sigma'$ are $DF$-equivalent if and only if there exists a Lie algebra isomorphism $\psi : \mathfrak{g} \to \mathfrak{g}'$ such that $\psi \cdot \Gamma = \Gamma'$.*

Accordingly, the classification of systems (where $\mathbf{d}\mathsf{Aut}(\mathsf{G}) = \mathsf{Aut}(\mathfrak{g})$) under $DF$-equivalence reduces to the classification of affine subspaces $\Gamma$ of $\mathfrak{g}$ under Lie algebra automorphisms. In this work, we shall find it convenient to use the above characterisation as its own definition of equivalence, which we call $\mathfrak{L}$-equivalence. More precisely, 2.2.12 DEFINITION. Two (not necessarily full-rank) systems $\Sigma$ and $\Sigma'$ are **$\mathfrak{L}$-equivalent** if there exists a Lie algebra isomorphism $\psi : \mathfrak{g} \to \mathfrak{g}'$ such that $\psi \cdot \Gamma = \Gamma'$. In particular, if $\mathsf{G} = \mathsf{G}'$ and $\Gamma = \Gamma'$, then we say that $\Sigma'$ is a *reparametrization* of $\Sigma$.

Notice that if two full-rank systems are $\mathfrak{L}$-equivalent, then they are detached feedback equivalent. (The converse, however, does not hold.) Any two $\mathfrak{L}$-equivalent systems are either both controllable or neither is controllable whenever the full-rank condition is equivalent to controllability.

### 2.2.3 Cost-extended equivalence

Now that we have a way of talking about two control systems being equivalent, we want to introduce a way of talking about when two associated optimal control problems are equivalent. This theory was introduced and investigated by Biggs and Remsing in [26] and this section is based upon their work. Where proofs for statements are not given they can be found in [26].

2.2.13 DEFINITION. Two cost-extended systems $(\Sigma = (\mathsf{G}, \Xi), \chi)$ and $(\Sigma' = (\mathsf{G}', \Xi'), \chi')$ are **cost equivalent (or C-equivalent)** if there exists a mapping $\Phi = (\phi, \varphi) : (\Sigma, \chi) \to (\Sigma', \chi')$ such that

$$\Phi : \mathsf{G} \times \mathbb{R}^{\ell} \to \mathsf{G}' \times \mathbb{R}^{\ell'}, \quad (g, u) \mapsto (\phi(g), \varphi(u))$$

where the state component $\phi : \mathsf{G} \to \mathsf{G}'$ is a Lie group isomorphism and the feedback component $\varphi : \mathbb{R}^{\ell} \to \mathbb{R}^{\ell'}$ is an affine isomorphism (in particular, $\ell = \ell'$), such that

$$T_g\phi \cdot \Xi\,(g, u) = \Xi'\,(\phi(g), \varphi(u)) \quad \text{and} \quad (\chi' \circ \varphi)(u) = r\chi(u)$$

for every $g \in \mathsf{G}$ and $u \in \mathbb{R}^{\ell}$.

In other words, the following diagrams

$$\begin{tikzcd}
\mathsf{G} \times \mathbb{R}^{\ell} \arrow[r, "\Phi"] \arrow[d, "\Xi"'] & \mathsf{G}' \times \mathbb{R}^{\ell'} \arrow[d, "\Xi'"] \\
T\mathsf{G} \arrow[r, "T\phi"'] & T\mathsf{G}'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{R}^{\ell} \arrow[r, "\varphi"] \arrow[d, "\chi"'] & \mathbb{R}^{\ell'} \arrow[d, "\chi'"] \\
\mathbb{R} \arrow[r, "\delta_r"'] & \mathbb{R}
\end{tikzcd}$$

commute for some $r > 0$. (Here $\delta_r$ denotes the dilation by $r$.) For any $\Phi$, the constant $r > 0$ is uniquely determined.

2.2.14 PROPOSITION. *Cost equivalence is an equivalence relation.*

PROOF. Let $(\Sigma, \chi)$, $(\Sigma', \chi')$, and $(\Sigma'', \chi'')$ be cost extended systems.

Let $\phi = \mathrm{id}_{\mathsf{G}}$ and $\varphi = \mathrm{id}_{\mathbb{R}^{\ell}}$. Then $T_g\phi = \mathrm{id}_{T_g\mathsf{G}}$ for every $g \in \mathsf{G}$. Hence $T_g\phi \cdot \Xi(g, u) = \Xi(g, u) = \Xi(\phi(g), \varphi(u))$ and $(\chi \circ \mathrm{id}_{\mathbb{R}^{\ell}})(u) = \chi(u)$. Thus, $(\Sigma, \chi)$ is $C$-equivalent to itself, i.e., $C$-equivalence is reflexive.

Suppose $(\Sigma, \chi)$ is $C$-equivalent to $(\Sigma', \chi')$. Then there exist a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ and $\varphi : \mathbb{R}^{\ell} \to \mathbb{R}^{\ell}$ such that $T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u))$ and $(\chi' \circ \varphi)(u) = r\chi(u)$ for every $g \in \mathsf{G}$ and $u \in \mathbb{R}^{\ell}$. Now, as $(T_g\phi)^{-1} = T_{\phi(g)}\phi^{-1}$, we have that $T_{\phi(g)}\phi^{-1} \cdot \Xi'(\phi(g), \varphi(u)) = \Xi(\phi^{-1}(\phi(g)), \varphi^{-1}(\varphi(u))) = \Xi(g, u)$ for every $\phi(g) \in \mathsf{G}$ and $\varphi(u) \in \mathbb{R}^{\ell}$. Also, $\frac{1}{r}(\chi' \circ \varphi^{-1}(\varphi))(u) = \frac{1}{r}\chi'(u) = (\chi \circ \varphi^{-1})(u)$. That is, $(\Sigma', \chi')$ is $C$-equivalent to $(\Sigma, \chi)$, i.e., $C$-equivalence is symmetric.

Finally, suppose $(\Sigma, \chi)$ is $C$-equivalent to $(\Sigma', \chi')$ and $(\Sigma', \chi')$ is $C$-equivalent to $(\Sigma'', \chi'')$. That is, there exist Lie group isomorphisms $\phi_1 : \mathsf{G} \to \mathsf{G}'$, $\phi_2 : \mathsf{G}' \to \mathsf{G}''$, affine isomorphisms $\varphi_1, \varphi_2 : \mathbb{R}^\ell \to \mathbb{R}^\ell$, and constants $r_1, r_2 > 0$, such that

$$\begin{aligned} T_g\phi_1 \cdot \Xi(g,u) &= \Xi'(\phi_1(g), \varphi_1(u)) \qquad \text{and} \qquad & T_h\phi_2 \cdot \Xi'(h,u) &= \Xi''(\phi_2(h), \varphi_2(u)) \\ (\chi' \circ \varphi_1)(u) &= r_1\chi(u) & (\chi'' \circ \varphi_2)(u) &= r_2\chi'(u) \end{aligned}$$

for every $g \in \mathsf{G}$, $h \in \mathsf{G}'$, and $u \in \mathbb{R}^\ell$. Let $\phi = \phi_2 \circ \phi_1$ and $\varphi = \varphi_2 \circ \varphi_1$. Then, for every $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$, we have $T_g\phi \cdot \Xi(g,u) = \Xi''(\phi(g), \varphi(u))$ (see proof of proposition 2.2.5). Also, $(\chi'' \circ \varphi_2 \circ \varphi_1)(u) = (\chi'' \circ \varphi_2)(\varphi_1(u)) = r_2\chi'(\varphi_1(u)) = r_2 r_1 \chi(u)$. Thus, $(\Sigma, \chi)$ is $C$-equivalent to $(\Sigma'', \chi'')$, i.e., $C$-equivalence is transitive. □

2.2.15 PROPOSITION. *If $(\Sigma, \chi)$ and $(\Sigma', \chi')$ are cost equivalent cost-extended systems, then the underlying control systems $\Sigma$ and $\Sigma'$ are detached feedback equivalent.*

This follows immediately from the definition of cost-equivalence. The following result also follows almost immediately from the definition of cost-equivalence and provides the relationship of cost-equivalence with state space equivalence and detached feedback equivalence.

2.2.16 PROPOSITION. *Let $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}', \Xi')$ be two full rank systems and let $\chi : \mathbb{R}^\ell \to \mathbb{R}$ be any admissible cost.*

*(i) If $\Sigma$ and $\Sigma'$ are state space equivalent, then $(\Sigma, \chi)$ and $(\Sigma', \chi)$ are cost equivalent.*

*(ii) If $\Sigma$ and $\Sigma'$ are detached feedback equivalent with respect to a feedback transformation $\varphi$, then $(\Sigma, \chi \circ \varphi)$ and $(\Sigma', \chi)$ are cost equivalent.*

Let $(g(\cdot), u(\cdot))$ be a trajectory-control pair, defined over an interval $[0, T]$, of a cost-extended system $(\Sigma, \chi)$. Then $(g(\cdot), u(\cdot))$ is an *optimal controlled trajectory* (shortly OCT) of $(\Sigma, \chi)$ if it is a solution for the associated optimal control problem.

2.2.17 THEOREM. ([26]) *Suppose $(\Sigma, \chi)$ and $(\Sigma', \chi')$ are cost equivalent (w.r.t $\Phi = (\phi, \varphi)$).*

*(i) If $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is an OCT of $(\Sigma', \chi')$, then $(g(\cdot), u(\cdot))$ is an OCT of $(\Sigma, \chi)$.*

*(ii) If $(g'(\cdot), u'(\cdot))$ is an OCT of $(\Sigma', \chi')$, then there exists an OCT $(g(\cdot), u(\cdot))$ of $(\Sigma, \chi)$ such that $(g'(\cdot), u'(\cdot)) = (\phi \circ g(\cdot), \varphi \circ u(\cdot))$.*

PROOF. Let $(g(\cdot), u(\cdot))$ be a trajectory-control pair of $\Sigma$.

(i) Assume the image $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is an OCT of $(\Sigma', \chi')$ but $(g(\cdot), u(\cdot))$ is not an OCT of $(\Sigma, \chi)$. Then there exists a trajectory-control pair $(h(\cdot), v(\cdot))$ such that $h(0) = g(0)$, $h(T) = g(T)$, and

$$\mathcal{J}(v(\cdot)) = \int_0^T \chi(v(t))dt < \int_0^T \chi(u(t))dt = \mathcal{J}(u(\cdot)).$$

Hence $(\phi \circ h(\cdot), \varphi \circ v(\cdot))$ is a trajectory-control pair of $(\Sigma', \chi')$ such that (for some $r > 0$)

$$\begin{aligned}\mathcal{J}'(\varphi(v(\cdot))) &= \int_0^T (\chi' \circ \varphi)(v(t))dt = r\int_0^T \chi(v(t))dt \\ &< r\int_0^T \chi(u(t))dt = \int_0^T (\chi' \circ \varphi)(u(t))dt = \mathcal{J}'(\varphi(u(\cdot))).\end{aligned}$$

This contradicts the fact that $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is an OCT of $(\Sigma'\chi')$.

Hence, if $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is an OCT of $(\Sigma', \chi')$, then $(g(\cdot), u(\cdot))$ is an OCT of $(\Sigma, \chi)$.

(ii) Suppose that $(g'(\cdot), u'(\cdot))$ is an OCT of $(\Sigma', \chi')$. As $\phi$ is surjective, there exists $g \in \mathsf{G}$ such that $\phi(g) = g'(0)$. Thus, there exists a trajectory-control pair $\big(g(\cdot), \varphi^{-1} \circ u'(\cdot)\big)$ of $\Sigma$ such that $g(0) = g$. We claim that the image of $\big(g(\cdot), \varphi^{-1} \circ u'(\cdot)\big)$ under $\Phi$ is $(g'(\cdot), u'(\cdot))$. Now

$$\begin{aligned}\frac{d}{dt}\phi(g(t)) &= T_{g(t)}\phi \cdot \Xi(g(t), \varphi^{-1}(u'(t))) \\ &= \Xi'(\phi(g(t)), (\varphi \circ \varphi^{-1})(u'(t))) \\ &= \Xi'(\phi(g(t)), u'(t)).\end{aligned}$$

Hence, as $\phi(g(t))$ and $g'(t)$ solve the same Cauchy problem, they are equal. By part (i) of the proof, it follows that $\big(g(\cdot), \varphi^{-1} \circ u'(\cdot)\big)$ is an OCT of $(\Sigma, \chi)$.

□

2.2.18 COROLLARY. *Suppose* $(\Sigma, \chi)$ *and* $(\Sigma', \chi')$ *are cost equivalent (w.r.t* $\Phi = (\phi, \varphi)$*). Then* $(g(\cdot), u(\cdot))$ *is an OCT of* $(\Sigma, \chi)$ *if and only if* $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ *is an OCT of* $(\Sigma', \chi')$.

Thus we have that optimal trajectories are in a one-to-one correspondence for cost-equivalent systems.

One can investigate cost equivalence when either the dynamics or costs, of two (cost-extended) systems, are identical. Let $\Sigma$ be a control system and $\chi : \mathbb{R}^\ell \to \mathbb{R}$ be an admissible cost.

For an $\ell$-input system $\Sigma = (\mathsf{G}.\Xi)$, let $\mathcal{T}_\Sigma$ denote the group of feedback transformations leaving $\Sigma$ invariant. More precisely,

$$\mathcal{T}_\Sigma = \left\{\varphi \in \mathsf{Aff}(\mathbb{R}^\ell) \ : \ \exists \psi \in \mathbf{dAut}(\mathsf{G}),\ \psi \cdot \Xi(\mathbf{1}, u) = \Xi(\mathbf{1}, \varphi(u))\right\}.$$

Likewise, for an admissible cost $\chi : \mathbb{R}^\ell \to \mathbb{R}$, let $\mathcal{T}_\chi$ denote the group of feedback transformations leaving $\chi$ invariant. More precisely,

$$\mathcal{T}_\chi = \left\{\varphi \in \mathsf{Aff}(\mathbb{R}^\ell) \ : \ \chi \circ \varphi = r\chi \text{ for some } r > 0\right\}.$$

2.2.19 THEOREM. ([26]) $(\Sigma, \chi)$ *and* $(\Sigma, \chi')$ *are cost equivalent if and only if there exists* $\varphi \in \mathcal{T}_\Sigma$ *such that* $\chi' \circ \varphi = r\chi$ *for some* $r > 0$.

2.2.20 THEOREM. ([26]) $(\Sigma, \chi)$ *and* $(\Sigma', \chi)$ *are cost equivalent if and only if there exists a Lie group isomorphism* $\phi : \mathsf{G} \to \mathsf{G}'$ *and* $\varphi \in \mathcal{T}_\chi$ *such that* $\Xi'(\mathbf{1}, u) = T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, \varphi(u))$.

The groups $\mathcal{T}_\Sigma$ and $\mathcal{T}_\chi$ may facilitate the classification of various distinguished subclasses of cost-extended systems. For the purposes of classification, one only considers classes of full-rank cost-extended systems on a fixed Lie group $\mathsf{G}$ (and a fixed input space $\mathbb{R}^\ell$). If one has a classification of systems under detached feedback equivalence, one need only consider feedback transformations $\mathcal{T}_\Sigma$ to complete a classification of cost-extended systems under cost equivalence. On the other hand, if one wished to fix a cost $\chi$, one need only consider feedback transformations $\mathcal{T}_\chi$ to complete a classification of cost-extended systems under cost equivalence. Both of these approaches are included here for completeness and for a better understanding of cost equivalence. In this thesis we will use the approach of classifying control systems under DF-equivalence and considering the class of feedback transformations $\mathcal{T}_\Sigma$.

Let $(\Sigma = (\mathsf{G}, \Xi), \chi)$ and $(\Sigma' = (\mathsf{G}', \Xi'), \chi')$ be two cost-extended control systems. Consider the cost function given by $\chi(u) = (u-\mu)^\top Q(u-\mu)$, where $Q$ is a positive-definite $\ell \times \ell$ matrix. Then for an affine transformation of the controls $\varphi : u \mapsto Fu + x$ it follows that

$$\begin{aligned}(\chi \circ \varphi)(u) &= (Fu + x - \mu)^\top Q(Fu + x - \mu)\\ &= (F(u + F^{-1}(x-\mu)))^\top Q(F(u + F^{-1}(x-\mu)))\\ &= (u - \mu')^\top F^\top Q F(u - \mu'), \quad \text{where} \quad \mu' = F^{-1}(x - \mu).\end{aligned}$$

To any cost-extended control system $(\Sigma, \chi)$ we can associate, via the Pontryagin Maximum Principle, a Hamilton-Poisson system on the corresponding Lie-Poisson space $\mathfrak{g}^*_-$. (See theorem 2.1.5.)

### 2.2.4 Linear and affine equivalence

We have seen that the investigation of the extremal controls of a cost extended system $(\Sigma = (\mathsf{G}, \Xi), \chi)$ reduces to the investigation of the associated Hamilton-Poisson system on the dual space of the Lie algebra $\mathfrak{g}^*$. We now introduce the notion of two Hamilton-Poisson systems being equivalent. This greatly simplifies the necessary work. (The definitions and basic results concerning Hamilton-Poisson systems are given in appendix B.)

2.2.21 Definition. Two Hamilton-Poisson systems $(\mathfrak{g}^*_-, G)$ and $(\mathfrak{h}^*_-, H)$ are said to be **affinely (resp. linearly) equivalent** if the associated vector fields $\vec{G}$ and $\vec{H}$ are compatible with an affine (resp. linear) isomorphism. More precisely, these two systems are equivalent provided there exists an affine (resp. linear) isomorphism $\psi : \mathfrak{g}^* \to \mathfrak{h}^*$ such that $T\psi \cdot \vec{G} = \vec{H} \circ \psi$.

2.2.22 Proposition. *Affine equivalence is an equivalence relation.*

Proof. Let $H_{A,\mathcal{Q}}$, $H_{B,\mathcal{R}}$, and $H_{C,\mathcal{S}}$ be quadratic Hamilton-Poisson systems on a (minus) Lie-Poisson space $\mathfrak{g}^*_-$.

We have $\mathrm{id}_{\mathfrak{g}^*} \cdot H_{A,\mathcal{Q}} = H_{A,\mathcal{Q}} \circ \mathrm{id}_{\mathfrak{g}^*}$, and so $H_{A,\mathcal{Q}}$ is equivalent to itself. Hence, affine equivalence is reflexive.

Suppose $H_{A,\mathcal{Q}}$ and $H_{B,\mathcal{R}}$ are equivalent. Thus there exists an affine isomorphism $\psi : p \mapsto \psi_0(p) + q$ such that $\psi_0 \cdot \vec{H}_{A,\mathcal{Q}} = \vec{H}_{B,\mathcal{R}} \circ \psi$. Then $\psi_0^{-1} \cdot \vec{H}_{B,\mathcal{R}} = \vec{H}_{A,\mathcal{Q}} \circ \psi^{-1}$, and so $H_{B,\mathcal{R}}$ is equivalent to $H_{A,\mathcal{Q}}$. Hence, affine equivalence is symmetric.

Now, suppose $H_{A,\mathcal{Q}}$ is equivalent to $H_{B,\mathcal{R}}$ and $H_{B,\mathcal{R}}$ is equivalent to $H_{C,\mathcal{S}}$. Then there exist affine isomorphisms $\psi : p \mapsto \psi_0(p) + q$ and $\psi' : p \mapsto \psi_0'(p) + q'$ such that $\psi_0 \cdot \vec{H}_{A,\mathcal{Q}} = \vec{H}_{B,\mathcal{R}} \circ \psi$

and $\psi_0' \cdot \vec{H}_{B,\mathcal{R}} = \vec{H}_{C,\mathcal{S}} \circ \psi'$. Then $\psi' \circ \psi : p \mapsto (\psi_0' \circ \psi_0)(p) + \psi_0'(q) + q'$ is an affine isomorphism such that $(\psi_0' \circ \psi_0) \cdot \vec{H}_{A,\mathcal{Q}} = \psi_0' \cdot \vec{H}_{B,\mathcal{R}} \circ \psi = \vec{H}_{C,\mathcal{S}} \circ (\psi' \circ \psi)$, and so $H_{A,\mathcal{Q}}$ is equivalent to $H_{C,\mathcal{S}}$. Hence, affine equivalence is transitive. □

The map $\psi$ establishes a one-to-one correspondence between the integral curves of $\vec{G}$ and $\vec{H}$, which we show below. Clearly, if two systems are linearly equivalent, then they are also affinely equivalent.

2.2.23 PROPOSITION. *If $H_{A,\mathcal{Q}}$ is affinely equivalent to $H_{B,\mathcal{R}}$, then the integral curves and equilibrium states of $H_{A,\mathcal{Q}}$ and $H_{B,\mathcal{R}}$ are in a one-to-one correspondence.*

PROOF. Let $p(\cdot)$ be an integral curve of $\vec{H}_{A,\mathcal{Q}}$, i.e., $\dot{p}(t) = \vec{H}_{A,\mathcal{Q}}(p(t))$. Since $H_{A,\mathcal{Q}}$ and $H_{B,\mathcal{R}}$ are affinely equivalent, there exists an affine isomorphism $\psi : p \mapsto \psi_0(p) + q$ such that $\psi_0 \cdot \vec{H}_{A,\mathcal{Q}} = \vec{H}_{B,\mathcal{R}} \circ \psi$. We will show that $\psi(p(\cdot))$ is the unique integral curve of $\vec{H}_{B,\mathcal{R}}$ corresponding to $p(\cdot)$. Indeed,

$$\begin{aligned}\frac{d}{dt}\psi(p(t)) &= T_{p(t)}\psi \cdot \dot{p}(t)\\ &= \psi_0 \cdot \vec{H}_{A,\mathcal{Q}}(p(t)) = \vec{H}_{B,\mathcal{R}}(\psi(p(t))).\end{aligned}$$

Therefore, $\psi(p(\cdot))$ is an integral curve of $\vec{H}_{B,\mathcal{R}}$. Suppose $\psi(p_1(\cdot)) = \psi(p_2(\cdot))$, where $p_1(\cdot)$ and $p_2(\cdot)$ are integral curves of $\vec{H}_{A,\mathcal{Q}}$. Then $p_1(\cdot) = \psi^{-1}(\psi(p_1(\cdot))) = \psi^{-1}(\psi(p_2(\cdot))) = p_2(\cdot)$. Thus, integral curves are mapped injectively from $\vec{H}_{A,\mathcal{Q}}$ to $\vec{H}_{B,\mathcal{R}}$. Now, let $p'(\cdot)$ be an integral curve of $\vec{H}_{B,\mathcal{R}}$. We have

$$\begin{aligned}\frac{d}{dt}\psi^{-1}(p'(t)) &= T_{p'(t)}\psi^{-1} \cdot p'(t)\\ &= \psi_0^{-1} \cdot \vec{H}_{B,\mathcal{R}} = \vec{H}_{A,\mathcal{Q}}(\psi^{-1}(p'(t))).\end{aligned}$$

Therefore, $p'(t)$ is an integral curve of $\vec{H}_{A,\mathcal{Q}}$. That is, integral curves are also mapped surjectively. Hence, the integral curves of $\vec{H}_{A,\mathcal{Q}}$ and $\vec{H}_{B,\mathcal{R}}$ are in a one-to-one correspondence.

The fact that the integral curves are in a one-to-one correspondence immediately implies that the equilibrium states of $\vec{H}_{A,\mathcal{Q}}$ and $\vec{H}_{B,\mathcal{R}}$ are also in a one-to-one correspondence. (Equilibrium states are just those points corresponding to constant trajectories.)

□

Given a fixed Lie-Poisson space $\mathfrak{g}_-^*$, the following lemma proves useful in classifying Hamilton-Poisson systems.

2.2.24 LEMMA. *The following Hamilton-Poisson systems (on $\mathfrak{g}_-^*$) are affinely equivalent to $H_{A,\mathcal{Q}}$:*

*($\mathfrak{E}$1) $H_{A,\mathcal{Q}} \circ \psi$, where $\psi$ is a linear Poisson automorphism;*

*($\mathfrak{E}$2) $H_{A,r\mathcal{Q}}$, where $r \neq 0$;*

*($\mathfrak{E}$3) $H_{A,\mathcal{Q}} + C$, where $C$ is a Casimir function.*

PROOF. $(\mathfrak{C}1)$ Let $F \in C^\infty(\mathfrak{g}^*)$ and let $G = H_{A,\mathcal{Q}} \circ \psi$. Then

$$\begin{aligned}(\vec{H}_{A,\mathcal{Q}} \circ \psi)[F] &= \vec{H}_{A,\mathcal{Q}}[F] \circ \psi = \{F, H_{A,\mathcal{Q}}\} \circ \psi \\ &= \{F \circ \psi, G\} = \vec{G}[F \circ \psi] = (\psi \cdot \vec{G})[F].\end{aligned}$$

Thus, it follows that $\psi \cdot \vec{G} = \vec{H}_{A,\mathcal{Q}} \circ \psi$, i.e., $H_{A,\mathcal{Q}}$ is affinely equivalent to $H_{A,\mathcal{Q}} \circ \psi$.

$(\mathfrak{C}2)$ Let $G = H_{A,\mathcal{Q}} + C$. Then for every $F \in C^\infty(\mathfrak{g}^*)$ we have

$$\begin{aligned}\vec{G}[F] &= \{F, H_{A,\mathcal{Q}} + C\} = \{F, H_{A,\mathcal{Q}}\} + \{F, C\} \\ &= \{F, H_{A,\mathcal{Q}}\} = \vec{H}_{A,\mathcal{Q}}[F].\end{aligned}$$

Thus, $\vec{G} = \vec{H}_{A,\mathcal{Q}}$, and so $H_{A,\mathcal{Q}}$ is affinely equivalent to $H_{A,\mathcal{Q}} + C$.

$(\mathfrak{C}3)$ Let $\psi$ be the linear isomorphism $\psi : p \mapsto \frac{1}{r}p$. Then

$$\begin{aligned}(\psi \cdot \vec{H}_{A,\mathcal{Q}})(p) - \vec{H}_{A,r\mathcal{Q}}(p) &= \frac{1}{r}\vec{H}_{A,\mathcal{Q}}(p) - \vec{H}_{A,r\mathcal{Q}}(\tfrac{1}{r}p) \\ &= \tfrac{1}{r}\vec{L}_A(p) + \tfrac{1}{r}\vec{H}_{\mathcal{Q}}(p) - \vec{L}_A(\tfrac{1}{r}p) + \vec{H}_{\mathcal{Q}}(\tfrac{1}{r}p).\end{aligned}$$

As $\vec{L}_A$ is linear, it follows that $\vec{L}_A(\frac{1}{r}p) = \frac{1}{r}\vec{L}_A(p)$. Moreover, as $\mathbf{d}H_{r\mathcal{Q}} = r\mathbf{d}H_{\mathcal{Q}}(p) = \mathbf{d}H_{\mathcal{Q}}(rp)$ and $\mathrm{ad}^*_{r\mathbf{d}H_{\mathcal{Q}}(p)}p = r\mathrm{ad}^*_{\mathbf{d}H_{\mathcal{Q}}(p)}(p)$, we get

$$\begin{aligned}\frac{1}{r}\vec{H}_{\mathcal{Q}}(p) - \vec{H}_{r\mathcal{Q}}(\frac{1}{r}(p)) &= \frac{1}{r}\mathrm{ad}^*_{\mathbf{d}H_{\mathcal{Q}}(p)}(p) - \mathrm{ad}^*_{\mathbf{d}H_{r\mathcal{Q}}(\frac{1}{r}p)}(\frac{1}{r}p) \\ &= \frac{1}{r}\mathrm{ad}^*_{\mathbf{d}H_{\mathcal{Q}}(p)}(p) - \frac{1}{r}\mathrm{ad}^*_{\mathbf{d}H_{\mathcal{Q}}(p)}(p) = 0.\end{aligned}$$

Thus, $\psi \cdot \vec{H}_{A,\mathcal{Q}} = \vec{H}_{A,r\mathcal{Q}} \circ \psi$, and so $H_{A,\mathcal{Q}}$ is affinely equivalent to $H_{A,r\mathcal{Q}}$.

$\square$

It turns out that the linear equivalence of quadratic Hamilton-Poisson systems on $\mathfrak{g}^*_-$ is closely related to the cost equivalence of cost-extended control systems on $\mathsf{G}$.

2.2.25 THEOREM. ([26]) *If two cost-extended control systems* $(\Sigma, \chi)$ *and* $(\Sigma', \chi')$ *are cost equivalent, then their associated Hamilton-Poisson systems, given by* (2.5)*, are linearly equivalent.*

2.2.26 REMARK. The converse of this statement is not true. (See [26] for a counter example.)

# Chapter 3

# Control systems on $\mathsf{SO}(3)$

## 3.1 Preliminaries

The orthogonal group

$$\mathsf{SO}(3) = \left\{ g \in \mathsf{GL}(3,\mathbb{R}) \ : \ g^\top g = \mathbf{1},\ \det g = 1 \right\}$$

is a three-dimensional, simple, compact Lie group. Its Lie algebra is given by

$$\mathfrak{so}(3) = \left\{ A \in \mathbb{R}^{3\times 3} \ : \ A^\top + A = \mathbf{0} \right\}.$$

Let

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

be the standard (ordered) basis for $\mathfrak{so}(3)$. These three basis elements represent the infinitesimal rotations about the three axes of $\mathbb{R}^3$. The basis elements satisfy the following commutator relations:

$$[E_2, E_3] = E_1, \qquad [E_3, E_1] = E_2, \quad [E_1, E_2] = E_3.$$

The Lie algebra $\mathfrak{so}(3)$ is isomorphic to the Lie algebra $(\mathbb{R}^3, \times)$, where $\times$ is the usual cross-product on $\mathbb{R}^3$. The group of Lie algebra automorphisms (w.r.t the standard basis) is just the Lie group $\mathsf{SO}(3)$, i.e.,

$$\mathsf{Aut}(\mathfrak{so}(3)) = \{A \mid A \in \mathsf{SO}(3)\}.$$

3.1.1 Proposition. ([34],[52]) *A control system* $\Sigma = (\mathsf{SO}(3), \Xi)$ *with trace* $\Gamma$ *is controllable if and only if* $\mathrm{Lie}(\Gamma) = \mathfrak{so}(3)$.

The above result follows from the fact that the full-rank condition is a necessary and sufficient condition for controllability when the state space is compact.

3.1.2 Proposition. *The group* $\mathsf{dAut}(\mathsf{SO}(3)) = \mathsf{Aut}(\mathfrak{so}(3))$.

Proof. This result follows immediately from the fact that every automorphism of $\mathfrak{so}(3)$ is an inner automorphism and $\mathsf{Int}(\mathfrak{so}(3)) \subset \mathsf{dAut}(\mathsf{SO}(3))$.

□

Let $(E_1^*, E_2^*, E_3^*)$ denote the dual of the standard basis. We shall write an element $p = p_1 E_1^* + p_2 E_2^* + p_3 E_3^* \in \mathfrak{so}(3)^*$ as $\begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}$. The group of linear Poisson automorphisms takes the form

$$\{p \mapsto p\Psi \,:\, \Psi \in \mathbb{R}^{3\times 3},\ \Psi\Psi^\top = \mathbf{1},\ \det\Psi = 1\} \cong \mathsf{SO}(3).$$

Note that $C(p) = p_1^2 + p_2^2 + p_3^2$ is a Casimir function.

3.1.3 REMARK. The Hamiltonian vector fields on $\mathfrak{so}(3)^*_-$ are complete as their integral curves evolve on the compact subsets $C^{-1}(c_0)$, $c_0 \geq 0$ (cf. [1]).

3.1.4 REMARK. The Hamiltonian vector field associated to a function $H \in C^\infty(\mathfrak{so}(3)^*)$ can be expressed as $\vec{H} = \frac{1}{2}\nabla C \times \nabla H$. Hence the (regular) level sets of $H$ and $C$ are tangent exactly at equilibria.

## 3.2 Equivalence of control systems

In this section we provide a classification of all cost-extended control systems on $\mathsf{SO}(3)$. We begin by recalling a classification of detached feedback equivalent systems on $\mathsf{SO}(3)$.

3.2.1 PROPOSITION. ([23]) *Any full-rank control system $\Sigma = (\mathsf{SO}(3), \Xi)$ is detached feedback equivalent to exactly one of the systems on $\mathsf{SO}(3)$ specified by*

$$\begin{aligned}
\Xi_\alpha^{(1,1)}(\mathbf{1}, u) &= \alpha E_1 + u_1 E_2 \\
\Xi^{(2,0)}(\mathbf{1}, u) &= u_1 E_2 + u_2 E_3 \\
\Xi_\alpha^{(2,1)}(\mathbf{1}, u) &= \alpha E_1 + u_1 E_2 + u_3 E_3 \\
\Xi^{(3,0)}(\mathbf{1}, u) &= u_1 E_1 + u_2 E_2 + u_3 E_3.
\end{aligned}$$

*Here $\alpha > 0$ parametrizes families of (non-equivalent) class representatives.*

In the remainder of this section we shall assume that all the control systems we consider have full rank.

### Single-input systems

Clearly no single-input homogeneous system can have full rank.

3.2.2 PROPOSITION. *Every (full-rank) single-input inhomogeneous cost-extended system is C-equivalent to exactly one of the systems $(\Sigma_\alpha^{(1,1)}, \chi_\eta)$, where $\Sigma_\alpha^{(1,1)} = (\mathsf{SO}(3), \Xi_\alpha^{(1,1)})$ and $\chi_\eta(u) = (u_1 - \eta)^2$, $\eta \geq 0$.*

PROOF. Every single-input inhomogeneous cost-extended control system $(\Sigma, \chi)$ has underlying system $\Sigma$ which is detached feedback equivalent to the system $\Sigma_\alpha^{(1,1)} = (\mathsf{SO}(3), \Xi_\alpha^{(1,1)})$ where

$$\Xi_\alpha^{(1,1)}(\mathbf{1}, u) = \alpha_1 E_1 + u_1 E_2.$$

Thus $(\Sigma, \chi)$ is cost equivalent to $(\Sigma_\alpha^{(1,1)}, \chi_0)$ for some $\chi_0$. A simple calculation shows that $\mathcal{T}_{\Sigma_\alpha^{(1,1)}} = \begin{bmatrix} 1 & 0 \\ 0 & \sigma \end{bmatrix}$, $\sigma \in \{-1, 1\}$. We have $\chi_0 : u \mapsto (u-\mu)^\top q(u-\mu)$ for some $q > 0$ and $\mu \in \mathbb{R}$. Clearly,

$$\chi_1(u) = \frac{1}{q}\chi_0(u) = (u-\mu)^\top(u-\mu).$$

In fact we may assume $\mu \geq 0$. If not, apply $\varphi_1 = \operatorname{diag}(1, -1)$. That is every such cost-extended system is equivalent to one of the systems

$$(\Sigma_\alpha^{(1,1)}, \chi_\beta) : \begin{cases} \Xi_\alpha^{(1,1)}(\mathbf{1}, u) = \alpha E_1 + u_1 E_2 \\ \chi_\eta(u) = (u_1 - \eta)^2, \ \eta \geq 0. \end{cases}$$

We now verify that no two systems $(\Sigma_\alpha^{(1,1)}, \chi_\eta)$ and $(\Sigma_\alpha^{(1,1)}, \chi_{\eta'})$ are equivalent. (Clearly, for different $\alpha$ and $\alpha'$ two systems cannot be equivalent us there underlying control systems are not detached feedback equivalent.) Assume these two systems are equivalent for some $\eta \neq \eta'$, $\eta, \eta' \geq 0$. Then there exists an affine transformation of the controls $\varphi = \begin{bmatrix} 1 & 0 \\ 0 & \sigma \end{bmatrix} \in \mathcal{T}_{\Sigma_\alpha^{(1,1)}}$, with $\sigma \in \{-1, 1\}$, such that $\chi_\eta \circ \varphi = r\chi_{\eta'}$, for some $r > 0$. Specifically, we have that $(u_1 - \sigma\eta)^2 = r(u_1 - \eta')^2$. It is straightforward to verify that this equation holds only if $r = 1$ and $\sigma\eta = \eta'$, a contradiction. Thus the representatives obtained are distinct.

□

### Two-input homogeneous systems

3.2.3 LEMMA. *The group of feedback transformations for* $\Sigma^{(2,0)}$ *is given by* $\mathcal{T}_{\Sigma^{(2,0)}} = \mathsf{O}(2)$.

PROOF. Let $\psi \in \mathsf{Aut}(\mathsf{SO}(3))$ and let $K \in \mathsf{GL}(2, \mathbb{R})$. We then consider the condition $\psi \cdot \Xi(\mathbf{1}, u) = \Xi(\mathbf{1}, Ku)$, i.e.,

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}.$$

This conditions immediately implies that $x_2 = x_3 = 0$. As $\psi \in \mathsf{SO}(3)$ it then follows that $x_4 = x_7 = 0$ and $x_1 = \sigma = \pm 1$. Thus we have that $\begin{bmatrix} x_5 & x_6 \\ x_8 & x_9 \end{bmatrix} = \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}$ which implies that $K \in \mathsf{O}(2)$. Hence $\mathcal{T}_{\Sigma^{(2,0)}} = \mathsf{O}(2)$.

□

3.2.4 PROPOSITION. *Every two-input homogeneous cost-extended system is C-equivalent to exactly one of the systems* $(\Sigma^{(2,0)}, \chi^1_{\boldsymbol{\eta}\beta})$ *or* $(\Sigma^{(2,0)}, \chi^2_\eta)$, *where* $\Sigma^{(2,0)} = (\mathsf{SO}\,(3), \Xi^{(2,0)})$ *and*

$$\chi^1_{\boldsymbol{\eta}\beta} = (u_1 - \eta_1)^2 + \beta(u_2 - \eta_2)^2$$
$$\chi^2_\eta(u) = (u_1 - \eta_1)^2 + u_2^2.$$

*Here* $\eta_1, \eta_2 \geq 0$ *and* $0 < \beta < 1$.

Proof. Every two-input homogeneous cost-extended control system $(\Sigma, \chi)$ is detached feedback equivalent to the system $\Sigma^{(2,0)} = (\mathsf{SO}\,(3), \Xi^{(2,0)})$ where

$$\Xi^{(2,0)}(\mathbf{1}, u) = u_1 E_2 + u_2 E_3.$$

Thus $(\Sigma, \chi)$ is cost equivalent to $(\Sigma^{(2,0)}, \chi_0)$ for some $\chi_0$. Recall by lemma 3.2.3 that $\mathcal{T}_{\Sigma^{(2,0)}} = \mathsf{O}(2)$. We have $\chi_0 : u \mapsto (u - \mu)^\top Q(u - \mu)$ for some positive definite matrix $Q = \begin{bmatrix} a_1 & b \\ b & a_2 \end{bmatrix}$ and $\mu \in \mathbb{R}^2$. Now there exists a $\varphi_1 \in \mathsf{O}\,(2) = \mathcal{T}_{\Sigma^{(2,0)}}$ such that

$$\chi_1(u) = (\chi_0 \circ \varphi_1)(u) = (u - \mu')^\top \mathrm{diag}(\alpha_1, \alpha_2)(u - \mu')$$

for some $\mu' \in \mathbb{R}^2$, where $\alpha_1 \geq \alpha_2 > 0$. It then follows that

$$\chi_2(u) = \frac{1}{\alpha_1}\chi_1(u) = (u - \mu')^\top \mathrm{diag}(1, \beta)(u - \mu')$$

where $0 < \beta \leq 1$. Suppose $\beta \neq 1$. The only transformations left preserving this quadratic form are the matrices

$$\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \in \mathsf{O}\,(2), \qquad \sigma_1, \sigma_2 \in \{-1, 1\}.$$

Thus we can transform $\mu'_1$ and $\mu'_2$ into $\eta_1$ and $\eta_2$, respectively, for some $\eta_1, \eta_2 \geq 0$. Hence, every such cost-extended system (with $\beta \neq 1$) is equivalent to one of the systems

$$(\Sigma^{(2,0)}, \chi^1_{\boldsymbol{\eta}\beta}) : \begin{cases} \Xi^{(2,0)}(\mathbf{1}, u) = u_1 E_2 + u_2 E_3 \\ \chi^1_{\boldsymbol{\eta}\beta}(u) = (u_1 - \eta_1)^2 + \beta(u_2 - \eta_2)^2, \quad \eta_1, \eta_2 \geq 0, \ 0 < \beta < 1. \end{cases}$$

We verify that each choice of constants leads to a distinct non-equivalent representative. Assume $(\Sigma^{(2,0)}, \chi^1_{\boldsymbol{\eta}\beta})$ and $(\Sigma^{(2,0)}, \chi^1_{\boldsymbol{\eta}'\beta'})$ are two C-equivalent systems. Then there exists a $\psi = \begin{bmatrix} \sigma\cos\theta & -\sigma\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in \mathsf{O}(2)$ and $r > 0$ such that $\chi^1_{\boldsymbol{\eta}\beta} \circ \psi = r\chi^1_{\boldsymbol{\eta}'\beta'}$. (Here $\sigma \in \{-1, 1\}$.) More specifically, we have

$$\begin{aligned} &\beta\left(u_1 \sin\theta + u_2\cos\theta - \eta_2\right)^2 + \left((u_2\sin\theta - u_1\cos\theta)\,\sigma + \eta_1\right)^2 \\ &= r\left(\left(u_1 - \eta'_1\right)^2 + \beta'\left(u_2 - \eta'_2\right)^2\right). \end{aligned}$$

It is then straightforward to verify that this equation holds only if $r = 1$, $\beta = \beta'$, $\eta_1 = \eta'_1$, and $\eta_2 = \eta'_2$. Hence each of these representatives are distinct and non-equivalent.

Now, if $\beta = 1$, then clearly any element of $\mathsf{O}\,(2)$ preserves the quadratic form. There then exists a $\eta \geq 0$ and $\theta \in \mathbb{R}$ such that $\mu'_1 = \eta\cos\theta$ and $\mu_2 = \eta\sin\theta$. Thus, for $\varphi_2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, it follows that

$$\chi_3(u) = (\chi_2 \circ \varphi_2)(u) = \left(u - \begin{bmatrix} \eta \\ 0 \end{bmatrix}\right)^\top \left(u - \begin{bmatrix} \eta \\ 0 \end{bmatrix}\right).$$

That is, every such cost extended system is equivalent to one of the systems

$$(\Sigma^{(2,0)},\chi^2_\eta) : \begin{cases} \Xi^{(2,0)}(\mathbf{1},u) = u_1E_2 + u_2E_3 \\ \chi^2_\eta(u) = (u_1-\eta)^2 + u_2^2, \quad \eta \geq 0. \end{cases}$$

We verify that each choice of constants leads to a distinct non-equivalent representative. Let $(\Sigma^{(2,0)},\chi^2_\eta)$ and $(\Sigma^{(2,0)},\chi^2_{\eta'})$ be two such systems. Let $\varphi = \begin{bmatrix} \sigma\cos\theta & -\sigma\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in \mathcal{T}_{\Sigma^{(2,0)}}$, where $\sigma \in \{-1,1\}$. Considering the equation $\chi^2_\eta \circ \varphi = r\chi^2_{\eta'}$, for some $r > 0$, it follows that we must have

$$\alpha^2 - u_1 2\eta\sigma\cos\theta + u_1^2 + u_2 2\eta\sigma\sin\theta + u_2^2 = r{\eta'}^2 - 2r\eta' u_1 + ru_1^2 + ru_2^2.$$

Comparing coefficients it is simple to check that the above equation holds only if $r = 1$ and $\eta = \eta'$. Hence, each of these representatives are distinct and non-equivalent.

Using the same arguments, it is straightforward to verify that no system $(\Sigma^{(2,0)},\chi^1_{\eta\beta})$ is equivalent to a system $(\Sigma^{(2,0)},\chi^2_{\eta'})$.

$\square$

### Two-input inhomogeneous systems

3.2.5 LEMMA. *The group of feedback transformations for* $\Sigma^{(2,1)}$ *is given by* $\mathcal{T}_{\Sigma^{(2,1)}_\alpha} = \begin{bmatrix} 1 & 0 \\ 0 & S \end{bmatrix}$, $S \in \mathsf{SO}\,(2)$.

PROOF. Let $\psi \in \mathsf{Aut}(\mathsf{SO}(3))$ and let $\varphi = \begin{bmatrix} 1 & 0 \\ \mathbf{v} & K \end{bmatrix}$, where $\mathbf{v} \in \mathbb{R}^2$ and $K \in \mathsf{GL}(2,\mathbb{R})$. We then consider the condition $\psi \cdot \Xi(\mathbf{1},u) = \Xi(\mathbf{1},\varphi(u))$, i.e.,

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix} \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ v_1 & k_1 & k_2 \\ v_2 & k_3 & k_4 \end{bmatrix}.$$

These conditions immediately imply that $x_2 = x_3 = 0$ and $x_1 = 1$. As $\psi \in \mathsf{SO}(3)$ it then follows that $x_4 = x_7 = 0$, which implies that $v_1 = v_2 = 0$. Thus, as $\begin{bmatrix} x_5 & x_6 \\ x_8 & x_9 \end{bmatrix} = \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}$ it follows that $K \in \mathsf{SO}(2)$.

$\square$

3.2.6 PROPOSITION. *Every two-input inhomogeneous cost-extended system is C-equivalent to exactly one of the systems* $(\Sigma^{(2,1)}_\alpha,\chi^1_{\eta\beta\gamma})$, $(\Sigma^{(2,1)}_\alpha,\chi^2_{\beta\eta})$ *or* $(\Sigma^{(2,1)}_\alpha,\chi^3_\eta)$, *where* $\Sigma^{(2,1)}_\alpha = (\mathsf{SO}\,(3),\Xi^{(2,1)}_\alpha)$ *and*

$$\begin{aligned} \chi^1_{\alpha\beta\gamma}(u) &= (u_1-\alpha_1)^2 + \beta(u_2-\gamma)^2 \\ \chi^2_{\beta\eta}(u) &= u_1^2 + \beta(u_2-\eta)^2 \\ \chi^3_\eta(u) &= (u_1-\eta)^2 + u_2^2. \end{aligned}$$

*Here* $\alpha_1 > 0$, $\eta \geq 0$, $0 < \beta < 1$, *and* $\gamma \in \mathbb{R}$.

PROOF. Every two-input inhomogeneous cost-extended control system $(\Sigma, \chi)$ is detached feedback equivalent to one of the systems $\Sigma^{(2,1)}_{\alpha} = (\mathsf{SO}\,(3), \Xi^{(2,1)}_{\alpha})$ where

$$\Xi^{(2,1)}_{\alpha}(\mathbf{1}, u) = \alpha E_1 + u_1 E_2 + u_2 E_3, \quad \alpha > 0.$$

Thus $(\Sigma, \chi)$ is cost equivalent to $(\Sigma^{(2,1)}_{\alpha}, \chi_0)$ for some $\chi_0$. Recall by lemma 3.2.5 that $\mathcal{T}_{\Sigma^{(2,1)}_{\alpha}} = \begin{bmatrix} 1 & 0 \\ 0 & S \end{bmatrix}$, $S \in \mathsf{SO}\,(2)$. We have that $\chi_0 : u \mapsto (u - \mu)^\top Q (u - \mu)$ for some positive definite matrix $Q = \begin{bmatrix} a_1 & b \\ b & a_2 \end{bmatrix}$ and $\mu \in \mathbb{R}^2$. Now there exists a $\varphi_1 \in \mathcal{T}_{\Sigma^{(2,1)}_{\alpha}}$ such that

$$\chi_1(u) = (\chi_0 \circ \varphi_1)(u) = (u - \mu')^\top \mathrm{diag}(\alpha_1, \alpha_2)(u - \mu')$$

for some $\mu' \in \mathbb{R}^2$, where $\alpha_1 \geq \alpha_2 > 0$. It then follows that

$$\chi_2(u) = \frac{1}{\alpha_1}\chi_1(u) = (u - \mu')^\top \mathrm{diag}(1, \beta)(u - \mu')$$

where $0 < \beta \leq 1$. Assume $0 < \beta < 1$. The only transformations preserving this quadratic form are those where $S = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix} \in \mathsf{SO}\,(2)$, $\sigma_1 \in \{-1, 1\}$. Thus $\mu'_1 = \eta$ and $\mu'_2 = \gamma$ for some $\eta \geq 0$, $\gamma \in \mathbb{R}$. If $\eta = 0$, then the same transformation can be applied such that we may assume that $\gamma \geq 0$. That is, every such cost-extended system is equivalent to one of the systems

$$(\Sigma^{(2,1)}_{\alpha}, \chi^1_{\alpha\beta\gamma}) : \begin{cases} \Xi^{(2,1)}_{\alpha}(\mathbf{1}, u) = \alpha E_1 + u_1 E_2 + u_2 E_3, & \alpha > 0 \\ \chi^1_{\alpha\beta\gamma}(u) = (u_1 - \alpha_1)^2 + \beta(u_2 - \gamma)^2, & \alpha_1 > 0,\ 0 < \beta < 1,\ \gamma \in \mathbb{R} \end{cases}$$

or

$$(\Sigma^{(2,1)}_{\alpha}, \chi^2_{\beta\eta}) : \begin{cases} \Xi^{(2,1)}_{\alpha}(\mathbf{1}, u) = \alpha E_1 + u_1 E_2 + u_2 E_3, & \alpha > 0 \\ \chi^2_{\beta\eta}(u) = u_1^2 + \beta(u_2 - \eta)^2, & 0 < \beta < 1,\ \eta \geq 0. \end{cases}$$

We verify that each choice of constants leads to a distinct non-equivalent representative. Let $(\Sigma^{(2,1)}, \chi^1_{\eta\beta\gamma})$ and $(\Sigma^{(2,1)}, \chi^1_{\eta'\beta'\gamma'})$ be two such systems and let

$$\varphi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \in \mathcal{T}_{\Sigma^{(2,1)}}. \tag{3.1}$$

Considering the equation $\chi^1_{\alpha\beta\gamma} \circ \varphi = r\chi^1_{\alpha'\beta'\gamma'}$, for some $r > 0$, it follows that we have (after simplifying)

$$\begin{aligned} &\beta\,(u_1 \sin\theta + u_2 \cos\theta - \gamma)^{\,2} + (\alpha_1 - u_1 \cos\theta + u_2 \sin\theta)^{\,2} \\ &= r\left(u_1 - \alpha'_1\right)^{\,2} + r\beta'\left(u_2 - \gamma'\right)^{\,2}. \end{aligned}$$

Expanding the above equation and comparing coefficients it is straightforward to determine that this equation holds only if $r = 1$, $\beta = \beta'$, $\alpha_1 = \alpha'_1$, and $\gamma = \gamma'$. Hence, each of these representatives are distinct and non-equivalent. (The verification for $\chi^2_{\beta\eta}$ is similar and thus shall be omitted.)

Now, if $\beta = 1$, then clearly any element of $\mathsf{SO}\,(2)$ preserves the quadratic form. Therefore, there exists an $\eta \geq 0$ and $\theta \in \mathbb{R}$ such that $\mu_1' = \eta\cos\theta$ and $\mu_2 = \eta\sin\theta$. Thus, for $\varphi$ (as in equation (3.1)), it follows that

$$\chi_3(u) = (\chi_2(u) \circ \varphi)(u) = \left(u - \begin{bmatrix}\eta \\ 0\end{bmatrix}\right)^\top \left(u - \begin{bmatrix}\eta \\ 0\end{bmatrix}\right).$$

That is every such cost-extended system is equivalent to one of the systems

$$(\Sigma_\alpha^{(2,1)}, \chi_\eta^3) : \begin{cases} \Xi_\alpha^{(2,1)}(\mathbf{1}, u) = \alpha E_1 + u_1 E_2 + u_2 E_3, & \alpha > 0 \\ \chi_\eta^3(u) = (u_1 - \eta)^2 + u_2^2, \quad \eta \geq 0. \end{cases}$$

We verify that each choice of constants leads to a distinct non-equivalent representative. Let $(\Sigma_\alpha^{(2,1)}, \chi_\eta^3)$ and $(\Sigma_\alpha^{(2,1)}, \chi_{\eta'}^3)$ be two such systems. Let $\varphi \in \mathcal{T}_{\Sigma^{(2,1)}}$. Considering the equation $\chi_\eta^3 \circ \varphi = r\chi_{\eta'}^3$, for some $r > 0$, it follows that we must have

$$\eta^2 + u_1 2\eta\cos\theta + u_1^2 - u_2 2\eta\sin\theta + u_2^2 = r{\eta'}^2 - 2r\eta' u_1 + ru_1^2 + ru_2^2.$$

It is straightforward to verify that this equation holds only if $r = 1$ and $\eta = \eta'$. Hence, each of these representatives are distinct and non-equivalent.

In a very similar fashion one can verify that none of the systems $(\Sigma_\alpha^{(2,1)}, \chi_{\eta\beta\gamma}^1)$, $(\Sigma_\alpha^{(2,1)}, \chi_{\beta\eta}^2)$, and $(\Sigma_\alpha^{(2,1)}, \chi_\eta^3)$ are equivalent.

□

**Three-input systems**

3.2.7 PROPOSITION. *Every three-input homogeneous cost-extended system is C-equivalent to exactly one of the systems* $(\Sigma^{(3,0)}, \chi)$, *where* $\Sigma^{(3,0)} = (\mathsf{SO}\,(3), \Xi^{(3,0)})$ *and* $\chi$ *is given by exactly one of the following:*

$$\begin{aligned}
\chi^1_{\boldsymbol{\alpha}\boldsymbol{\beta}\gamma}(u) &= (u_1 - \alpha_1)^2 + \beta_1(u_2 - \alpha_2)^2 + \beta_2(u_3 - \gamma)^2 \\
\chi^2_{\alpha\boldsymbol{\beta}\eta}(u) &= u_1^2 + \beta_1(u_2 - \alpha_1)^2 + \beta_2(u_3 - \eta_1)^2 \\
\chi^3_{\boldsymbol{\eta}\boldsymbol{\beta}}(u) &= (u_1 - \eta_1)^2 + \beta_1 u_2^2 + \beta_2(u_3 - \eta_2)^2 \\
\chi^4_{\boldsymbol{\eta}\beta}(u) &= (u_1 - \eta_1)^2 + u_2^2 + \beta_1(u_3 - \eta_2)^2. \\
\chi^5_{\boldsymbol{\eta}\beta}(u) &= (u_1 - \eta_1)^2 + \beta_1 u_2^2 + \beta_1(u_3 - \eta_2)^2 \\
\chi^6_{\eta}(u) &= (u_1 - \eta_1)^2 + u_2^2 + u_3^2.
\end{aligned}$$

*Here* $\alpha_1, \alpha_2 > 0$, $\eta_1, \eta_2 \geq 0$, $0 < \beta_2 < \beta_1 < 1$, *and* $\gamma \in \mathbb{R}$.

PROOF. Every three-input homogeneous cost-extended control system $(\Sigma, \chi)$ is detached feedback equivalent to the system $\Sigma^{(3,0)} = (\mathsf{SO}\,(3), \Xi^{(3,0)})$ where

$$\Xi^{(3,0)}(\mathbf{1}, u) = u_1 E_1 + u_2 E_2 + u_3 E_3.$$

Thus $(\Sigma, \chi)$ is cost equivalent to $(\Sigma^{(3,0)}, \chi_0)$ for some $\chi_0$. A simple computation (very similar to lemmas 3.2.3 and 3.2.5) shows that $\mathcal{T}_{\Sigma^{(3,0)}} = \mathsf{SO}\,(3)$. Now, $\chi_0 : u \mapsto (u-\mu)^\top Q(u-\mu)$ for some positive definite matrix $Q \in \mathbb{R}^{3\times 3}$ and $\mu \in \mathbb{R}^3$. Then, there exists a $\varphi_1 \in \mathsf{SO}\,(3) = \mathcal{T}_{\Sigma^{(3,0)}}$ such that

$$\chi_1(u) = (\chi_0 \circ \varphi_1)(u) = (u-\mu')^\top \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3)(u-\mu')$$

for some $\mu' \in \mathbb{R}^3$, where $\alpha_1 \geq \alpha_2 \geq \alpha_3 > 0$. It then follows that

$$\chi_2(u) = \frac{1}{\alpha_1}\chi_1 = (u-\mu')^\top \mathrm{diag}(1, \beta_1, \beta_2)(u-\mu')$$

where $0 < \beta_2 \leq \beta_1 \leq 1$.

If $0 < \beta_2 < \beta_1 < 1$ then the only transformations preserving this quadratic form are the matrices $\mathrm{diag}\,(\sigma_1\sigma_2, \sigma_1, \sigma_2)$, where $\sigma_1, \sigma_2 \in \{-1, 1\}$. Thus $\mu_1' = \eta_1$, $\mu_1' = \eta_2$, and $\mu_3' = \gamma$, where $\eta_1, \eta_2 \geq 0$, $\gamma \in \mathbb{R}$. If $\eta_1 = 0$ or $\eta_2 = 0$, then the same transformation can be applied such that we may assume that $\gamma \geq 0$. Hence, every such cost extended system is equivalent to one of the systems

$$(\Sigma^{(3,0)}, \chi^1_{\boldsymbol{\alpha\beta}\gamma}) : \begin{cases} \Xi^{(3,0)}(\mathbf{1}, u) = u_1 E_1 + u_2 E_2 + u_3 E_3 \\ \chi^1_{\boldsymbol{\alpha\beta}\gamma}(u) = (u_1 - \alpha_1)^2 + \beta_1(u_2 - \alpha_2)^2 + \beta_2(u_3 - \gamma)^2 \end{cases} \tag{3.2}$$

or

$$(\Sigma^{(3,0)}, \chi^2_{\alpha\boldsymbol{\beta}\eta}) : \begin{cases} \Xi^{(3,0)}(\mathbf{1}, u) = u_1 E_1 + u_2 E_2 + u_3 E_3 \\ \chi^2_{\alpha\boldsymbol{\beta}\eta}(u) = u_1^2 + \beta_1(u_2 - \alpha_1)^2 + \beta_2(u_3 - \eta_1)^2. \end{cases} \tag{3.3}$$

or

$$(\Sigma^{(3,0)}, \chi^3_{\boldsymbol{\eta\beta}}) : \begin{cases} \Xi^{(3,0)}(\mathbf{1}, u) = u_1 E_1 + u_2 E_2 + u_3 E_3 \\ \chi^3_{\boldsymbol{\eta\beta}}(u) = (u_1 - \eta_1)^2 + \beta_1 u_2^2 + \beta_2(u_3 - \eta_2)^2. \end{cases} \tag{3.4}$$

Here $\alpha_1, \alpha_2 > 0$, $\eta_1, \eta_2 \geq 0$, $\gamma \in \mathbb{R}$ and $0 < \beta_2 < \beta_1 < 1$. We verify that each different choice of parameters leads to a distinct non-equivalent representative. Let $(\Sigma^{(3,0)}, \chi^1_{\boldsymbol{\alpha\beta}\gamma})$ and $(\Sigma^{(3,0)}, \chi^1_{\boldsymbol{\alpha'\beta'}\gamma'})$ be two such systems and let $\varphi \in \mathsf{SO}(3)$. Considering the equation $\chi^1_{\boldsymbol{\alpha\beta}\gamma} \circ \varphi = r\chi^1_{\boldsymbol{\alpha'\beta'}\gamma'}$, for some $r > 0$, we need only check this condition for $\varphi = \mathrm{diag}\,(\sigma_1\sigma_2, \sigma_1, \sigma_2)$, where $\sigma_1, \sigma_2 \in \{-1, 1\}$. Indeed, by lemma 4.2.11, it follows that if $R\mathrm{diag}\,(1, \beta_1, \beta_2)R^\top = \mathrm{diag}\,(r, r\beta_1', r\beta_2')$ for some $R \in \mathsf{SO}(3)$ then $r = 1$, $\beta_1 = \beta_1'$ and $\beta_2 = \beta_2'$. Then the matrices $\mathrm{diag}\,(\sigma_1\sigma_2, \sigma_1, \sigma_2)$, where $\sigma_1, \sigma_2 \in \{-1, 1\}$, are the only matrices such that $R\mathrm{diag}\,(1, \beta_1, \beta_2)R^\top = \mathrm{diag}\,(1, \beta_1, \beta_2)$. It then follows that

$$\beta_1\,(u_2\sigma_1 - \alpha_2)^2 + \beta_2\,(u_3\sigma_2 - \gamma)^2 + (u_1\sigma_1\sigma_2 - \alpha_1)^2 = r\left((u_1 - \alpha_1')^2 + (u_2 - \alpha_2')^2\beta_1' + (\gamma_2' - u_3)^2\beta_2'\right).$$

Expanding the above equation and comparing coefficients it is straightforward to determine that this equation holds only if $r = 1$, $\boldsymbol{\beta} = \boldsymbol{\beta}'$, $\boldsymbol{\alpha} = \boldsymbol{\alpha}'$, and $\gamma = \gamma'$. Hence, each of these representatives are distinct and non-equivalent. (The verifications for $\chi^2_{\alpha\boldsymbol{\beta}\eta}$ and $\chi^3_{\boldsymbol{\eta\beta}}$ are similar and thus shall be omitted.)

On the other hand, if $0 < \beta_2 < \beta_1 = 1$, then the transformations preserving this quadratic form are the all matrices of the form $\begin{bmatrix} S & 0 \\ 0 & \det S \end{bmatrix} \in \mathsf{SO}\,(3)$. There then exist $\eta \geq 0$ and $\theta, z \in \mathbb{R}$ such that

$$\mu_1' = \eta\cos\theta, \qquad \mu_2' = \eta\sin\theta \quad \text{and} \quad \mu_3' = z.$$

Now, for $\varphi_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$, we have

$$\chi_3(u) = (\chi_2 \circ \varphi_2)(u) = \left( u - \begin{bmatrix} \eta \\ 0 \\ z \end{bmatrix} \right)^\top \operatorname{diag}(1, 1, \beta) \left( u - \begin{bmatrix} \eta \\ 0 \\ z \end{bmatrix} \right).$$

In fact, we may assume that $z \geq 0$. If not, apply the automorphism $\operatorname{diag}(1, -1, -1)$. That is every such cost-extended system is cost equivalent to the system

$$(\Sigma^{(3,0)}, \chi^4_{\boldsymbol{\eta}\beta}) : \begin{cases} \Xi^{(3,0)}(\mathbf{1}, u) = u_1 E_1 + u_2 E_2 + u_3 E_3 \\ \chi^4_{\boldsymbol{\eta}\beta}(u) = (u_1 - \eta_1)^2 + u_2^2 + \beta(u_3 - \eta_2)^2. \end{cases} \tag{3.5}$$

Here $\eta_1, \eta_2 \geq 0$ and $0 < \beta < 1$. We verify that each different choice of parameters leads to a distinct non-equivalent representative. Considering the equation $\chi^4_{\boldsymbol{\eta}\beta} \circ \varphi = r\chi^4_{\boldsymbol{\eta}'\beta'}$, for some $r > 0$, we need only check this condition for $\varphi = \begin{bmatrix} S & 0 \\ 0 & \det S \end{bmatrix} \in \mathsf{SO}\,(3)$. It then follows that

$$u_1^2 + u_2^2 + \eta_1^2 + \beta\left(u_3^2 + \eta_2^2\right) - 2\left(\left(u_1 \cos\theta - u_2 \sin\theta\right)\eta_1 + \beta u_3 \eta_2\right)\sigma_1 = r\left(u_2^2 + \left(u_1 - \eta_1'\right)^2 + \beta'\left(u_3 - \eta_2'\right)^2\right).$$

Expanding the above equation and comparing coefficients it is straightforward to determine that this equation holds only if $r = 1$, $\beta = \beta'$, and $\boldsymbol{\eta} = \boldsymbol{\eta}'$. Hence, each of these representatives are distinct and non-equivalent.

Similarly, if $0 < \beta_1 = \beta_2 < 1$, then the transformations preserving this quadratic form are all the matrices of the form $\begin{bmatrix} \det S & 0 \\ 0 & S \end{bmatrix} \in \mathsf{SO}\,(3)$. Then there exist $\eta > 0$ and $\theta, z \in \mathbb{R}$ such that $\mu_1' = z$, $\mu_2' = \eta \sin\theta$, and $\mu_3' = \eta\cos\theta$. Now, for $\varphi_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$, we have that

$$\chi_3(u) = (\chi_2 \circ \varphi_2)(u) = \left( u - \begin{bmatrix} z \\ 0 \\ \eta \end{bmatrix} \right)^\top \operatorname{diag}(1, \beta, \beta) \left( u - \begin{bmatrix} z \\ 0 \\ \eta \end{bmatrix} \right).$$

Again, we may in fact assume that $z \geq 0$. That is, every such cost-extended system is cost equivalent to the system

$$(\Sigma^{(3,0)}, \chi^5_{\boldsymbol{\eta}\beta}) : \begin{cases} \Xi^{(3,0)}(\mathbf{1}, u) = u_1 E_1 + u_2 E_2 + u_3 E_3 \\ \chi^5_{\boldsymbol{\eta}\beta}(u) = (u_1 - \eta_1)^2 + \beta u_2^2 + \beta(u_3 - \eta_2)^2. \end{cases} \tag{3.6}$$

Here $\eta_1, \eta_2 \geq 0$ and $0 < \beta < 1$. The verification is almost exactly the same as for the case when $0 < \beta_2 < \beta_1 = 1$.

Lastly, if $0 < \beta_2 = \beta_1 = 1$, then any $R \in \mathsf{SO}\,(3)$ preserves this quadratic form. There then exist

$\eta \geq 0$ and $\varphi_2 \in \mathsf{SO}\,(3)$ such that

$$\chi_3(u) = (\chi_2 \circ \varphi_2)(u) = \left( u - \begin{bmatrix} \eta \\ 0 \\ 0 \end{bmatrix} \right)^\top \left( u - \begin{bmatrix} \eta \\ 0 \\ 0 \end{bmatrix} \right).$$

That is every such cost-extended system is cost equivalent to the system

$$(\Sigma^{(3,0)}, \chi^6_\eta) : \begin{cases} \Xi^{(3,0)}(\mathbf{1}, u) = u_1 E_1 + u_2 E_2 + u_3 E_3 \\ \chi^6_\eta(u) = (u_1 - \eta)^2 + u_2^2 + u_3^2, \quad \eta \geq 0. \end{cases} \tag{3.7}$$

Again, straightforward calculations show that each of these representatives are distinct.

Finally, we verify that none of the systems (3.2), (3.3), (3.4), (3.5), (3.6), (3.7) are equivalent to one another. It follows almost immediately that $(\Sigma^{(3,0)}, \chi^4_{\boldsymbol{\eta}\beta})$, $(\Sigma^{(3,0)}, \chi^5_{\boldsymbol{\eta}\beta})$, and $(\Sigma^{(3,0)}, \chi^6_\eta)$ cannot be equivalent to any other system as their respective quadratic parts cannot be the same. That is, from lemma 4.2.11, it follows that if $R\mathrm{diag}\,(a_1, a_2, a_3)R^\top = \mathrm{diag}\,(a'_1, a'_2, a'_3)$ for some $R \in \mathsf{SO}(3)$ then $\boldsymbol{a} = \boldsymbol{a}'$. Thus we need only consider the systems $(\Sigma^{(3,0)}, \chi^1_{\boldsymbol{\alpha\beta}\gamma})$, $(\Sigma^{(3,0)}, \chi^2_{\boldsymbol{\alpha\beta}\eta)}$, and $(\Sigma^{(3,0)}, \chi^3_{\boldsymbol{\eta\beta}})$. As in the verification of (3.2), we need only consider those transformations of the controls of the form $\varphi = \mathrm{diag}\,(\sigma_1\sigma_2, \sigma_1, \sigma_2)$, where $\sigma_1, \sigma_2 \in \{-1, 1\}$. Considering the condition $\chi^1_{\boldsymbol{\alpha\beta}\gamma} \circ \varphi = r\chi^2_{\alpha'\boldsymbol{\beta}'\eta'}$, for some $r > 0$, it follows that

$$\beta_1 \left(u_2\sigma_1 - \alpha_2\right)^2 + \beta_2 \left(u_3\sigma_2 - \gamma\right)^2 + \left(u_1\sigma_1\sigma_2 - \alpha_1\right)^2 = r\left(u_1^2 + \left(u_2 - \alpha'_1\right)^2\beta'_1 + \left(u_3 - \eta'_1\right)^2\beta'_2\right).$$

It is straightforward to verify that there is no solution of this equation, which satisfying the conditions on the parameters. In a similar fashion, we get that none of the systems $(\Sigma^{(3,0)}, \chi^1_{\boldsymbol{\alpha\beta}\gamma})$, $(\Sigma^{(3,0)}, \chi^2_{\boldsymbol{\alpha\beta}})$, or $(\Sigma^{(3,0)}, \chi^3_{\boldsymbol{\alpha\beta}})$ are equivalent.

□

| **Cost-equivalence classification** | | |
|---|---|---|
| | $\Xi(\mathbf{1},u)$ | Cost function $\chi$ |
| $\Sigma_{\alpha}^{(1,1)}$ | $\alpha E_1 + u_1 E_2$ | $\chi_{\eta}(u) = (u_1 - \eta_1)^2$ |
| $\Sigma^{(2,0)}$ | $u_1 E_2 + u_2 E_3$ | $\chi^1_{\boldsymbol{\eta}\beta}(u) = (u_1 - \eta_1)^2 + \beta_1(u_2 - \eta_2)^2$ |
| | | $\chi^2_{\eta}(u) = (u_1 - \eta_1)^2 + u_2^2$ |
| $\Sigma_{\alpha}^{(2,1)}$ | $\alpha E_1 + u_1 E_2 + u_2 E_3$ | $\chi^1_{\alpha\beta\gamma}(u) = (u_1 - \alpha_1)^2 + \beta_1(u_2 - \gamma)^2$ |
| | | $\chi^2_{\beta\eta}(u) = u_1^2 + \beta_1(u_2 - \eta_1)^2$ |
| | | $\chi^3_{\eta}(u) = (u_1 - \eta_1)^2 + u_2^2$ |
| $\Sigma^{(3,0)}$ | $u_1 E_1 + u_2 E_2 + u_3 E_3$ | $\chi^1_{\boldsymbol{\alpha\beta}\gamma}(u) = (u_1 - \alpha_1)^2 + \beta_1(u_2 - \alpha_2)^2 + \beta_2(u_3 - \gamma)^2$ |
| | | $\chi^2_{\alpha\boldsymbol{\beta\eta}}(u) = u_1^2 + \beta_1(u_2 - \alpha_1)^2 + \beta_2(u_3 - \eta_1)^2$ |
| | | $\chi^3_{\boldsymbol{\eta\beta}}(u) = (u_1 - \eta_1)^2 + \beta_1 u_2^2 + \beta_2(u_3 - \eta_2)^2$ |
| | | $\chi^4_{\boldsymbol{\eta}\beta}(u) = (u_1 - \eta_1)^2 + u_2^2 + \beta_1(u_3 - \eta_2)^2$ |
| | | $\chi^5_{\boldsymbol{\eta}\beta}(u) = (u_1 - \eta_1)^2 + \beta_1 u_2^2 + \beta_1(u_3 - \eta_2)^2$ |
| | | $\chi^6_{\eta}(u) = (u_1 - \eta_1)^2 + u_2^2 + u_3^2$ |
| $\alpha, \alpha_1, \alpha_2 > 0, \quad \eta_1, \eta_2 \geq 0, \quad \gamma \in \mathbb{R}, \quad 0 < \beta_2 < \beta_1 < 1$ | | |

Table 3.1: Classification of cost-extended systems on $\mathsf{SO}\,(3)$

## 3.3 Hamilton-Poisson systems

In this section we provide a classification of all (quadratic) Hamilton-Poisson systems on $\mathfrak{so}(3)^*_-$ under affine equivalence. This work has been published in [6]. It is interesting to note that Tudoran, in [60], introduced a stronger form of equivalence, called orthogonal equivalence, in the investigation of homogeneous quadratic Hamilton-Poisson systems on the Lie-Poisson space $\mathfrak{so}\,(3)^*_-$. Essentially, in [60], two systems are equivalent if there exists an orthogonal matrix, i.e., an element in $\mathsf{SO}(3)$, such that the solutions of one system are mapped into the solutions of the other system, under this orthogonal map. In our work, orthogonal equivalence corresponds to two systems being equivalent under a linear Poisson automorphism. Although our equivalence relation is much weaker, the stability nature of the equilibria and the integral curves of the Hamiltonian vector fields are still preserved under our equivalence.

One of the main contributions of this section is our investigation of inhomogeneous quadratic Hamilton-Poisson systems on the Lie-Poisson space $\mathfrak{so}\,(3)^*_-$. There are nine (families of) such systems, under affine equivalence. A system will be referred to as a system of type I if its set of equilibria is a union of lines or planes; otherwise, it will be referred to as a system of type II. For the sake of completeness, a brief treatment of the homogeneous systems is included.

For each system we investigate the (Lyapunov) stability nature of the equilibria. Energy-Casimir methods (see appendix C) are used to prove stability; note that for any system on $\mathfrak{so}(3)^*_-$ the origin is a stable equilibrium state. On the other hand, instability usually follows from spectral instability; however, a direct approach is required in some cases. Throughout this section, we graph the stable and unstable equilibria in blue and red, respectively.

We obtain explicit expressions for the integral curves of systems of type I (but not of type II). In each case we partition the set of initial conditions so as to distinguish between integral curves with different qualitative behaviour. The equations of motion are reduced (using the constants of motion) to a single separable differential equation, which is then transformed into a standard form (see appendix D). An appropriate elliptic integral is used to obtain (after some manipulation) an explicit expression for the integral curve in terms of Jacobi elliptic functions.

We distinguish between integral curves with different qualitative behaviour by determining when the level surfaces of the Hamiltonian and Casimir are tangent to one another. These surfaces are tangent exactly at equilibria. Hence we get a set of critical values (corresponding to equilibria) for the energy states $(h_0, c_0)$ of the Hamiltonian and Casimir. This set partitions the space of energy states into a number of regions. (Within each region, the corresponding nonconstant integral curves can be continuously deformed into one another.) Each region usually corresponds to different explicit expressions for the integral curves.

For each system we graph the critical energy states. We select some typical values for $(h_0, c_0)$ from each region (as well as some typical critical values) for which we then graph the corresponding level surfaces of the Hamiltonian and Casimir. For convenience, we shall refer to this as a typical configuration. The intersection of these surfaces (i.e., the traces of the corresponding integral curves) and the equilibria are also graphed.

### 3.3.1 Affine equivalence

We now provide a classification of all quadratic Hamilton-Poisson systems on $\mathfrak{so}(3)^*_-$ under affine equivalence.

MATHEMATICA. *For detailed calculations of the verification procedure of the following theorem refer to the Mathematica file:*

*Thesis Mathematica\SO(3)\Affine equivalence\AequivVerification.nb*

3.3.1 THEOREM. *Let $H$ be a quadratic Hamilton-Poisson system on $\mathfrak{so}(3)^*_-$. If $H$ is homogeneous, then it is equivalent to exactly one of the systems:*

$$H^0(p) = 0 \qquad \text{(type I)}$$
$$H^1(p) = \tfrac{1}{2}p_1^2 \qquad \text{(type I)}$$
$$H^2(p) = p_1^2 + \tfrac{1}{2}p_2^2 \qquad \text{(type I).}$$

*If $H$ is inhomogeneous, then it is equivalent to exactly one of the systems:*

$$\begin{array}{lll}
H^0_{1,\alpha}(p) = \alpha p_1 & \alpha > 0 & \textit{(type I)} \\
H^1_0(p) = \tfrac{1}{2}p_1^2 & & \textit{(type I)} \\
H^1_1(p) = p_2 + \tfrac{1}{2}p_1^2 & & \textit{(type I)} \\
H^1_{2,\alpha}(p) = p_1 + \alpha p_2 + \tfrac{1}{2}p_1^2 & \alpha > 0 & \textit{(type II)} \\
H^2_{1,\alpha}(p) = \alpha p_1 + p_1^2 + \tfrac{1}{2}p_2^2 & \alpha > 0 & \textit{(type I)} \\
H^2_{2,\alpha}(p) = \alpha p_2 + p_1^2 + \tfrac{1}{2}p_2^2 & \alpha > 0 & \textit{(type I)} \\
H^2_{3,\alpha}(p) = \alpha_1 p_1 + \alpha_2 p_2 + p_1^2 + \tfrac{1}{2}p_2^2 & \alpha_1, \alpha_2 > 0 & \textit{(type II)} \\
H^2_{4,\alpha}(p) = \alpha_1 p_1 + \alpha_3 p_3 + p_1^2 + \tfrac{1}{2}p_2^2 & \alpha_1 \geq \alpha_3 > 0 & \textit{(type II)} \\
H^2_{5,\alpha}(p) = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + p_1^2 + \tfrac{1}{2}p_2^2 & & \\
\qquad \alpha_2 > 0,\ \alpha_1 > |\alpha_3| > 0 \quad \textit{or} \quad \alpha_2 > 0,\ \alpha_1 = \alpha_3 > 0 & & \textit{(type II).}
\end{array}$$

*Here $\alpha, \alpha_1, \alpha_2, \alpha_3$ parametrize families of class representatives, each different value corresponding to a distinct (non-equivalent) representative.*

PROOF. Let $H(p) = pA + pQp^\top$, where $Q$ is a symmetric $3 \times 3$ matrix. Here $A = a_1E_1 + a_2E_2 + a_3E_3 \in \mathfrak{so}(3)$ is identified with $\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}^\top$. We may assume that $Q$ is positive definite. (If $Q$ is not positive definite, then $H$ is equivalent to a system $H + \mu C$ for which the quadratic form is positive definite, for some sufficiently large $\mu$.)

Given a linear Poisson automorphism $\psi : p \mapsto p\Psi$, we have

$$(H \circ \psi)(p) = p\Psi A + p\Psi Q \Psi^\top p^\top.$$

As any symmetric matrix can be diagonalized by an orthogonal matrix (see, e.g., [56]), it follows that there exists a linear Poisson automorphism $\psi$ such that $(H \circ \psi)(p) = p\Psi A + p\,\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)\,p^\top$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$. Thus

$$(H \circ \psi)(p) - \lambda_3 C(p) = p\Psi A + p\,\mathrm{diag}(\lambda_1 - \lambda_3, \lambda_2 - \lambda_3, 0)\,p^\top$$

with $\lambda_1 - \lambda_3 \geq \lambda_2 - \lambda_3 \geq 0$. If $\lambda_1 - \lambda_3 = 0$, then (by ($\mathfrak{E}1$) and ($\mathfrak{E}3$)) $H$ is equivalent to an intermediate system $G^0_B(p) = pB$, where $B = \Psi A$. On the other hand, if $\lambda_1 - \lambda_3 > 0$, then

$$(H \circ \psi)(p) - \lambda_3 C(p) = p\Psi A + (\lambda_1 - \lambda_3)\,p\,\mathrm{diag}\left(1, \tfrac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3}, 0\right)\,p^\top$$

and so $H$ is equivalent to $H'(p) = p\Psi A + p_1^2 + \alpha p_2^2$, with $\alpha = \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3}$. If $\alpha = 0$, then $H'(p) = p\Psi A + p_1^2$ and so $H'$ is equivalent to (an intermediate system) $G^1_B(p) = pB + \frac{1}{2}p_1^2$ with $B = \Psi A$. Suppose $\alpha = 1$. Then

$$\psi' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

is a linear Poisson automorphism such that $(H' \circ \psi' - C)(p) = p\Psi'\Psi A - p_1^2$. So $H'$ is equivalent to $G^1_B(p) = pB + \frac{1}{2}p_1^2$, where $B = \Psi'\Psi A$. On the other hand, suppose $0 < \alpha < 1$. Then the vector fields associated to

$$H'(p) = a_1'p_1 + a_2'p_2 + a_3'p_3 + p_1^2 + \alpha p_2^2, \quad A' = \Psi A$$

and

$$G_B^2(p) = b_1p_1 + b_2p_2 + b_3p_3 + p_1^2 + \tfrac{1}{2}p_2^2$$

are compatible with the affine isomorphism

$$p \mapsto p \begin{bmatrix} -\sqrt{2(1-\alpha)} & 0 & 0 \\ 0 & 2\sqrt{\alpha(1-\alpha)} & 0 \\ 0 & 0 & -\sqrt{2\alpha} \end{bmatrix} + \begin{bmatrix} -\frac{1-2\alpha}{\sqrt{2(1-\alpha)}}a_1' \\ \frac{1-2\alpha}{2\sqrt{\alpha(1-\alpha)}}a_2' \\ -\frac{1-2\alpha}{\sqrt{2\alpha}}a_3' \end{bmatrix}$$

provided $b_1 = -\frac{\alpha\sqrt{2(1-\alpha)}}{1-\alpha}a_1'$, $b_2 = \frac{1}{2\sqrt{\alpha(1-\alpha)}}a_2'$, and $b_3 = -\frac{\sqrt{2}(1-\alpha)}{\sqrt{\alpha}}a_3'$.

Suppose that $H$ is homogeneous, i.e., $A = 0$. Then, by the above argument, $H$ is equivalent to $G_{\mathbf{0}}^0 = H^0$, $G_{\mathbf{0}}^1 = H^1$ or $G_{\mathbf{0}}^2 = H^2$. No two of the systems $H^0$, $H^1$ and $H^2$ are equivalent. Indeed, if two systems are equivalent, then there exists an affine bijection between their equilibria. However, the set of equilibria for $H^1$ is the union of a plane and a line, whereas the set of equilibria for $H^2$ is the union of three lines.

On the other hand, suppose that $H$ is inhomogeneous. Then, by the above argument, $H$ is equivalent to one of the intermediate systems

$$G_B^0(p) = pB \qquad G_B^1(p) = pB + \tfrac{1}{2}p_1^2 \qquad G_B^2(p) = pB + p_1^2 + \tfrac{1}{2}p_2^2$$

for some $B \in \mathfrak{so}(3)$. (Here $B$ is the image of $A$ under some linear isomorphism and so $B \neq \mathbf{0}$). We note that $G_B^1$ and $G_{B'}^2$ are not equivalent for any $B, B' \in \mathfrak{so}(3)$. Indeed, if they were equivalent, then a simple calculation shows that their homogeneous parts $H^1$ and $H^2$ would be equivalent, a contradiction. Likewise, $G_B^0$ cannot be equivalent to $G_{B'}^1$ or $G_{B'}^2$ (for any $B, B' \in \mathfrak{so}(3)$).

Suppose $H$ is equivalent to $G_B^0$. As $\mathsf{SO}(3)$ acts transitively on any sphere, there exists a linear Poisson automorphism $\psi$ such that $(G_B^0 \circ \psi)(p) = \alpha p_1$ for some $\alpha > 0$. Thus $H$ is equivalent to $H_{1,\alpha}^0$. We claim that $H_{1,\alpha}^0$ and $H_{1,\beta}^0$ are equivalent only if $\alpha = \beta$ (i.e., no further reduction is possible). Indeed, if they are equivalent, then there exists an affine isomorphism $\psi : p \mapsto p\Psi + q$ such that $T\psi \cdot \vec{H}_{1,\alpha}^0 = \vec{H}_{1,\beta}^0 \circ \psi$, i.e.,

$$\begin{aligned} -\alpha\Psi_{31}p_2 + \alpha\Psi_{21}p_3 &= 0 \\ -\alpha\Psi_{32}p_2 + \alpha\Psi_{22}p_3 - \beta(\Psi_{31}p_1 + \Psi_{32}p_2 + \Psi_{33}p_3 + q_3) &= 0 \\ -\alpha\Psi_{33}p_2 + \alpha\Psi_{23}p_3 - \beta(\Psi_{21}p_1 + \Psi_{22}p_2 + \Psi_{23}p_3 + q_2) &= 0 \end{aligned}$$

for all $p \in \mathfrak{so}(3)^*$; however this implies that $\alpha = \beta$. (Here $\Psi = [\Psi_{ij}]$.)

Next, suppose $H$ is equivalent to $G_B^1$. Given a linear Poisson automorphism $\psi : p \mapsto p\Psi$, we have that $(G_B^1 \circ \psi)(p) = p\Psi B + p\Psi \mathrm{diag}\,(\tfrac{1}{2}, 0, 0)\Psi^\top p^\top$. Now $\Psi$ leaves $\mathrm{diag}\,(\tfrac{1}{2}, 0, 0)$ invariant, i.e., $\Psi \mathrm{diag}\,(\tfrac{1}{2}, 0, 0)\Psi^\top = \mathrm{diag}\,(\tfrac{1}{2}, 0, 0)$ if and only if $\Psi = \begin{bmatrix} \det(S) & 0 \\ 0 & S \end{bmatrix}$, $S \in \mathrm{O}(2)$. Thus, there exists a linear Poisson automorphism $\psi$ such that

$$H'(p) = (G_B^1 \circ \psi)(p) = \gamma_1 p_1 + \gamma_2 p_2 + \tfrac{1}{2}p_1^2$$

for some $\gamma_1, \gamma_2 \geq 0$, $(\gamma_1, \gamma_2) \neq (0,0)$. Assume $\gamma_1 = 0$. Then the vector fields associated to $H'(p) = \gamma_2 p_2 + \frac{1}{2}p_1^2$ and $H_1^1(p) = p_2 + \frac{1}{2}p_1^2$ are compatible with the affine isomorphism

$$p \mapsto p \begin{bmatrix} 1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 - \gamma_2^2 \\ 0 \end{bmatrix} \cdot$$

Assume $\gamma_2 = 0$. Then the vector fields associated to $H'(p) = \gamma_1 p_1 + \frac{1}{2}p_1^2$ and $H^1(p) = \frac{1}{2}p_1^2$ are compatible with the affine isomorphism

$$p \mapsto p \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} -\gamma_1 \\ 0 \\ 0 \end{bmatrix} \cdot$$

Assume $\gamma_1, \gamma_2 > 0$. Then the vector fields associated to $H'(p) = \gamma_1 p_1 + \gamma_2 p_2 + \frac{1}{2}p_1^2$ and $H_{2,\alpha}^1(p) = p_1 + \alpha p_2 + \frac{1}{2}p_1^2$, $\alpha = \gamma_2\sqrt{\gamma_1}$ are compatible with the affine isomorphism

$$p \mapsto p \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{\gamma_1}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\gamma_1}} \end{bmatrix} + \begin{bmatrix} \gamma_1 - 1 \\ \frac{(\gamma_1 - 1)\gamma_2}{\sqrt{\gamma_1}} \\ 0 \end{bmatrix} \cdot$$

By direct computation, it is straightforward to show that $T\psi \cdot \vec{H}_{2,\alpha}^1 = \vec{H}_{2,\beta}^1 \circ \psi$ for some affine isomorphism $\psi$ only if $\alpha = \beta$. Thus $H_{2,\alpha}^1$ is equivalent to $H_{2,\beta}^1$ only if $\alpha = \beta$.

Lastly, suppose $H$ is equivalent to $G_B^2(p) = b_1 p_1 + b_2 p_2 + b_3 p_3 + p_1^2 + \frac{1}{2}p_2^2$. Assume $b_1, b_2, b_3$ are all nonzero. Note that

$$\psi : p \mapsto p\Psi, \qquad \Psi = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

is a linear Poisson automorphism such that $(G_B^2 \circ \psi - C)(p) = b_3 p_1 + b_2 p_2 - b_1 p_3 - p_1^2 - \frac{1}{2}p_2^2$. Thus $G_B^2$ is equivalent to a system $H'(p) = b_1' p_1 + b_2' p_2 + b_3' p_3 + p_1^2 + \frac{1}{2}p_2^2$, where $|b_1'| \geq |b_3'| > 0$. The linear Poisson automorphisms $\operatorname{diag}(-1, 1, -1)$, $\operatorname{diag}(1, -1, -1)$, and $\operatorname{diag}(-1, -1, 1)$ allow us to change the signs of $b_1$ and $b_2$. Thus $H'$ is equivalent to $H_{5,\boldsymbol{\beta}}^2(p) = \beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3 + p_1^2 + \frac{1}{2}p_2^2$ for some $\boldsymbol{\beta}$, where $\beta_1 \geq |\beta_3| > 0$ and $\beta_2 > 0$. If $\beta_1 = |\beta_3|$, then there exists an automorphism such that $H'$ is equivalent to $H_{5,\boldsymbol{\beta}}^2(p) = \beta_1 p_1 + \beta_2 p_2 + \beta_1 p_3 + p_1^2 + \frac{1}{2}p_2^2$ for some $\beta_1, \beta_2 > 0$. Likewise, if $b_1 = 0$, $b_2 = 0$, $b_3 = 0$, $b_1 = b_2 = 0$, $b_1 = b_3 = 0$, or $b_2 = b_3 = 0$, then $H$ is equivalent to $H_{3,\boldsymbol{\beta}}^2$, $H_{4,\boldsymbol{\beta}}^2$, $H_{3,\boldsymbol{\beta}}^2$, $H_{1,\beta}^2$, $H_{2,\beta}^2$, or $H_{1,\beta}^2$ (respectively) for some $\beta_1 \geq |\beta_3| > 0$, $\beta_2 > 0$.

Direct computations shows that $H_{i,\boldsymbol{\beta}}^2$ is equivalent to $H_{i,\boldsymbol{\beta}'}^2$ $(i = 1, \ldots 5)$ only if $\boldsymbol{\beta} = \boldsymbol{\beta}'$.

□

3.3.2 REMARK. For any Hamilton-Poisson system on $\mathfrak{so}(3)_-^*$, the origin is a stable equilibrium state. This follows trivially from the continuous energy-Casimir method since $C(p) = p_1^2 + p_2^2 + p_3^2$ is a Casimir function for $\mathfrak{so}(3)_-^*$.

It turns out that any homogeneous system on $\mathfrak{so}\,(3)^*_-$ is equivalent to a system on $\mathfrak{se}\,(2)^*_-$, see [22]. The Euclidean Lie algebra

$$\mathfrak{se}\,(2) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix} = x_1\tilde{E}_1 + x_2\tilde{E}_2 + x_3\tilde{E}_3 \ : \ x_1, x_2, x_3 \in \mathbb{R} \right\}$$

has nonzero commutators $[\tilde{E}_2, \tilde{E}_3] = \tilde{E}_1$ and $[\tilde{E}_3, \tilde{E}_1] = \tilde{E}_2$. The Lie-Poisson space $\mathfrak{se}\,(2)^*_-$ has Casimir function $\widetilde{C}(\tilde{p}) = \tilde{p}_1^2 + \tilde{p}_2^2$. The systems

$$\left(\mathfrak{so}\,(3)^*_-,\ \tfrac{1}{2}p_1^2\right) \ : \ \begin{cases} \dot{p}_1 = 0 \\ \dot{p}_2 = p_1 p_3 \\ \dot{p}_3 = -p_1 p_2 \end{cases} \qquad \text{and} \qquad \left(\mathfrak{se}\,(2)^*_-,\ \tfrac{1}{2}\tilde{p}_3^2\right) \ : \ \begin{cases} \dot{\tilde{p}}_1 = \tilde{p}_2\tilde{p}_3 \\ \dot{\tilde{p}}_2 = -\tilde{p}_1\tilde{p}_3 \\ \dot{\tilde{p}}_3 = 0 \end{cases}$$

are compatible with the linear isomorphism

$$\psi : \mathfrak{so}\,(3)^* \to \mathfrak{se}\,(2)^*, \qquad \psi = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

On the other hand, the systems

$$\left(\mathfrak{so}\,(3)^*_-,\ p_1^2 + \tfrac{1}{2}p_2^2\right) \ : \ \begin{cases} \dot{p}_1 = -p_2 p_3 \\ \dot{p}_2 = 2p_1 p_3 \\ \dot{p}_3 = -p_1 p_2 \end{cases} \qquad \text{and} \qquad \left(\mathfrak{se}\,(2)^*_-,\ \tilde{p}_2^2 + \tilde{p}_3^2\right) \ : \ \begin{cases} \dot{\tilde{p}}_1 = 2\tilde{p}_2\tilde{p}_3 \\ \dot{\tilde{p}}_2 = -2\tilde{p}_1\tilde{p}_3 \\ \dot{\tilde{p}}_3 = 2\tilde{p}_1\tilde{p}_2 \end{cases}$$

are compatible with the linear isomorphism

$$\psi : \mathfrak{so}\,(3)^* \to \mathfrak{se}\,(2)^*, \qquad \psi = \begin{bmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}.$$

We shall make use of such an equivalence in the investigation of the system $H^2_{1,\alpha}$ to relate to some results previously obtained.

3.3.3 REMARK. Throughout the following sections we present results on the stability nature of the equilibrium states for the systems corresponding to the different normal forms obtained in our classification of Hamilton-Poisson systems on $\mathfrak{so}(3)^*_-$. For each case we provide what we consider the important details required to justify our claims but omit the technical calculations and details. (See below.)

MATHEMATICA. *The explicit calculations and technical details for the stability nature of each of these system's equilibria are contained in the Mathematica file:*

*Thesis Mathematica\SO(3)\Stability\StabilitySO(3).nb*

### 3.3.2 Homogeneous systems

We consider the two homogeneous systems $H^1$ and $H^2$ (see theorem 3.3.1). The integral curves of the system $H^1$ can easily be found in terms of elementary functions; it is then a simple matter to determine the stability nature of the equilibria. On the other hand, the integral curves of the system $H^2$ can be found in terms of basic Jacobi elliptic functions. The stability nature of the equilibria can be determined via the energy-Casimir methods (and the investigation of spectral stability). Somewhat less refined versions of these results were obtained elsewhere (cf. [29], see also [5]).

Throughout, we shall parametrize the equilibrium states by $\mu, \nu, \eta \in \mathbb{R},\ \nu \neq 0$.

#### System $H^1$

The system $H^1(p) = \frac{1}{2}p_1^2$ has equations of motion

$$\begin{cases} \dot{p}_1 = 0 \\ \dot{p}_2 = p_1 p_3 \\ \dot{p}_3 = -p_1 p_2. \end{cases}$$

The equilibria are $\mathsf{e}_1^{\mu,\eta} = (0, \mu, \eta)$ and $\mathsf{e}_2^{\mu} = (\mu, 0, 0)$. In Figure 3.1 we graph the critical energy states $(c_0, h_0)$ and a corresponding typical configuration.

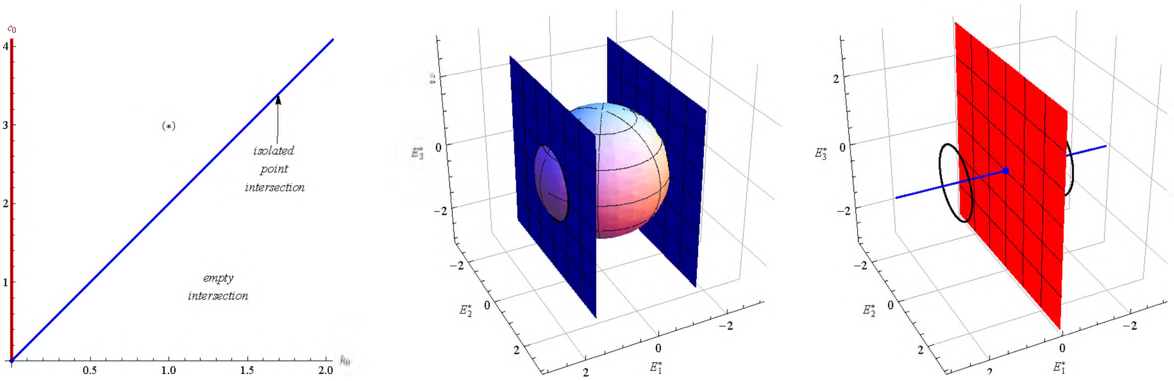


Figure 3.1: Critical energy states for $H^1$ and a corresponding typical configuration

3.3.4 THEOREM. *The equilibrium states have the following behaviour:*

*(i) Each equilibrium state* $\mathsf{e}_1^{\mu,\nu}$ *is unstable*

*(ii) Each equilibrium state* $\mathsf{e}_2^{\mu}$ *is stable.*

PROOF. (i) Consider the equilibrium state $\mathsf{e}_1^{\mu,\nu}$. Let $\mathcal{B}_\epsilon$ be the open ball of radius $\epsilon = \sqrt{\mu^2+\nu^2} > 0$ around the point $e_1^{\mu,\nu} = (0, \mu, \nu)$. We have that $p(t) = (\delta, \mu\cos(\delta t) + \nu\sin(\delta t), \nu\cos(\delta t) - \mu\sin(\delta t))$ is an integral curve of $\vec{H}_1$ (for $\delta > 0$) such that $\|p(0) - \mathsf{e}_1^{\mu,\nu}\| = \delta$. Accordingly, for any neighbourhood $V \subset \mathcal{B}_\epsilon$ of $\mathsf{e}_1^{\mu,\nu}$ there exists $\delta > 0$ such that $p(0) \in V$. Furthermore, $\|p(\frac{\pi}{3\delta}) - \mathsf{e}_1^{\mu,\nu}\| =$

$\sqrt{\delta^2+\mu^2+\nu^2} > \epsilon$. Thus, there exists a $t_1 = \frac{\pi}{3\delta} > 0$ such that $p(t_1) \notin \mathcal{B}_\epsilon$. Hence, by definition, the states $\mathsf{e}_1^{\mu,\nu}$ are unstable.

(ii) Let $H_\lambda(p) = \lambda_1 H^1(p) + \lambda_2 C(p)$. Let $\mu \neq 0$ and let $\lambda_1 = 1$ and $\lambda_2 = -\frac{1}{2}$. We have that $\mathbf{d}H_\lambda(\mathsf{e}_2^\mu) = 0$ and that the Hessian $\mathbf{d}^2 H_\lambda(e_2^\mu) = \operatorname{diag}(0,-1,-1)$ is definite when restricted to the subspace $W = \operatorname{span}\{(0,1,0),(0,0,1)\}$. Hence, by the generalized energy-Casimir method, the states $\mathsf{e}_2^\mu$ are stable. □

MATHEMATICA. *The verification calculations for the integral curves of the system $H^1$ are contained in the Mathematica file:*

*Thesis Mathematica\SO(3)\Integration\H1(p)\H1int.nb*

3.3.5 THEOREM. *Suppose $p(\cdot) : (-\epsilon, \epsilon) \to \mathfrak{so}(3)^*_-$ is an integral curve of $\vec{H}^1$ such that $H^1(p(0)) = h_0 > 0$, $C(p(0)) = c_0 > 0$, and $c_0 > 2h_0$. Then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t+t_0)$, where*

$$\begin{aligned} \bar{p}_1(t) &= \sigma\sqrt{2h_0} \\ \bar{p}_2(t) &= \sigma\sqrt{c_0 - 2h_0}\,\sin(\sqrt{2h_0}\,t) \\ \bar{p}_3(t) &= \sqrt{c_0 - 2h_0}\,\cos(\sqrt{2h_0}\,t). \end{aligned}$$

We note that for $c_0 = 2h_0$ the intersection of the level surfaces is two isolated points, and for $c_0 < 2h_0$ the intersection is empty. We omit the proof in this case as this is a straightforward result.

### System $H^2$

The system $H^2(p) = p_1^2 + \frac{1}{2}p_2^2$ has equations of motion

$$\begin{cases} \dot{p}_1 = -p_2 p_3 \\ \dot{p}_2 = 2p_1 p_3 \\ \dot{p}_3 = -p_1 p_2. \end{cases}$$

The equilibria are $\mathsf{e}_1^\mu = (\mu, 0, 0)$, $\mathsf{e}_2^\nu = (0, \nu, 0)$, and $\mathsf{e}_3^\nu = (0, 0, \nu)$.

There are three qualitatively different cases for the intersection of a parabolic cylinder $(H^2)^{-1}(h_0)$ and a sphere $C^{-1}(c_0)$, corresponding to (a) $c_0 < 2h_0$, (b) $c_0 = 2h_0$, and (c) $c_0 > 2h_0$. In Figure 3.2 we graph the critical energy states $(h_0, c_0)$; in Figure 3.3 we graph the corresponding typical configurations.

3.3.6 THEOREM. *The equilibrium states have the following behaviour:*

*(i) The states $\mathsf{e}_1^\mu$ are stable.*

*(ii) The states $\mathsf{e}_2^\nu$ are (spectrally) unstable.*

*(iii) The states $\mathsf{e}_3^\nu$ are stable.*

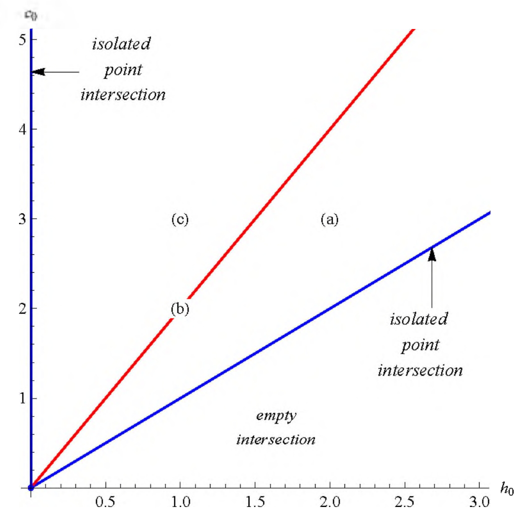
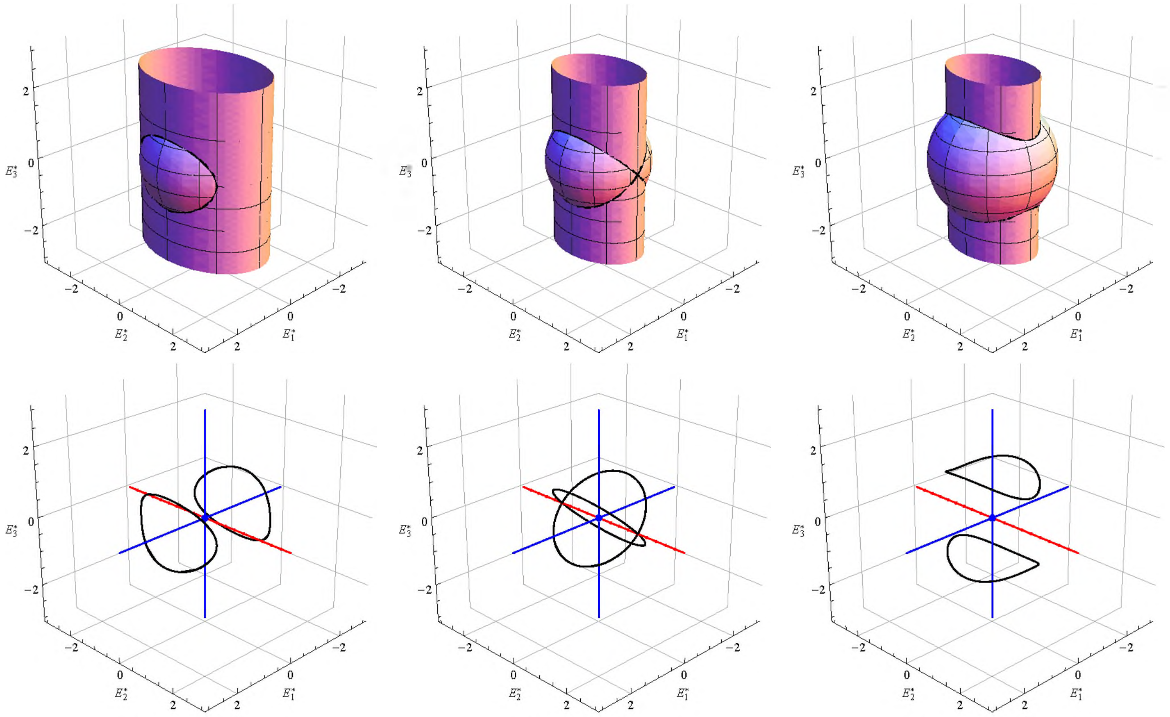


Figure 3.2: Critical energy states for $H^2$



(a) $c_0 < 2h_0$ (b) $c_0 = 2h_0$ (c) $c_0 > 2h_0$

Figure 3.3: Typical configurations for $H^2$

PROOF. Let $H_\lambda(p) = \lambda_1 H^2(p) + \lambda_2 C(p)$. (i) Suppose $\mu \neq 0$ and let $\lambda_1 = -\lambda_2 = 1$. We have $\mathbf{d}^2 H_\lambda(\mathsf{e}_1^\mu) = 0$ and that the Hessian $\mathbf{d}^2 H_\lambda(\mathsf{e}_1^\mu) = \operatorname{diag}(0, -1, -2)$ is definite when restricted to the subspace $W = \operatorname{span}\{(0,1,0),(0,0,1)\}$. Hence, by the generalized energy-Casimir method, the states $\mathsf{e}_1^\mu$ are stable.

(ii) The linearization of the system at $\mathsf{e}_2^\nu$ has eigenvalues $\lambda_1 = 0$ and $\lambda_{2,3} = \pm\nu$. Thus, the states $\mathsf{e}_2^\nu$ are spectrally unstable.

(iii) Let $\lambda_1 = 1$ and $\lambda_2 = 0$. We have $\mathbf{d}^2 H_\lambda(\mathsf{e}_3^\nu) = 0$ and that the Hessian $\mathbf{d}^2 H_\lambda(\mathsf{e}_3^\nu) = \operatorname{diag}(2, 1, 0)$ is definite when restricted to the subspace $W = \operatorname{span}\{(1,0,0),(0,1,0)\}$. Hence, by the generalized energy-Casimir method, the states $\mathsf{e}_3^\nu$ are stable.

□

MATHEMATICA. *The verification calculations for the integral curves of the system $H^2$ are contained in the Mathematica file:*

*Thesis Mathematica\SO(3)\Integration\H2(p)\H2int.nb*

We now derive the expressions for the integral curves of the system $H^2$. The proof for case (a) is similar to that of case (c). A full proof is provided. However, we only explain how the expressions for the integral curves were found in case (c).

3.3.7 THEOREM. *Let $p(\cdot)$ be an integral curve of the system $H^2$ through $p(0)$. Let $h_0 = H^2(p(0))$ and $c_0 = C(p(0))$.*

*(a) If $0 < c_0 < 2h_0$, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$, where*

$$\begin{aligned} \bar{p}_1(t) &= \sigma\sqrt{h_0}\,\operatorname{dn}(\Omega\, t,\, k) \\ \bar{p}_2(t) &= \sqrt{2}\sqrt{c_0 - h_0}\,\operatorname{sn}(\Omega\, t,\, k) \\ \bar{p}_3(t) &= \sigma\sqrt{c_0 - h_0}\,\operatorname{cn}(\Omega\, t,\, k)\,. \end{aligned}$$

*Here $\Omega = \sqrt{2h_0}$ and $k = \sqrt{\frac{c_0 - h_0}{h_0}}$.*

*(b) If $c_0 = 2h_0 > 0$, then there exist $t_0 \in \mathbb{R}$ and $\sigma_1, \sigma_2 \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$, where*

$$\begin{aligned} \bar{p}_1(t) &= \sigma_1\sqrt{h_0}\,\operatorname{sech}\left(\sqrt{2h_0}\, t\right) \\ \bar{p}_2(t) &= \sigma_1\sigma_2\sqrt{2h_0}\,\tanh\left(\sqrt{2h_0}\, t\right) \\ \bar{p}_3(t) &= \sigma_2\sqrt{h_0}\,\operatorname{sech}\left(\sqrt{2h_0}\, t\right). \end{aligned}$$

*(c) If* $c_0 > 2h_0 > 0$*, then there exist* $t_0 \in \mathbb{R}$ *and* $\sigma \in \{-1, 1\}$ *such that* $p(t) = \bar{p}(t + t_0)$*, where*

$$\begin{aligned}\bar{p}_1(t) &= \sqrt{h_0}\,\mathrm{cn}\,(\Omega\, t,\, k)\\ \bar{p}_2(t) &= \sigma\sqrt{2h_0}\,\mathrm{sn}\,(\Omega\, t,\, k)\\ \bar{p}_3(t) &= \sigma\tfrac{\Omega}{\sqrt{2}}\,\mathrm{dn}\,(\Omega\, t,\, k)\,.\end{aligned}$$

*Here* $\Omega = \sqrt{2}\sqrt{c_0 - h_0}$ *and* $k = \sqrt{\frac{h_0}{c_0 - h_0}}$.

PROOF. (a) The expressions for the integral curves in this case can be found by the method of reduction to standard form and applying the elliptic integral (D.2). We show that $p(\cdot)$ (as stated in the theorem) is an integral curve for $\sigma \in \{-1, 1\}$. Indeed, after some simplification we get

$$\frac{d}{dt}\bar{p}_2(t) - 2\bar{p}_1(t)\bar{p}_3(t) = 2\left(1 - \sigma^2\right)\sqrt{h_0(c_0 - h_0)}\,\mathrm{cn}\,(\Omega\, t,\, k)\,\mathrm{dn}\,(\Omega\, t,\, k)\,.$$

Therefore, $\frac{d}{dt}\bar{p}_2(t) = 2\bar{p}_1(t)\bar{p}_3(t)$ whenever $\sigma \in \{-1, 1\}$. (It is not difficult to verify that $H^2(\bar{p}(t)) = h_0$ and $C(\bar{p}(t)) = c_0$ for $t \in \mathbb{R}$.) It is then easy to show that $\bar{p}(t)$ satisfies $\frac{d}{dt}\bar{p}(t) = \vec{H}^2(\bar{p}(t))$ whenever $\sigma \in \{-1, 1\}$. Hence $\bar{p}(\cdot)$ is an integral curve; it is straightforward to verify that $0 < k < 1$ and that $\bar{p}(\cdot)$ is defined for all $t \in \mathbb{R}$.

Let $p(\cdot)$ be an integral curve through $p(0)$, let $h_0 = H^2(p(0))$, $c_0 = C(p(0))$, and suppose $2h_0 > c_0 > 0$. We claim that $p(t) = \bar{p}(t + t_0)$ for some $\sigma \in \{-1, 1\}$ and $t_0 \in \mathbb{R}$. Let $\sigma = \mathrm{sgn}(p_3(0))$. (We may assume $\sigma \neq 0$.) Note that $(\bar{p}_1(t), \bar{p}_2(t))$ parametrizes the ellipses $S = \left\{(x, y) \mid x^2 + \frac{1}{2}y^2 = h_0\right\}$. We have $p_1(0)^2 + \frac{1}{2}p_2(0)^2 = h_0$, i.e, $(p_1(0), p_2(0)) \in S$. Therefore, there exists $t_0 \in \mathbb{R}$ such that $\bar{p}_1(t_0) = p_1(0)$ and $\bar{p}_2(t_0) = p_2(0)$. Accordingly, we get

$$p_3(0)^2 = c_0 - p_1(0)^2 - p_2(0)^2 = c_0 - \bar{p}_1(t_0)^2 - \bar{p}_2(t_0)^2 = \bar{p}_3(t_0)^2.$$

Hence, as $\mathrm{sgn}(\bar{p}_3(t_0)) = \sigma = \mathrm{sgn}(p_3(0))$, we have $p_3(0) = \bar{p}_3(t_0)$. Thus there exists $t_0 \in \mathbb{R}$ such that $p(0) = \bar{p}(t_0)$. Consequently, the integral curves $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ solve the same Cauchy problem, and therefore are identical.

(b) By limiting $c_0 \to 2h_0$ in case (a) or case (c), and allowing for possible changes in sign, we obtain the following prospective integral curve for $\vec{H}_2$:

$$\begin{aligned}\bar{p}_1(t) &= \zeta_1\sqrt{h_0}\,\mathrm{sech}\left(\sqrt{2h_0}\,t\right)\\ \bar{p}_2(t) &= \zeta_2\sqrt{2h_0}\,\tanh\left(\sqrt{2h_0}\,t\right)\\ \bar{p}_3(t) &= \zeta_3\sqrt{h_0}\,\mathrm{sech}\left(\sqrt{2h_0}\,t\right)\end{aligned}$$

where $\zeta_1, \zeta_2, \zeta_3 \in \{-1, 1\}$. We investigate under which conditions $\bar{p}(\cdot)$ is an integral curve. Now

$$\frac{d}{dt}\bar{p}_1(t) + \bar{p}_2(t)\bar{p}_3(t) = (-\zeta_1 + \zeta_2\zeta_3)\sqrt{2}h_0\,\mathrm{sech}\left(\sqrt{2h_0}\,t\right)\tanh\left(\sqrt{2h_0}\,t\right).$$

Therefore, if $\zeta_1 = \sigma_1$, $\zeta_2 = \sigma_1\sigma_2$, $\zeta_3 = \sigma_2$, and $\sigma_1, \sigma_2 \in \{-1, 1\}$, then $\frac{d}{dt}\bar{p}_1(t) = -\bar{p}_2(t)\bar{p}_3(t)$. It is easy to verify for these choices of signs that $\frac{d}{dt}\bar{p}(t) = \vec{H}^2(\bar{p}(t))$ for all $t \in \mathbb{R}$. Hence $\bar{p}(\cdot)$ is an integral curve.

Let $p(\cdot)$ be an integral curve through $p(0)$. Let $\sigma_1 = \operatorname{sgn}(p_1(0))$, $\sigma_2 = \operatorname{sgn}(p_3(0))$, let $h_0 = H^2(p(0))$, $c_0 = C(p(0))$, and suppose that $c_0 = 2h_0$. We claim that $p(t) = \bar{p}(t+t_0)$ for some $t_0 \in \mathbb{R}$. As $p_1(0)^2 + \frac{1}{2}p_2(0)^2 = h_0$ and $p_1(0)^2 + p_2(0)^2 + p_3(0)^2 = 2h_0$, we have $-\sqrt{2h_0} \leq p_2(0) \leq \sqrt{2h_0}$. Now, $\lim_{t\to\pm\infty} \bar{p}_2(t) = \pm\sigma_1\sigma_2\sqrt{2h_0}$. Thus there exists $t_0 \in \mathbb{R}$ such that $p_2(0) = \bar{p}_2(t_0)$. As

$$p_1(0)^2 = h_0 - \frac{1}{2}p_2(0)^2 = h_0 - \frac{1}{2}\bar{p}_2(t_0)^2 = \bar{p}_1(t_0)^2$$

and $\operatorname{sgn}(p_1(0)) = \operatorname{sgn}(\bar{p}_1(t_0)) = \sigma_1$, it follows that $p_1(0) = \bar{p}_1(t_0)$. Similarly, as

$$p_3(0)^2 = 2h_0 - p_1(0)^2 - p_2(0)^2 = 2h_0 - \bar{p}_1(t_0)^2 - \bar{p}_2(t_0)^2 = \bar{p}_3(t_0)^2$$

and $\operatorname{sgn}(p_3(0)) = \operatorname{sgn}(\bar{p}_3(t_0)) = \sigma_2$, it follows that $p_3(0) = \bar{p}_3(t_0)$. Thus there exists $t_0 \in \mathbb{R}$ such that $p(0) = \bar{p}(t_0)$. Consequently, the integral curves $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ solve the same Cauchy problem and therefore are identical.

(c) We start by explaining how the expression for $\bar{p}(\cdot)$ was found. A number of convenient assumptions are made implicitly and translations in the independent variable are discarded. We shall verify that $\bar{p}(\cdot)$ is a maximal integral curve (defined for any $c_0$ and $h_0$ satisfying the conditions of the theorem) only at the end of the construction. Suppose $\bar{p}(\cdot)$ is an integral curve of $H^2$ such that $c_0 > 2h_0$, where $h_0 = H^2(\bar{p}(0))$ and $c_0 = C(\bar{p}(0))$. As $(\frac{d\bar{p}_1}{dt})^2 = \bar{p}_2^2\bar{p}_3^2$, $H^2(\bar{p}(t)) = h_0$, and $C(\bar{p}(t)) = c_0$, we have

$$\frac{d\bar{p}_1}{dt} = \sqrt{2}\sqrt{(h_0 - \bar{p}_1^2)(c_0 - 2h_0 + \bar{p}_1^2)}.$$

Therefore,

$$\sqrt{2}t = \int \frac{d\bar{p}_1}{\sqrt{(h_0 - \bar{p}_1^2)(c_0 - 2h_0 + \bar{p}_1^2)}}.$$

By applying the elliptic integral formula (D.1), we obtain $\bar{p}_1(t) = \sqrt{h_0}\operatorname{cn}(\Omega\, t,\, k)$, where $\Omega = \sqrt{2c_0 - 2h_0}$ and $k = \sqrt{\frac{h_0}{c_0 - h_0}}$. As $h_0 = \bar{p}_1(t)^2 + \frac{1}{2}\bar{p}_2(t)^2$, we have $\bar{p}_2(t) = \sigma\sqrt{2h_0}\operatorname{sn}(\Omega\, t,\, k)$ for some $\sigma \in \{-1, 1\}$. Then as $\frac{d}{dt}\bar{p}_3(t) = -\sigma\sqrt{2}h_0\operatorname{cn}(\Omega\, t,\, k)\operatorname{sn}(\Omega\, t,\, k)$, it follows that $\bar{p}_3(t) = \sigma\frac{\sqrt{2}h_0}{k^2\Omega}\operatorname{dn}(\Omega\, t,\, k) = \sigma\frac{\Omega}{\sqrt{2}}\operatorname{dn}(\Omega\, t,\, k)$.

An easy calculation shows that $\frac{d}{dt}\bar{p}(t) = \vec{H}^2(\bar{p}(t))$ whenever $\sigma \in \{-1, 1\}$. Thus $\bar{p}(\cdot)$ is an integral curve; it is not difficult to verify that $0 < k < 1$ and that $\bar{p}(t)$ is defined for all $t \in \mathbb{R}$ .

Let $p(\cdot)$ be an integral curve through $p(0)$, let $h_0 = H^2(p(0))$, $c_0 = C(p(0))$, and suppose $c_0 > 2h_0 > 0$. We claim that $p(t) = \bar{p}(t + t_0)$ for some $\sigma \in \{-1, 1\}$ and $t_0 \in \mathbb{R}$. let $\sigma = \operatorname{sgn}(p_3(0))$. (We may assume $\sigma \neq 0$.) Note that $(\bar{p}_1(t), \bar{p}_2(t))$ parametrizes the ellipses $S = \big\{(x, y) \mid x^2 + \frac{1}{2}y^2 = h_0\big\}$. We have $p_1(0)^2 + \frac{1}{2}p_2(0)^2 = h_0$, i.e, $(p_1(0), p_2(0)) \in S$. Therefore, there exists $t_0 \in \mathbb{R}$ such that $\bar{p}_1(t_0) = p_1(0)$ and $\bar{p}_2(t_0) = p_2(0)$. Accordingly, we get

$$p_3(0)^2 = c_0 - p_1(0)^2 - p_2(0)^2 = c_0 - \bar{p}_1(t_0)^2 - \bar{p}_2(t_0)^2 = \bar{p}_3(t_0)^2.$$

Hence, as $\operatorname{sgn}(\bar{p}_3(t_0)) = \sigma = \operatorname{sgn}(p_3(0))$, we have $p_3(0) = \bar{p}_3(t_0)$. Thus there exists $t_0 \in \mathbb{R}$ such that $p(0) = \bar{p}(t_0)$. Consequently, the integral curves $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ solve the same Cauchy problem, and therefore are identical. ☐

### 3.3.3 Inhomogeneous systems of type I

In this section we consider those inhomogeneous systems whose equilibria are unions of lines and planes (type I). There are five such systems (in fact two systems and three one-parameter families of systems; see theorem 3.3.1). Note that the systems which are equivalent to $H_0^1$ are homogeneous systems in disguise. For each system we obtain explicit expressions for the integral curves: for $H_{1,\alpha}^0$ in terms of elementary functions and for the remaining systems in terms of rational functions of (possibly square roots of) Jacobi elliptic functions. For each system the stability nature of all equilibria is determined (again via the energy-Casimir methods). We note that the system $H_{1,\alpha}^2$ is equivalent to a system on $\mathfrak{se}(2)_-^*$ which has been considered previously (see [7]). Again, the equilibria are parametrized by $\mu, \nu, \eta \in \mathbb{R}$, $\nu \neq 0$.

#### System $H_{1,\alpha}^0$

The system $H_{1,\alpha}^0(p) = \alpha p_1$, $\alpha > 0$ has equations of motion

$$\begin{cases} \dot{p}_1 = 0 \\ \dot{p}_2 = \alpha p_3 \\ \dot{p}_3 = -\alpha p_2. \end{cases}$$

The equilibria are $\mathbf{e}_1^\mu = (\mu, 0, 0)$; all equilibria are stable. In Figure 3.4 we graph the critical energy states $(h_0, c_0)$ and a corresponding typical configuration. (The value $\alpha = 1$ was used in Figure 3.4.)

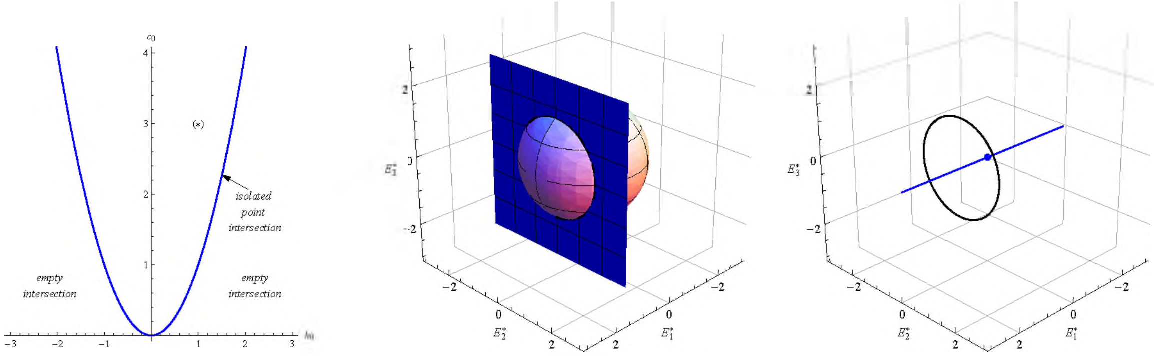


Figure 3.4: Critical energy states for $H_{1,\alpha}^0$ and a corresponding typical configuration

The integral curves of this system are given by

$$\begin{aligned} p_1(t) &= p_1(0) \\ p_2(t) &= p_2(0)\cos(\alpha t) + p_3(0)\sin(\alpha t) \\ p_3(t) &= p_3(0)\cos(\alpha t) - p_2(0)\sin(\alpha t). \end{aligned}$$

### System $H_1^1$

The system $H_1^1(p) = p_2 + \frac{1}{2}p_1^2$ has equations of motion

$$\begin{cases} \dot{p}_1 = -p_3 \\ \dot{p}_2 = p_1 p_3 \\ \dot{p}_3 = p_1 - p_1 p_2. \end{cases}$$

The equilibria are $\mathsf{e}_1^\mu = (0, \mu, 0)$ and $\mathsf{e}_2^\nu = (\nu, 1, 0)$.

There are three qualitatively different cases for the intersection of a parabolic cylinder $(H_1^1)^{-1}(h_0)$ and a sphere $C^{-1}(c_0)$, corresponding to (a) $c_0 < h_0^2$, (b) $c_0 = h_0^2$, and (c) $c_0 > h_0^2$. In Figure 3.5 we graph the critical energy states $(h_0, c_0)$ ; in Figure 3.6 we graph the corresponding typical configurations.

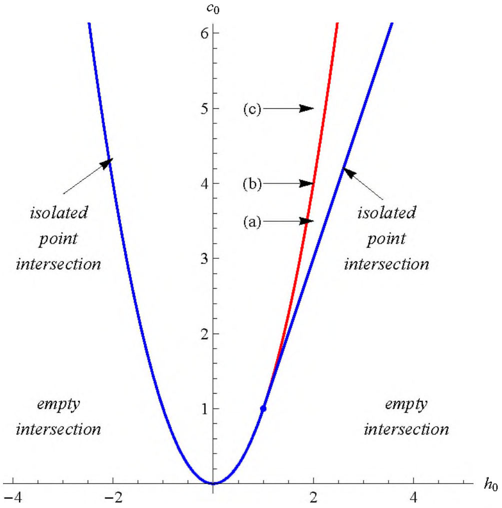


Figure 3.5: Critical energy states for $H_1^1$

3.3.8 Theorem. *The equilibrium states have the following behaviour:*

*(i) The states $\mathsf{e}_1^\mu$, $\mu \leq 1$ are stable.*

*(ii) The states $\mathsf{e}_1^\mu$, $\mu > 1$ are (spectrally) unstable.*

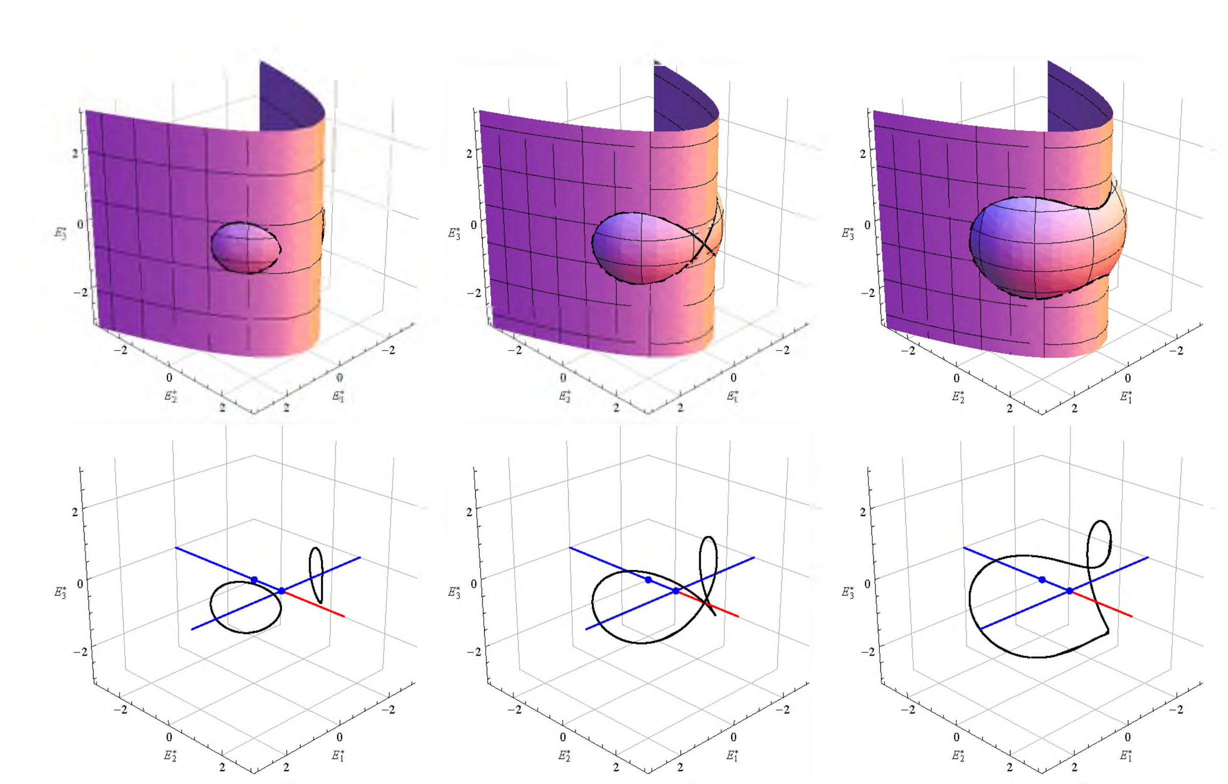
*(iii) The states $\mathsf{e}_2^\nu$ are stable.*

Proof. Let $H_\lambda(p) = \lambda_1 H_1^1(p) + \lambda_2 C(p)$. (i) Suppose $\mu < 1$, $\mu \neq 0$, and let $\lambda_1 = 1$ and $\lambda_2 = -\frac{1}{2\mu}$. We have $\mathbf{d}H_\lambda(\mu, 0, 0) = 0$ and that the Hessian $\mathbf{d}^2 H_\lambda(\mu, 0, 0) = \operatorname{diag}(\frac{\mu-1}{\mu}, -\frac{1}{\mu}, -\frac{1}{\mu})$ is definite. Thus, by the generalized energy-Casimir method, the states $\mathsf{e}_1^\mu$, $\mu < 1$, $\mu \neq 0$ are stable. Suppose $\mu = 1$. Then $H(\mathsf{e}_1^1) = C(\mathsf{e}_1^1) = 1$. It is a simple matter to show that $(H_1^1)^{-1}(1) \cap C^{-1}(1) = \{\mathsf{e}_1^1\}$. Thus, by the continuous energy-Casimir method, the state $\mathsf{e}_1^1$ is stable. Likewise, the origin is stable.

(ii) The linearization of the system at $\mathsf{e}_1^{\mu}$ has eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = \pm\sqrt{\mu - 1}$. Thus the states $\mathsf{e}_1^{\mu}$, $\mu > 1$ are spectrally unstable.

(iii) Let $\lambda_1 = 1$ and $\lambda_2 = -\frac{1}{2}$. We have $\mathbf{d}H_{\lambda}(\nu, 1, 0) = 0$. Also, $\mathbf{d}^2 H_{\lambda}(\nu, 1, 0) = \operatorname{diag}(0, -1, -1)$ is definite when restricted to the subspace $W = \operatorname{span}\{(1, -\nu, 0), (0, 0, 1)\}$. Hence, by the generalized energy-Casimir method, the states $\mathsf{e}_2^{\nu}$ are stable.

□



(a) $c_0 < h_0^2$ (b) $c_0 = h_0^2$ (c) $c_0 > h_0^2$

Figure 3.6: Typical configurations for $H_1^1$

MATHEMATICA. *The explicit details for the integration of the system* $H_1^1$ *are contained in the directory:*

*Thesis Mathematica\SO(3)\Integration\H1-1(p)*

*The individual files for each case are as follows: case (a) - H11Int.clth2.nb; case (b) - H11Int.ceqh2.nb; and case (c) - H11Int.cgth2.nb.*

3.3.9 THEOREM. ([6]) *Let* $p(\cdot)$ *be an integral curve of the system* $H_1^1$ *through* $p(0)$. *Let* $h_0 = H_1^1(p(0))$ *and* $c_0 = C(p(0))$.

*(a) If* $c_0 < h_0^2$*, then there exist* $t_0 \in \mathbb{R}$ *and* $\sigma \in \{-1, 1\}$ *such that* $p(t) = \bar{p}(t + t_0)$*, where*

$$\begin{aligned}
\bar{p}_1(t) &= \sigma\sqrt{2\delta}\,\frac{1 + k\,\mathrm{sn}\,(\Omega\,t,\,k)}{\mathrm{dn}\,(\Omega\,t,\,k)}\\
\bar{p}_2(t) &= h_0 + \delta - \frac{2\delta}{1 - k\,\mathrm{sn}\,(\Omega\,t,\,k)}\\
\bar{p}_3(t) &= -\sigma k\Omega\sqrt{2\delta}\,\frac{\mathrm{cn}\,(\Omega\,t,\,k)}{1 - k\,\mathrm{sn}\,(\Omega\,t,\,k)}.
\end{aligned}$$

*Here* $\Omega = \sqrt{h_0 - 1 + \delta}$*,* $k = \sqrt{\frac{h_0 - 1 - \delta}{h_0 - 1 + \delta}}$*, and* $\delta = \sqrt{h_0^2 - c_0}$*.*

*(b) If* $c_0 = h_0^2$*, then there exist* $t_0 \in \mathbb{R}$ *and* $\sigma \in \{-1, 1\}$ *such that* $p(t) = \bar{p}(t + t_0)$*, where*

$$\begin{aligned}
\bar{p}_1(t) &= 2\sigma\sqrt{h_0 - 1}\,\mathrm{sech}\left(\sqrt{h_0 - 1}\,t\right)\\
\bar{p}_2(t) &= h_0 - 2(h_0 - 1)\,\mathrm{sech}\left(\sqrt{h_0 - 1}\,t\right)^2\\
\bar{p}_3(t) &= 2\sigma(h_0 - 1)\,\mathrm{sech}\left(\sqrt{h_0 - 1}\,t\right)\,\tanh\left(\sqrt{h_0 - 1}\,t\right).
\end{aligned}$$

*(c) If* $c_0 > h_0^2$*, then there exists* $t_0 \in \mathbb{R}$ *such that* $p(t) = \bar{p}(t + t_0)$*, where*

$$\begin{aligned}
\bar{p}_1(t) &= \sqrt{2(h_0 + \delta - 1)}\,\mathrm{cn}\,(\Omega\,t,\,k)\\
\bar{p}_2(t) &= h_0 - (h_0 + \delta - 1)\,\mathrm{cn}\,(\Omega\,t,\,k)^2\\
\bar{p}_3(t) &= \sqrt{2\delta(h_0 + \delta - 1)}\,\mathrm{dn}\,(\Omega\,t,\,k)\,\mathrm{sn}\,(\Omega\,t,\,k).
\end{aligned}$$

*Here* $\Omega = \sqrt{\delta}$*,* $k = \sqrt{\frac{h_0 + \delta - 1}{2\delta}}$*, and* $\delta = \sqrt{1 + c_0 - 2h_0}$*.*

PROOF. In [6], details were only provided for case (a). In order to illustrate the method used we provide details for case (c). (The result for case (b) is found by limiting $c_0$ to $h_0^2$.)

We start by explaining how the expression for $\bar{p}(\cdot)$ was found. A number of convenient assumptions are made implicitly and translations in the independent variable are discarded. We shall verify that $\bar{p}(\cdot)$ is a maximal integral curve (defined for any $c_0$ and $h_0$ satisfying the conditions of the theorem) only at the end of the construction. Suppose $\bar{p}(\cdot)$ is an integral curve of $H_1^1$ such that $c_0 > h_0^2$, where $h_0 = H_1^1(\bar{p}(0))$ and $c_0 = C(\bar{p}(0))$. As $(\frac{d\bar{p}_1}{dt})^2 = \bar{p}_3^2$, $H_1^1(\bar{p}(t)) = h_0$, and $C(\bar{p}(t)) = c_0$, we have

$$\frac{d\bar{p}_1}{dt} = \tfrac{1}{2}\sqrt{(2 + 2\delta - 2h_0 + \bar{p}_1^2)(2h_0 + 2\delta - 2 - \bar{p}_1^2)}$$

where $\delta = \sqrt{1 + c_0 - 2h_0}$. Therefore

$$\int \tfrac{1}{2}dt = \int \frac{d\bar{p}_1}{\sqrt{(a^2 + \bar{p}_1^2)(b^2 - \bar{p}_1^2)}}$$

where $a = \sqrt{2}\sqrt{1 + \delta - h_0}$ and $b = \sqrt{2}\sqrt{h_0 + \delta - 1}$. By applying the elliptic integral formula (D.1), we obtain

$$\bar{p}_1(t) = \sqrt{2(h_0 + \delta - 1)}\,\mathrm{cn}\,(\Omega\,t,\,k)$$

where $\Omega = \sqrt{\delta}$ and $k = \sqrt{\frac{h_0+\delta-1}{2\delta}}$. As $h_0 = \bar{p}_2(t) + \frac{1}{2}\bar{p}_1(t)^2$, we have

$$\bar{p}_2(t) = h_0 - (h_0+\delta-1)\,\mathrm{cn}\,(\Omega\, t,\, k)^2\,.$$

Then as $\frac{d}{dt}\bar{p}_1(t) = -\bar{p}_3(t)$ it follows that

$$\bar{p}_3(t) = \sqrt{2\delta(h_0+\delta-1)}\,\mathrm{dn}\,(\Omega\, t,\, k)\;\mathrm{sn}\,(\Omega\, t,\, k)\,.$$

It is easy to show that $\bar{p}(t)$ satisfies $\frac{d}{dt}\bar{p}(t) = \vec{H}_1^1(\bar{p}(t))$. Hence $p(\cdot)$ is an integral curve; it is not difficult to verify that $0 < k < 1$ and that $\bar{p}(t)$ is defined for all $t \in \mathbb{R}$.

Let $p(\cdot)$ be an integral curve through $p(0)$, let $h_0 = H_1^1(p(0))$, $c_0 = C(p(0))$, and suppose that $c_0 > h_0^2$. We claim that $p(t) = \bar{p}(t+t_0)$ for some $t_0 \in \mathbb{R}$. As $p_2(0) + \frac{1}{2}p_1(0)^2 = h_0$ and $p_1(0)^2 + p_2(0)^2 + p_3(0)^2 = c_0$, we have $(2h_0 - c_0) - 2p_2(0) + p_2^2(0) \leq 0$ and $p_2(0) \leq h_0$. It follows that

$$1 - \sqrt{1 + c_0 - 2h_0} \leq p_2(0) \leq h_0 \leq 1 + \sqrt{1 + c_0 - 2h_0}.$$

Now $\bar{p}_2(0) = 1 - \sqrt{1 + c_0 - 2h_0}$ and $\bar{p}_2(\frac{K}{\Omega}) = h_0$. Thus there exists $t_2 \in [0, \frac{K}{\Omega}]$ such that $p_2(0) = \bar{p}_2(t_2)$. As

$$\tfrac{1}{2}p_1(0)^2 = h_0 - p_2(0) = h_0 - \bar{p}_2(t_2) = \tfrac{1}{2}\bar{p}_1(t_2)^2$$

it follows that $p_1(0) = \pm\bar{p}_1(t_2)$. Furthermore $\bar{p}_1(t + \frac{2K}{\Omega}) = -\bar{p}_1(t)$ and $\bar{p}_2(t + \frac{2K}{\Omega}) = \bar{p}_2(t)$. Thus there exists $t_1 \in \mathbb{R}$ ($t_1 = t_2$ or $t_1 = t_2 + \frac{2K}{\Omega}$) such that $p_1(0) = \bar{p}_1(t_1)$ and $p_2(0) = \bar{p}_2(t_1)$. On the other hand

$$p_3(0)^2 = c_0 - p_1(0)^2 - p_2(0)^2 = c_0 - \bar{p}_1(t_1)^2 - \bar{p}_2(t_1)^2 = \bar{p}_3(t_1)^2$$

and so $p_3(0) = \pm\bar{p}_3(t_1)$. Furthermore $\bar{p}_1(-t) = \bar{p}_1(t)$, $\bar{p}_2(-t) = \bar{p}_2(t)$, and $\bar{p}_3(-t) = -\bar{p}_3(t)$. Thus there exists $t_0 \in \mathbb{R}$ ($t_0 = t_1$ or $t_0 = -t_1$) such that $p(0) = \bar{p}(t_0)$. Consequently, the integral curves $t \mapsto p(t)$ and $t \mapsto \bar{p}(t+t_0)$ solve the same Cauchy problem and therefore are identical. ☐

### System $H_{1,\alpha}^2$

The system $H_{1,\alpha}^2(p) = \alpha p_1 + p_1^2 + \frac{1}{2}p_2^2$, $\alpha > 0$ has equations of motion

$$\begin{cases} \dot{p}_1 = -p_2p_3 \\ \dot{p}_2 = (\alpha + 2p_1)p_3 \\ \dot{p}_3 = -(\alpha + p_1)p_2. \end{cases}$$

The equilibria are $\mathsf{e}_1^\mu = (\mu, 0, 0)$, $\mathsf{e}_2^\nu = (-\alpha, \nu, 0)$, and $\mathsf{e}_3^\nu = (-\frac{\alpha}{2}, 0, \nu)$.

The system $(\mathfrak{so}(3)_-^*, H_{1,\alpha}^2)$ is equivalent to the system $(\mathfrak{se}(2)_-^*, \widetilde{H}_\alpha)$, where $\widetilde{H}_\alpha(\tilde{p}) = \tilde{p}_1 + \frac{1}{\alpha^2}\tilde{p}_2^2 + \frac{1}{2}\tilde{p}_3^2$. Stability and integration of $\widetilde{H}_\alpha$ were treated in [7]. Explicitly, the systems $(\mathfrak{se}(2)_-^*, \widetilde{H}_\alpha)$ and $(\mathfrak{so}(3)_-^*, H_{1,\alpha}^2)$ are compatible with the affine isomorphism $\psi : \mathfrak{se}(2)^* \to \mathfrak{so}(3)^*$ given by

$$\tilde{p} \mapsto \tilde{p}\begin{bmatrix} -\frac{1}{\alpha} & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{\alpha} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} + \begin{bmatrix} -\frac{\alpha}{2} & 0 & 0 \end{bmatrix}.$$

Hence, any integral curve of $(\mathfrak{so}\,(3)_-^*, H_{1,\alpha}^2)$ is just the image under $\psi$ of an integral curve of

$(\mathfrak{se}(2)^*_-, \widetilde{H}_\alpha)$. The expressions for the integral curves split into a number of cases. (Some divisions are based on qualitative grounds, whereas others where retrospectively made to facilitate integration.) An index of the conditions defining these cases appears in Table 3.2. In Figure 3.7 we graph the critical energy states $(h_0, c_0)$; in Figure 3.8 we graph the corresponding typical configurations. (The value $\alpha = \frac{3}{2}$ was used in both these figures.)

| Conditions $(\omega_\pm = 2h_0 + \alpha(\alpha \pm \sqrt{\alpha^2 + 4h_0}))$ | | | Index |
|---|---|---|---|
| $h_0 \leq 0$ | $2c_0 > \omega_+$ | $\alpha^2 + h_0 > c_0 + \sqrt{c_0^2 + h_0^2 - c_0(\alpha^2 + 2h_0)}$ | 1a(i) |
| | | $\alpha^2 + h_0 = c_0 + \sqrt{c_0^2 + h_0^2 - c_0(\alpha^2 + 2h_0)}$ | 1a(ii) |
| | | $\alpha^2 + h_0 < c_0 + \sqrt{c_0^2 + h_0^2 - c_0(\alpha^2 + 2h_0)}$ | 1a(iii) |
| | $2c_0 = \omega_+$ | $c_0 < \alpha^2 + h_0$ | 1b(i) |
| | | $c_0 = \alpha^2 + h_0$ | 1b(ii) |
| | $\omega_- < 2c_0 < \omega_+$ | | 1c |
| $h_0 > 0$ | $c_0 > \alpha^2 + 2h_0$ | | 2a |
| | $c_0 = \alpha^2 + 2h_0$ | | 2b |
| | $c_0 < \alpha^2 + 2h_0$ | $2c_0 > \omega_+$ | 2c(i) |
| | | $2c_0 = \omega_+$ | 2c(ii) |
| | | $\omega_- < 2c_0 < \omega_+$ | 2c(iii) |

Table 3.2: Index of cases for integral curves of $H^2_{1,\alpha}$

We provide the details for how the expressions of the integral curves on $\mathfrak{so}(3)^*_-$ are obtained from those on $\mathfrak{se}(2)^*_-$ only for case 1a(i). (The remaining cases follow a similar argument and thus the proofs are omitted.) We provide an independent proof for the stability nature of the equilibria.

MATHEMATICA. *The verification of the expressions for the integral curves of the system* $H^2_{1,\alpha}$ *are contained in the Mathematica file:*

*Thesis Mathematica\SO(3)\Integration\H21a(p)\H21aSysVer.nb*

3.3.10 THEOREM. *The equilibrium states have the following behaviour:*

*(i) The states* $\mathsf{e}_1^\mu$, $\mu \in (-\infty, -\alpha) \cup [-\frac{\alpha}{2}, \infty)$ *are stable.*

*(ii) The states* $\mathsf{e}_1^\mu$, $-\alpha < \mu < -\frac{\alpha}{2}$ *are (spectrally) unstable.*

*(iii) The state* $\mathsf{e}_1^{-\alpha}$ *is unstable.*

*(iv) The states* $\mathsf{e}_2^\nu$ *are (spectrally) unstable.*

*(v) The states* $\mathsf{e}_3^\nu$ *are stable.*

PROOF. Let $H_\lambda(p) = \lambda_1 H^2_{1,\alpha} + \lambda_2 C(p)$. (i) Suppose $\mu \in (-\infty, -\alpha) \cup (-\frac{\alpha}{2}, \infty)$ and let $\lambda_1 = 1$ and $\lambda_2 = -\frac{2\mu}{\alpha+2\mu}$. We have $\mathbf{d}H_\lambda(\mathsf{e}_1^\mu) = 0$ and that $\mathbf{d}^2 H_\lambda(\mathsf{e}_1^\mu) = \operatorname{diag}(\frac{2\alpha}{\alpha+2\mu}, 1 + \frac{\alpha}{\alpha+2\mu}, 2)$ is definite when restricted to the subspace $W = \operatorname{span}\{(0,1,0),(0,0,1)\}$. Hence the states $\mathsf{e}_1^\mu$, $\mu \in (-\infty, -\alpha) \cup (-\frac{\alpha}{2}, \infty)$ are stable. Suppose $\mu = -\frac{\alpha}{2}$. We have $H^2_{1,\alpha}(\mathsf{e}_1^{-\frac{\alpha}{2}}) = -\frac{\alpha^2}{4}$ and $C(\mathsf{e}_1^{-\frac{\alpha}{2}}) = \frac{\alpha^2}{4}$. It is straightforward to show that $(H^2_{1,\alpha})^{-1}(-\frac{\alpha^2}{4}) \cap C^{-1}(\frac{\alpha^2}{4}) = \{\mathsf{e}_1^{-\frac{\alpha}{2}}\}$. Hence the state $\mathsf{e}_1^{-\frac{\alpha}{2}}$ is stable.

(ii) The linearization of the system at $\mathsf{e}_1^\mu$ has eigenvalues $\lambda_{1,2} = \pm\sqrt{-(\alpha+\mu)(\alpha+2\mu)}$, $\lambda_3 = 0$. Thus the states $\mathsf{e}_1^\mu$, $-\alpha < \mu < -\frac{\alpha}{2}$ are spectrally unstable.

(iii) Consider the equilibrium state $\mathsf{e}_1^{-\alpha}$. We have that $p(t) = (\frac{-\alpha^3 t^2}{2+t^2\alpha^2}, \frac{-2\alpha^2 t}{2+t^2\alpha^2}, \frac{-2\alpha}{2+t^2\alpha^2})$ is an integral curve of the system $H^2_{1,\alpha}$ such that $\lim_{t\to-\infty} p(t) = \mathsf{e}_1^{-\alpha}$. Let $\mathcal{B}_\varepsilon$ be the open ball of radius $\varepsilon = \alpha$ centred at the point $\mathsf{e}_1^{-\alpha}$. For any neighbourhood $V \subset \mathcal{B}_\varepsilon$ of $\mathsf{e}_1^{-\alpha}$ there exists $t_0 < 0$ such that $p(t_0) \in V$. Furthermore $\|p(0) - \mathsf{e}_1^{-\alpha}\| = \sqrt{2}\alpha > \varepsilon$, i.e., $p(0) \notin \mathcal{B}_\varepsilon$. Hence the state $\mathsf{e}_1^{-\alpha}$ is unstable.

(iv) The linearization of the system at $\mathsf{e}_2^\nu$ has eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = \pm\nu$. Thus the states $\mathsf{e}_2^\nu$ are spectrally unstable.

(v) Let $\lambda_1 = 1$ and $\lambda_2 = 0$. We have $\mathbf{d}H_\lambda(\mathsf{e}_3^\nu) = 0$ and $\mathbf{d}^2 H_\lambda(\mathsf{e}_3^\nu) = \operatorname{diag}(2,1,0)$ is definite when restricted $W = \operatorname{span}\big\{(\nu, 0, \frac{\alpha}{2}), (0,1,0)\big\}$. Therefore the states $\mathsf{e}_3^\nu$ are stable. □

3.3.11 REMARK. In theorems 3.3.12–3.3.20 we shall find it convenient to use $\eta_0 = \sqrt{\alpha^2 + 4h_0}$ instead of $h_0$; also, we shall make use of the following notation

$$\delta = \tfrac{1}{4}\sqrt{(\alpha^2 - 4c_0)^2 - 2(\alpha^2 + 4c_0)\eta_0^2 + \eta_0^4} \quad \text{and} \quad \rho_\pm = \tfrac{1}{\sqrt{2}}\sqrt{4c_0 - \alpha^2 - \eta_0^2 \pm 4\delta}.$$

3.3.12 THEOREM. (CASE 1$a(i)$) *Let $p(\cdot)$ be an integral curve of the system $H^2_{1,\alpha}$ through $p(0)$. Let $h_0 = H^2_{1,\alpha}(p(0))$ and $c_0 = C(p(0))$. If the conditions of case 1a(i) are satisfied, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$, where*

$$\begin{aligned}
\bar{p}_1(t) &= -\tfrac{\alpha}{2} - \tfrac{\eta_0}{2}\,\frac{\rho_- - \rho_+ \operatorname{sn}(\Omega\, t,\, k)}{\rho_+ - \rho_- \operatorname{sn}(\Omega\, t,\, k)}\\
\bar{p}_2(t) &= \sigma\eta_0\sqrt{2\delta}\,\frac{\operatorname{cn}(\Omega\, t,\, k)}{\rho_+ - \rho_- \operatorname{sn}(\Omega\, t,\, k)}\\
\bar{p}_3(t) &= -\tfrac{2\sigma\delta}{k'}\,\frac{\operatorname{dn}(\Omega\, t,\, k)}{\rho_+ - \rho_- \operatorname{sn}(\Omega\, t,\, k)}.
\end{aligned}$$

*Here $\Omega = \frac{1}{2}\sqrt{4c_0 - \alpha^2 - 3\eta_0^2 + 4\delta}$, $k = \sqrt{\frac{\alpha^2+4\delta-4c_0+3\eta_0^2}{\alpha^2-4\delta-4c_0+3\eta_0^2}}$, and $k' = 2\sqrt{\frac{2\delta}{4c_0-\alpha^2-3\eta_0^2+4\delta}}$.*

PROOF. Any integral curve of $H^2_{1,\alpha}$ is the image under $\psi$ of an integral curve of $\tilde{H}_\alpha$. In [7], explicit expressions for all integral curves of $\tilde{H}_\alpha$ are determined; there are a number of cases (corresponding to different explicit expressions). The expression for the integral curve $\tilde{p}(\cdot)$ of $\tilde{H}_\alpha$ through a point $\tilde{p}(0) \in \mathfrak{se}(2)^*$ involves the constants $\tilde{h}_0 = \tilde{H}_\alpha(\tilde{p}(0))$ and $\tilde{c}_0 = \tilde{C}(\tilde{p}(0))$. The various cases are expressed in terms of inequalities in $\tilde{h}_0$ and $\tilde{c}_0$. We wish to find the image $\psi(\tilde{p}(\cdot))$ of each such integral curve and to express $\tilde{c}_0$ and $\tilde{h}_0$ in terms of the constants $h_0 = H^2_{1,\alpha}(\psi(\tilde{p}(0)))$

and $c_0 = C(\psi(\tilde{p}(0)))$. Moreover, we wish to find the corresponding conditions for the various cases on $\mathfrak{so}(3)^*$ in terms of inequalities in $h_0$ and $c_0$.

Let $\tilde{p} \in \mathfrak{se}(2)^*$ and let

$$\tilde{h}_0 = \widetilde{H}_\alpha(\tilde{p}) = \tilde{p}_1 + \tfrac{1}{\alpha^2}\tilde{p}_2^2 + \tfrac{1}{2}\tilde{p}_3^2 \quad \text{and} \quad \tilde{c}_0 = \widetilde{C}(\tilde{p}) = \tilde{p}_1^2 + \tilde{p}_2^2.$$

Correspondingly, let $h_0 = H^2_{1,\alpha}(\psi(\tilde{p}))$ and $c_0 = C(\psi(\tilde{p}))$; we have

$$h_0 = \tfrac{1}{\alpha^2}(\tilde{p}_1^2 + \tilde{p}_2^2) - \tfrac{1}{4}\alpha^2 \quad \text{and} \quad c_0 = \tilde{p}_1 + \tfrac{1}{\alpha^2}\tilde{p}_2^2 + \tfrac{1}{2}\tilde{p}_3^2 + \tfrac{1}{\alpha^2}(\tilde{p}_1^2 + \tilde{p}_2^2) + \tfrac{1}{4}\alpha^2.$$

Hence

$$h_0 = \tfrac{1}{\alpha^2}\tilde{c}_0 - \tfrac{1}{4}\alpha^2 \quad \text{and} \quad c_0 = \tilde{h}_0 + \tfrac{1}{4}\alpha^2 + \tfrac{1}{\alpha^2}\tilde{c}_0. \tag{3.8}$$

We can invert these relations to get

$$\tilde{c}_0 = \alpha^2 h_0 + \tfrac{1}{4}\alpha^4 \quad \text{and} \quad \tilde{h}_0 = c_0 - h_0 - \tfrac{1}{2}\alpha^2. \tag{3.9}$$

Therefore, $\tilde{p} \in (\widetilde{H}_\alpha)^{-1}(\tilde{h}_0) \cap \widetilde{C}^{-1}(\tilde{c}_0)$ if and only if $\psi(p) \in (H^2_{1,\alpha})^{-1}(h_0) \cap C^{-1}(c_0)$ whenever (3.8) or (3.9) holds.

We consider the first case for the integral curves of $\widetilde{H}_\alpha$ treated in [7]. Let $\tilde{p}(\cdot)$ be an integral curve of $\widetilde{H}_\alpha$ and let $\tilde{h}_0 = \widetilde{H}_\alpha(\tilde{p}(0))$ and $\tilde{c}_0 = \widetilde{C}(\tilde{p}(0))$. If the conditions

$$\tilde{c}_0 - \tfrac{1}{4}\alpha^4 \leq 0, \quad \tilde{h}_0 > \sqrt{\tilde{c}_0}, \quad \tfrac{1}{2}\alpha^2 - \tilde{h}_0 > \sqrt{\tilde{h}_0^2 - \tilde{c}_0} \tag{3.10}$$

hold, then there exist $\sigma \in \{-1, 1\}$ and $t_0 \in \mathbb{R}$ such that $\tilde{p}(t) = \bar{\bar{p}}(t + t_0)$, where

$$\begin{cases} \bar{\bar{p}}_1(t) = \sqrt{\tilde{c}_0}\, \dfrac{\sqrt{\tilde{h}_0 - \tilde{\delta}} - \sqrt{\tilde{h}_0 + \tilde{\delta}}\,\mathrm{sn}(\tilde{\Omega}\, t,\, \tilde{k})}{\sqrt{\tilde{h}_0 + \tilde{\delta}} - \sqrt{\tilde{h}_0 - \tilde{\delta}}\,\mathrm{sn}(\tilde{\Omega}\, t,\, \tilde{k})} \\ \bar{\bar{p}}_2(t) = -\sigma\sqrt{2\tilde{c}_0\tilde{\delta}}\, \dfrac{\mathrm{cn}(\tilde{\Omega}\, t,\, \tilde{k})}{\sqrt{\tilde{h}_0 + \tilde{\delta}} - \sqrt{\tilde{h}_0 - \tilde{\delta}}\,\mathrm{sn}(\tilde{\Omega}\, t,\, \tilde{k})} \\ \bar{\bar{p}}_3(t) = \dfrac{2\sigma\tilde{\delta}}{\tilde{k}'}\, \dfrac{\mathrm{dn}(\tilde{\Omega}\, t,\, \tilde{k})}{\sqrt{\tilde{h}_0 + \tilde{\delta}} - \sqrt{\tilde{h}_0 - \tilde{\delta}}\,\mathrm{sn}(\tilde{\Omega}\, t,\, \tilde{k})}. \end{cases}$$

Here $\tilde{\delta} = \sqrt{\tilde{h}_0^2 - \tilde{c}_0}$, $\tilde{\Omega} = \sqrt{\frac{2}{\alpha^2}(\tilde{h}_0 + \tilde{\delta})(\frac{1}{2}\alpha^2 - \tilde{h}_0 + \tilde{\delta})}$, $\tilde{k} = \sqrt{\frac{(\tilde{h}_0 - \tilde{\delta})(\frac{1}{2}\alpha^2 - \tilde{h}_0 - \tilde{\delta})}{(\tilde{h}_0 + \tilde{\delta})(\frac{1}{2}\alpha^2 - \tilde{h}_0 + \tilde{\delta})}}$ and $\tilde{k}' = \sqrt{\frac{2\alpha^2\tilde{\delta}}{(\alpha^2 + 2\tilde{\delta} - 2\tilde{h}_0)(\tilde{\delta} + \tilde{h}_0)}}$. We now find the corresponding integral curves of $H^2_{1,\alpha}$. Let $p(\cdot)$ be an integral curve of $H^2_{1,\alpha}$ and let $h_0 = H^2_{1,\alpha}(p(0))$ and $c_0 = C(p(0))$. We have that $\psi^{-1}(p(\cdot))$ is an integral curve of $\widetilde{H}_\alpha$. By (3.9) we have that $\psi^{-1}(p(\cdot))$ satisfies the requisite conditions (3.10) of the above result if and only if the conditions

$$h_0 \leq 0, \qquad 2c_0 > 2h_0 + \alpha(\alpha + \sqrt{\alpha^2 + 4h_0}), \qquad \alpha^2 + h_0 > c_0 + \sqrt{c_0^2 + h_0^2 - c_0\,(\alpha^2 + 2h_0)}$$

hold. Supposing these conditions hold, there exist $\sigma \in \{-1, 1\}$ and $t_0 \in \mathbb{R}$ such that $\psi^{-1}(p(t)) =$

$\bar{p}(t+t_0)$, i.e., $p(t) = \psi(\bar{p}(t+t_0))$. Finally, we let $\bar{p}(t) = \psi(\bar{\bar{p}}(t))$ and replace $\widetilde{h}_0$ and $\widetilde{c}_0$ with expressions in $h_0$ and $c_0$ (using (3.9)) and simplify to obtain the result. ☐

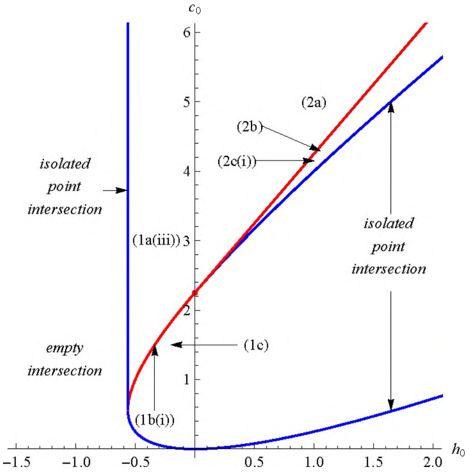


Figure 3.7: Critical energy states for $H^2_{1,\alpha}$

3.3.13 THEOREM. (CASE 1a(*ii*)) *Let $p(\cdot)$ be an integral curve of the system $H^2_{1,\alpha}$ through $p(0)$. Let $h_0 = H^2_{1,\alpha}(p(0))$ and $c_0 = C(p(0))$. If the conditions of case 1a(ii) are satisfied, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$, where*

$$\bar{p}_1(t) = -\tfrac{\alpha}{2} - \tfrac{\eta_0}{2}\,\frac{\rho_- - \rho_+\,\sin(\Omega\,t)}{\rho_+ - \rho_-\,\sin(\Omega\,t)}$$

$$\bar{p}_2(t) = \sigma\eta_0\sqrt{2\delta}\,\frac{\cos(\Omega\,t)}{\rho_+ - \rho_-\,\sin(\Omega\,t)}$$

$$\bar{p}_3(t) = -\frac{2\sigma\delta}{\rho_+ - \rho_-\,\sin(\Omega\,t)}.$$

*Here* $\Omega = \sqrt{\frac{3\alpha^2 - 4c_0 + \eta_0^2}{2}}$.

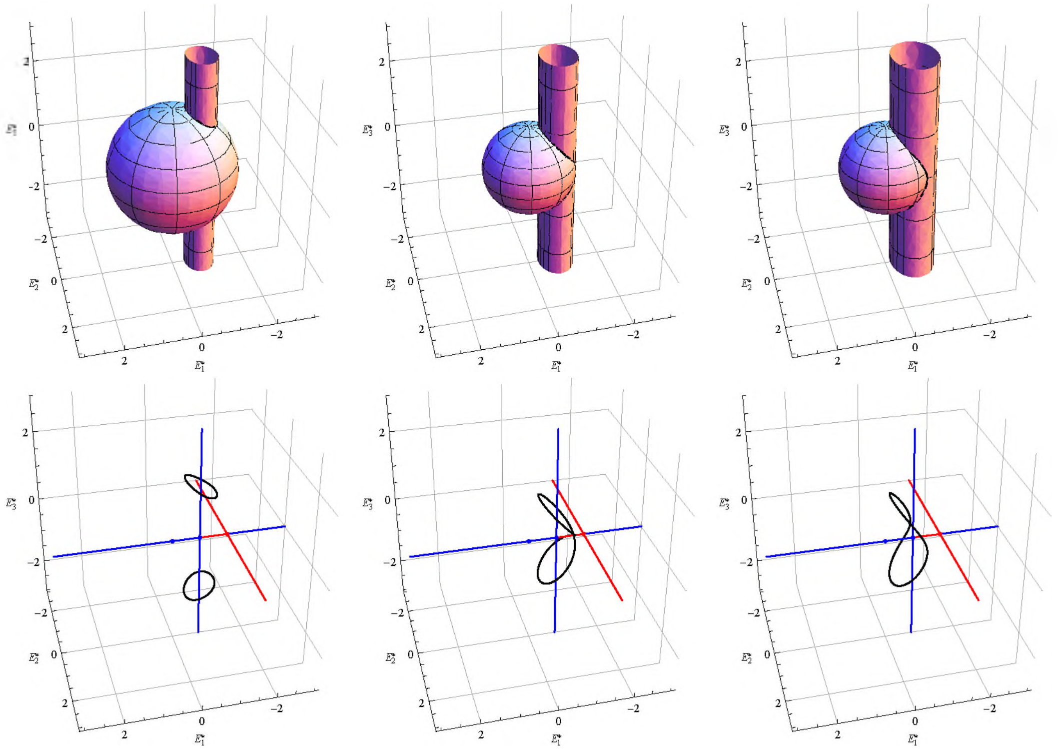
3.3.14 THEOREM. (CASES 1a(*iii*) & 2a) *Let $p(\cdot)$ be an integral curve of the system $H^2_{1,\alpha}$ through $p(0)$. Let $h_0 = H^2_{1,\alpha}(p(0))$ and $c_0 = C(p(0))$. If the conditions of case 1a(iii) or 2a are satisfied, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$, where*

$$\bar{p}_1(t) = -\tfrac{\alpha}{2} - \tfrac{\eta_0}{2}\,\frac{\rho_- - \rho_+\,\mathrm{cn}\,(\Omega\,t,\,k)}{\rho_+ - \rho_-\,\mathrm{cn}\,(\Omega\,t,\,k)}$$

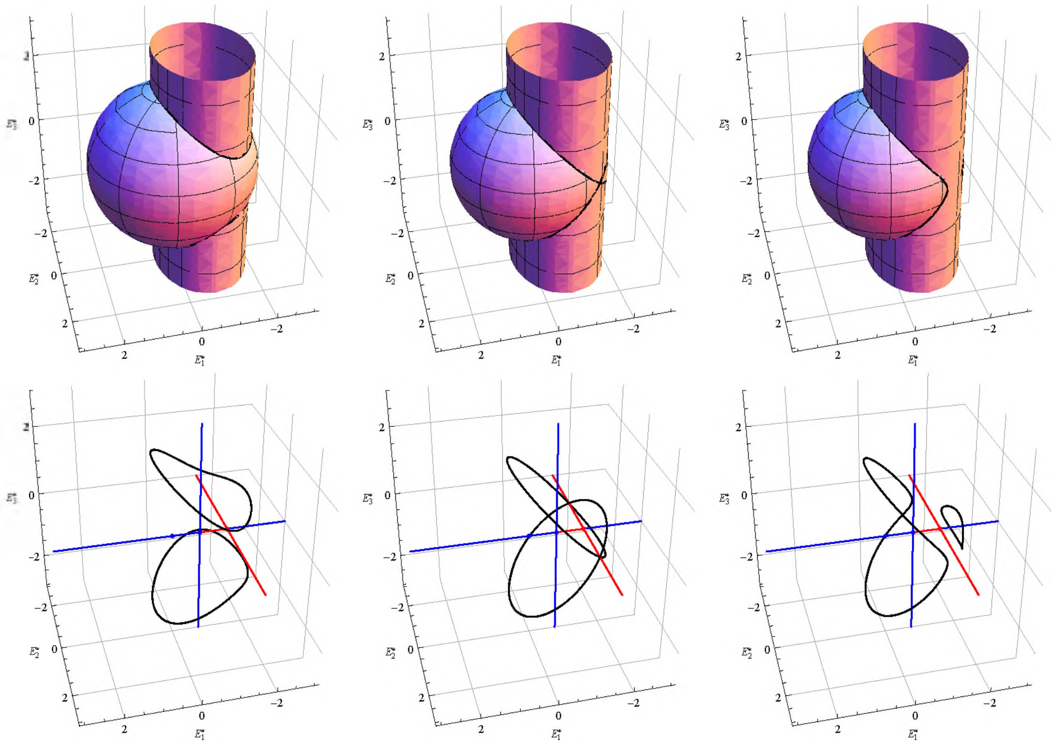
$$\bar{p}_2(t) = -\sigma\eta_0\sqrt{2\delta}\,\frac{\mathrm{sn}\,(\Omega\,t,\,k)}{\rho_+ - \rho_-\,\mathrm{cn}\,(\Omega\,t,\,k)}$$

$$\bar{p}_3(t) = -2\sigma\delta\frac{\mathrm{dn}\,(\Omega\,t,\,k)}{\rho_+ - \rho_-\,\mathrm{cn}\,(\Omega\,t,\,k)}.$$

*Here* $\Omega = \sqrt{2\delta}$ *and* $k = \sqrt{\frac{(3\alpha^2 - 4\delta - 4c_0 + \eta_0^2)(\alpha^2 + 4\delta - 4c_0 + \eta_0^2)}{2\alpha^2\delta}}$.



Case 1a(iii) Case 1b(i) Case 1c



Case 2a Case 2b Case 2c(i)

Figure 3.8: Typical cases of $\vec{H}^2_{1,\alpha}$

3.3.15 THEOREM. (CASE 1$b$($i$)) *Let $p(\cdot)$ be an integral curve of the system $H^2_{1,\alpha}$ through $p(0)$. Let $h_0 = H^2_{1,\alpha}(p(0))$ and $c_0 = C(p(0))$. If the conditions of case 1b(i) are satisfied, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$, where*

$$\begin{aligned}
\bar{p}_1(t) &= -\tfrac{\alpha+\eta_0}{2} - \frac{(\alpha - \eta_0)\,\eta_0}{\eta_0 - \alpha\cosh(\Omega\,t)^2}\\
\bar{p}_2(t) &= \sigma\eta_0\sqrt{2\alpha\,(\alpha-\eta_0)}\,\frac{\sinh(\Omega\,t)}{\eta_0 - \alpha\cosh(\Omega\,t)^2}\\
\bar{p}_3(t) &= \sigma\,(\alpha-\eta_0)\,\sqrt{\alpha\eta_0}\,\frac{\cosh(\Omega\,t)}{\eta_0 - \alpha\cosh(\Omega\,t)^2}.
\end{aligned}$$

*Here* $\Omega = \sqrt{\frac{(\alpha-\eta_0)\eta_0}{2}}$.

3.3.16 THEOREM. (CASE 1$b$($ii$)) *Let $p(\cdot)$ be an integral curve of the system $H^2_{1,\alpha}$ through $p(0)$. Let $h_0 = H^2_{1,\alpha}(p(0))$ and $c_0 = C(p(0))$. If the conditions of case 1b(ii) are satisfied, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$, where*

$$\begin{aligned}
\bar{p}_1(t) &= -\frac{\alpha^3 t^2}{2 + \alpha^2 t^2}\\
\bar{p}_2(t) &= -\frac{2\sigma\alpha^2 t}{2 + \alpha^2 t^2}\\
\bar{p}_3(t) &= -\frac{2\sigma\alpha}{2 + \alpha^2 t^2}.
\end{aligned}$$

3.3.17 THEOREM. (CASES 1$c$ & 2$c$($iii$)) *Let $p(\cdot)$ be an integral curve of the system $H^2_{1,\alpha}$ through $p(0)$. Let $h_0 = H^2_{1,\alpha}(p(0))$ and $c_0 = C(p(0))$. If the conditions of case 1c or 2c(iii) are satisfied, then there exists $t_0 \in \mathbb{R}$ such that $p(t) = \bar{p}(t + t_0)$, where*

$$\begin{aligned}
\bar{p}_1(t) &= -\tfrac{\alpha}{2} - \varepsilon_1\,\frac{\frac{\alpha^3\eta_0 - 2\zeta_2}{\alpha^3\eta_0 + 2\zeta_2}\,\sqrt{\zeta_1+\zeta_2} - \sqrt{\zeta_1-\zeta_2}\,\mathrm{cd}\,(\Omega\,t,\,k)}{\sqrt{\zeta_1+\zeta_2} - \sqrt{\zeta_1-\zeta_2}\,\mathrm{cd}\,(\Omega\,t,\,k)}\\
\bar{p}_2(t) &= \varepsilon_2\,\frac{\mathrm{sd}\left(\tfrac{1}{2}\Omega\,t,\,k\right)\sqrt{1+k\,\mathrm{cd}\,(\Omega\,t,\,k)}\sqrt{1+\,\mathrm{nd}\,(\Omega\,t,\,k)}}{\sqrt{\zeta_1+\zeta_2} - \sqrt{\zeta_1-\zeta_2}\,\mathrm{cd}\,(\Omega\,t,\,k)}\\
\bar{p}_3(t) &= \varepsilon_3\,\frac{\mathrm{cn}\left(\tfrac{1}{2}\Omega\,t,\,k\right)\sqrt{1-k\,\mathrm{cd}\,(\Omega\,t,\,k)}\sqrt{1+\,\mathrm{nd}\,(\Omega\,t,\,k)}}{\sqrt{\zeta_1+\zeta_2} - \sqrt{\zeta_1-\zeta_2}\,\mathrm{cd}\,(\Omega\,t,\,k)}.
\end{aligned}$$

*Here*

$$\begin{aligned}
\Omega &= \tfrac{1}{\alpha}\sqrt{\zeta_1+\zeta_2-\tfrac{1}{8}\tau^2} & \tau &= \alpha\left(\alpha+\eta_0-\sqrt{2}\sqrt{\alpha^2+\eta_0^2-2c_0}\right)\\
k &= \sqrt{\frac{\zeta_1-\zeta_2-\frac{1}{8}\tau^2}{\zeta_1+\zeta_2-\frac{1}{8}\tau^2}} & \zeta_1 &= \tfrac{1}{4}\left(4\alpha^2\,(\alpha+\eta_0)\,\eta_0 - \alpha\,(\alpha+4\eta_0)\,\tau + \tau^2\right)\\
k' &= \sqrt{\frac{2\zeta_2}{\zeta_1+\zeta_2-\frac{1}{8}\tau^2}} & \zeta_2 &= \tfrac{1}{2}\sqrt{\alpha\eta_0\,(\alpha\,(\alpha+\eta_0)-\tau)\,(2\alpha\,(\alpha+\eta_0)-\tau)\,(2\alpha\eta_0-\tau)}
\end{aligned}$$

*and*

$$\varepsilon_1 = \frac{\alpha^3\eta_0 + 2\zeta_2}{2\alpha^2\left(\alpha + 2\eta_0\right) - 2\alpha\tau}, \qquad \varepsilon_2 = \tfrac{k'}{2\sqrt{2k\alpha}}\sqrt{\tau\eta_0\left(4\zeta_2 - 4\zeta_1 + \alpha^2\tau\right)}$$
$$\varepsilon_3 = \sqrt{\frac{\zeta_2(\zeta_1 - \zeta_2)\left(\alpha^2\left(\zeta_2 + \alpha^2\eta_0(\alpha + \eta_0)\right) - \left(\zeta_2 + \alpha^3\eta_0\right)\tau\right)}{k\alpha^3\eta_0\left(\alpha^2 + 2\alpha\eta_0 - \tau\right)^2}}.$$

3.3.18 THEOREM. (CASE $2b$) *Let $p(\cdot)$ be an integral curve of the system $H^2_{1,\alpha}$ through $p(0)$. Let $h_0 = H^2_{1,\alpha}(p(0))$ and $c_0 = C(p(0))$. If the conditions of case 2b are satisfied, then there exist $t_0 \in \mathbb{R}$ and $\sigma_1, \sigma_2 \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$, where*

$$\begin{aligned}
\bar{p}_1(t) &= -\alpha - \frac{\Omega^2}{\alpha - \sigma_1\eta_0\cosh(\Omega\, t)}\\
\bar{p}_2(t) &= -\sigma_1\sigma_2\eta_0\Omega\,\frac{\sinh(\Omega\, t)}{\alpha - \sigma_1\eta_0\cosh(\Omega\, t)}\\
\bar{p}_3(t) &= -\frac{\sigma_2\Omega^2}{\alpha - \sigma_1\eta_0\cosh(\Omega\, t)}.
\end{aligned}$$

*Here* $\Omega = \sqrt{\frac{\eta_0^2 - \alpha^2}{2}}$.

3.3.19 THEOREM. (CASE $2c(i)$) *Let $p(\cdot)$ be an integral curve of the system $H^2_{1,\alpha}$ through $p(0)$. Let $h_0 = H^2_{1,\alpha}(p(0))$ and $c_0 = C(p(0))$. If the conditions of case 2c(i) are satisfied, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$, where*

$$\begin{aligned}
\bar{p}_1(t) &= -\tfrac{\alpha}{2} - \tfrac{\eta_0}{2}\,\frac{k'\rho_+ - \sigma\rho_-\,\mathrm{dn}\,(\Omega\, t,\, k)}{k'\rho_- - \sigma\rho_+\,\mathrm{dn}\,(\Omega\, t,\, k)}\\
\bar{p}_2(t) &= \tfrac{2\delta\eta_0}{\Omega}\,\frac{\mathrm{cn}\,(\Omega\, t,\, k)}{k'\rho_- - \sigma\rho_+\,\mathrm{dn}\,(\Omega\, t,\, k)}\\
\bar{p}_3(t) &= -2\sigma\delta k'\,\frac{\mathrm{sn}\,(\Omega\, t,\, k)}{k'\rho_- - \sigma\rho_+\,\mathrm{dn}\,(\Omega\, t,\, k)}.
\end{aligned}$$

*Here* $\Omega = \frac{1}{2}\sqrt{\alpha^2 - 4c_0 + 3\eta_0^2 + 4\delta}$, $k = 2\sqrt{\frac{2\delta}{\alpha^2 - 4c_0 + 3\eta_0^2 + 4\delta}}$, *and* $k' = \sqrt{\frac{\alpha^2 - 4c_0 + 3\eta_0^2 - 4\delta}{\alpha^2 - 4c_0 + 3\eta_0^2 + 4\delta}}$.

3.3.20 THEOREM. (CASE $2c(ii)$) *Let $p(\cdot)$ be an integral curve of the system $H^2_{1,\alpha}$ through $p(0)$. Let $h_0 = H^2_{1,\alpha}(p(0))$ and $c_0 = C(p(0))$. If the conditions of case 2c(ii) are satisfied, then there exists $t_0 \in \mathbb{R}$ such that $p(t) = \bar{p}(t + t_0)$, where*

$$\begin{aligned}
\bar{p}_1(t) &= -\tfrac{\alpha + \eta_0}{2} - \frac{(\alpha - \eta_0)\,\eta_0}{\eta_0 - \alpha\cos(\Omega\, t)^2}\\
\bar{p}_2(t) &= -\eta_0\sqrt{2\alpha\left(\eta_0 - \alpha\right)}\,\frac{\sin(\Omega\, t)}{\eta_0 - \alpha\cos(\Omega\, t)^2}\\
\bar{p}_3(t) &= (\alpha - \eta_0)\,\sqrt{\alpha\eta_0}\,\frac{\cos(\Omega\, t)}{\eta_0 - \alpha\cos(\Omega\, t)^2}.
\end{aligned}$$

*Here* $\Omega = \sqrt{\frac{\eta_0(\eta_0 - \alpha)}{2}}$.

**System $H^2_{2,\alpha}$**

The system $H^2_{2,\alpha}(p) = \alpha p_2 + p_1^2 + \frac{1}{2}p_2^2$, $\alpha > 0$ has equations of motion

$$\begin{cases} \dot{p}_1 = -(\alpha + p_2)p_3 \\ \dot{p}_2 = 2p_1p_3 \\ \dot{p}_3 = p_1(\alpha - p_2). \end{cases}$$

The equilibria are $\mathsf{e}_1^\nu = (\nu, \alpha, 0)$, $\mathsf{e}_2^\mu = (0, \mu, 0)$, and $\mathsf{e}_3^\nu = (0, -\alpha, \nu)$.

3.3.21 Theorem. *The equilibrium states have the following behaviour:*

*(i) The states $\mathsf{e}_1^\nu$ are stable.*

*(ii) The states $\mathsf{e}_2^\mu$, $\mu \in (-\infty, -\alpha) \cup (\alpha, \infty)$ are (spectrally) unstable.*

*(iii) The states $\mathsf{e}_2^\mu$, $-\alpha \leq \mu \leq \alpha$ are stable.*

*(iv) The states $\mathsf{e}_3^\nu$ are stable.*

Proof. Let $H_\lambda(p) = \lambda_1 H^2_{2,\alpha}(p) + \lambda_2 C(p)$.

(i) If $\lambda_1 = 1$ and $\lambda_2 = -1$, then we have that $\mathbf{d}H_\lambda(\mathsf{e}_1^\nu) = 0$ and that $\mathbf{d}^2H_\lambda(\mathsf{e}_1^\nu) = \operatorname{diag}(0, -1, -2)$ is definite when restricted to $W = \operatorname{span}\{(\alpha, -\nu, 0), (0, 0, 1)\}$. Hence the states $\mathsf{e}_1^\nu$ are stable.

(ii) The linearization of the system at $\mathsf{e}_2^\mu$ has eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = \pm\sqrt{\mu^2 - \alpha^2}$. Thus the states $\mathsf{e}_2^\mu$, $\mu \in (-\infty, -\alpha) \cup (\alpha, \infty)$ are spectrally unstable.

(iii) Suppose $-\alpha < \mu < \alpha$, $\mu \neq 0$ and let $\lambda_1 = 1$ and $\lambda_2 = -\frac{\alpha+\mu}{2\mu}$. We have $\mathbf{d}H_\lambda(\mathsf{e}_2^\mu) = 0$ and that $\mathbf{d}^2H_\lambda(\mathsf{e}_2^\mu) = \operatorname{diag}(-\frac{\alpha-\mu}{\mu}, -\frac{\alpha}{\mu}, -\frac{\alpha+\mu}{\mu})$ is definite when restricted to the subspace $W = \operatorname{span}\{(1, 0, 0), (0, 0, 1)\}$. Thus the states $\mathsf{e}_2^\mu$, $-\alpha < \mu < \alpha$, $\mu \neq 0$ are stable. Suppose $\mu = -\alpha$. Then $H^2_{2,\alpha}(\mathsf{e}_2^{-\alpha}) = -\frac{\alpha^2}{2}$ and $C(\mathsf{e}_2^{-\alpha}) = \alpha^2$. It is easy to show that $(H^2_{2,\alpha})^{-1}(-\frac{\alpha^2}{2}) \cap C^{-1}(\alpha^2) = \{\mathsf{e}_2^{-\alpha}\}$. Hence the state $\mathsf{e}_2^{-\alpha}$ is stable. Similarly, the state $\mathsf{e}_2^\alpha$ is stable.

(iv) Let $\lambda_1 = 1$ and $\lambda_2 = 0$. We have $\mathbf{d}H_\lambda(\mathsf{e}_3^\nu) = 0$ and that $\mathbf{d}^2H_\lambda(\mathsf{e}_3^\nu) = \operatorname{diag}(2, 1, 0)$ is definite when restricted to the subspace $W = \operatorname{span}\{(1, 0, 0), (0, \nu, \alpha)\}$. Hence the states $\mathsf{e}_3^\nu$ are stable.

□

There are five cases for the intersection of a parabolic cylinder $(H^2_{2,\alpha})^{-1}(h_0)$ and a sphere $C^{-1}(c_0)$. (We note that if the intersection is nonempty, then $h_0 \leq c_0 + \frac{1}{2}\alpha^2$.) We further subdivide one of these cases into two subcases to facilitate integration. An index of the conditions defining these cases appears in Table 3.3. In Figure 3.9 we graph the critical energy states $(h_0, c_0)$; in Figure 3.10 we graph the corresponding typical configurations. (The value $\alpha = 1$ was used in both these figures.)

We will find the following lemma useful when verifying that a given curve is an integral curve.

3.3.22 Lemma. *If $p(\cdot)$ is a curve such that $H^2_{2,\alpha}(p(t)) = h_0$, $C(p(t)) = c_0$, and $\dot{p}_2(t) = 2p_1(t)p_3(t)$ for $t \in \mathbb{R}$, then $p(\cdot)$ is an integral curve of the system $H^2_{2,\alpha}$.*

Proof. As $\frac{d}{dt}C(p(t)) = 2p_1\dot{p}_1 + 2p_2\dot{p}_2 + 2p_3\dot{p}_3 = 0$, $\frac{d}{dt}H^2_{2,\alpha}(p(t)) = \alpha\dot{p}_2 + 2p_1\dot{p}_1 + p_2\dot{p}_2 = 0$, and $\dot{p}_2 = 2p_1p_3$, we have $p_1\dot{p}_1 = -(\alpha + p_2)p_1p_3$ and $p_3\dot{p}_3 = (\alpha - p_2)p_1p_3$. It follows that $p(\cdot)$ is an integral curve of the system $H^2_{2,\alpha}$.

□

| Conditions | | Index |
|---|---|---|
| $\frac{c_0}{2} + \alpha\sqrt{c_0} < h_0$ | | a |
| $\frac{c_0}{2} + \alpha\sqrt{c_0} = h_0$ | | b |
| $\frac{c_0}{2} - \alpha\sqrt{c_0} < h_0 < \frac{c_0}{2} + \alpha\sqrt{c_0}$ | $c_0 = 2h_0$ | c(i) |
| | $c_0 \neq 2h_0$ | c(ii) |
| $\frac{c_0}{2} - \alpha\sqrt{c_0} = h_0$ | | d |
| $\frac{c_0}{2} - \alpha\sqrt{c_0} > h_0$ | | e |

Table 3.3: Index of cases for integral curves of $H^2_{2,\alpha}$

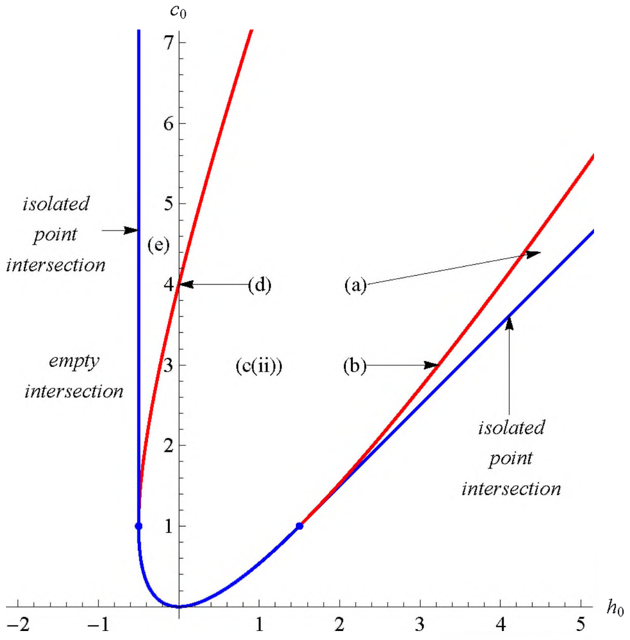
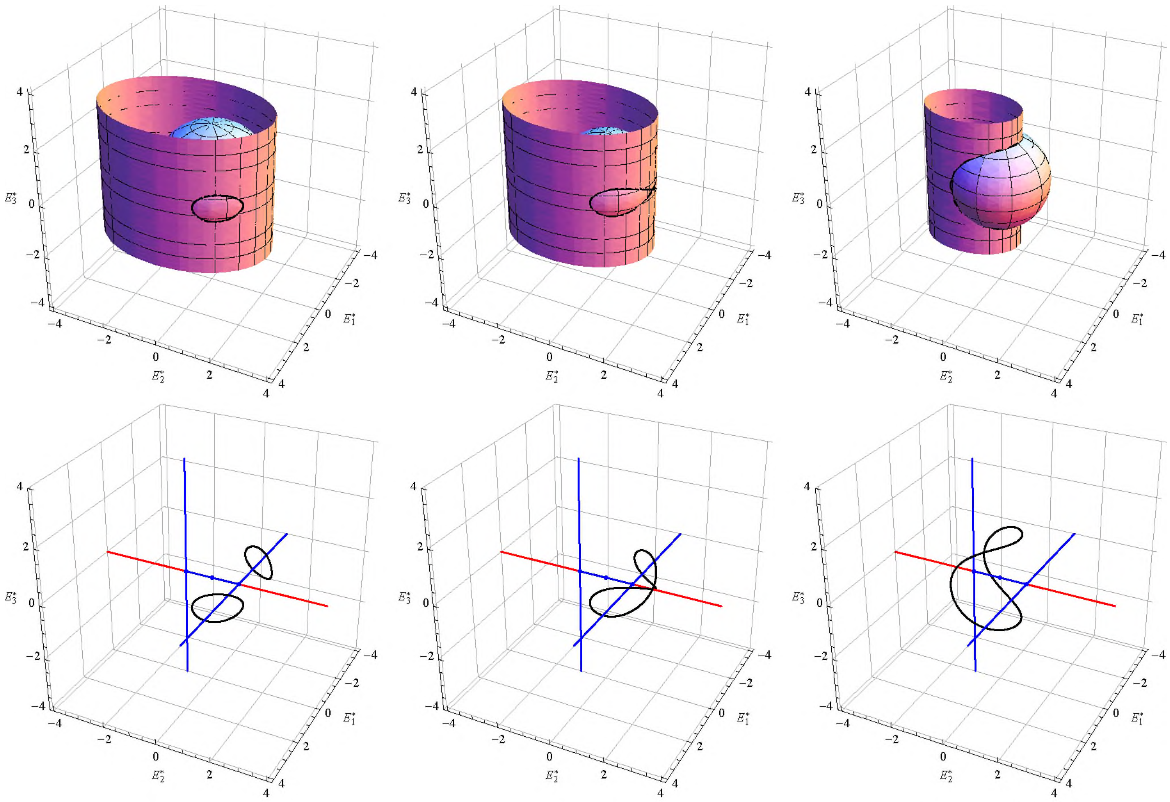
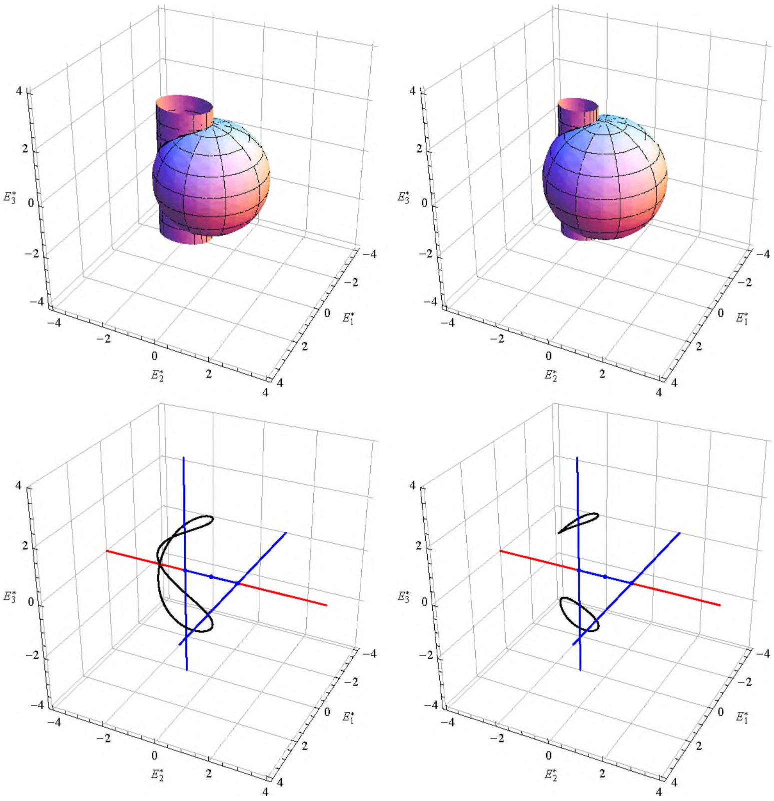


Figure 3.9: Critical energy states for $H^2_{2,\alpha}$



Case a Case b Case c(ii)



Case d Case e

Figure 3.10: Typical configurations for $H^2_{2,\alpha}$

MATHEMATICA. *The explicit calculations for the integration of the system $H^2_{2,\alpha}$ are contained in the directory:*

*Thesis Mathematica\SO(3)\Integration\H22a(p)*

*The individual files for each case are as follows: case (a) - H22aIntA.nb; case (b) - H22aIntB.nb; case c (i) - H22aIntC(i).nb; case c(ii) - H22aIntC(ii).nb; case (d) - H22aIntD.nb; and case (e) - H22aIntE.nb.*

We now derive the expressions for the integral curves in the first case.

3.3.23 THEOREM. (CASE *a*) *Let $p(\cdot)$ be an integral curve of the system $H^2_{2,\alpha}$ through $p(0)$. Let $h_0 = H^2_{2,\alpha}(p(0))$ and $c_0 = C(p(0))$. If $\frac{c_0}{2} + \alpha\sqrt{c_0} < h_0$, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$, where*

$$\begin{aligned}
\bar{p}_1(t) &= \sigma\sqrt{\delta}\sqrt{c_0 + \delta - \alpha^2}\,\frac{\operatorname{dn}(\Omega\, t, k)}{\sqrt{\rho+\delta} - \sqrt{\rho-\delta}\operatorname{sn}(\Omega\, t, k)}\\
\bar{p}_2(t) &= -\tfrac{1}{2\alpha}\frac{(\delta + c_0 - 2h_0)\sqrt{\rho+\delta} + (\delta - c_0 + 2h_0)\sqrt{\rho-\delta}\operatorname{sn}(\Omega\, t, k)}{\sqrt{\rho+\delta} - \sqrt{\rho-\delta}\operatorname{sn}(\Omega\, t, k)}\\
\bar{p}_3(t) &= -\sigma\sqrt{\delta}\sqrt{\alpha^2 + 2c_0 - 2h_0}\,\frac{\operatorname{cn}(\Omega\, t, k)}{\sqrt{\rho+\delta} - \sqrt{\rho-\delta}\operatorname{sn}(\Omega\, t, k)}.
\end{aligned}$$

*Here $\delta = \sqrt{c_0^2 + 4h_0^2 - 4c_0(\alpha^2 + h_0)}$, $\Omega = \sqrt{c_0 + \delta - \alpha^2}$, $k = \sqrt{\frac{\alpha^2+\delta-c_0}{\alpha^2-\delta-c_0}}$, and $\rho = 2h_0 - c_0 - 2\alpha^2$.*

3.3.24 REMARK. If we take the limit of the expression for $\bar{p}(t)$, as $\alpha$ tends to 0, then we obtain integral curves for the first case of the system $H^2$.

PROOF. We start by explaining how the expression for $\bar{p}(\cdot)$ was found. (Again, a number of convenient assumptions are made implicitly and translations in the independent variable are discarded.) Suppose $\bar{p}(\cdot)$ is an integral curve of $H^2_{2,\alpha}$ such that $h_0 > \alpha\sqrt{c_0} + \frac{c_0}{2}$, where $h_0 = H^2_{2,\alpha}(\bar{p}(0))$ and $c_0 = C(\bar{p}(0))$. As $\bar{p}(\cdot)$ satisfies $(\frac{d\bar{p}_2}{dt})^2 = 4\bar{p}_1^2\bar{p}_3^2$, $H^2_{2,\alpha}(\bar{p}(t)) = h_0$, and $C(\bar{p}(t)) = c_0$, we have

$$\frac{d\bar{p}_2}{dt} = \sqrt{(2h_0 - 2\alpha p_2 - p_2^2)(2c_0 - 2h_0 + 2\alpha p_2 - p_2^2)}. \tag{3.11}$$

We transform (3.11) into standard form (see appendix D.2). First, we can rewrite (3.11) as

$$\frac{d\bar{p}_2}{dt} = \sqrt{\left(A_1(\bar{p}_2 - r_1)^2 + B_1(\bar{p}_2 - r_2)^2\right)\left(A_2(\bar{p}_2 - r_1)^2 + B_2(\bar{p}_2 - r_2)^2\right)}$$

where

$$\begin{aligned}
r_1 &= \frac{\delta - c_0 + 2h_0}{2\alpha} & r_2 &= -\frac{\delta + c_0 - 2h_0}{2\alpha}\\
A_1 &= \frac{2\alpha^2 - \delta - c_0 + 2h_0}{2\delta} > 0 & A_2 &= \frac{\rho - \delta}{2\delta} > 0\\
B_1 &= \frac{-2\alpha^2 - \delta + c_0 - 2h_0}{2\delta} < 0 & B_2 &= -\frac{\rho + \delta}{2\delta} < 0.
\end{aligned}$$

Here $\delta = \sqrt{c_0^2 + 4h_0^2 - 4c_0\,(\alpha^2 + h_0)}$ and $\rho = 2h_0 - c_0 - 2\alpha^2$.

Making the change of variable $s = \frac{\bar{p}_2 - r_1}{\bar{p}_2 - r_2}$ yields

$$t = \frac{1}{(r_1 - r_2)\sqrt{A_1 A_2}} \int_{\frac{\bar{p}_2 - r_1}{\bar{p}_2 - r_2}}^{\infty} \frac{ds}{\sqrt{\left(s^2 - (-\frac{B_2}{A_2})\right)\left(s^2 - (-\frac{B_1}{A_1})\right)}}.$$

By applying the elliptic integral formula (D.3) we obtain

$$\bar{p}_2(t) = \frac{r_2\sqrt{-\frac{B_2}{A_2}}\ \mathrm{ns}\left((r_1 - r_2)\sqrt{A_1A_2}\sqrt{-\frac{B_2}{A_2}}\,t, \frac{\sqrt{-\frac{B_1}{A_1}}}{\sqrt{-\frac{B_2}{A_2}}}\right) - r_1}{\sqrt{-\frac{B_2}{A_2}}\ \mathrm{ns}\left((r_1 - r_2)\sqrt{A_1A_2}\sqrt{-\frac{B_2}{A_2}}\,t, \frac{\sqrt{-\frac{B_1}{A_1}}}{\sqrt{-\frac{B_2}{A_2}}}\right) - 1}.$$

Substituting the values for $A_1, A_2, B_1, B_2, r_1, r_2$ and simplifying then yields

$$\bar{p}_2(t) = -\frac{1}{2\alpha}\,\frac{(\delta + c_0 - 2h_0)\sqrt{\rho + \delta} + (\delta - c_0 + 2h_0)\sqrt{\rho - \delta}\,\mathrm{sn}\,(\Omega\,t, k)}{\sqrt{\rho + \delta} - \sqrt{\rho - \delta}\,\mathrm{sn}\,(\Omega\,t, k)}$$

where $\Omega = \sqrt{c_0 + \delta - \alpha^2}$ and $k = \sqrt{\frac{\alpha^2 + \delta - c_0}{\alpha^2 - \delta - c_0}}$. As $h_0 = \bar{p}_2(t) + \bar{p}_1(t)^2 + \frac{1}{2}\bar{p}_2(t)^2$, we have $\bar{p}_1(t) = \sigma_1\sqrt{h_0 - \alpha\bar{p}_2(t) - \frac{1}{2}\bar{p}_2(t)^2}$ for some $\sigma_1 \in \{-1, 1\}$. Accordingly, we get

$$\bar{p}_1(t) = \sigma_1\sqrt{\frac{\delta\,(\alpha^2 + 2h_0)\,(\alpha^2 + 2c_0 - 2h_0)}{c_0 - \alpha^2 - \delta}}\,\frac{\mathrm{dn}\,(\Omega\,t, k)}{\sqrt{\rho + \delta} - \sqrt{\rho - \delta}\,\mathrm{sn}\,(\Omega\,t, k)}.$$

Then, as $\bar{p}_3(t)^2 = c_0 - \bar{p}_1(t)^2 - \bar{p}_2(t)^2$, it follows that

$$\bar{p}_3(t) = \sigma_2\frac{\sqrt{\delta\,(\alpha^2 + 2c_0 - 2h_0)}\,\mathrm{cn}\,(\Omega\,t, k)}{\sqrt{\rho + \delta} - \sqrt{\rho - \delta}\,\mathrm{sn}\,(\Omega\,t, k)}$$

for some $\sigma_2 \in \{-1, 1\}$. We show that $\bar{p}(\cdot)$ is an integral curve for certain values of $\sigma_1$ and $\sigma_2$. We have

$$\frac{d}{dt}\bar{p}_2(t) - 2\bar{p}_1(t)\bar{p}_3(t) = -\frac{\left(2\alpha(1 + \sigma_1\sigma_2)\,(\alpha^2 + 2c_0 - 2h_0)\,\sqrt{\frac{\alpha^2 + 2h_0}{c_0 - \alpha^2 - \delta}}\right)\,\mathrm{cn}\,(\Omega\,t, k)\ \mathrm{dn}\,(\Omega\,t, k)}{\alpha\left(\sqrt{\rho + \delta} - \sqrt{\rho - \delta}\,\mathrm{sn}\,(\Omega\,t, k)\right)^2}.$$

Therefore $\frac{d}{dt}\bar{p}_2(t) = 2\bar{p}_1(t)\bar{p}_3(t)$ whenever $\sigma_1 = -\sigma_2 = \sigma$. By construction, $H^2_{2,\alpha}(\bar{p}(t)) = h_0$ and $C(\bar{p}(t)) = c_0$. Consequently, by lemma 3.3.22, it follows that $\bar{p}(\cdot)$ (as stated in the theorem) is an integral curve; it is not difficult to show that $0 < k < 1$ and that $\bar{p}(t)$ is defined for all $t \in \mathbb{R}$.

Let $p(\cdot)$ be an integral curve through $p(0)$, let $h_0 = H^2_{2,\alpha}(p(0))$, $c_0 = C(p(0))$, and suppose that $h_0 > \alpha\sqrt{c_0} + \frac{c_0}{2}$. We claim that $p(t) = \bar{p}(t + t_0)$ for some $\sigma \in \{-1, 1\}$ and $t_0 \in \mathbb{R}$. As $\alpha p_2(0) + p_1(0)^2 + \frac{1}{2}p_2(0)^2 = h_0$ and $p_1(0)^2 + p_2(0)^2 + p_3(0)^2 = c_0$, we have $h_0 - c_0 - \alpha p_2(0) + \frac{1}{2}p_2(0)^2 =$

$-p_3(0)^2 \leq 0$. It follows that

$$\alpha - \sqrt{\alpha^2 + 2c_0 - 2h_0} \leq p_2(0) \leq \alpha + \sqrt{\alpha^2 + 2c_0 - 2h_0}.$$

Now $\bar{p}_2(\frac{K}{\Omega}) = \alpha - \sqrt{\alpha^2 + 2c_0 - 2h_0}$ and $\bar{p}_2(\frac{3K}{\Omega}) = \alpha + \sqrt{\alpha^2 + 2c_0 - 2h_0}$. Thus there exists $t_1 \in [\frac{K}{\Omega}, \frac{3K}{\Omega}]$ such that $p_2(0) = \bar{p}_2(t_1)$. As $p_1(0)^2 = h_0 - \alpha p_2(0) - \frac{1}{2}p_2(0)^2$ and

$$\min_{\substack{\alpha-\sqrt{\alpha^2+2c_0-2h_0}\leq p_2 \\ \alpha+\sqrt{\alpha^2+2c_0-2h_0}\geq p_2}} (h_0 - \alpha p_2 - \tfrac{1}{2}p_2^2) = 2h_0 - 2\alpha^2 - c_0 - 2\alpha\sqrt{\alpha^2 + 2c_0 - 2h_0} > 0$$

it follows that $p_1(0) \neq 0$. Let $\sigma = \operatorname{sgn}(p_1(0))$. We have

$$p_1(0)^2 = h_0 - \alpha p_2(0) - \tfrac{1}{2}p_2(0)^2 = h_0 - \alpha\bar{p}_2(t_1) - \tfrac{1}{2}\bar{p}_2(t_1)^2 = \bar{p}_1(t_1)^2.$$

Hence, as $\operatorname{sgn}(\bar{p}_1(t_1)) = \sigma$, we get $p_1(0) = \bar{p}_1(t_1)$. On the other hand

$$p_3(0)^2 = c_0 - p_1(0)^2 - p_2(0)^2 = c_0 - \bar{p}_1(t_1)^2 - \bar{p}_2(t_1)^2 = \bar{p}_3(t_1)^2$$

and so $p_3(0) = \pm\bar{p}_3(t_1)$. Furthermore $\bar{p}_1(-t+\frac{2K}{\Omega}) = \bar{p}_1(t)$, $\bar{p}_2(-t+\frac{2K}{\Omega}) = \bar{p}_2(t)$, and $\bar{p}_3(-t+\frac{2K}{\Omega}) = -\bar{p}_3(t)$. Thus there exists $t_0 \in \mathbb{R}$ ($t_0 = t_1$ or $t_0 = -t_1 + \frac{2K}{\Omega}$) such that $p(0) = \bar{p}(t_0)$. Consequently, the integral curves $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ solve the same Cauchy problem and therefore are identical. □

One might consider limiting $h_0$ to $\frac{c_0}{2} + \alpha\sqrt{c_0}$ in case (a) in order to produce integral curves for case (b). However, this limit degenerates and so another approach is required. The proof for case (b) is similar to that of case (d). Thus for case (b), we do not explain how the expressions for the integral curve were found. However, we do verify that the expressions given are indeed integral curves of the system.

3.3.25 THEOREM. (CASE *b*) *Let $p(\cdot)$ be an integral curve of the system $H^2_{2,\alpha}$ through $p(0)$. Let $h_0 = H^2_{2,\alpha}(p(0))$ and $c_0 = C(p(0))$. If $\frac{c_0}{2} + \alpha\sqrt{c_0} = h_0$, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$, where*

$$\begin{aligned}
\bar{p}_1(t) &= \tfrac{2\sigma\left(c_0-\alpha^2\right)\sqrt{\alpha}}{\sqrt{\sqrt{c_0}-\alpha}}\, \frac{\cosh(\frac{1}{2}\Omega\, t)}{\sqrt{c_0} + \alpha\, \cosh(\Omega\, t)} \\
\bar{p}_2(t) &= \sqrt{c_0} - \frac{2(c_0 - \alpha^2)}{\sqrt{c_0} + \alpha\, \cosh(\Omega\, t)} \\
\bar{p}_3(t) &= \tfrac{2\sigma\left(c_0-\alpha^2\right)\sqrt{\alpha}}{\sqrt{\sqrt{c_0}+\alpha}}\, \frac{\sinh(\frac{1}{2}\Omega\, t)}{\sqrt{c_0} + \alpha\, \cosh(\Omega\, t)}.
\end{aligned}$$

*Here $\Omega = 2\sqrt{c_0 - \alpha^2}$.*

PROOF. We show that $\bar{p}(\cdot)$ (as stated in the theorem) is an integral curve for $\sigma \in \{-1, 1\}$. After some simplification, we get

$$\frac{d}{dt}\bar{p}_2(t) - 2\bar{p}_1(t)\bar{p}_3(t) = \frac{4\alpha\sinh(\Omega\, t)\left(\sigma^2 - 1\right)\left(\alpha^2 - c_0\right)\sqrt{c_0 - \alpha^2}}{\left(\alpha\cosh(\Omega\, t) + \sqrt{c_0}\right)^2}.$$

Therefore $\frac{d}{dt}\bar{p}_2(t) = 2\bar{p}_1(t)\bar{p}_3(t)$ whenever $\sigma \in \{-1, 1\}$. (It is not difficult to verify that $H^2_{2,\alpha}(\bar{p}(t)) = h_0$ and $C(\bar{p}(t)) = c_0$ for $t \in \mathbb{R}$.) Consequently, by lemma 3.3.22, it follows that $\bar{p}(\cdot)$ is an integral curve; it is not difficult to show that $0 < k < 1$ and that $\bar{p}(t)$ is defined for all $t \in \mathbb{R}$.

Let $p(\cdot)$ be an integral curve through $p(0)$, let $h_0 = H^2_{2,\alpha}(p(0))$, $c_0 = C(p(0))$, and suppose that $h_0 = \alpha\sqrt{c_0} + \frac{c_0}{2}$. We claim that $p(t) = \bar{p}(t + t_0)$ for some $\sigma \in \{-1, 1\}$ and $t_0 \in \mathbb{R}$. We have $\alpha\, p_2(0) + p_1(0)^2 + \frac{1}{2}p_2(0)^2 = \alpha\sqrt{c_0} + \frac{c_0}{2}$ and $p_1(0)^2 + p_2(0)^2 + p_3(0)^2 = c_0$. If $p_1(0) = 0$, then $p(\cdot)$ is constant. We assume $p_1(0) \neq 0$. Let $\operatorname{sgn}(p_1(0)) = \sigma$. As $\alpha\sqrt{c_0} + \frac{c_0}{2} - \alpha p_2(0) + \frac{1}{2}p_2(0)^2 \leq c_0$, it follows that

$$2\alpha - \sqrt{c_0} \leq p_2(0) \leq \sqrt{c_0}.$$

Now $\bar{p}_2(0) = 2\alpha + \sqrt{c_0}$ and $\lim_{t\to\infty} \bar{p}_2(t) = \sqrt{c_0}$. Thus there exists $t_1 \in [0, \infty)$ such that $p_2(0) = \bar{p}_2(t_1)$. As

$$p_3(0)^2 = c_0 - p_1(0)^2 - p_2(0)^2 = c_0 - \bar{p}_1(t_0)^2 - \bar{p}_2(t_0)^2 = \bar{p}_3(t_0)^2$$

it follows that $p_3(0) = \pm\bar{p}_3(t_1)$. Furthermore $\bar{p}_3(-t) = -\bar{p}_3(t)$ and $\bar{p}_2(-t) = \bar{p}_2(t)$. Thus there exists $t_0 \in \mathbb{R}$ ( $t_0 = t_1$ or $t_0 = -t_1$) such that $p_3(0) = \bar{p}_3(t_0)$. On the other hand

$$p_1(0)^2 = h_0 - 2\alpha p_2(0) - \tfrac{1}{2}p_2(0)^2 = h_0 - 2\alpha\bar{p}_2(t_1) - \tfrac{1}{2}\bar{p}_2(t_1)^2 = \bar{p}_1(t_1)^2$$

and so, as $\operatorname{sgn}(p_1(0)) = \sigma = \operatorname{sgn}(\bar{p}_1(t_1))$, we have $p_1(0) = \bar{p}_1(t_0)$. Thus there exists $t_0 \in \mathbb{R}$ such that $p(0) = \bar{p}(t_0)$. Consequently, the integral curves $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ solve the same Cauchy problem, and therefore are identical. □

In the reduction to standard form in case c(i) the roots of the two quadratics need to be deinterlaced. Consequently, the expressions for the corresponding integral curves are more involved.

3.3.26 THEOREM. (CASE *c(i)*) *Let $p(\cdot)$ be an integral curve of the system $H^2_{2,\alpha}$ through $p(0)$. Let $h_0 = H^2_{2,\alpha}(p(0))$ and $c_0 = C(p(0))$. If $c_0 = 2h_0$, then there exists $t_0 \in \mathbb{R}$ such that $p(t) = \bar{p}(t + t_0)$, where*

$$\bar{p}_1(t) = \alpha k\sqrt{\alpha}\sqrt[4]{c_0}\,\frac{\operatorname{cn}(\Omega\, t, k)}{\sqrt{\alpha^2 + c_0} - \sqrt{c_0} + \alpha\operatorname{dn}(\Omega\, t, k)}\sqrt{\frac{1 + \operatorname{dn}(\Omega\, t, k)}{k' + \operatorname{dn}(\Omega\, t, k)}}$$

$$\bar{p}_2(t) = \sqrt{c_0}\,\frac{\sqrt{\alpha^2 + c_0} - \sqrt{c_0} - \alpha\operatorname{dn}(\Omega\, t, k)}{\sqrt{\alpha^2 + c_0} - \sqrt{c_0} + \alpha\operatorname{dn}(\Omega\, t, k)}$$

$$\bar{p}_3(t) = \alpha k\sqrt{\alpha}\sqrt[4]{c_0}\,\frac{\operatorname{sn}(\Omega\, t, k)}{\sqrt{\alpha^2 + c_0} - \sqrt{c_0} + \alpha\operatorname{dn}(\Omega\, t, k)}\sqrt{\frac{k' + \operatorname{dn}(\Omega\, t, k)}{1 + \operatorname{dn}(\Omega\, t, k)}}.$$

*Here* $\Omega = \frac{\alpha^2}{\sqrt{\alpha^2+c_0}-\sqrt{c_0}}$, $k = \frac{2\sqrt[4]{c_0}\sqrt[4]{\alpha^2+c_0}}{\sqrt{\alpha^2+c_0}+\sqrt{c_0}}$, *and* $k' = \frac{\sqrt{\alpha^2+c_0}-\sqrt{c_0}}{\sqrt{\alpha^2+c_0}+\sqrt{c_0}}$.

PROOF. We start by explaining how the expression for $\bar{p}(\cdot)$ was found. Suppose $\bar{p}(\cdot)$ is an integral curve of $H^2_{2,\alpha}$ such that $c_0 = 2h_0$, where $h_0 = H^2_{2,\alpha}(\bar{p}(0))$ and $c_0 = C(\bar{p}(0))$. Note that $\frac{c_0}{2} - \alpha\sqrt{c_0} < h_0 < \frac{c_0}{2} + \alpha\sqrt{c_0}$ is trivially satisfied when $c_0 > 0$. As $\bar{p}(\cdot)$ satisfies $(\frac{d\bar{p}_2}{dt})^2 = 4\bar{p}_1^2\bar{p}_3^2$, $H^2_{2,\alpha}(\bar{p}(t)) = \frac{c_0}{2}$, and $C(\bar{p}(t)) = c_0$, we have

$$\frac{d\bar{p}_2}{dt} = \sqrt{(c_0 - 2\alpha p_2 - p_2^2)(c_0 + 2\alpha p_2 - p_2^2)}.$$

After deinterlacing the roots of the two quadratics we get

$$\frac{d\bar{p}_2}{dt} = \sqrt{(c_0 + 2\sqrt{\alpha^2 + c_0}\, p_2 + p_2^2)(c_0 - 2\sqrt{\alpha^2 + c_0}\, p_2 + p_2^2)}.$$

We transform this equation into standard form. Making the change of variables $s = -\frac{\bar{p}_2 - r_1}{\bar{p}_2 - r_2}$ yields

$$t = \frac{1}{(r_1 - r_2)\sqrt{-A_1 A_2}} \int_{\frac{r_1 - \bar{p}_2}{\bar{p}_2 - r_2}}^{\sqrt{-\frac{B_1}{A_1}}} \frac{ds}{\sqrt{(-\frac{B_1}{A_1} - s^2)(s^2 - (-\frac{B_2}{A_2}))}}.$$

Here

$$A_1 = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 + \tfrac{\alpha^2}{c_0}} < 0 \qquad A_2 = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 + \tfrac{\alpha^2}{c_0}} > 0$$

$$B_1 = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 + \tfrac{\alpha^2}{c_0}} > 0 \qquad B_2 = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 + \tfrac{\alpha^2}{c_0}} < 0$$

$$r_1 = \sqrt{c_0} \qquad r_2 = -\sqrt{c_0}.$$

By applying the elliptic integral formula (D.2) we obtain

$$\bar{p}_2(t) = \sqrt{c_0}\frac{\sqrt{\alpha^2 + c_0} - \sqrt{c_0} - \alpha\,\mathrm{dn}\,(\Omega\, t, k)}{\sqrt{\alpha^2 + c_0} - \sqrt{c_0} + \alpha\,\mathrm{dn}\,(\Omega\, t, k)}$$

where $\Omega = \frac{\alpha^2}{\sqrt{\alpha^2 + c_0} - \sqrt{c_0}}$ and $k = \frac{2\sqrt[4]{c_0}\,\sqrt[4]{\alpha^2 + c_0}}{\sqrt{\alpha^2 + c_0} + \sqrt{c_0}}$.
As $\bar{p}_1(t)^2 = \frac{c_0}{2} - \alpha\bar{p}_2(t) - \frac{1}{2}\bar{p}_2(t)^2$, we have

$$\begin{aligned}
\bar{p}_1(t)^2 &= \frac{\alpha\sqrt{c_0}\,(1 + \,\mathrm{dn}\,(\Omega\, t, k))\left(2\sqrt{c_0\,(\alpha^2 + c_0)} - 2c_0 - \alpha^2 + \alpha^2\,\mathrm{dn}\,(\Omega\, t, k)\right)}{\left(\sqrt{\alpha^2 + c_0} - \sqrt{c_0} + \alpha\,\mathrm{dn}\,(\Omega\, t, k)\right)^2} \\
&= \frac{\alpha^3\sqrt{c_0}\,(1 + \,\mathrm{dn}\,(\Omega\, t, k))\,(\,\mathrm{dn}\,(\Omega\, t, k) - k')}{\left(\sqrt{\alpha^2 + c_0} - \sqrt{c_0} + \alpha\,\mathrm{dn}\,(\Omega\, t, k)\right)^2}
\end{aligned}$$

where $k' = \frac{\sqrt{\alpha^2 + c_0} - \sqrt{c_0}}{\sqrt{\alpha^2 + c_0} + \sqrt{c_0}}$. We now multiply this equation by

$$\frac{\mathrm{cn}\,(\Omega\, t, k)^2}{\mathrm{cn}\,(\Omega\, t, k)^2} = \frac{k^2\,\mathrm{cn}\,(\Omega\, t, k)^2}{\mathrm{dn}\,(\Omega\, t, k)^2 - (k')^2} = \frac{k^2\,\mathrm{cn}\,(\Omega\, t, k)^2}{(\mathrm{dn}\,(\Omega\, t, k) - k')(\mathrm{dn}\,(\Omega\, t, k) + k')}$$

and take the square root to obtain

$$\bar{p}_1(t) = \sigma_1 \frac{\alpha k\sqrt{\alpha\sqrt{c_0}}\,\mathrm{cn}\,(\Omega\, t, k)}{\sqrt{\alpha^2 + c_0} - \sqrt{c_0} + \alpha\,\mathrm{dn}\,(\Omega\, t, k)}\sqrt{\frac{1 + \mathrm{dn}\,(\Omega\, t, k)}{k' + \mathrm{dn}\,(\Omega\, t, k)}}$$

for some $\sigma_1 \in \{-1, 1\}$. Similarly, using $c_0 = \bar{p}_1(t)^2 + \bar{p}_2(t)^2 + \bar{p}_3(t)^2$ and multiplying by

$$\frac{\operatorname{sn}(\Omega\, t, k)^2}{\operatorname{sn}(\Omega\, t, k)^2} = \frac{k^2 \operatorname{sn}(\Omega\, t, k)^2}{(1 - \operatorname{dn}(\Omega\, t, k))(1 + \operatorname{dn}(\Omega\, t, k))}$$

yields

$$\bar{p}_3(t) = \sigma_2 \frac{\alpha k \sqrt{\alpha \sqrt{c_0}} \operatorname{sn}(\Omega\, t, k)}{\sqrt{\alpha^2 + c_0} - \sqrt{c_0} + \alpha \operatorname{dn}(\Omega\, t, k)} \sqrt{\frac{k' + \operatorname{dn}(\Omega\, t, k)}{1 + \operatorname{dn}(\Omega\, t, k)}}$$

for some $\sigma_2 \in \{-1, 1\}$.

We show that $\bar{p}(\cdot)$ is an integral curve for certain values of $\sigma_1$ and $\sigma_2$. We have

$$\frac{d}{dt}\bar{p}_2(t) - 2\bar{p}_1(t)\bar{p}_3(t) = \frac{2k^2\alpha^3\sqrt{c_0}(1 - \sigma_1\sigma_2) \operatorname{cn}(\Omega\, t, k) \operatorname{sn}(\Omega\, t, k)}{\left(\sqrt{\alpha^2 + c_0} - \sqrt{c_0} + \alpha \operatorname{dn}(\Omega\, t, k)\right)^2}.$$

Therefore $\frac{d}{dt}\bar{p}_2(t) = 2\bar{p}_1(t)\bar{p}_3(t)$ whenever $\sigma_1 = \sigma_2 = 1$. By construction, $H^2_{2,\alpha}(\bar{p}(t)) = h_0$ and $C(\bar{p}(t)) = c_0$. Consequently, by lemma 3.3.22, it follows that $\bar{p}(\cdot)$ (as stated in the theorem) is an integral curve; it is not difficult to show that $0 < k < 1$ and that $\bar{p}(t)$ is defined for all $t \in \mathbb{R}$.

Let $p(\cdot)$ be an integral curve through $p(0)$, let $h_0 = H^2_{2,\alpha}(p(0))$, $c_0 = C(p(0))$, and suppose that $c_0 = 2h_0$. We claim that $p(t) = \bar{p}(t + t_0)$ for some $t_0 \in \mathbb{R}$. We have $\alpha\, p_2(0) + p_1(0)^2 + \frac{1}{2}p_2(0)^2 = \frac{c_0}{2}$ and $p_1(0)^2 + p_2(0)^2 + p_3(0)^2 = c_0$. Therefore $\alpha p_2(0) + \frac{1}{2}p_2(0)^2 \leq \frac{c_0}{2}$ and so $-\alpha - \sqrt{\alpha^2 + c_0} \leq p_2(0) \leq -\alpha + \sqrt{\alpha^2 + c_0}$. We also have $p_1(0)^2 + p_2(0)^2 \leq c_0$, which implies that $\alpha - \sqrt{\alpha^2 + c_0} \leq p_2(0) \leq \alpha + \sqrt{\alpha^2 + c_0}$. Thus

$$\alpha - \sqrt{\alpha^2 + c_0} \leq p_2(0) \leq -\alpha + \sqrt{\alpha^2 + c_0}.$$

Now $\bar{p}_2(0) = \alpha - \sqrt{\alpha^2 + c_0}$ and $\bar{p}_2(\frac{K}{\Omega}) = -\alpha + \sqrt{\alpha^2 + c_0}$. Thus there exists $t_2 \in [0, \frac{K}{\Omega}]$ such that $p_2(0) = \bar{p}_2(t_2)$. As

$$p_1(0)^2 = \tfrac{c_0}{2} - 2\alpha p_2(0) - \tfrac{1}{2}p_2(0)^2 = \tfrac{c_0}{2} - 2\alpha\bar{p}_2(t_2) - \tfrac{1}{2}\bar{p}_2(t_2)^2 = \bar{p}_1(t_2)^2$$

it follows that $p_1(0) = \pm\bar{p}_1(t_2)$. Furthermore $\bar{p}_1(t + \frac{2K}{\Omega}) = -\bar{p}_1(t)$ and $\bar{p}_2(t + \frac{2K}{\Omega}) = \bar{p}_2(t)$. Thus there exists $t_1 \in \mathbb{R}$ ($t_1 = t_2$ or $t_1 = t_2 + \frac{2K}{\Omega}$) such that $p_1(0) = \bar{p}_1(t_1)$ and $p_2(0) = \bar{p}_2(t_1)$. On the other hand

$$p_3(0)^2 = c_0 - p_1(0)^2 - p_2(0)^2 = c_0 - \bar{p}_1(t_1)^2 - \bar{p}_2(t_1)^2 = \bar{p}_3(t_1)^2$$

and so $p_3(0) = \pm\bar{p}_3(t_1)$. Furthermore $\bar{p}_1(-t) = \bar{p}_1(t)$, $\bar{p}_2(-t) = \bar{p}_2(t)$, and $\bar{p}_3(-t) = -\bar{p}_3(t)$. Thus there exists $t_0 \in \mathbb{R}$ ($t_0 = t_1$ or $t_0 = -t_1$) such that $p(0) = \bar{p}(t_0)$. Consequently, the integral curves $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ solve the same Cauchy problem, and therefore are identical. □

Case c(ii) is very similar to case c(i), although the computations are more involved. Thus, we do not provide details for how the expressions of the integral curves were found. (Refer to the Mathematica file mentioned in this section for the details on how the expressions for the integral curves were found.) The identity $\operatorname{cn}\left(\frac{1}{2}\Omega\, t + \frac{1}{2}K, k\right)^2 = \frac{k'(1 - \operatorname{sn}(\Omega\, t, k))}{k' + \operatorname{dn}(\Omega\, t, k)}$ proved to be useful in deriving the below expression for $\bar{p}_1(t)$.

3.3.27 THEOREM. (CASE *c(ii)*) *Let* $p(\cdot)$ *be an integral curve of the system* $H^2_{2,\alpha}$ *through* $p(0)$. *Let* $h_0 = H^2_{2,\alpha}(p(0))$ *and* $c_0 = C(p(0))$. *If* $\frac{c_0}{2} - \alpha\sqrt{c_0} < h_0 < \frac{c_0}{2} + \alpha\sqrt{c_0}$ *and* $c_0 \neq 2h_0$, *then there exists* $t_0 \in \mathbb{R}$ *such that* $p(t) = \bar{p}(t + t_0)$, *where*

$$\begin{aligned}
\bar{p}_1(t) &= \varsigma\,\varepsilon_1\,\frac{\operatorname{cn}\left(\frac{1}{2}\Omega\,t + \frac{1}{2}K, k\right)\sqrt{1 + k\operatorname{sn}(\Omega\,t, k)}\sqrt{k' + \operatorname{dn}(\Omega\,t, k)}}{\sqrt{\omega + \rho} - \varsigma\sqrt{\omega - \rho}\operatorname{sn}(\Omega\,t, k)}\\
\bar{p}_2(t) &= \varepsilon_2\,\frac{\frac{\rho - 2\alpha(\delta + \eta_0)}{\rho + 2\alpha(\delta + \eta_0)}\sqrt{\omega + \rho} + \varsigma\sqrt{\omega - \rho}\operatorname{sn}(\Omega\,t, k)}{\sqrt{\omega + \rho} - \varsigma\sqrt{\omega - \rho}\operatorname{sn}(\Omega\,t, k)}\\
\bar{p}_3(t) &= \varsigma\,\varepsilon_3\,\frac{\operatorname{cn}\left(\frac{1}{2}\Omega\,t - \frac{1}{2}K, k\right)\sqrt{1 - k\operatorname{sn}(\Omega\,t, k)}\sqrt{k' + \operatorname{dn}(\Omega\,t, k)}}{\sqrt{\omega + \rho} - \varsigma\sqrt{\omega - \rho}\operatorname{sn}(\Omega\,t, k)}.
\end{aligned}$$

*Here*

$$\begin{aligned}
\Omega &= \tfrac{1}{2}\sqrt{2\rho - \tau} & \eta_0 &= \sqrt{\alpha^2 + 2h_0}\\
k &= \sqrt{\frac{\tau + 2\rho}{\tau - 2\rho}} & \tau &= \delta^2 - 4\alpha^2 - 6\delta\eta_0 + \eta_0^2\\
k' &= \sqrt{\frac{4\rho}{2\rho - \tau}} & \rho &= 2\sqrt{\delta\eta_0\,(2\alpha + \delta - \eta_0)\,(2\alpha - \delta + \eta_0)}\\
\delta &= \sqrt{2\,(\alpha^2 + c_0) - \eta_0^2} & \omega &= 2\alpha(\delta + \eta_0) - (\delta - \eta_0)^2\\
\varsigma &= \operatorname{sgn}(\delta - \eta_0)
\end{aligned}$$

*and*

$$\begin{aligned}
\varepsilon_1 &= \tfrac{1}{(\delta - \eta_0)\sqrt{2k'}}\sqrt{(\omega + \rho)\left(\eta_0^2 + (2\alpha - \delta)\eta_0 - \tfrac{1}{2}\rho\right)\left(\eta_0^2 - (2\alpha + \delta)\eta_0 + \tfrac{1}{2}\rho\right)}\\
\varepsilon_2 &= \frac{\rho + 2\alpha(\delta + \eta_0)}{2(\delta - \eta_0)}\\
\varepsilon_3 &= \tfrac{1}{(\delta - \eta_0)\sqrt{2k'}}\sqrt{(\omega + \rho)\left(\delta^2 + (2\alpha - \eta_0)\,\delta - \tfrac{1}{2}\rho\right)\left(\delta^2 - (2\alpha + \eta_0)\,\delta + \tfrac{1}{2}\rho\right)}.
\end{aligned}$$

PROOF. Using MATHEMATICA one can verify that $p(\cdot)$ is an integral curve for $\varsigma \in \{-1, 1\}$.

Let $p(\cdot)$ be an integral curve through $p(0)$, let $h_0 = H^2_{2,\alpha}(p(0))$, $c_0 = C(p(0))$, and suppose that the conditions of case c(ii) hold. We claim that $p(t) = \bar{p}(t + t_0)$ for some $t_0 \in \mathbb{R}$. We have $\alpha\,p_2(0) + p_1(0)^2 + \frac{1}{2}p_2(0)^2 = h_0$ and $p_1(0)^2 + p_2(0)^2 + p_3(0)^2 = c_0$. Therefore $\alpha p_2(0) + \frac{1}{2}p_2(0)^2 \leq h_0$ and so $-\alpha - \sqrt{\alpha^2 + 2h_0} \leq p_2(0) \leq -\alpha + \sqrt{\alpha^2 + 2h_0}$. We also have $\frac{1}{2}p_2(0)^2 - \alpha p_2(0) + h_0 \leq c_0$, which implies that $\alpha - \sqrt{\alpha^2 + 2c_0 - 2h_0} \leq p_2(0) \leq \alpha + \sqrt{\alpha^2 + 2c_0 - 2h_0}$. Thus

$$\alpha - \sqrt{\alpha^2 + 2c_0 - 2h_0} \leq p_2(0) \leq -\alpha + \sqrt{\alpha^2 + 2h_0}.$$

Now $\bar{p}_2(-\frac{K}{\Omega}) = \alpha - \sqrt{\alpha^2 + 2c_0 - 2h_0}$ and $\bar{p}_2(\frac{K}{\Omega}) = -\alpha + \sqrt{\alpha^2 + 2h_0}$. Thus there exists $t_2 \in [-\frac{K}{\Omega}, \frac{K}{\Omega}]$ such that $p_2(0) = \bar{p}_2(t_2)$. As

$$p_1(0)^2 = h_0 - 2\alpha p_2(0) - \tfrac{1}{2}p_2(0)^2 = h_0 - 2\alpha\bar{p}_2(t_2) - \tfrac{1}{2}\bar{p}_2(t_2)^2 = \bar{p}_1(t_2)^2$$

it follows that $p_1(0) = \pm\bar{p}_1(t_2)$. Furthermore $\bar{p}_1(t + \frac{4K}{\Omega}) = -\bar{p}_1(t)$ and $\bar{p}_2(t + \frac{4K}{\Omega}) = \bar{p}_2(t)$. Thus there exists $t_1 \in \mathbb{R}$ ($t_1 = t_2$ or $t_1 = t_2 + \frac{4K}{\Omega}$) such that $p_1(0) = \bar{p}_1(t_1)$ and $p_2(0) = \bar{p}_2(t_1)$. On the other hand

$$p_3(0)^2 = c_0 - p_1(0)^2 - p_2(0)^2 = c_0 - \bar{p}_1(t_1)^2 - \bar{p}_2(t_1)^2 = \bar{p}_3(t_1)^2$$

and so $p_3(0) = \pm\bar{p}_3(t_1)$. Furthermore $\bar{p}_1(-t - \frac{2K}{\Omega}) = \bar{p}_1(t)$, $\bar{p}_2(-t - \frac{2K}{\Omega}) = \bar{p}_2(t)$, and $\bar{p}_3(-t - \frac{2K}{\Omega}) = -\bar{p}_3(t)$. Thus there exists $t_0 \in \mathbb{R}$ ($t_0 = t_1$ or $t_0 = -t_1 - \frac{2K}{\Omega}$) such that $p(0) = \bar{p}(t_0)$. Consequently, the integral curves $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ solve the same Cauchy problem, and therefore are identical. ☐

One might consider limiting $h_0$ to $\frac{c_0}{2} - \alpha\sqrt{c_0}$ in case (e) in order to produce integral curves for case (d). However, as in case (b), this limit degenerates. Thus a more direct approach is required.

3.3.28 THEOREM. (CASE *d*) *Let $p(\cdot)$ be an integral curve of the system $H^2_{2,\alpha}$ through $p(0)$. Let $h_0 = H^2_{2,\alpha}(p(0))$ and $c_0 = C(p(0))$. If $\frac{c_0}{2} - \alpha\sqrt{c_0} = h_0$, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$, where*

$$\begin{aligned}
\bar{p}_1(t) &= -\tfrac{2\sigma(c_0-\alpha^2)\sqrt{\alpha}}{\sqrt{\sqrt{c_0}+\alpha}}\, \frac{\sinh(\frac{1}{2}\Omega\, t)}{\sqrt{c_0} + \alpha\, \cosh(\Omega\, t)} \\
\bar{p}_2(t) &= -\sqrt{c_0} + \frac{2(c_0 - \alpha^2)}{\sqrt{c_0} + \alpha\, \cosh(\Omega\, t)} \\
\bar{p}_3(t) &= \tfrac{2\sigma(c_0-\alpha^2)\sqrt{\alpha}}{\sqrt{\sqrt{c_0}-\alpha}}\, \frac{\cosh(\frac{1}{2}\Omega\, t)}{\sqrt{c_0} + \alpha\, \cosh(\Omega\, t)}.
\end{aligned}$$

*Here $\Omega = 2\sqrt{c_0 - \alpha^2}$.*

PROOF. We start by explaining how the expression for $\bar{p}(\cdot)$ was found. Suppose $\bar{p}(\cdot)$ is an integral curve of $H^2_{2,\alpha}$ such that $h_0 > \alpha\sqrt{c_0} + \frac{c_0}{2}$, where $h_0 = H^2_{2,\alpha}(\bar{p}(0))$ and $c_0 = C(\bar{p}(0))$. As $\bar{p}(\cdot)$ satisfies $(\frac{d\bar{p}_2}{dt})^2 = 4\bar{p}_1^2\bar{p}_2^2$, $h_0 = H^2_{2,\alpha}(\bar{p}(t))$, and $c_0 = C(\bar{p}(t))$, we have

$$\frac{d\bar{p}_2}{dt} = \sqrt{(\sqrt{c_0} + \bar{p}_2)^2(-4\alpha^2 + (\sqrt{c_0} - \bar{p}_2)^2)}.$$

We transform this equation into standard form. Making the change of variables $s = -\frac{\bar{p}_2 - r_1}{\bar{p}_2 - r_2}$ yields

$$-\tfrac{2\alpha\sqrt{c_0-\alpha^2}}{\sqrt{c_0}}\, t = \int_{\frac{r_1-\bar{p}_2}{\bar{p}_2-r_2}}^{a} \frac{ds}{\sqrt{(a^2 - s^2)(s^2 - b^2)}}$$

where $a = \sqrt{\frac{c_0}{\alpha^2}}$ and $b = 0$. By applying the integral formula (D.2) we obtain

$$\bar{p}_2(t) = -\sqrt{c_0} + \frac{2(c_0 - \alpha^2)}{\sqrt{c_0} + \alpha\, \cosh(\Omega\, t)}.$$

Here $\Omega = 2\sqrt{c_0 - \alpha^2}$. As $h_0 = \alpha\bar{p}_2(t) + \bar{p}_1(t)^2 + \frac{1}{2}\bar{p}_2^2(t)$ we have

$$\bar{p}_1(t)^2 = \frac{2\alpha\left(\alpha - \sqrt{c_0}\right)^2\left(\alpha + \sqrt{c_0}\right)\left(\cosh(\Omega\, t) - 1\right)}{\left(\sqrt{c_0} + \alpha\cosh(\Omega\, t)\right)^2}.$$

Accordingly, we get

$$\bar{p}_1(t) = \frac{2\sigma_1(c_0-\alpha^2)\sqrt{\alpha}}{\sqrt{\sqrt{c_0}+\alpha}}\,\frac{\sinh(\frac{1}{2}\Omega\,t)}{\sqrt{c_0}+\alpha\,\cosh(\Omega\,t)}$$

for some $\sigma_1 \in \{-1,1\}$.
Likewise, as $\bar{p}_3(t)^2 = c_0 - \bar{p}_1(t)^2 - \bar{p}_2(t)^2$, we get

$$\bar{p}_3(t) = \frac{2\sigma_2(c_0-\alpha^2)\sqrt{\alpha}}{\sqrt{\sqrt{c_0}-\alpha}}\,\frac{\cosh(\frac{1}{2}\Omega\,t)}{\sqrt{c_0}+\alpha\,\cosh(\Omega\,t)}$$

for some $\sigma_2 \in \{-1,1\}$.

We show that $\bar{p}(\cdot)$ (as stated in the theorem) is an integral curve for certain values of $\sigma_1$ and $\sigma_2$. We have

$$\frac{d}{dt}\bar{p}_2(t) - 2\bar{p}_1(t)\bar{p}_3(t) = -\frac{4\alpha\,(1+\sigma_1\sigma_2)\left(c_0-\alpha^2\right)\sqrt[3]{c_0-\alpha^2}\sinh(\Omega\,t)}{\left(\sqrt{c_0}+\alpha\cosh(\Omega\,t)\right)^{\,2}}.$$

Therefore $\frac{d}{dt}\bar{p}_2(t) = 2\bar{p}_1(t)\bar{p}_3(t)$ whenever $-\sigma_1 = \sigma_2 = \sigma$. By construction, $H^2_{2,\alpha}(\bar{p}(t)) = h_0$ and $C(\bar{p}(t)) = c_0$. Consequently, by lemma 3.3.22, it follows that $\bar{p}(\cdot)$ (as stated in the theorem) is an integral curve.

Let $p(\cdot)$ be an integral curve through $p(0)$, let $h_0 = H^2_{2,\alpha}(p(0))$, $c_0 = C(p(0))$, and suppose that $h_0 = -\alpha\sqrt{c_0} + \frac{c_0}{2}$. We claim that $p(t) = \bar{p}(t+t_0)$ for some $\sigma \in \{-1,1\}$ and $t_0 \in \mathbb{R}$. We have $\alpha\, p_2(0) + p_1(0)^2 + \frac{1}{2}p_2(0)^2 = -\alpha\sqrt{c_0} + \frac{c_0}{2}$ and $p_1(0)^2 + p_2(0)^2 + p_3(0)^2 = c_0$. If $p_3(0) = 0$, then $p(\cdot)$ is constant. We assume $p_3(0) \neq 0$. Let $\operatorname{sgn}(p_3(0)) = \sigma$. As $\alpha\, p_2(0) + \frac{1}{2}p_2(0)^2 \leq -\alpha\sqrt{c_0} + \frac{1}{2}c_0$, it follows that

$$-\sqrt{c_0} \leq p_2(0) \leq -2\alpha + \sqrt{c_0}.$$

Now $\bar{p}_2(0) = -2\alpha + \sqrt{c_0}$ and $\lim_{t\to\infty}\bar{p}_2(t) = -\sqrt{c_0}$. Thus there exists $t_1 \in [0,\infty)$ such that $p_2(0) = \bar{p}_2(t_1)$. As

$$p_1(0)^2 = h_0 - 2\alpha p_2(0) - \tfrac{1}{2}p_2(0)^2 = h_0 - 2\alpha\bar{p}_2(t_1) - \tfrac{1}{2}\bar{p}_2(t_1)^2 = \bar{p}_1(t_1)^2$$

it follows that $p_1(0) = \pm\bar{p}_1(t_1)$. Furthermore $\bar{p}_1(-t) = -\bar{p}_1(t)$ and $\bar{p}_2(-t) = \bar{p}_2(t)$. Thus there exists $t_0 \in \mathbb{R}$ ($t_0 = t_1$ or $t_0 = -t_1$) such that $p_1(0) = \bar{p}_1(t_0)$. On the other hand

$$p_3(0)^2 = c_0 - p_1(0)^2 - p_2(0)^2 = c_0 - \bar{p}_1(t_0)^2 - \bar{p}_2(t_0)^2 = \bar{p}_3(t_0)^2$$

and so, as $\operatorname{sgn}(p_3(0)) = \sigma = \operatorname{sgn}(\bar{p}_3(t_1))$, we have $p_3(0) = \bar{p}_3(t_0)$. Thus there exists $t_0 \in \mathbb{R}$ such that $p(0) = \bar{p}(t_0)$. Consequently, the integral curves $t \mapsto p(t)$ and $t \mapsto \bar{p}(t+t_0)$ solve the same Cauchy problem, and therefore are identical.

□

The proof for case (e) is similar to that of case (a). A full proof is provided. However, this time we shall not explain how the expression for $\bar{p}(t)$ was found.

3.3.29 THEOREM. (CASE *e*) *Let $p(\cdot)$ be an integral curve of the system $H^2_{2,\alpha}$ through $p(0)$. Let $h_0 = H^2_{2,\alpha}(p(0))$ and $c_0 = C(p(0))$. If $\frac{c_0}{2} - \alpha\sqrt{c_0} > h_0$, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1,1\}$ such* *that* $p(t) = \bar{p}(t + t_0)$, *where*

$$\begin{aligned}
\bar{p}_1(t) &= \sigma\sqrt{\delta}\sqrt{\alpha^2 + 2h_0}\,\frac{\mathrm{cn}\,(\Omega\, t, k)}{\sqrt{\rho+\delta} - \sqrt{\rho-\delta}\,\mathrm{sn}\,(\Omega\, t, k)}\\
\bar{p}_2(t) &= \tfrac{1}{2\alpha}\frac{(\delta - c_0 + 2h_0)\sqrt{\rho+\delta} + (\delta + c_0 - 2h_0)\sqrt{\rho-\delta}\,\mathrm{sn}\,(\Omega\, t, k)}{\sqrt{\rho+\delta} - \sqrt{\rho-\delta}\,\mathrm{sn}\,(\Omega\, t, k)}\\
\bar{p}_3(t) &= \sigma\sqrt{\delta}\sqrt{\delta + c_0 - \alpha^2}\,\frac{\mathrm{dn}\,(\Omega\, t, k)}{\sqrt{\rho+\delta} - \sqrt{\rho-\delta}\,\mathrm{sn}\,(\Omega\, t, k)}.
\end{aligned}$$

*Here* $\delta = \sqrt{c_0^2 + 4h_0^2 - 4c_0\,(\alpha^2 + h_0)}$, $\Omega = \sqrt{\delta + c_0 - \alpha^2}$, $k = \sqrt{\frac{\alpha^2+\delta-c_0}{\alpha^2-\delta-c_0}}$, *and* $\rho = c_0 - 2\alpha^2 - 2h_0$.

3.3.30 REMARK. If we take the limit of the expression for $\bar{p}(t)$, as $\alpha$ tends to $0$, then we obtain integral curves for the third case of the system $H^2$.

PROOF. We show that $\bar{p}(\cdot)$ (as stated in the theorem) is an integral curve for $\sigma \in \{-1, 1\}$. After some simplification, we get

$$\frac{d}{dt}\bar{p}_2(t) - 2\bar{p}_1(t)\bar{p}_3(t) = \frac{2\delta(1-\sigma^2)\sqrt{(\delta + c_0 - \alpha^2)\,(\alpha^2 + 2h_0)}\,\mathrm{cn}\,(\Omega\, t, k)\,\mathrm{dn}\,(\Omega\, t, k)}{\left(\sqrt{\delta+\rho} - \sqrt{\rho-\delta}\,\mathrm{sn}\,(\Omega\, t, k)\right)^2}.$$

Therefore $\frac{d}{dt}\bar{p}_2(t) = 2\bar{p}_1(t)\bar{p}_3(t)$ whenever $\sigma \in \{-1, 1\}$. (It is not difficult to verify that $H^2_{2,\alpha}(\bar{p}(t)) = h_0$ and $C(\bar{p}(t)) = c_0$ for $t \in \mathbb{R}$.) Consequently, by lemma 3.3.22, it follows that $\bar{p}(\cdot)$ is an integral curve; it is not difficult to show that $0 < k < 1$ and that $\bar{p}(t)$ is defined for all $t \in \mathbb{R}$.

Let $p(\cdot)$ be an integral curve through $p(0)$, let $h_0 = H^2_{2,\alpha}(p(0))$, $c_0 = C(p(0))$, and suppose $h_0 < -\alpha\sqrt{c_0} + \frac{c_0}{2}$. We claim that $p(t) = \bar{p}(t + t_0)$ for some $\sigma \in \{-1, 1\}$ and $t_0 \in \mathbb{R}$. As $\alpha\, p_2(0) + p_1(0)^2 + \frac{1}{2}p_2(0)^2 = h_0$ we have

$$-\alpha - \sqrt{\alpha^2 + 2h_0} \le p_2(0) \le -\alpha + \sqrt{\alpha^2 + 2h_0}.$$

Now $\bar{p}_2(\frac{K}{\Omega}) = -\alpha - \sqrt{\alpha^2 + 2h_0}$ and $\bar{p}_2(\frac{3K}{\Omega}) = -\alpha + \sqrt{\alpha^2 + 2h_0}$. Thus there exists $t_1 \in [\frac{K}{\Omega}, \frac{3K}{\Omega}]$ such that $p_2(0) = \bar{p}_2(t_1)$. As $p_3(0)^2 = c_0 - p_1(0)^2 - p_2(0)^2 = c_0 - (h_0 - \alpha p_2(0) + \frac{1}{2}p_2(0)^2)$ and

$$\max_{\substack{-\alpha - \sqrt{\alpha^2 + 2h_0} \le p_2 \\ -\alpha + \sqrt{\alpha^2 + 2h_0} \ge p_2}} (h_0 - \alpha p_2 + \tfrac{1}{2}p_2^2) = 2\left(h_0 + \alpha\left(\alpha + \sqrt{\alpha^2 + 2h_0}\right)\right) > 0$$

it follows that $p_3(0) \neq 0$. Let $\sigma = \mathrm{sgn}\,(p_3(0))$. We have

$$p_3(0)^2 = c_0 - p_1(0)^2 - p_2(0)^2 = c_0 - \bar{p}_1(t_1)^2 - \bar{p}_2(t_1)^2 = \bar{p}_3(t_1)^2.$$

Hence, as $\mathrm{sgn}\,(\bar{p}_3(t_1)) = \sigma$, we get $p_3(0) = \bar{p}_3(t_1)$. On the other hand

$$p_1(0)^2 = h_0 - \alpha p_2(0) - \tfrac{1}{2}p_2(0)^2 = h_0 - \alpha\bar{p}_2(t_1) - \tfrac{1}{2}\bar{p}_2(t_1)^2 = \bar{p}_1(t_1)^2$$

and so $p_1(0) = \pm\bar{p}_1(t_1)$. Furthermore

$$\bar{p}_1(-t + \tfrac{2K}{\Omega}) = -\bar{p}_1(t), \quad \bar{p}_2(-t + \tfrac{2K}{\Omega}) = \bar{p}_2(t), \quad \bar{p}_3(-t + \tfrac{2K}{\Omega}) = \bar{p}_3(t).$$

Thus there exists $t_0 \in \mathbb{R}$ ($t_0 = t_1$ or $t_0 = -t_1 + \frac{2K}{\Omega}$) such that $p(0) = \bar{p}(t_0)$. Consequently, the integral curves $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ solve the same Cauchy problem and therefore are identical.

$\square$

### 3.3.4 Inhomogeneous systems of type II

Among the inhomogeneous systems on $\mathfrak{so}(3)^*_-$, there are four kinds of systems whose equilibria cannot be expressed as unions of lines and planes (type II). In fact, there is one one-parameter family of systems, two two-parameter families of systems, and one three-parameter family of systems (see theorem 3.3.1). The stability nature of all equilibria is determined for the system $H^1_{2,\alpha}$. On the other hand, for the remaining systems (i.e., those with homogeneous part $H^2$) we were unable to determine the stability nature of a few isolated equilibrium states. We found it unfeasible to compute expressions for the integral curves, due to computational complexity. Some indication of this complexity can be inferred from the graphs of the critical energy states.

#### System $H^1_{2,\alpha}$

The system $H^1_{2,\alpha}(p) = p_1 + \alpha p_2 + \frac{1}{2}p_1^2$, $\alpha > 0$ has equations of motion

$$\begin{cases} \dot{p}_1 = -\alpha p_3 \\ \dot{p}_2 = (1 + p_1)p_3 \\ \dot{p}_3 = \alpha p_1 - (1 + p_1)p_2. \end{cases}$$

The equilibria are $\mathsf{e}_1^\mu = (\mathrm{e}^\mu - 1,\, \alpha(1 - \mathrm{e}^{-\mu}), 0)$ and $\mathsf{e}_2^\mu = (-\mathrm{e}^\mu - 1,\, \alpha(1 + \mathrm{e}^\mu),\, 0)$.

In Figure 3.11 we graph the critical energy states $(h_0, c_0)$; in Figure 3.12 we graph the corresponding typical configurations. (The value $\alpha = \frac{1}{2}$ was used for both these figures.)

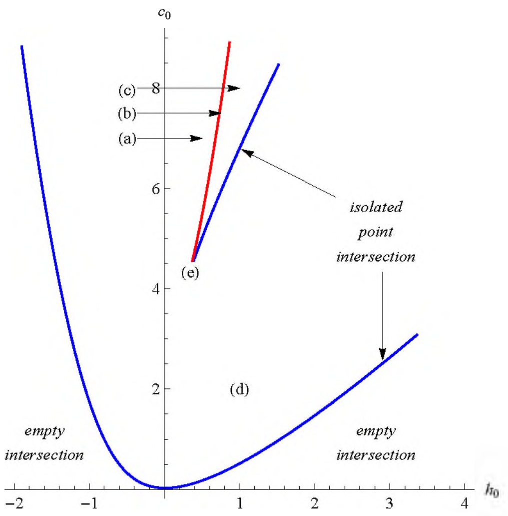
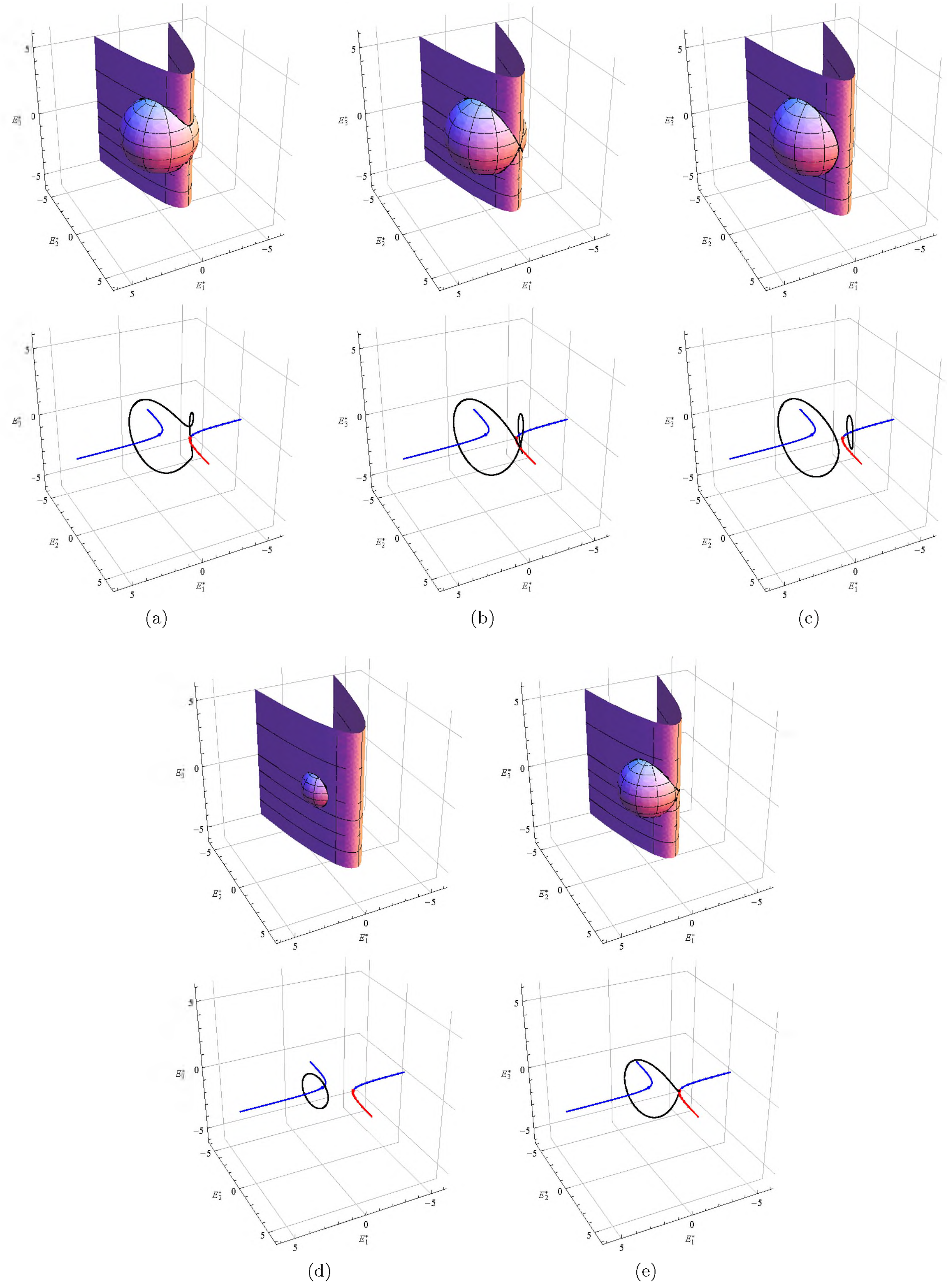


Figure 3.11: Critical energy states for $H^1_{2,\alpha}$



(a) (b) (c)

(d) (e)

Figure 3.12: Typical configurations for $H_{2,\alpha}^{1}$

3.3.31 Theorem. *The equilibrium states have the following behaviour:*

*(i) The states* $\mathsf{e}_1^\mu$ *are stable.*

*(ii) The states* $\mathsf{e}_2^\mu$, $\mu < \frac{2}{3}\ln\alpha$ *are (spectrally) unstable.*

*(iii) The state* $\mathsf{e}_2^\mu$, $\mu = \frac{2}{3}\ln\alpha$ *is unstable.*

*(iv) The states* $\mathsf{e}_2^\mu$, $\mu > \frac{2}{3}\ln\alpha$ *are stable.*

Proof. Let $H_\lambda(p) = \lambda_1 H^1_{2,\alpha}(p) + \lambda_2 C(p)$. (i) Let $\lambda_1 = 2\,(\mathrm{e}^{-\mu} - 1)$ and $\lambda_2 = 1$. We have $\mathbf{d}H_\lambda(\mathsf{e}_1^\mu) = 0$ and that $\mathbf{d}^2 H_\lambda(\mathsf{e}_1^\mu) = \operatorname{diag}(2\,\mathrm{e}^{-\mu}, 2, 2)$ is definite. Thus the states $\mathsf{e}_1^\mu$ are stable.

(ii) The linearization of the system at $\mathsf{e}_2^\mu$ has eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = \pm\sqrt{\mathrm{e}^{-\mu}\,(\alpha^2 - \mathrm{e}^{3\mu})}$. Hence the states $\mathsf{e}_2^\mu$, $\mu < \frac{2}{3}\ln\alpha$ are spectrally unstable.

(iii) Let $\mu = \frac{2}{3}\ln\alpha$; we consider the equilibrium state $\mathsf{e}_2^\mu = (-1 - \alpha^{\frac{2}{3}}, \alpha^{\frac{1}{3}} + \alpha, 0)$. We have that

$$p(t) = \left( \frac{4\alpha^{\frac{2}{3}}}{1 + \alpha^{\frac{4}{3}} t^2} - 1 - \alpha^{\frac{2}{3}},\ \alpha^{\frac{1}{3}} + \alpha - \frac{12\alpha^{\frac{1}{3}} + 4\alpha^{\frac{5}{3}} t^2}{(1 + \alpha^{\frac{4}{3}} t^2)^2},\ \frac{8\alpha t}{(1 + \alpha^{\frac{4}{3}} t^2)^2} \right)$$

is an integral curve of the system $H^1_{2,\alpha}$ such that $\lim_{t\to-\infty} p(t) = \mathsf{e}_2^\mu$. Let $\mathcal{B}_\varepsilon$ be the open ball of radius $\varepsilon = \alpha^{\frac{1}{3}}$ centred at $\mathsf{e}_2^\mu$. For any neighbourhood $V \subset \mathcal{B}_\varepsilon$ of $\mathsf{e}_2^\mu$ there exists $t_0 < 0$ such that $p(t_0) \in V$. Furthermore $\|p(0) - \mathsf{e}_2^\mu\| = 4\alpha^{\frac{1}{3}}\sqrt{1 + \alpha^{\frac{2}{3}}} > \varepsilon$, i.e., $p(0) \notin \mathcal{B}_\varepsilon$. Thus $\mathsf{e}_2^\mu$, $\mu = \frac{2}{3}\ln\alpha$ is unstable.

(iv) Assume $\mu > \frac{2}{3}\ln\alpha$ and let $\lambda_1 = -2\,(\mathrm{e}^{-\mu} + 1)$ and $\lambda_2 = 1$. We have $\mathbf{d}H_\lambda(\mathsf{e}_2^\mu) = 0$ and that $\mathbf{d}^2 H_\lambda(\mathsf{e}_2^\mu) = \operatorname{diag}(-2\,\mathrm{e}^{-\mu}, 2, 2)$ is definite when restricted to $W = \operatorname{span}\{(\alpha, \mathrm{e}^\mu, 0), (0, 0, 1)\}$. Hence the states $\mathsf{e}_2^\mu$, $\mu > \frac{2}{3}\ln\alpha$ are stable. □

### System $H^2_{3,\alpha}$

The system $H^2_{3,\alpha}(p) = \alpha_1 p_1 + \alpha_2 p_2 + p_1^2 + \frac{1}{2}p_2^2$, $\alpha_1, \alpha_2 > 0$ has equations of motion

$$\begin{cases} \dot{p}_1 = -(\alpha_2 + p_2)p_3 \\ \dot{p}_2 = (\alpha_1 + 2p_1)p_3 \\ \dot{p}_3 = \alpha_2 p_1 - (\alpha_1 + p_1)p_2. \end{cases}$$

The equilibria are $\mathsf{e}_1^\mu = (\mathrm{e}^\mu - \alpha_1,\ \alpha_2(1 - \alpha_1\,\mathrm{e}^{-\mu}),\ 0)$, $\mathsf{e}_2^\mu = (-\mathrm{e}^\mu - \alpha_1,\ \alpha_2(1 + \alpha_1\,\mathrm{e}^{-\mu}),\ 0)$, and $\mathsf{e}_3^\nu = \left(-\frac{\alpha_1}{2},\ -\alpha_2,\ \nu\right)$. In Figure 3.13 we graph the critical energy states $(h_0, c_0)$; in Figure 3.14 we graph the corresponding typical configurations. (The values $\alpha_1 = 1$ and $\alpha_2 = \frac{1}{2}$ were used in both the figures.)

3.3.32 Theorem. *The equilibrium states have the following behaviour:*

*(i) The states* $\mathsf{e}_1^\mu$, $\mu < \ln\frac{\alpha_1}{2}$ *are (spectrally) unstable.*

*(ii) The states* $\mathsf{e}_1^\mu$, $\ln\frac{\alpha_1}{2} \leq \mu$ *are stable.*

*(iii) The states* $\mathsf{e}_2^\mu$, $\mu < \frac{1}{3}\ln\alpha_1\alpha_2^2$ *are (spectrally) unstable.*

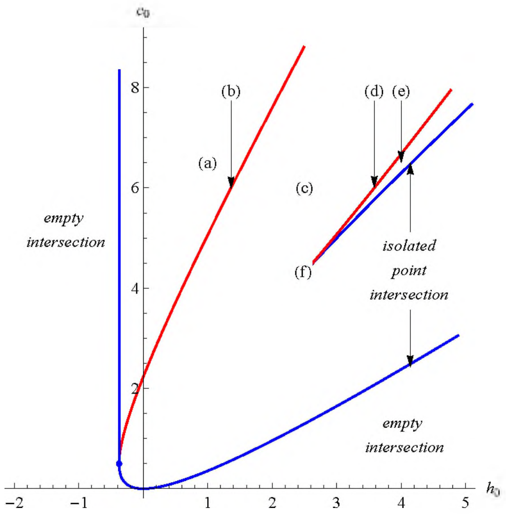


Figure 3.13: Critical energy states for $H^2_{3,\alpha}$

*(iv) The states* $\mathsf{e}_2^\mu$, $\mu > \frac{1}{3}\ln\alpha_1\alpha_2^2$ *are stable.*

*(v) The states* $\mathsf{e}_3^\nu$ *are stable.*

3.3.33 REMARK. The equilibrium state $\mathsf{e}_2^\mu$, $\mu = \frac{1}{3}\ln\alpha_1\alpha_2^2$ is spectrally stable. However, we were unable to determine its Lyapunov stability nature. We suspect that this state is unstable (see Figure 3.14f).

PROOF. Let $H_\lambda(p) = \lambda_1 H^2_{3,\alpha}(p) + \lambda_2 C(p)$. (i) The linearization of the system at $\mathsf{e}_1^\mu$ has eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = \pm\,\mathrm{e}^{-\mu}\sqrt{\alpha_1 - 2\,\mathrm{e}^{\mu}}\sqrt{\mathrm{e}^{3\mu} + \alpha_1\alpha_2^2}$. Hence the states $\mathsf{e}_1^\mu$, $\mu < \ln\frac{\alpha_1}{2}$ are spectrally unstable.

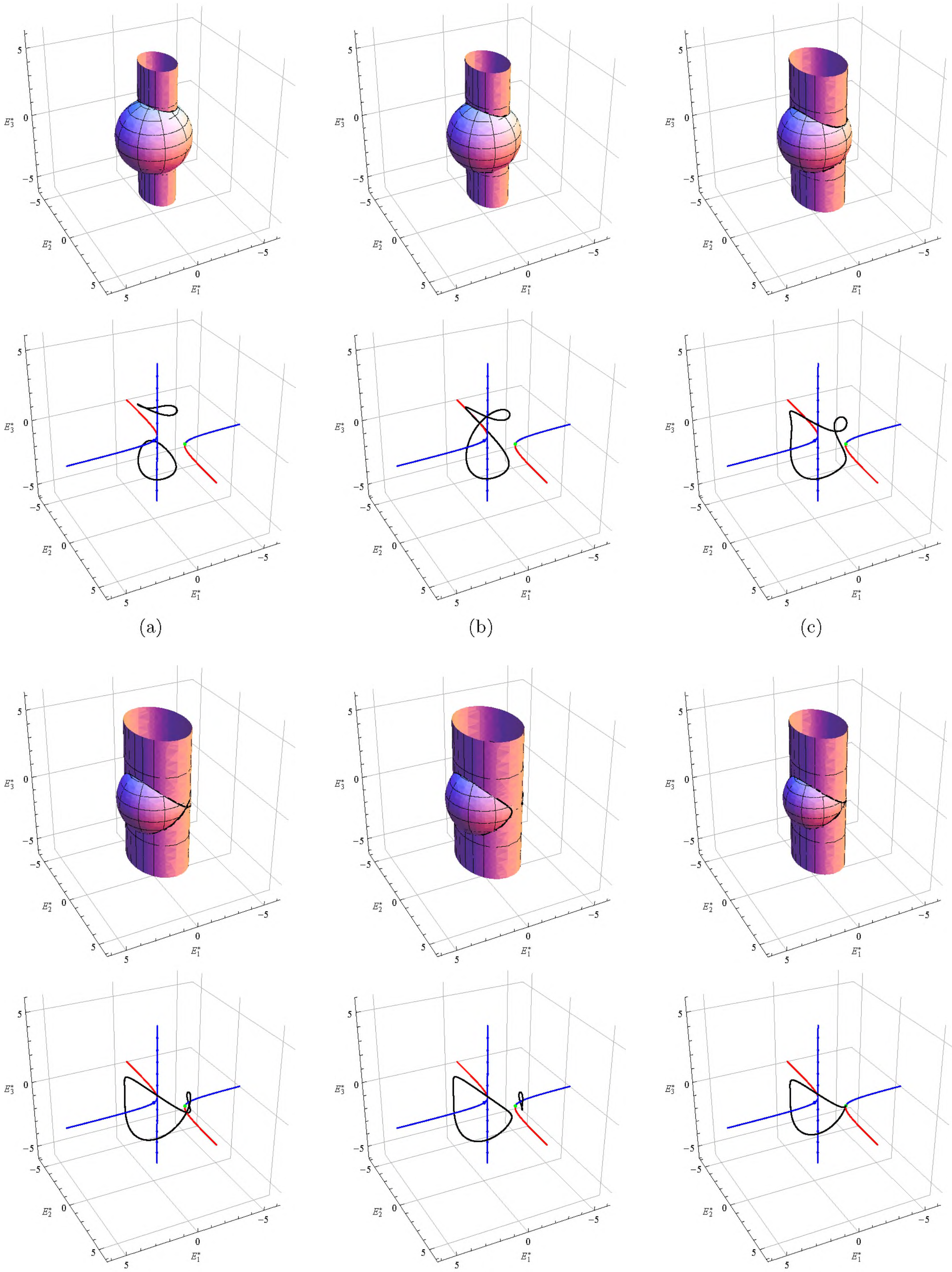
(ii) We consider that states $\mathsf{e}_1^\mu$, $\ln\frac{\alpha_1}{2} < \mu$, $\mu \neq \ln\alpha_1$. Let $\lambda_1 = 1$ and $\lambda_2 = -1 - \frac{\alpha_1}{2\,\mathrm{e}^{\mu} - 2\alpha_1}$. We have $\mathbf{d}H_\lambda(\mathsf{e}_1^\mu) = 0$ and that $\mathbf{d}^2 H_\lambda(\mathsf{e}_1^\mu) = \mathrm{diag}\,(\frac{\alpha_1}{\alpha_1 - \mathrm{e}^{\mu}}, \frac{1}{\alpha_1\,\mathrm{e}^{-\mu} - 1}, \frac{\alpha_1}{\alpha_1 - \mathrm{e}^{\mu}} - 2)$ is definite when restricted to the subspace $W = \mathrm{span}\,\{(-\alpha_2, \mathrm{e}^{\mu}, 0), (0,0,1)\}$. Hence the states $\mathsf{e}_1^\mu$, $\ln\frac{\alpha_1}{2} < \mu$, $\mu \neq \ln\alpha_1$ are stable. The state $\mathsf{e}_1^{\ln\alpha_1}$ is the origin and is therefore stable. Next, we consider the state $\mathsf{e}_1^{\ln\frac{\alpha_1}{2}}$. We have $H^2_{3,\alpha}(\mathsf{e}_1^{\ln\frac{\alpha_1}{2}}) = -\frac{\alpha_1^2}{4} - \frac{\alpha_2^2}{2}$ and $C(\mathsf{e}_1^{\ln\frac{\alpha_1}{2}}) = \frac{\alpha_1^2}{4} + \alpha_2^2$. It is straightforward to show that $(H^2_{3,\alpha})^{-1}(-\frac{\alpha_1^2}{4} - \frac{\alpha_2^2}{2}) \cap C^{-1}(\frac{\alpha_1^2}{4} + \alpha_2^2) = \{\mathsf{e}_1^{\ln\frac{\alpha_1}{2}}\}$. Hence the state $\mathsf{e}_1^{\ln\frac{\alpha_1}{2}}$ is stable.

(iii) The linearization of the system at $\mathsf{e}_2^\mu$ has eigenvalues $\lambda_{1,2} = \pm\frac{\sqrt{2\,\mathrm{e}^{\mu} + \alpha_1}\sqrt{\alpha_1\alpha_2^2 - \mathrm{e}^{3\mu}}}{\mathrm{e}^{\mu}}$ and $\lambda_3 = 0$. Therefore the states $\mathsf{e}_2^\mu$, $\mu < \frac{1}{3}\ln\alpha_1\alpha_2^2$ are spectrally unstable.

(iv) Suppose $\mu > \frac{1}{3}\ln\alpha_1\alpha_2^2$ and let $\lambda_1 = -1 - \frac{\alpha_1}{2\,\mathrm{e}^{\mu} + \alpha_1}$ and $\lambda_2 = 1$. We have $\mathbf{d}H_\lambda(\mathsf{e}_2^\mu) = 0$ and that $\mathbf{d}^2 H_\lambda(\mathsf{e}_2^\mu) = \mathrm{diag}\,(-\frac{2\alpha_1}{\alpha_1 + 2\,\mathrm{e}^{\mu}}, \frac{2\,\mathrm{e}^{\mu}}{\alpha_1 + 2\,\mathrm{e}^{\mu}}, 2)$ is definite when restricted to the subspace $W = \mathrm{span}\,\{(\alpha_2, \mathrm{e}^{\mu}, 0), (0,0,1)\}$. Hence the states $\mathsf{e}_2^\mu$, $\mu > \frac{1}{3}\ln\alpha_1\alpha_2^2$ are stable.

(v) Let $\lambda_1 = 1$ and $\lambda_2 = 0$. We have $\mathbf{d}H_\lambda(\mathsf{e}_3^\nu) = 0$ and that $\mathbf{d}^2 H_\lambda(\mathsf{e}_3^\nu) = \mathrm{diag}\,(2,1,0)$ is definite when restricted to $W = \mathrm{span}\,\{(2\nu, 0, \alpha_1), (0, \nu, \alpha_2)\}$. Hence the states $\mathsf{e}_3^\nu$ are stable.

$\square$



(a) (b) (c)

(d) (e) (f)

Figure 3.14: Typical configurations for $H^2_{3,\alpha}$

### System $H^2_{4,\alpha}$

The system $H^2_{4,\alpha}(p) = \alpha_1 p_1 + \alpha_3 p_3 + p_1^2 + \frac{1}{2}p_2^2$, $\alpha_1 \geq \alpha_3 > 0$ has equations of motion

$$\begin{cases} \dot{p}_1 = (\alpha_3 - p_3)p_2 \\ \dot{p}_2 = -\alpha_3 p_1 + (\alpha_1 + 2p_1)p_3 \\ \dot{p}_3 = -(\alpha_1 + p_1)p_2. \end{cases}$$

The equilibria are $\mathsf{e}_1^\mu = \left(\frac{1}{2}(\mathrm{e}^\mu - \alpha_1),\ 0,\ \frac{\alpha_3}{2}(1 - \alpha_1 \mathrm{e}^{-\mu})\right)$, $\mathsf{e}_2^\mu = \left(-\frac{1}{2}(\mathrm{e}^\mu + \alpha_1),\ 0,\ \frac{\alpha_3}{2}(1 + \alpha_1 \mathrm{e}^{-\mu})\right)$, and $\mathsf{e}_3^\nu = (-\alpha_1,\ \nu,\ \alpha_3)$. When $\alpha_1 = \alpha_3$ the set of unstable equilibria degenerates (see Figure 3.15); we treat this case separately. In Figures 3.17 and 3.18 we graph the critical energy states $(h_0,\ c_0)$ and the corresponding typical configurations. (We used the values $\alpha_1 = 1$, $\alpha_3 = \frac{1}{5}$ for Figures 3.15i, 3.16, 3.17, and 3.18 and the values $\alpha_1 = \alpha_3 = 1$ for Figure 3.15ii.)

3.3.34 Theorem. *If $\alpha_1 > \alpha_3 > 0$, then the equilibrium states have the following behaviour:*

*(i) The states $\mathsf{e}_1^\mu$ are stable.*

*(ii) The states $\mathsf{e}_2^\mu$, $\frac{1}{3}\ln \alpha_1\alpha_3^2 < \mu < \ln \alpha_1$ are (spectrally) unstable.*

*(iii) The state $\mathsf{e}_2^\mu$, $\mu = \ln \alpha_1$ is unstable.*

*(iv) The states $\mathsf{e}_2^\mu$, $\mu \in \left(-\infty, \frac{1}{3}\ln \alpha_1\alpha_3^2\right) \cup (\ln \alpha_1, \infty)$ are stable.*

*(v) The states $\mathsf{e}_3^\nu$ are (spectrally) unstable.*

*If $\alpha_1 = \alpha_3 > 0$, then the equilibrium states have the following behaviour:*

*(vi) The states $\mathsf{e}_1^\mu$ are stable.*

*(vii) The state $\mathsf{e}_2^\mu$, $\mu = \ln \alpha_1$ is unstable.*

*(viii) The states $\mathsf{e}_2^\mu$, $\mu \neq \ln \alpha_1$ are stable.*

*(ix) The states $\mathsf{e}_3^\nu$ are (spectrally) unstable.*

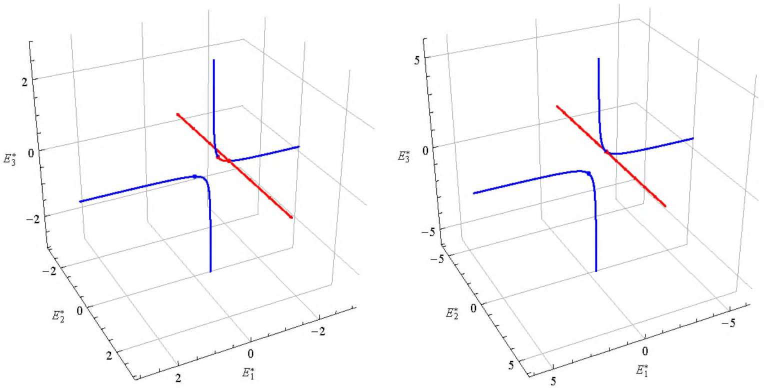
3.3.35 Remark. The equilibrium state $\mathsf{e}_2^\mu$, $\mu = \frac{1}{3}\ln \alpha_1\alpha_3^2$, $\alpha_1 \neq \alpha_3$ is spectrally stable. However, we were unable to determine its Lyapunov stability nature. We suspect it is unstable (see Figure 3.18e).

Proof. Suppose $\alpha_1 > \alpha_3 > 0$. Let $H_\lambda(p) = \lambda_1 H^2_{4,\alpha}(p) + \lambda_2 C(p)$. (i) Let $\lambda_1 = \alpha_1 \mathrm{e}^{-\mu} - 1$ and $\lambda_2 = 1$. We have $\mathbf{d}H_\lambda(\mathsf{e}_1^\mu) = 0$ and that $\mathbf{d}^2 H_\lambda(\mathsf{e}_1^\mu) = \operatorname{diag}(2\alpha_1 \mathrm{e}^{-\mu}, 1 + \alpha_1 \mathrm{e}^{-\mu}, 2)$ is definite. Thus the states $\mathsf{e}_1^\mu$ are stable.

(ii) The linearization of the system at $\mathsf{e}_2^\mu$ has eigenvalues $\lambda_{1,2} = \pm\frac{\mathrm{e}^{-\mu}}{\sqrt{2}}\sqrt{\alpha_1 - \mathrm{e}^\mu}\sqrt{\mathrm{e}^{3\mu} - \alpha_1\alpha_3^2}$, $\lambda_3 = 0$. Hence the states $\mathsf{e}_2^\mu$, $\frac{1}{3}\ln \alpha_1\alpha_3^2 < \mu < \ln \alpha_1$ are spectrally unstable.

(iii) Consider the equilibrium state $\mathsf{e}_2^{\ln \alpha_1} = (-\alpha_1, 0, \alpha_3)$. We have that

$$p(t) = \left(\frac{4(\alpha_1 + \alpha_3)}{4 + 2(\alpha_1 + \alpha_3)^2 t^2} - \alpha_1,\ \frac{-2(\alpha_1 + \alpha_3)^2 t}{2 + (\alpha_1 + \alpha_3)^2 t^2},\ \alpha_3 - \frac{4(\alpha_1 + \alpha_3)}{4 + 2(\alpha_1 + \alpha_3)^2 t^2}\right)$$



(i) $\alpha_1 > \alpha_3$ (ii) $\alpha_1 = \alpha_3$

Figure 3.15: Equilibria of $H^{2}_{4,\boldsymbol{\alpha}}$

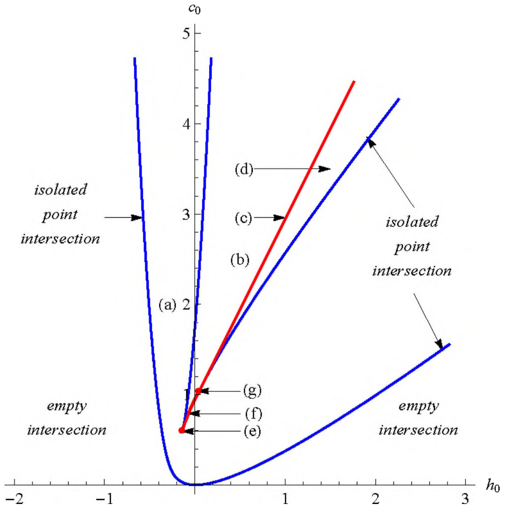
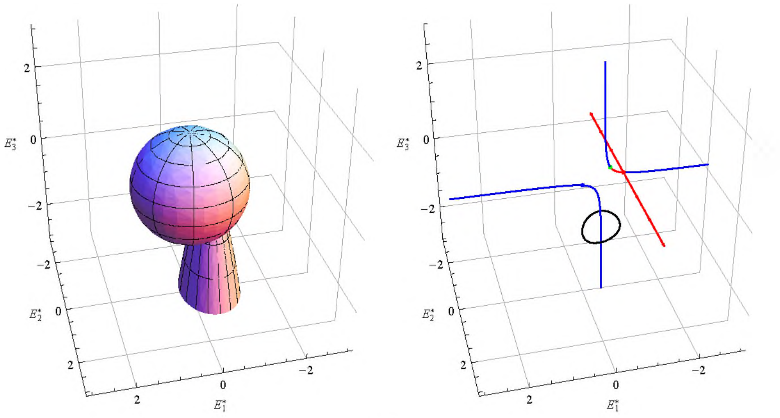
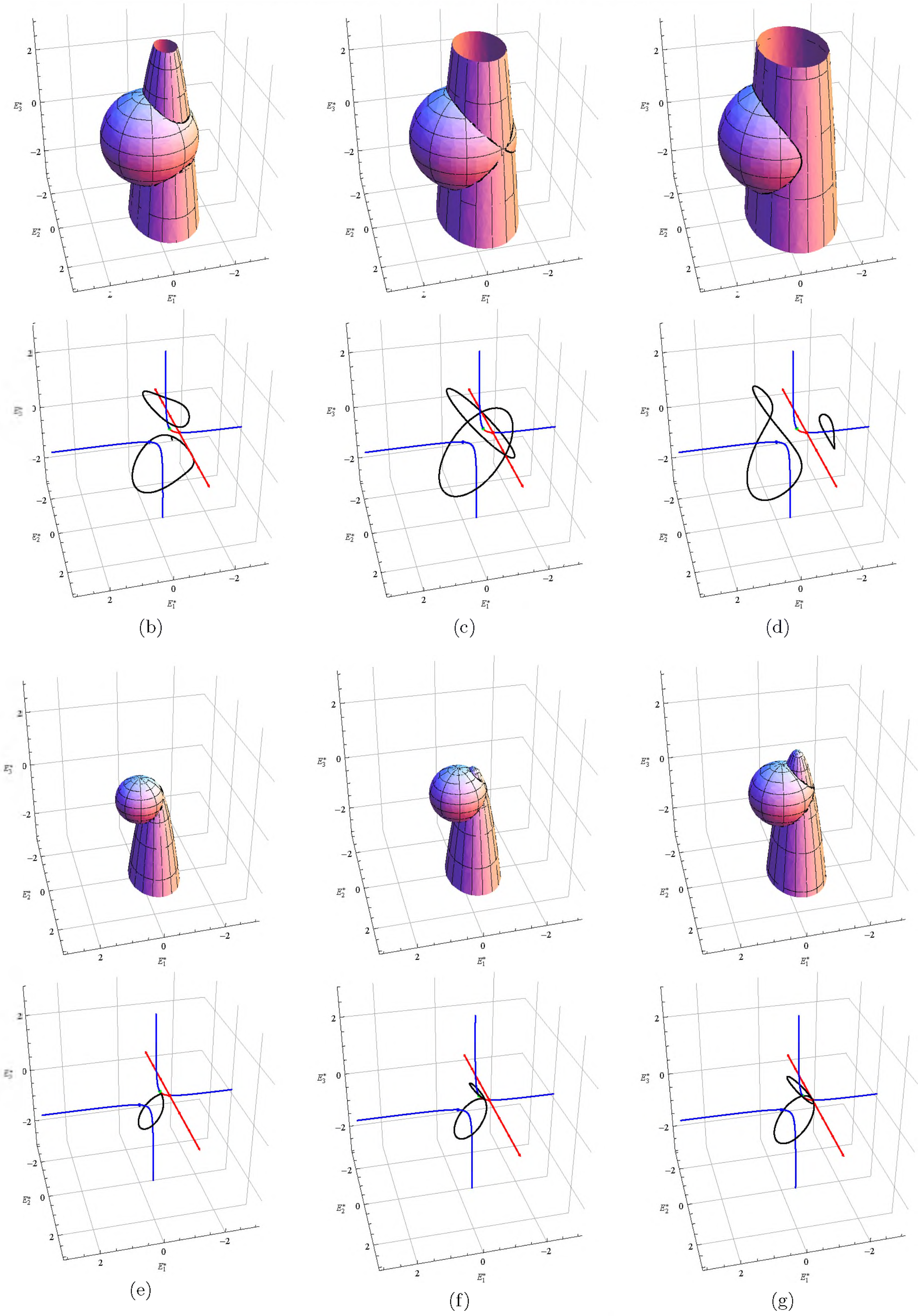


Figure 3.16: Critical energy states for $H^{2}_{4,\boldsymbol{\alpha}}$



(a)

Figure 3.17: Typical configurations for $H^{2}_{4,\boldsymbol{\alpha}}$



(b) (c) (d)

(e) (f) (g)

Figure 3.18: Typical configurations for $H^2_{1,\alpha}$, cont.

is an integral curve of the system $H^2_{4,\boldsymbol{\alpha}}$ such that $\lim_{t\to-\infty} p(t) = \mathsf{e}_2^{\ln\alpha_1}$. Let $\mathcal{B}_\varepsilon$ be the open ball of radius $\varepsilon = \alpha_1 + \alpha_3$ centred at the point $\mathsf{e}_2^{\ln\alpha_1}$. For any neighbourhood $V \subset \mathcal{B}_\varepsilon$ of $\mathsf{e}_2^{\ln\alpha_1}$ there exists $t_0 < 0$ such that $p(t_0) \in V$. Furthermore $\|p(0) - \mathsf{e}_2^{\ln\alpha_1}\| = \sqrt{2}(\alpha_1 + \alpha_3) > \varepsilon$, i.e., $p(0) \notin \mathcal{B}_\varepsilon$. Hence the state $\mathsf{e}^{\ln\alpha_1}$ is unstable.

(iv) Suppose $\mu \in \left(-\infty, \frac{1}{3}\ln\alpha_1\alpha_3^2\right) \cup (\ln\alpha_1, \infty)$ and let $\lambda_1 = -1 - \alpha_1\,\mathrm{e}^{-\mu}$ and $\lambda_2 = 1$. We have $\mathbf{d}H_\lambda(\mathsf{e}_2^\mu) = 0$ and that $\mathbf{d}^2H_\lambda(\mathsf{e}_2^\mu) = \operatorname{diag}(-2\alpha_1\,\mathrm{e}^{-\mu}, 1 - \alpha_1\,\mathrm{e}^{-\mu}, 2)$ is definite when restricted to the subspace $W = \operatorname{span}\{(\alpha_3, 0, \mathrm{e}^\mu), (0, 1, 0)\}$. Hence the states $\mathsf{e}_2^\mu$, $\mu \in \left(-\infty, \frac{1}{3}\ln\alpha_1\alpha_3^2\right) \cup (\ln\alpha_1, \infty)$ are stable.

(v) The linearization of the system at $\mathsf{e}_3^\nu$ has eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = \pm\nu$. Thus the states $\mathsf{e}_3^\nu$ are spectrally unstable.

When $\alpha_1 = \alpha_3$ the same arguments apply; item (ii), however, falls away.

□

### System $H^2_{5,\boldsymbol{\alpha}}$

The system

$$H^2_{5,\boldsymbol{\alpha}}(p) = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + p_1^2 + \tfrac{1}{2}p_2^2, \qquad \alpha_2 > 0,\ \alpha_1 > |\alpha_3| > 0 \quad \text{or} \quad \alpha_2 > 0,\ \alpha_1 = \alpha_3 > 0$$

has equations of motion

$$\begin{cases} \dot{p}_1 = \alpha_3\, p_2 - (\alpha_2 + p_2)\, p_3 \\ \dot{p}_2 = -\alpha_3\, p_1 + (\alpha_1 + 2p_1)\, p_3 \\ \dot{p}_3 = \alpha_2\, p_1 - (\alpha_1 + p_1)\, p_2. \end{cases}$$

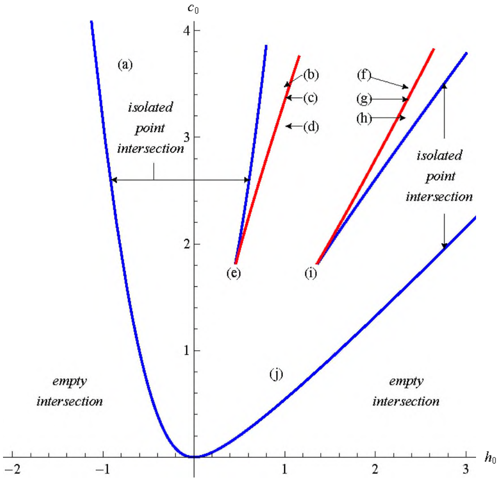


Figure 3.19: Critical energy states for $H^2_{5,\boldsymbol{\alpha}}$

The equilibria are $\left(x, \frac{\alpha_2 x}{\alpha_1 + x}, \frac{\alpha_3 x}{\alpha_1 + 2x}\right)$, $x \neq -\alpha_1$, $x \neq -\frac{1}{2}\alpha_1$. These points are the union of three curves which have respective parametrizations

$$\begin{aligned}
\mathsf{e}_1^{\mu} &= \left(-\mathrm{e}^{\mu}-\alpha_1,\ \alpha_2+\alpha_1\alpha_2\,\mathrm{e}^{-\mu},\ \frac{\alpha_3(\alpha_1+\mathrm{e}^{\mu})}{\alpha_1+2\,\mathrm{e}^{\mu}}\right)\\
\mathsf{e}_2^{\mu} &= \left(\tfrac{1}{4}\alpha_1\tanh(\mu)-\tfrac{3}{4}\alpha_1,\ \alpha_2-\frac{4\alpha_2}{1+\tanh(\mu)},\ \tfrac{1}{2}\alpha_3(2+\mathrm{e}^{2\mu})\right)\\
\mathsf{e}_3^{\mu} &= \left(\mathrm{e}^{\mu}-\tfrac{1}{2}\alpha_1,\ \alpha_2-\frac{2\alpha_1\alpha_2}{\alpha_1+2\,\mathrm{e}^{\mu}},\ \tfrac{1}{4}\alpha_3(2-\alpha_1\,\mathrm{e}^{-\mu})\right).
\end{aligned}$$

The first case corresponds to $x<-\alpha_1$, the second to $-\alpha_1<x<-\frac{1}{2}\alpha_1$, and the third to $x>-\frac{1}{2}\alpha_1$.

The paraboloid $(H^2_{5,\boldsymbol{\alpha}})^{-1}(h_0)$ and sphere $C^{-1}(c_0)$ are tangent at a point $p\in\mathfrak{so}(3)^*$ if and only if $h_0=H^2_{5,\boldsymbol{\alpha}}(p)$, $c_0=C(p)$, and $\begin{bmatrix}\alpha_1+2p_1 & \alpha_2+p_2 & \alpha_3\end{bmatrix}=\kappa\begin{bmatrix}2p_1 & 2p_2 & 2p_3\end{bmatrix}$ for some $\kappa\in\mathbb{R}$. Assuming $p\neq 0$, this yields $p=\left(\frac{\alpha_1}{2(\kappa-1)},\ \frac{\alpha_2}{2(\kappa-\frac{1}{2})},\ \frac{\alpha_3}{2\kappa}\right)$, $\kappa\neq 0,\frac{1}{2},1$. In other words, apart from at the origin, the level surfaces of $H^2_{5,\boldsymbol{\alpha}}$ and $C$ are tangent at the points

$$\mathsf{e}^{\kappa}=\left(\frac{\alpha_1}{2(\kappa-1)},\frac{\alpha_2}{2(\kappa-\frac{1}{2})},\frac{\alpha_3}{2\kappa}\right),\qquad \kappa\neq 0,\tfrac{1}{2},1.$$

We shall find it more convenient to use this parametrization of the equilibria (covering all equilibrium points except the origin) in determining the stability nature of the equilibria. We note that

$$\begin{aligned}
&\mathsf{e}^{\kappa}=\mathsf{e}_1^{\mu} &&\text{for}\quad \tfrac{1}{2}<\kappa=\frac{2\,\mathrm{e}^{\mu}+\alpha_1}{2\,(\mathrm{e}^{\mu}+\alpha_1)}<1,\ \mu\in\mathbb{R}\\
&\mathsf{e}^{\kappa}=\mathsf{e}_2^{\mu} &&\text{for}\quad 0<\kappa=\frac{1}{2+\mathrm{e}^{2\mu}}<\tfrac{1}{2},\ \mu\in\mathbb{R}\\
&\mathsf{e}^{\kappa}=\mathsf{e}_3^{\mu} &&\text{for}\quad \kappa=\frac{2\,\mathrm{e}^{\mu}}{2\,\mathrm{e}^{\mu}-\alpha_1}<0,\ \mu<\ln\frac{\alpha_1}{2}\quad\text{or}\quad \kappa=\frac{2\,\mathrm{e}^{\mu}}{2\,\mathrm{e}^{\mu}-\alpha_1}>1,\ \mu>\ln\frac{\alpha_1}{2}.
\end{aligned}$$

In Figure 3.19 we graph the critical energy states $(h_0,c_0)$; in Figures 3.20 and 3.21 we graph the corresponding typical configurations. (We used the values $\alpha_1=\frac{1}{2}$, $\alpha_2=\frac{2}{5}$, $\alpha_3=\frac{1}{2}$ for these figures.)

The polynomial following will be central to our discussion of the stability nature of the equilibria:

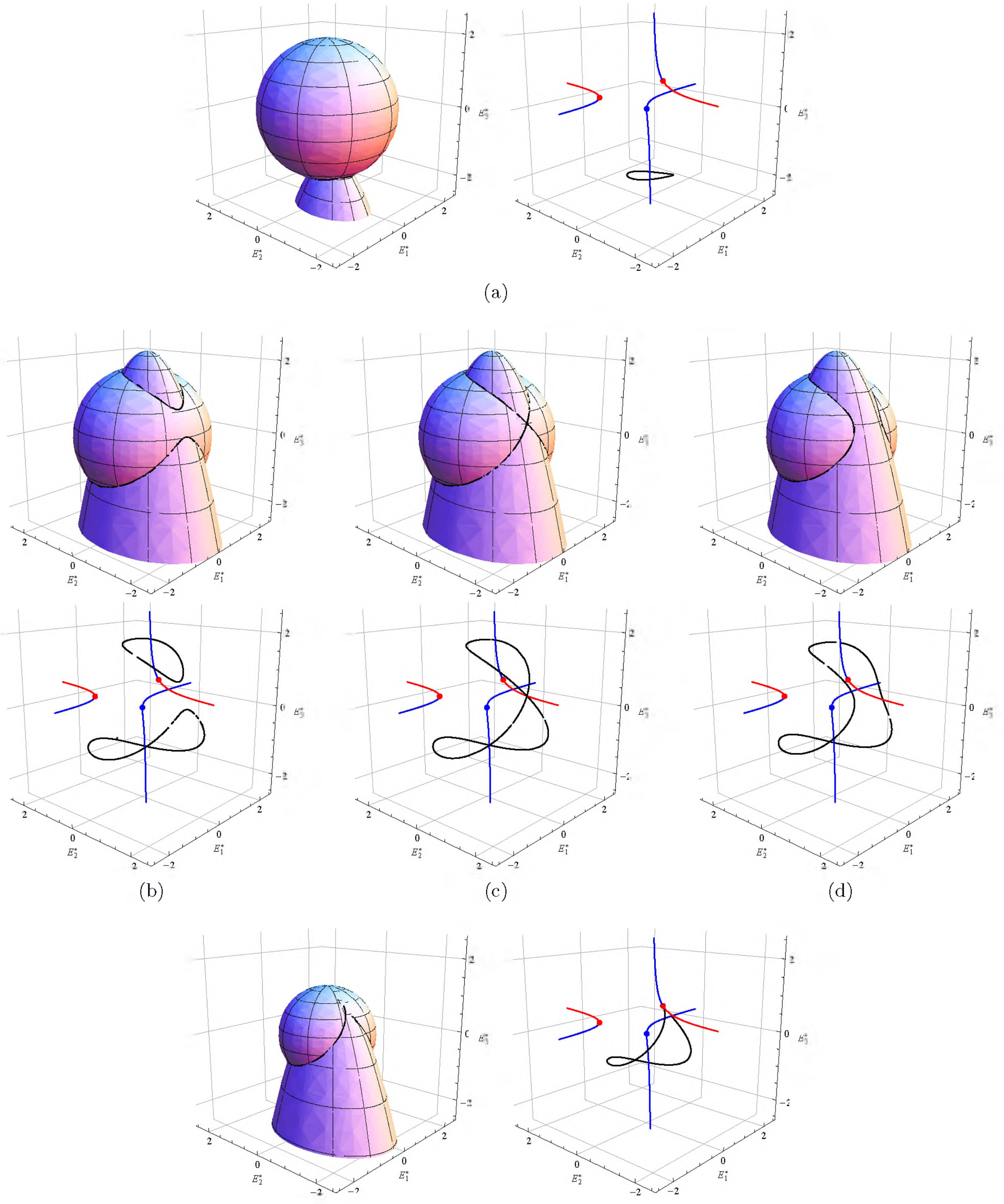
$$P_{\boldsymbol{\alpha}}(\kappa)=-\alpha_3^2\left(1-3\kappa+2\kappa^2\right)^3+\kappa^3\left(-8\alpha_2^2(\kappa-1)^3-\alpha_1^2(2\kappa-1)^3\right).$$

3.3.36 LEMMA. *The polynomial* $P_{\boldsymbol{\alpha}}(\kappa)$ *has exactly two real roots* $\kappa_1\in(0,\frac{1}{2})$ *and* $\kappa_2\in(\frac{1}{2},1)$.

PROOF. We have that

$$\begin{aligned}
P_{\boldsymbol{\alpha}}(-\kappa)=&-\alpha_3^2-9\alpha_3^2\kappa-33\alpha_3^2\kappa^2-\left(\alpha_1^2+8\alpha_2^2+63\alpha_3^2\right)\kappa^3-\left(6\alpha_1^2+24\alpha_2^2+66\alpha_3^2\right)\kappa^4\\
&-\left(12\alpha_1^2+24\alpha_2^2+36\alpha_3^2\right)\kappa^5-\left(8\alpha_1^2+8\alpha_2^2+8\alpha_3^2\right)\kappa^6.
\end{aligned}$$

Thus $P_{\boldsymbol{\alpha}}(-\kappa)<0$ for $\kappa\geq 0$ and so $P_{\boldsymbol{\alpha}}$ has no nonpositive real roots. Furthermore, $P_{\boldsymbol{\alpha}}(0)=-\alpha_3^2<0$, $P_{\boldsymbol{\alpha}}(\frac{1}{2})=\frac{\alpha_2^2}{8}>0$, and $P_{\boldsymbol{\alpha}}(1)=-\alpha_1^2<0$. Therefore $P_{\boldsymbol{\alpha}}$ has at least one root in $(0,\frac{1}{2})$



(a)

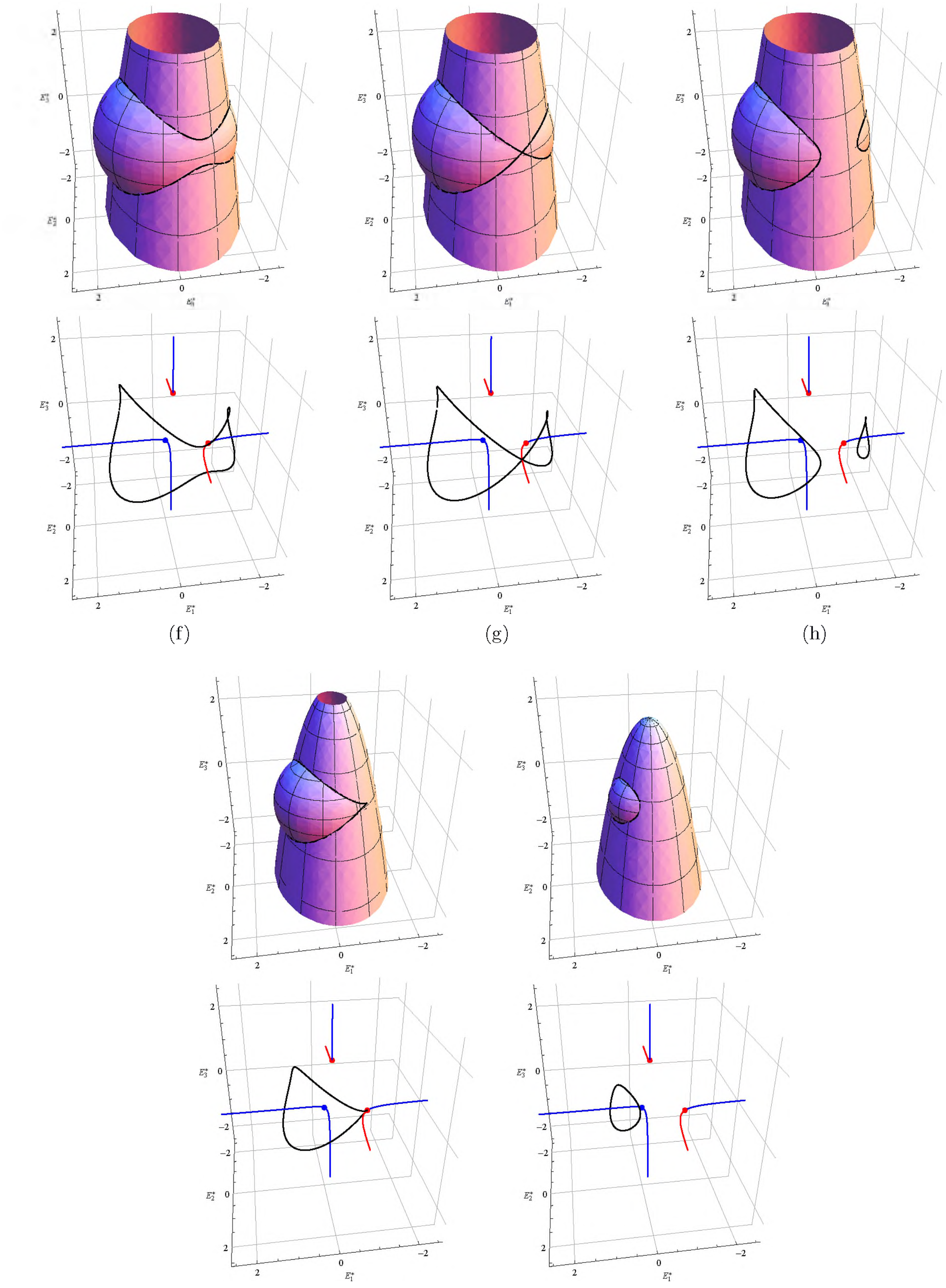
(b)

(c)

(d)

(e)

Figure 3.20: Typical configurations for $H^2_{5,\alpha}$



(f) (g) (h)

(i) (j)

Figure 3.21: Typical configurations for $H^2_{5,\boldsymbol{\alpha}}$, cont.

and at least one root in $(\frac{1}{2}, 1)$. As $\left(1-3\kappa+2\kappa^2\right)^3 > 0$, $(\kappa-1)^3 > 0$, and $(2\kappa-1)^3 > 0$ for $\kappa > 1$, it follows that $P_{\boldsymbol{\alpha}}(\kappa) < 0$ for $\kappa > 1$. Thus $P_{\boldsymbol{\alpha}}$ has no real roots in $(1, \infty)$.

It remains to be shown that $P_{\boldsymbol{\alpha}}$ has at most one real root in $(0, \frac{1}{2})$ and at most one real root in $(\frac{1}{2}, 1)$. Suppose $\kappa \in (\frac{1}{2}, 1)$. Then we have that $1 - 6\kappa + 8\kappa^2 \geq 0$ and so

$$\begin{aligned} \frac{\frac{d}{d\kappa}P_{\boldsymbol{\alpha}}(\kappa)}{\kappa-\frac{1}{2}} &= -48\alpha_2^2(\kappa-1)^2\kappa^2 - 6\alpha_1^2\kappa^2\left(1-6\kappa+8\kappa^2\right) - 6\alpha_3^2(\kappa-1)^2(3+2\kappa(4\kappa-5)) \\ &< -6\alpha_3^2\kappa^2\left(1-6\kappa+8\kappa^2\right) - 6\alpha_3^2(-1+\kappa)^2(3+2\kappa(-5+4\kappa)) \\ &= -6(\alpha_3-2\alpha_3\kappa)^2(3+4(\kappa-1)\kappa) < 0. \end{aligned}$$

Hence $P_{\boldsymbol{\alpha}}$ is strictly decreasing on $(\frac{1}{2}, 1)$. Therefore $P_{\boldsymbol{\alpha}}$ has at most one real root in $(\frac{1}{2}, 1)$. Similar computations, although somewhat more involved, show that $P_{\boldsymbol{\alpha}}$ is strictly increasing on $(0, \frac{1}{2})$; hence $P_{\boldsymbol{\alpha}}$ has at most one real root in $(0, \frac{1}{2})$.

□

3.3.37 THEOREM. *The equilibrium states have the following behaviour:*

*(i) The states* $\mathsf{e}^{\kappa}$, $\kappa \in (\frac{1}{2}, \kappa_2)$, *or correspondingly* $\mathsf{e}_1^{\mu}$, $\mu < \ln\frac{\alpha_1(2\kappa_2-1)}{2(1-\kappa_2)}$, *are (spectrally) unstable.*

*(ii) The states* $\mathsf{e}^{\kappa}$, $\kappa \in (\kappa_2, 1)$, *or correspondingly* $\mathsf{e}_1^{\mu}$, $\mu > \ln\frac{\alpha_1(2\kappa_2-1)}{2(1-\kappa_2)}$, *are stable.*

*(iii) The states* $\mathsf{e}^{\kappa}$, $\kappa \in (0, \kappa_1)$, *or correspondingly* $\mathsf{e}_2^{\mu}$, $\mu > \frac{1}{2}\ln\frac{1-2\kappa_1}{\kappa_1}$, *are stable.*

*(iv) The states* $\mathsf{e}^{\kappa}$, $\kappa \in (\kappa_1, \frac{1}{2})$, *or correspondingly* $\mathsf{e}_2^{\mu}$, $\mu < \frac{1}{2}\ln\frac{1-2\kappa_1}{\kappa_1}$, *are (spectrally) unstable.*

*(v) The states* $\mathsf{e}^{\kappa}$, $\kappa \in (-\infty, 0) \cup (1, \infty)$, *or correspondingly* $\mathsf{e}_3^{\mu}$, $\mu \neq \ln\frac{\alpha_1}{2}$, *are stable.*

*(vi) The origin* $\mathsf{e}_3^{\mu}$, $\mu = \ln\frac{\alpha_1}{2}$ *is stable.*

3.3.38 REMARK. The states $\mathsf{e}^{\kappa_1}$ and $\mathsf{e}^{\kappa_2}$ are spectrally stable. However, we were unable to determine their Lyapunov stability nature. We suspect that they are unstable (see Figures 3.20e and 3.21i).

PROOF. The linearization of the system at $\mathsf{e}^{\kappa}$ has eigenvalues 0 and $\frac{\pm\sqrt{P_{\boldsymbol{\alpha}}(\kappa)}}{\sqrt{2}\sqrt{\kappa^2(1-3\kappa+2\kappa^2)^2}}$. Hence, as $\kappa^2\left(1-3\kappa+2\kappa^2\right)^2 > 0$ for $\kappa \neq 0$, $\kappa \neq \frac{1}{2}$, $\kappa \neq 1$, we have a positive real eigenvalue if and only if $P_{\boldsymbol{\alpha}}(\kappa) > 0$. We have that $P_{\boldsymbol{\alpha}}(0) = -\alpha_3^2 < 0$, $P_{\boldsymbol{\alpha}}(\frac{1}{2}) = \frac{\alpha_2^2}{8} > 0$, and $P_{\boldsymbol{\alpha}}(1) = -\alpha_1^2 < 0$. Furthermore, by the foregoing lemma, $P_{\boldsymbol{\alpha}}$ has exactly two real roots $\kappa_1 \in (0, \frac{1}{2})$ and $\kappa_2 \in (\frac{1}{2}, 1)$. Therefore $P_{\boldsymbol{\alpha}}(\kappa) > 0$ for $\kappa \in (\kappa_1, \kappa_2)$ and $P_{\boldsymbol{\alpha}}(\kappa) \leq 0$ for $\kappa \in (-\infty, \kappa_1] \cup [\kappa_2, \infty)$. Consequently, the equilibrium states $\mathsf{e}^{\kappa}$, $\kappa \in (\kappa_1, \frac{1}{2})$ and $\mathsf{e}^{\kappa}$, $\kappa \in (\frac{1}{2}, \kappa_2)$ are spectrally unstable; all other states are spectrally stable.

Consider the energy function $H_\lambda = \lambda H_{5,\alpha}^2 - \lambda\kappa C$. We have $\mathbf{d}H_\lambda(\mathsf{e}^{\kappa}) = 0$ and $\mathbf{d}^2 H_\lambda(\mathsf{e}^{\kappa}) = \operatorname{diag}(2(1-\kappa)\lambda,\ \lambda(1-2\kappa),\ -2\kappa\lambda)$. Suppose $\kappa \in (-\infty, 0) \cup (1, \infty)$ and let $\lambda = -\kappa$. Then $\mathbf{d}^2 H_\lambda(\mathsf{e}^{\kappa}) = \operatorname{diag}(2(\kappa-1)\kappa,\ \kappa(2\kappa-1),\ 2\kappa^2\lambda)$ is positive definite. Therefore the states $\mathsf{e}^{\kappa}$, $\kappa \in (-\infty, 0) \cup (1, \infty)$ are stable.

On the other hand, assume that $\kappa \in (0,1)$. It is a simple matter to show that $p \in \ker \mathbf{d}C(\mathsf{e}^{\kappa})$ if and only if $p_1 = \frac{(1-\kappa)(2\alpha_2\kappa p_2+\alpha_3(2\kappa-1)p_3)}{\alpha_1\kappa(2\kappa-1)}$, i.e., $\ker \mathbf{d}C(\mathsf{e}^{\kappa})$ has basis

$$\left(\left(\frac{2\alpha_2(1-\kappa)}{\alpha_1(2\kappa-1)}, 1, 0\right), \left(\frac{\alpha_3(1-\kappa)}{\alpha_1\kappa}, 0, 1\right)\right).$$

The restriction of $\mathbf{d}^2 H_\lambda(\mathsf{e}^{\kappa})$ to $\ker \mathbf{d}C(\mathsf{e}^{\kappa})$ is

$$Q = \begin{bmatrix} -\frac{\left(8\alpha_2^2(\kappa-1)^3+\alpha_1^2(2\kappa-1)^3\right)\lambda}{\alpha_1^2(1-2\kappa)^2} & -\frac{4\alpha_2\alpha_3(\kappa-1)^3\lambda}{\alpha_1^2\kappa(2\kappa-1)} \\ -\frac{4\alpha_2\alpha_3(\kappa-1)^3\lambda}{\alpha_1^2\kappa(2\kappa-1)} & -\frac{2\left(\alpha_3^2(\kappa-1)^3+\alpha_1^2\kappa^3\right)\lambda}{\alpha_1^2\kappa^2} \end{bmatrix}.$$

Suppose $\kappa \in (0, \kappa_1)$ and let $\lambda = 1$. Then the first minor $-\frac{8\alpha_2^2(\kappa-1)^3+\alpha_1^2(2\kappa-1)^3}{\alpha_1^2(1-2\kappa)^2} > 0$ and $\det Q = -\frac{2P_\alpha(\kappa)}{\alpha_1^2(1-2\kappa)^2\kappa^2}$. Hence, as $P_{\boldsymbol{\alpha}}$ is negative on $(0,\kappa_1)$, we have $\det Q > 0$ and so the states $\mathsf{e}^{\kappa}$, $\kappa \in (0,\kappa_1)$ are stable. Suppose $\kappa \in (\kappa_2, 1)$ and let $\lambda = -1$. We have that $\frac{2\left(\alpha_3^2(\kappa-1)^3+\alpha_1^2\kappa^3\right)}{\alpha_1^2\kappa^2} > 0$ and $\det Q = -\frac{2P_\alpha(\kappa)}{\alpha_1^2(1-2\kappa)^2\kappa^2}$. Hence, as $P_{\boldsymbol{\alpha}}$ is negative on $(\kappa_2, 1)$, we have $\det Q > 0$ and so the states $\mathsf{e}^{\kappa}$, $\kappa \in (\kappa_2, 1)$ are stable. ☐

## 3.4 Relating cost extended systems to Hamilton-Poisson systems

In this section we give the Hamiltonian functions associated to each of the cost-extended systems obtained in our classification, propositions 3.2.2 to 3.2.7. We also show which normal form, obtained in our classification under affine equivalence, each of the associated Hamiltonians are equivalent to. The proofs are in a sense constructive and provide the specific affine isomorphisms under which the Hamiltonian systems are equivalent.

Mathematica. *The verification of the results of this section are contained in the Mathematica file:*

*Thesis Mathematica\SO(3)\Cost-equivalence\SO(3) Pontryagin functor.nb*

3.4.1 Proposition. *The family of Hamiltonians corresponding to* $(\Sigma_\alpha^{(1,1)}, \chi_\eta)$ *is given by*

$$H_{\alpha\eta}^{(1,1)}(p) = \alpha p_1 + \eta p_2 + \tfrac{1}{2}p_2^2$$

*for* $\alpha > 0,\ \eta \geq 0$.

*(i) If* $\eta = 0$, *then* $H_{\alpha\eta}^{(1,1)}(p)$ *is (affinely) equivalent to* $H_1^1(p) = p_2 + \frac{1}{2}p_1^2$.

*(ii) If* $\eta > 0$, *then* $H_{\alpha\eta}^{(1,1)}(p)$ *is equivalent to* $H_{2,\alpha\sqrt{\eta}}^1(p) = p_1 + \alpha\sqrt{\eta}p_2 + \frac{1}{2}p_1^2$.

PROOF. Consider the Hamiltonian $H^{(1,1)}_{\alpha\eta}(p)$. If $\eta = 0$, then $T\psi \cdot \vec{H}^1_1 = \vec{H}^{(1,1)}_{\alpha\eta} \circ \psi$, where $H^1_1(p) = p_2 + \frac{1}{2}p_1^2$, given that

$$\psi : p \mapsto p \begin{bmatrix} 0 & \frac{1}{\alpha} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{\alpha} \end{bmatrix} + \begin{bmatrix} \frac{\alpha^2-1}{\alpha} & 0 & 0 \end{bmatrix}.$$

On the other hand, if $\eta > 0$, then $T\psi \cdot \vec{H}^1_{2,\alpha\sqrt{\eta}} = \vec{H}^{(1,1)}_{\alpha\eta} \circ \psi$, where $H^1_{2,\alpha\sqrt{\eta}}(p) = p_1 + \alpha\sqrt{\eta}p_2 + \frac{1}{2}p_1^2$, given that

$$\psi : p \mapsto p \begin{bmatrix} 0 & \sqrt{\eta} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\sqrt{\eta} \end{bmatrix} + \begin{bmatrix} \alpha(1-\eta) & 1-\eta & 0 \end{bmatrix}.$$

□

3.4.2 PROPOSITION. *The families of Hamiltonians corresponding to* $(\Sigma^{(2,0)}, \chi^1_{\boldsymbol{\eta}\beta})$ *and* $(\Sigma^{(2,0)}, \chi^2_{\eta})$ *are given, respectively, by*

$$H^{(2,0)}_{1,\boldsymbol{\eta}\beta}(p) = \eta_1 p_1 + \eta_2 p_2 + \tfrac{1}{2}\left(p_1^2 + \tfrac{1}{\beta}p_2^2\right), \qquad \eta_1, \eta_2 \geq 0,\ 0 < \beta < 1$$

$$H^{(2,0)}_{2,\eta}(p) = \eta p_1 + \frac{1}{2}\left(p_1^2 + p_2^2\right), \qquad \eta \geq 0.$$

*Let* $G(p) = H^{(2,0)}_{1,\boldsymbol{\eta}\beta}(p)$ *and* $\tilde{G}(p) = H^{(2,0)}_{2,\eta}(p)$.

*(i) If* $\eta_1 = \eta_2 = 0$, *then* $G(p)$ *is equivalent to* $H^2(p) = p_1^2 + \frac{1}{2}p_2^2$.

*(ii) If* $\eta_1 = 0$, $\eta_2 > 0$, *then* $G(p)$ *is equivalent to* $H^2_{1,\delta}(p) = \delta p_1 + p_1^2 + \frac{1}{2}p_2^2$, *where* $\delta = \frac{\eta_2\beta\sqrt{2}}{\sqrt{1-\beta}}$.

*(iii) If* $\eta_1 > 0$, $\eta_2 = 0$, *then* $G(p)$ *is equivalent to* $H^2_{2,\delta}(p) = \delta p_2 + p_1^2 + \frac{1}{2}p_2^2$, *where* $\delta = \frac{\eta_1}{2\sqrt{(1-\beta)\beta}}$.

*(iv) If* $\eta_1, \eta_2 > 0$, *then* $G(p)$ *is equivalent to* $H^2_{3,\boldsymbol{\delta}}(p) = \delta_1 p_1 + \delta_2 p_2 + p_1^2 + \frac{1}{2}p_2^2$, *where* $\delta_1 = \frac{\eta_2\beta\sqrt{2}}{\sqrt{1-\beta}}$ *and* $\delta_2 = \frac{\eta_1}{2\sqrt{(1-\beta)\beta}}$.

*(v) If* $\eta = 0$, *then* $\tilde{G}(p)$ *is equivalent to* $H^1(p) = \frac{1}{2}p_1^2$.

*(vi) If* $\eta > 0$, *then* $\tilde{G}(p)$ *is equivalent to* $H^1_1(p) = p_2 + \frac{1}{2}p_1^2$.

PROOF. Consider the Hamiltonian $H^{(2,0)}_{1,\boldsymbol{\eta}\beta}(p)$. If $\eta_1 = \eta_2 = 0$, then $T\psi \cdot \vec{H}^2 = \vec{H}^{(2,0)}_{1,\beta} \circ \psi$, where $H^2(p) = p_1^2 + \frac{1}{2}p_2^2$, given that

$$\psi : p \mapsto \begin{bmatrix} 0 & \sqrt{\frac{\beta}{1-\beta}} & 0 \\ \frac{\sqrt{2}\beta}{\sqrt{1-\beta}} & 0 & 0 \\ 0 & 0 & -\sqrt{2\beta} \end{bmatrix} p.$$

If $\eta_1 = 0,\ \eta_2 > 0$, then $T\psi \cdot \vec{H}^2_{1,\delta} = \vec{H}^{(2,0)}_{1,\boldsymbol{\eta}\beta} \circ \psi$, where $H^2_{1,\delta}(p) = \frac{\eta_2\beta\sqrt{2}}{\sqrt{1-\beta}}p_1 + p_1^2 + \frac{1}{2}p_2^2$, given that

$$\psi : p \mapsto p \begin{bmatrix} 0 & \sqrt{\frac{\beta}{1-\beta}} & 0 \\ \frac{\sqrt{2}\beta}{\sqrt{1-\beta}} & 0 & 0 \\ 0 & 0 & -\sqrt{2\beta} \end{bmatrix} + \begin{bmatrix} 0 & \frac{\eta_2\beta(2\beta-1)}{1-\beta} & 0 \end{bmatrix}.$$

If $\eta_1 > 0,\ \eta_2 = 0$, then $T\psi \cdot \vec{H}^2_{2,\delta} = \vec{H}^{(2,0)}_{1,\boldsymbol{\eta}\beta} \circ \psi$, where $H^2_{2,\delta}(p) = \frac{\eta_1}{2\sqrt{(1-\beta)\beta}}p_2 + p_1^2 + \frac{1}{2}p_2^2$, given that

$$\psi : p \mapsto p \begin{bmatrix} 0 & \sqrt{\frac{\beta}{1-\beta}} & 0 \\ \frac{\sqrt{2}\beta}{\sqrt{1-\beta}} & 0 & 0 \\ 0 & 0 & -\sqrt{2\beta} \end{bmatrix} + \begin{bmatrix} \frac{\eta_1(2\beta-1)}{2(1-\beta)} & 0 & 0 \end{bmatrix}.$$

If $\eta_1, \eta_2 > 0$, then $T\psi \cdot \vec{H}^2_{3,\boldsymbol{\delta}} = \vec{H}^{(2,0)}_{1,\boldsymbol{\eta}\beta} \circ \psi$, where $H^2_{3,\boldsymbol{\delta}}(p) = \frac{\eta_2\beta\sqrt{2}}{\sqrt{1-\beta}}p_1 + \frac{\eta_1}{2\sqrt{(1-\beta)\beta}}p_2 + p_1^2 + \frac{1}{2}p_2^2$, given that

$$\psi : p \mapsto p \begin{bmatrix} 0 & \sqrt{\frac{\beta}{1-\beta}} & 0 \\ \frac{\sqrt{2}\beta}{\sqrt{1-\beta}} & 0 & 0 \\ 0 & 0 & -\sqrt{2\beta} \end{bmatrix} + \begin{bmatrix} \frac{\eta_1(2\beta-1)}{2(1-\beta)} & \frac{\eta_2\beta(2\beta-1)}{1-\beta} & 0 \end{bmatrix}.$$

On the other hand, consider the Hamiltonian $H^{(2,0)}_{2,\eta}$. If $\eta = 0$, then $T\psi \cdot \vec{H}^1 = \vec{H}^{(2,0)}_{2,\eta} \circ \psi$, where $H^1(p) = \frac{1}{2}p_1^2$, given that

$$\psi : p \mapsto p \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

If $\eta > 0$, then $T\psi \cdot \vec{H}^1_1 = \vec{H}^{(2,0)}_{2,\eta} \circ \psi$, where $H^1_1(p) = p_2 + \frac{1}{2}p_1^2$, given that

$$\psi : p \mapsto p \begin{bmatrix} 0 & -\frac{1}{\eta} & 0 \\ 0 & 0 & \frac{1}{\eta} \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{\eta} - \eta & 0 & 0 \end{bmatrix}.$$

3.4.3 Remark. Note that in the proof above we have actually provided more details than necessary. More specifically, considering the Hamiltonian $H^{(2,0)}_{1.\boldsymbol{\eta}\beta}$, one can notice that the linear part of the affine isomorphism is the same in each case. Also, the affine part is completely captured by the case when $\eta_1, \eta_2 > 0$. That is, if we allow $\eta_1 = 0$ and/or $\eta_2 = 0$ in this case, we get back the previous cases. In the remaining cases we will not repeat the isomorphism for every subcase.

3.4.4 Proposition. *The families of Hamiltonians corresponding to* $(\Sigma^{(2,1)}_\alpha, \chi^1_{\eta\beta\gamma})$ *and* $(\Sigma^{(2,1)}_\alpha, \chi^3_\eta)$ *are given, respectively, by*

$$H^{(2,1)}_{1,\alpha\eta\beta\gamma}(p) = \alpha p_1 + \eta p_2 + \gamma p_3 + \tfrac{1}{2}\left(p_2^2 + \tfrac{1}{\beta}p_3^2\right), \qquad \alpha > 0,\ \eta \geq 0,\ \gamma \in \mathbb{R},\ 0 < \beta < 1$$
$$H^{(2,1)}_{2,\alpha\eta}(p) = \alpha p_1 + \eta p_2 + \tfrac{1}{2}\left(p_2^2 + p_3^2\right), \qquad \alpha > 0,\ \eta \geq 0.$$

*Then*

*(i)* $H^{(2,1)}_{1,\alpha\eta\beta\gamma}(p)$ *is equivalent to* $H^2_{5,\boldsymbol{\delta}}(p) = \delta_1 p_1 + \delta_2 p_2 + \delta_3 p_3 + p_1^2 + \frac{1}{2}p_2^2$, *where* $\delta_1 = \frac{\sqrt{2}\alpha(1-\beta)}{\sqrt{\beta}}$, $\delta_2 = \frac{\eta}{2\sqrt{(1-\beta)\beta}}$, *and* $\delta_3 = \frac{\sqrt{2}\beta\gamma}{\sqrt{1-\beta}}$.

*(ii) If* $\eta = 0$, *then* $H^{(2,1)}_{2,\alpha\eta}(p)$ *is equivalent to* $H^1(p) = \frac{1}{2}p_1^2$.

*(iii) If* $\eta > 0$, *then* $H^{(2,1)}_{2,\alpha\eta}(p)$ *is equivalent to* $H^1_{2,\sqrt{\alpha}\eta}(p) = p_1 + \eta\sqrt{\alpha}p_2 + \frac{1}{2}p_1^2$.

3.4.5 **Remark.** Note that we do not consider the cost-extended systems $(\Sigma^{(2,1)}_\alpha, \chi^2_{\eta\beta})$ as, by slightly relaxing the conditions on the parameters, they form a sub-case of the Hamiltonians corresponding to the systems $(\Sigma^{(2,1)}_\alpha, \chi^1_{\eta\beta\gamma})$.

**Proof.** Consider the Hamiltonian $H^{(2,1)}_{1,\alpha\eta\beta\gamma}$. Then $T\psi \cdot \vec{H}^2_{5,\boldsymbol{\delta}} = \vec{H}^{(2,1)}_{1,\alpha\eta\beta\gamma} \circ \psi$, where

$$\begin{aligned} H^2_{5,\boldsymbol{\delta}}(p) &= \delta_1 p_1 + \delta_2 p_2 + \delta_3 p_3 + p_1^2 + \tfrac{1}{2}p_2^2 \\ &= \tfrac{\sqrt{2}\alpha(1-\beta)}{\sqrt{\beta}} p_1 + \tfrac{\eta}{2\sqrt{(1-\beta)\beta}} p_2 + \tfrac{\sqrt{2}\beta\gamma}{\sqrt{1-\beta}} p_3 + p_1^2 + \tfrac{1}{2}p_2^2 \end{aligned}$$

given that

$$\psi : p \mapsto p \begin{bmatrix} -\sqrt{2\beta} & 0 & 0 \\ 0 & -\sqrt{\frac{\beta}{1-\beta}} & 0 \\ 0 & 0 & -\frac{\sqrt{2\beta}}{\sqrt{1-\beta}} \end{bmatrix} + \begin{bmatrix} \alpha(2\beta - 1) & \frac{\eta(2\beta-1)}{2(1-\beta)} & \frac{(2\beta-1)\beta\gamma}{1-\beta} \end{bmatrix}.$$

Note that $\delta_1 = \frac{\sqrt{2}\alpha(1-\beta)}{\sqrt{\beta}} > 0$ and $\delta_2 = \frac{\eta}{2\sqrt{(1-\beta)\beta}} \geq 0$. Also, we have that $\operatorname{sign}(\delta_3) = \operatorname{sign}(\gamma)$. If $\delta_1 < |\delta_3|$, we can compose the affine isomorphism above with the linear isomorphism

$$\psi_2 : p \mapsto p \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

to obtain the Hamiltonian $H'(p) = \delta_3 p_1 + \delta_2 p_2 - \delta_1 p_3 + p_1^2 + \frac{1}{2}p_2^2$. (We can then use $\operatorname{diag}(-1, 1, -1)$ to adjust the sign of $\delta_3$ if necessary.)

Now, consider the Hamiltonian $H^{(2,1)}_{2,\alpha\eta}(p)$. If $\eta = 0$, then $T\psi \cdot \vec{H}^1 = \vec{H}^{(2,1)}_{2,\alpha\eta} \circ \psi$, where $H^1(p) = \frac{1}{2}p_1^2$, given that

$$\psi : p \mapsto p \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} \alpha & 0 & 0 \end{bmatrix}.$$

If $\eta > 0$, then $T\psi \cdot \vec{H}^1_{2,\alpha\eta} = \vec{H}^{(2,1)}_{2,\alpha\eta} \circ \psi$, where $H^1_{2,\alpha\eta}(p) = p_1 + \sqrt{\alpha}\eta\, p_2 + \frac{1}{2}p_1^2$, given that

$$\psi : p \mapsto p \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\sqrt{\alpha} & 0 \\ 0 & 0 & -\sqrt{\alpha} \end{bmatrix} + \begin{bmatrix} \alpha - 1 & (\alpha - 1)\eta & 0 \end{bmatrix}.$$

□

3.4.6 PROPOSITION. *The families of Hamiltonians corresponding to the three-input cost-extended systems* $(\Sigma^{(3,0)}, \chi^1_{\boldsymbol{\alpha\beta}\gamma})$, $(\Sigma^{(3,0)}, \chi^4_{\boldsymbol{\eta}\gamma})$, $(\Sigma^{(3,0)}, \chi^5_{\boldsymbol{\eta}\beta})$, $(\Sigma^{(3,0)}, \chi^6_{\eta})$ *are given, respectively, by*

$$\begin{aligned} H^{(3,0)}_{1,\boldsymbol{\eta\beta}\gamma}(p) &= \eta_1 p_1 + \eta_2 p_2 + \gamma p_3 + \tfrac{1}{2}\left(p_1^2 + \tfrac{1}{\beta_1}p_2^2 + \tfrac{1}{\beta_2}p_3^2\right) \\ H^{(3,0)}_{2,\boldsymbol{\eta}\beta}(p) &= \eta_1 p_1 + \eta_2 p_3 + \tfrac{1}{2}\left(p_1^2 + p_2^2 + \tfrac{1}{\beta_2}p_3^2\right) \\ H^{(3,0)}_{3,\boldsymbol{\eta\beta}}(p) &= \eta_1 p_1 + \eta_2 p_3 + \tfrac{1}{2}\left(p_1^2 + \tfrac{1}{\beta_1}p_2^2 + \tfrac{1}{\beta_1}p_3^2\right) \\ H^{(3,0)}_{4,\eta}(p) &= \eta_1 p_1 + \tfrac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right). \end{aligned}$$

*Here* $\eta_1, \eta_2 \geq 0$, $0 < \beta_2 < \beta_1 < 1$, *and* $\gamma \in \mathbb{R}$. *It follows that:*

- $H^{(3,0)}_{1,\boldsymbol{\eta\beta}\gamma}(p)$ *is equivalent to* $H^2_{5,\boldsymbol{\delta}}(p) = \delta_1 p_1 + \delta_2 p_2 + \delta_3 p_3 + p_1^2 + \frac{1}{2}p_2^2$, *where* $\delta_1 = \frac{\sqrt{2}\eta_1(\beta_1-\beta_2)}{\sqrt{\beta_1\beta_2(1-\beta_1)(1-\beta_2)}}$, $\delta_2 = \frac{\eta_2\beta_1(1-\beta_2)}{2\sqrt{\beta_2(1-\beta_1)(\beta_1-\beta_2)}}$, *and* $\delta_3 = \frac{\sqrt{2}\gamma(1-\beta_1)\beta_2}{\sqrt{\beta_1(\beta_1-\beta_2)(1-\beta_2)}}$.
- $H^{(3,0)}_{4,\eta}(p)$ *is equivalent to* $H^0_{1,\eta_1}(p) = \eta_1 p_1$.

*Let* $G(p) = H^{(3,0)}_{2,\boldsymbol{\eta}\beta}(p)$ *and* $\tilde{G} = H^{(3,0)}_{3,\boldsymbol{\eta\beta}}(p)$.

- *If* $\eta_1, \eta_2 > 0$, *then* $G(p)$ *is equivalent to* $H^1_{2,\eta_1\sqrt{\eta_2}}(p) = p_1 + \eta_1\sqrt{\eta_2}p_2 + \frac{1}{2}p_1^2$.
- *If* $\eta_1 = 0$ *and* $\eta_2 > 0$, *then* $G(p)$ *is equivalent to* $H^1_0(p) = \frac{1}{2}p_1^2$.
- *If* $\eta_1 > 0$ *and* $\eta_2 = 0$, *then* $G(p)$ *is equivalent to* $H^1_1(p) = p_2 + \frac{1}{2}p_1^2$.
- *If* $\eta_1 = \eta_2 = 0$, *then* $G(p)$ *is equivalent to* $H^1_0(p) = \frac{1}{2}p_1^2$.
- *If* $\eta_1, \eta_2 > 0$, *then* $G(p)$ *is equivalent to* $H^1_{2,\eta_2\sqrt{\eta_1}}(p) = p_1 + \eta_2\sqrt{\eta_1}p_2 + \frac{1}{2}p_1^2$.
- *If* $\eta_1 = 0$ *and* $\eta_2 > 0$, *then* $\tilde{G}(p)$ *is equivalent to* $H^1_1(p) = p_2 + \frac{1}{2}p_1^2$.
- *If* $\eta_1 \geq 0$ *and* $\eta_2 = 0$, *then* $\tilde{G}(p)$ *is equivalent to* $H^1_0(p) = \frac{1}{2}p_1^2$.

3.4.7 REMARK. Note that we do not consider the cost-extended systems $(\Sigma^{(3,0)}, \chi^2_{\alpha\boldsymbol{\beta}\eta})$ or $(\Sigma^{(3,0)}, \chi^3_{\boldsymbol{\eta\beta}})$. By slightly relaxing the conditions on the parameters, these systems correspond to a sub-case of the Hamiltonians corresponding to the systems $(\Sigma^{(3,0)}, \chi^1_{\boldsymbol{\alpha\beta}\gamma})$.

PROOF. First, consider the Hamiltonian $H^{(3,0)}_1(p)$. Then $T\psi \cdot \vec{H}^2_{5,\boldsymbol{\delta}} = \vec{H}^{(3,0)}_{1,\boldsymbol{\eta\beta}\gamma} \circ \psi$, where $H^2_{5,\boldsymbol{\delta}}(p) = \frac{\sqrt{2}\eta_1(\beta_1-\beta_2)}{\sqrt{\beta_1\beta_2(1-\beta_1)(1-\beta_2)}}\,p_1 + \frac{\eta_2\beta_1(1-\beta_2)}{2\sqrt{\beta_2(1-\beta_1)(\beta_1-\beta_2)}}\,p_2 + \frac{\sqrt{2}\gamma(1-\beta_1)\beta_2}{\sqrt{\beta_1(\beta_1-\beta_2)(1-\beta_2)}}\,p_3 + p_1^2 + \frac{1}{2}p_2^2$ , given that

$$\psi : p \mapsto p \begin{bmatrix} -\sqrt{\frac{2\beta_1\beta_2}{(1-\beta_1)(1-\beta_2)}} & 0 & 0 \\ 0 & -\frac{\beta_1\sqrt{\beta_2}}{\sqrt{(1-\beta_1)(\beta_1-\beta_2)}} & 0 \\ 0 & 0 & -\frac{\beta_2\sqrt{2\beta_1}}{\sqrt{(1-\beta_2)(\beta_1-\beta_2)}} \end{bmatrix} + \begin{bmatrix} \frac{\eta_1(2\beta_2-\beta_1-\beta_1\beta_2)}{(1-\beta_1)(1-\beta_2)} \\ \frac{\eta_2\beta_1(2\beta_2-\beta_1-\beta_1\beta_2)}{2(1-\beta_1)(\beta_1-\beta_2)} \\ \frac{\gamma\beta_2(2\beta_2-\beta_1-\beta_1\beta_2)}{(1-\beta_2)(\beta_1-\beta_2)} \end{bmatrix}^\top .$$

Note that the linear part of the isomorphism is independent of the variables $\eta_1, \eta_2$ and $\gamma$. Thus the affine isomorphism given above, is the correct one for any allowable choice of the constants $\eta_1, \eta_2, \gamma, \beta_1$, and $\beta_2$. For example, if $\eta_1 = \eta_2 = \gamma = 0$, then the system $H_1^{(3,0)}(p)$ is equivalent to the system $H^2(p) = p_1^2 + \frac{1}{2}p_2^2$ under the above affine isomorphism (with $\eta_1 = \eta_2 = \gamma = 0$).

Now, consider the Hamiltonian $H_{2,\boldsymbol{\eta}\beta}^{(3,0)}(p)$. Assume $\eta_1, \eta_2 > 0$. Then $T\psi \cdot \vec{H}^1_{2,\eta_1\sqrt{\eta_2}} = \vec{H}^{(3,0)}_{2,\boldsymbol{\eta}\beta} \circ \psi$, where $H^1_{2,\eta_1\sqrt{\eta_2}}(p) = p_1 + \eta_1\sqrt{\eta_2}p_2 + \frac{1}{2}p_1^2$ , given that

$$\psi : p \mapsto p \begin{bmatrix} 0 & \frac{\sqrt{\eta_2}\beta_2}{1-\beta_2} & 0 \\ 0 & 0 & \frac{\sqrt{\eta_2}\beta_2}{1-\beta_2} \\ \frac{\beta_2}{1-\beta_2} & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{\eta_1(1-\eta_2)\beta_2}{1-\beta_2} & 0 & \frac{(1-\eta_2)\beta_2}{1-\beta_2} \end{bmatrix}.$$

Note that the linear part of this isomorphism depends on $\eta_2$. Thus we cannot use this isomorphism in the other sub-cases. If $\eta_1 = 0$ and $\eta_2 > 0$, then $T\psi \cdot \vec{H}^1_0 = \vec{H}^{(3,0)}_{2,\boldsymbol{\eta}\beta} \circ \psi$, where $H^1_0(p) = \frac{1}{2}p_1^2$, given that

$$\psi : p \mapsto p \begin{bmatrix} 0 & \frac{\sqrt{\eta_2}\beta_2}{1-\beta_2} & 0 \\ 0 & 0 & \frac{\sqrt{\eta_2}\beta_2}{1-\beta_2} \\ \frac{\beta_2}{1-\beta_2} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -\frac{\eta_2\beta_2}{1-\beta_2} \end{bmatrix}.$$

If $\eta_1 > 0$ and $\eta_2 = 0$, then $T\psi \cdot \vec{H}^1_1 = \vec{H}^{(3,0)}_{2,\boldsymbol{\eta}\beta} \circ \psi$, where $H^1_1(p) = p_2 + \frac{1}{2}p_1^2$ , given that

$$\psi : p \mapsto p \begin{bmatrix} 0 & \frac{\beta_2}{\eta_1(1-\beta_2)} & 0 \\ 0 & 0 & \frac{\beta_2}{\eta_1(1-\beta_2)} \\ \frac{\beta_2}{1-\beta_2} & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{\beta_2(\eta_1^2-1)}{\eta_1(1-\beta_2)} & 0 & 0 \end{bmatrix}.$$

If $\eta_1 = \eta_2 = 0$, then $T\psi \cdot \vec{H}^1_0 = \vec{H}^{(3,0)}_{2,\boldsymbol{\eta}\beta} \circ \psi$, where $H^1_0(p) = \frac{1}{2}p_1^2$, given that

$$\psi : p \mapsto p \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\beta_2}{1-\beta_2} & 0 & 0 \end{bmatrix}.$$

Next, consider the Hamiltonian $H^{(3,0)}_{3,\boldsymbol{\eta\beta}}(p)$. Assume $\eta_1, \eta_2 > 0$. Then $T\psi \cdot \vec{H}^1_{2,\eta_2\sqrt{\eta_1}} = \vec{H}^{(3,0)}_{3,\boldsymbol{\eta\beta}} \circ \psi$, where $H^1_{2,\eta_2\sqrt{\eta_1}}(p) = p_1 + \eta_2\sqrt{\eta_1}p_2 + \frac{1}{2}p_1^2$ , given that

$$\psi : p \mapsto p \begin{bmatrix} -\frac{\beta_1}{1-\beta_1} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{\eta_1}\beta_1}{1-\beta_1} \\ 0 & -\frac{\sqrt{\eta_1}\beta_1}{1-\beta_1} & 0 \end{bmatrix} + \begin{bmatrix} \frac{\beta_1(\eta_1-1)}{1-\beta_1} & 0 & \frac{\eta_2\beta_1(\eta_1-1)}{1-\beta_1} \end{bmatrix}.$$

Again, note that the linear part of this isomorphism depends on $\eta_1$ and thus we cannot use this isomorphism in the other subcases. If $\eta_1 = 0$ and $\eta_2 > 0$, then $T\psi \cdot \vec{H}^1_1(p) = \vec{H}^{(3,0)}_{2,\boldsymbol{\eta}\beta}(p) \circ \psi$, where $H^1_1(p) = p_2 + \frac{1}{2}p_1^2$, given that

$$\psi : p \mapsto p \begin{bmatrix} \frac{\beta_1}{1-\beta_1} & 0 & 0 \\ 0 & 0 & -\frac{\beta_1}{\eta_2(1-\beta_1)} \\ 0 & -\frac{\beta_1}{\eta_2(1-\beta_1)} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{\beta_1\left(1-\eta_2^2\right)}{\eta_2(1-\beta_1)} \end{bmatrix}.$$

If $\eta_1 \geq 0$ and $\eta_2 = 0$, then $T\psi \cdot \vec{H}_0^1(p) = \vec{H}^{(3,0)}_{2,\boldsymbol{\eta}\beta}(p) \circ \psi$, where $H_0^1(p) = \frac{1}{2}p_1^2$, given that

$$\psi : p \mapsto p \begin{bmatrix} \frac{\beta_1}{1-\beta_1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} \frac{\eta_1\beta_1}{1-\beta_1} & 0 & 0 \end{bmatrix}.$$

Lastly, consider the Hamiltonian $H^{(3,0)}_{4,\eta}(p)$. Clearly this system is equivalent to $H^0_{1,\eta_1}(p) = \eta_1 p_1$ as

$$\left(H^{(3,0)}_{4,\eta} - \tfrac{1}{2}C\right)(p) = H^0_{1,\eta_1}(p).$$

□

## 3.5 Trajectories on $\mathsf{SO}(3)$, $\mathsf{SU}(2)$ and $\mathbb{S}^2$

We begin this section with a presentation of the results for obtaining the extremal trajectories on the Lie groups $\mathsf{SO}(3)$ and $\mathsf{SU}(2)$, and their associated homogeneous space $\mathbb{S}^2$. This work recalls the results of Jurdjevic [33, 34] and Biggs and Holderbaum [15]. The results of these works are given here for completeness and to fix notation. The approach outlined in this section will be used in chapter 6 to investigate several illustrative examples of optimal control problems on $\mathsf{SO}(3)$.

There exist isomorphisms between the vector spaces $\mathbb{R}^3 \cong \mathfrak{so}(3) \cong \mathfrak{su}(2)$, explicitly given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \mapsto \frac{1}{2} \begin{bmatrix} ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & -ix_1 \end{bmatrix}. \tag{3.12}$$

As $\mathsf{SO}(3)$ (and $\mathsf{SU}(2)$) is a simple Lie group it follows that any element $p(t) \in \mathfrak{so}(3)^*$ can be identified with an element $p(t) = p_1(t)E_1 + p_2(t)E_2 + p_3(t)E_3 \in \mathfrak{so}(3)$. This identification is obtained using the nondegenerate Killing form, with exists for all (semi) simple Lie groups. Using this identification, it follows that the extremal equations $\frac{d}{dt}p(t) = \{p(t), H\}$ (of a given Hamilton-Poisson system) can be expressed in the dual form

$$\dot{p}(t) = [p(t), \mathbf{d}H(t)]. \tag{3.13}$$

The equations of motion on the group can then be expressed using the equation

$$\frac{dg(t)}{dt} = g(t)\mathbf{d}H(t) = g(t)\Xi(\mathbf{1}, u(t)). \tag{3.14}$$

The equations (3.13) and (3.14) are what are known as the Lax-pair equations.

On $\mathsf{SO}(3)$ it follows that

$$\mathbf{d}H(t) = \begin{bmatrix} 0 & -\frac{\partial H}{\partial p_3} & \frac{\partial H}{\partial p_2} \\ \frac{\partial H}{\partial p_3} & 0 & -\frac{\partial H}{\partial p_1} \\ -\frac{\partial H}{\partial p_2} & \frac{\partial H}{\partial p_1} & 0 \end{bmatrix}, \qquad p(t) = \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix}. \tag{3.15}$$

To derive the equations for the optimal trajectories $g(t) \in \mathsf{SO}(3)$ it is useful to consider a particular solution of $g(t)p(t)g(t)^{-1} = p(0)$. As $\mathsf{SO}(3)$ acts transitively on the sphere there always exists a $g_0 \in \mathsf{SO}(3)$ such that $p(0)$ can be conjugated to $g_0 p(0) g_0^{-1} = \sqrt{c_0}E_1$. (Recall that $c_0 = C(p(t)) = p_1^2 + p_2^2 + p_3^2$.) This conjugation, which corresponds to the adjoint action $\mathrm{Ad}_{g_0}$, does not change the orbit we are on, and thus by fixing our initial point on the sphere (adjoint orbit) to be $\sqrt{c_0}E_1$ we are in essence solving the same problem which is just computationally simpler.

3.5.1 REMARK. In this section we will assume that $p_2^2 + p_3^2 \neq 0$. This assumption turns out to be true for any Hamilton-Poisson system associated to one of the controllable systems on $\mathsf{SO}(3)$ we obtained in our classification of systems under detached feedback equivalence.

### Extremal curves on $\mathsf{SO}(3)$

Let $\phi_1, \phi_2, \phi_3$ denote the coordinates (Euler angles) of a point in $\mathsf{SO}(3)$ given by the formula

$$\begin{aligned} g(t) &= \exp(\phi_1 E_1)\exp(\phi_2 E_2)\exp(\phi_3 E_1) \qquad (3.16) \\ &= \begin{bmatrix} \cos\phi_2 & \sin\phi_2 \sin\phi_3 & \cos\phi_3 \sin\phi_2 \\ \sin\phi_1 \sin\phi_2 & \cos\phi_1\cos\phi_3 - \cos\phi_2 \sin\phi_1 \sin\phi_3 & -\cos\phi_2\cos\phi_3\sin\phi_1 - \cos\phi_1\sin\phi_3 \\ -\cos\phi_1\sin\phi_2 & \cos\phi_3\sin\phi_1 + \cos\phi_1\cos\phi_2\sin\phi_3 & \cos\phi_1\cos\phi_2\cos\phi_3 - \sin\phi_1\sin\phi_3 \end{bmatrix}. \end{aligned}$$

It then follows that

$$\begin{aligned} p(t) &= \exp(-\phi_3 E_1)\exp(-\phi_2 E_2)\sqrt{c_0}E_1 \exp(\phi_2 E_2)\exp(\phi_3 E_1) \\ &= \sqrt{c_0}\begin{bmatrix} 0 & -\cos(\phi_3)\sin(\phi_2) & \sin(\phi_2)\sin(\phi_3) \\ \cos(\phi_3)\sin(\phi_2) & 0 & -\cos(\phi_2) \\ -\sin(\phi_2)\sin(\phi_3) & \cos(\phi_2) & 0 \end{bmatrix}. \end{aligned}$$

Equating the above expression for $p(t)$ with equation (3.15) then gives that

$$\begin{aligned} \cos(\phi_2) &= \frac{p_1}{\sqrt{c_0}}, & \sin(\phi_2) &= \pm\frac{\sqrt{p_2^2 + p_3^2}}{\sqrt{c_0}} \\ \cos(\phi_3) &= \pm\frac{p_3}{\sqrt{p_2^2 + p_3^2}}, & \sin(\phi_3) &= \pm\frac{p_2}{\sqrt{p_2^2 + p_3^2}}. \end{aligned}$$

In order to obtain an expression for $\phi_1$ we substitute the coordinate expression for $g(t)$ into equation (3.14), which yields

$$\begin{aligned}
g(t)^{-1}\frac{dg(t)}{dt} &= \dot{\phi}_1 \begin{bmatrix} 0 & -\cos(\phi_3)\sin(\phi_2) & \sin(\phi_2)\sin(\phi_3) \\ \cos(\phi_3)\sin(\phi_2) & 0 & -\cos(\phi_2) \\ -\sin(\phi_2)\sin(\phi_3) & \cos(\phi_2) & 0 \end{bmatrix} \\
&+ \dot{\phi}_2 \begin{bmatrix} 0 & \sin(\phi_3) & \cos(\phi_3) \\ -\sin(\phi_3) & 0 & 0 \\ -\cos(\phi_3) & 0 & 0 \end{bmatrix} + \dot{\phi}_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.
\end{aligned}$$

Equating this to $\mathbf{d}H(t)$ in equation (3.15), and simplifying, then gives

$$\dot{\phi}_1 = \frac{\sqrt{c_0}}{p_2^2 + p_3^2}\left(\frac{\partial H}{\partial p_2}p_2 + \frac{\partial H}{\partial p_3}p_3\right).$$

**Extremal curves on SU(2)**

Again, as $\mathsf{SU}(2)$ is (semi) simple, we can express the equations of motion in Lax-pair form, where now

$$\mathbf{d}H(t) = \frac{1}{2}\begin{bmatrix} i\frac{\partial H}{\partial p_1} & \frac{\partial H}{\partial p_2} + i\frac{\partial H}{\partial p_3} \\ -\frac{\partial H}{\partial p_2} + i\frac{\partial H}{\partial p_3} & -i\frac{\partial H}{\partial p_1} \end{bmatrix}, \qquad p(t) = \frac{1}{2}\begin{bmatrix} ip_1 & p_2 + ip_3 \\ -p_2 + ip_3 & -ip_1 \end{bmatrix}. \tag{3.17}$$

(See appendix A.2 for details of $\mathsf{SU}(2)$ and its Lie algebra $\mathfrak{su}(2)$.) Let $\varphi_1, \varphi_2, \varphi_3$ denote the coordinates of a point in $\mathsf{SU}(2)$ given by the formula

$$g(t) = \exp(\varphi_1 F_1)\exp(\varphi_2 F_2)\exp(\varphi_3 F_1). \tag{3.18}$$

From the particular solution we are interested in it then follows that

$$\begin{aligned}
p(t) &= \exp(-\varphi_3 F_1)\exp(-\varphi_2 F_2)\sqrt{c_0}F_1\exp(\varphi_2 F_2)\exp(\varphi_3 F_1) \\
&= \frac{i\sqrt{c_0}}{2}\begin{bmatrix} \cos(\varphi_2) & \mathrm{e}^{-i\varphi_3} \\ \mathrm{e}^{i\varphi_3}\sin(\varphi_2) & -\cos(\varphi_2) \end{bmatrix}
\end{aligned}$$

Equating the above with equation (3.17) then gives that

$$\begin{aligned}
\cos(\varphi_2) &= \frac{p_1}{\sqrt{c_0}}, & \sin(\varphi_2) &= \pm\frac{\sqrt{p_2^2 + p_3^2}}{\sqrt{c_0}} \\
\cos(\varphi_3) &= \pm\frac{p_3}{\sqrt{p_2^2 + p_3^2}}, & \sin(\varphi_3) &= \pm\frac{p_2}{\sqrt{p_2^2 + p_3^2}}.
\end{aligned}$$

In order to obtain an expression for $\varphi_1$ we substitute the coordinate expression for $g(t)$ into equation (3.14), which yields

$$g(t)^{-1}\frac{dg(t)}{dt} = \frac{\dot{\varphi}_1}{2}\begin{bmatrix} i\cos(\varphi_2) & i\,\mathrm{e}^{-i\varphi_3}\sin(\varphi_2) \\ i\,\mathrm{e}^{i\varphi_3}\sin(\varphi_2) & -i\cos(\varphi_2) \end{bmatrix} + \frac{\dot{\varphi}_2}{2}\begin{bmatrix} 0 & \mathrm{e}^{-i\varphi_3} \\ -\mathrm{e}^{i\varphi_3} & 0 \end{bmatrix} + \frac{\dot{\varphi}_3}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Equating this to $\mathbf{d}H(t)$ in equation (3.17), and simplifying, then gives

$$\dot{\varphi}_1 = \frac{\sqrt{c_0}}{p_2^2 + p_3^2}\left(\frac{\partial H}{\partial p_2} p_2 + \frac{\partial H}{\partial p_3} p_3\right).$$

The above function is a meromorphic function of time and can thus be integrated to obtain $\varphi_1$. This result illustrates that integrating on $\mathsf{SU}(2)$ gives exactly the same expressions for optimal paths in local coordinates (or, Euler angles) as integrating on $\mathsf{SO}(3)$, i.e., $\phi_i = \varphi_i$. However, the solutions $g(t)$ are expressed much more compactly on $\mathsf{SU}(2)$.

Explicitly, we obtain the expression

$$g(t) = \begin{bmatrix} \mathrm{e}^{\frac{1}{2}i\varphi_1}\,\mathrm{e}^{\frac{1}{2}i\varphi_3}\cos(\frac{\varphi_2}{2}) & \mathrm{e}^{\frac{1}{2}i\varphi_1}\,\mathrm{e}^{-\frac{1}{2}i\varphi_3}\sin(\frac{\varphi_2}{2}) \\ -\,\mathrm{e}^{-\frac{1}{2}i\varphi_1}\,\mathrm{e}^{\frac{1}{2}i\varphi_3}\sin(\frac{\varphi_2}{2}) & \mathrm{e}^{-\frac{1}{2}i\varphi_1}\,\mathrm{e}^{-\frac{1}{2}i\varphi_3}\cos(\frac{\varphi_2}{2}) \end{bmatrix}$$

with

$$\begin{aligned} \cos(\frac{\varphi_2}{2}) &= \sqrt{\frac{1+\cos(\varphi_2)}{2}} = \sqrt{\frac{\sqrt{c_0}+p_1}{2\sqrt{c_0}}} \\ \sin(\frac{\varphi_2}{2}) &= \sqrt{\frac{1-\cos(\varphi_2)}{2}} = \sqrt{\frac{\sqrt{c_0}-p_1}{2\sqrt{c_0}}} \\ \mathrm{e}^{\frac{1}{2}i\varphi_3} &= \left(\frac{\sqrt{p_2^2+p_3^2}+p_3}{2\sqrt{p_2^2+p_3^2}}\right)^{\frac{1}{2}} + i\left(\frac{\sqrt{p_2^2+p_3^2}-p_3}{2\sqrt{p_2^2+p_3^2}}\right)^{\frac{1}{2}}. \end{aligned}$$

### Extremal curves on $\mathbb{S}^2$

Having solved for the optimal curves in the group it is also of interest to study the projections on to the base space (homogeneous space) $\mathbb{S}^2 \cong \mathsf{SO}(3)/\mathsf{SO}(2)$. For example, given a sub-Riemannian structure on $\mathsf{SO}(3)$ one can project it down to its homogeneous space $\mathbb{S}^2$. This projection endows $\mathbb{S}^2$ with the structure of an almost-Riemannian manifold. These structures arise in problems of population transfer in quantum mechanics and in the problem of orbital transfer in space mechanics (see [14] and references within). The projections onto $\mathbb{S}^2$ are the same for $\mathsf{SO}(3)$ and $\mathsf{SU}(2)$. For $\mathsf{SO}(3)$ this projection is done by multiplying the matrix (3.16) on the right-hand side by the vector $\vec{e}_1 = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^\top$, which yields $\vec{x} = g(t)\vec{e}_1 = \begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}^\top$, where

$$\begin{aligned} x_1 &= \cos(\varphi_2) \\ x_2 &= \sin(\varphi_1)\sin(\varphi_2) \\ x_3 &= -\cos(\varphi_1)\sin(\varphi_2). \end{aligned}$$

In the case of $\mathsf{SU}(2)$, with $g(t)$ given by equation (3.18), the equivalent projection

$$g(t)\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} g(t)^{-1} \to \mathbb{S}^2$$

gives an element of $\mathsf{SU}(2)$ isomorphic to $\mathbb{S}^2$ through equation (3.12). Therefore, substituting in the expressions obtained for $\mathsf{SO}(3)$ (or, $\mathsf{SU}(2)$) we obtain the extremal curves $\vec{x} \in \mathbb{S}^2$, explicitly given by:

$$\begin{aligned}
x_1 &= \frac{p_1}{\sqrt{c_0}} \\
x_2 &= \pm\frac{\sqrt{p_2^2+p_3^2}}{\sqrt{c_0}} \sin\left(\sqrt{c_0}\int_0^t \left(\frac{\frac{\partial H}{\partial p_2}p_2 + \frac{\partial H}{\partial p_3}p_3}{p_2^2+p_3^2}\right) dt\right) \\
x_3 &= \mp\frac{\sqrt{p_2^2+p_3^2}}{\sqrt{c_0}} \cos\left(\sqrt{c_0}\int_0^t \left(\frac{\frac{\partial H}{\partial p_2}p_2 + \frac{\partial H}{\partial p_3}p_3}{p_2^2+p_3^2}\right) dt\right)
\end{aligned}$$

Clearly this projection is onto the unit sphere as $p_1^2 + p_2^2 + p_3^2 = c_0$.

# Chapter 4

# Control systems on $\mathsf{SO}(4)$

This chapter is dedicated to the investigation of control systems on the orthogonal group $\mathsf{SO}(4)$. We begin with an introduction of the algebraic, topological and geometric properties of the Lie group $\mathsf{SO}(4)$ and its Lie algebra $\mathfrak{so}(4)$. A characterisation of the adjoint (and thus the coadjoint) orbits is given, as well as the relations between the three isomorphic Lie algebras $\mathfrak{su}(2) \times \mathfrak{su}(2) \cong \mathfrak{so}(4) \cong \mathfrak{so}(3) \times \mathfrak{so}(3)$. We also classify, under $\mathfrak{L}$-equivalence, all left-invariant control affine systems on the group $\mathsf{SO}(4)$.

## 4.1 Preliminaries

The orthogonal Lie group $\mathsf{SO}(4)$ is given by

$$\mathsf{SO}(4) = \left\{ g \in \mathsf{GL}(4, \mathbb{R}) \;:\; g^\top g = g g^\top = \mathbf{1},\; \det g = 1 \right\}.$$

This group is a (real) six-dimensional, semi-simple, compact Lie group. Its Lie algebra is given by

$$\mathfrak{so}(4) = \left\{ A \in \mathbb{R}^{4\times 4} \;:\; A + A^\top = \mathbf{0} \right\}$$

which has **standard basis**

$$E_1^s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad E_2^s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \qquad E_3^s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E_4^s = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad E_5^s = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad E_6^s = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The basis elements satisfy the commutator relations given in Table 4.1.

| | $E_1^s$ | $E_2^s$ | $E_3^s$ | $E_4^s$ | $E_5^s$ | $E_6^s$ |
|---|---|---|---|---|---|---|
| $E_1^s$ | 0 | $E_3^s$ | $-E_2^s$ | 0 | $E_6^s$ | $-E_5^s$ |
| $E_2^s$ | $-E_3^s$ | 0 | $E_1^s$ | $-E_6^s$ | 0 | $E_4^s$ |
| $E_3^s$ | $E_2^s$ | $-E_1^s$ | 0 | $E_5^s$ | $-E_4^s$ | 0 |
| $E_4^s$ | 0 | $E_6^s$ | $-E_5^s$ | 0 | $E_3^s$ | $-E_2^s$ |
| $E_5^s$ | $-E_6^s$ | 0 | $E_4^s$ | $-E_3^s$ | 0 | $E_1^s$ |
| $E_6^s$ | $E_5^s$ | $-E_4^s$ | 0 | $E_2^s$ | $-E_1^s$ | 0 |

Table 4.1: Commutator relations for the standard basis of $\mathfrak{so}(4)$

**Relation to** $\mathfrak{so}(3) \times \mathfrak{so}(3)$

Let

$$\mathbf{E}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{E}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad \mathbf{E}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

be the standard (ordered) basis for $\mathfrak{so}\,(3)$. The map $\varsigma : \mathfrak{so}\,(3) \oplus \mathfrak{so}\,(3) \to \mathfrak{so}\,(4)$, given by

$$\left( \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix} \right) \mapsto \frac{1}{2} \begin{bmatrix} 0 & -x_1+y_1 & -x_2+y_2 & -x_3+y_3 \\ x_1-y_1 & 0 & -x_3-y_3 & x_2+y_2 \\ x_2-y_2 & x_3+y_3 & 0 & -x_1-y_1 \\ x_3-y_3 & -x_2-y_2 & x_1+y_1 & 0 \end{bmatrix}$$

is a Lie algebra isomorphism. The **natural basis** of $\mathfrak{so}\,(4)$ is given by

$$\begin{aligned} E_i &= \varsigma \cdot (\mathbf{E}_i, \mathbf{0}) && i = 1, 2, 3 \\ E_j &= \varsigma \cdot (\mathbf{0}, \mathbf{E}_{j-3}) && j = 4, 5, 6. \end{aligned}$$

Explicitly, we have

$$E_1 = \begin{bmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} \qquad E_5 = \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \end{bmatrix} \qquad E_6 = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \end{bmatrix}.$$

This choice of basis proves to be the most convenient for our investigation. The commutator relations for the natural basis are given below (see Table 4.2).

| | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ | $E_6$ |
|---|---|---|---|---|---|---|
| $E_1$ | $0$ | $E_3$ | $-E_2$ | $0$ | $0$ | $0$ |
| $E_2$ | $-E_3$ | $0$ | $E_1$ | $0$ | $0$ | $0$ |
| $E_3$ | $E_2$ | $-E_1$ | $0$ | $0$ | $0$ | $0$ |
| $E_4$ | $0$ | $0$ | $0$ | $0$ | $E_6$ | $-E_5$ |
| $E_5$ | $0$ | $0$ | $0$ | $-E_6$ | $0$ | $E_4$ |
| $E_6$ | $0$ | $0$ | $0$ | $E_5$ | $-E_4$ | $0$ |

Table 4.2: Commutator relations for the natural basis of $\mathfrak{so}(4)$

4.1.1 PROPOSITION. *The map* $\psi_{ns} : \mathfrak{so}(4) \to \mathfrak{so}(4)$ *given by*

$$\psi_{ns}(x_1E_1 + x_2E_2 + x_3E_3 + x_4E_4 + x_5E_5 + x_6E_6)$$

$$\mapsto \frac{1}{2}\left((x_1 + x_4)E_1^s + (x_2 + x_5)E_2^s + (x_3 + x_6)E_3^s + (x_1 - x_4)E_4^s + (x_2 - x_5)E_5^s + (x_3 - x_6)E_6^s\right)$$

*is a Lie algebra isomorphism between the natural and standard basis of* $\mathfrak{so}(4)$.

**Relation to $\mathfrak{su}(2) \times \mathfrak{su}(2)$**

Let

$$F_1 = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad F_2 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad F_3 = \frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

be the standard (ordered) basis for the Lie algebra $\mathfrak{su}(2)$. The commutator relations are given by

$$[F_1, F_2] = F_3, \quad [F_2, F_3] = F_1, \quad [F_3, F_1] = F_2.$$

The map $\varsigma_2 : \mathfrak{so}(4) \to \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ given by

$$\begin{bmatrix} 0 & -b_1 & -b_2 & -b_3 \\ b_1 & 0 & -a_3 & a_2 \\ b_2 & a_3 & 0 & -a_1 \\ b_3 & -a_2 & a_1 & 0 \end{bmatrix} \mapsto \left(\frac{1}{2}\begin{bmatrix} (a_1 + b_1)i & (a_2 + b_2) + (a_3 + b_3)i \\ -(a_2 + b_2) + (a_3 + b_3)i & -(a_1 + b_1)i \end{bmatrix},\right.$$

$$\left.\frac{1}{2}\begin{bmatrix} (a_1 - b_1)i & (a_2 - b_2) + (a_3 - b_3)i \\ -(a_2 - b_2) + (a_3 - b_3)i & -(a_1 - b_1)i \end{bmatrix}\right)$$

where $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$, is a Lie algebra isomorphism. Again, the natural basis for $\mathfrak{so}(4)$ can be given by

$$E_i = \varsigma_2^{-1} \cdot (F_i, \mathbf{0}) \qquad i = 1, 2, 3$$

$$E_j = \varsigma_2^{-1} \cdot (\mathbf{0}, F_{j-3}) \qquad j = 4, 5, 6.$$

4.1.2 PROPOSITION. *The following Lie algebras are isomorphic*

$$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \cong \mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

*In fact,* $\varsigma_2 \circ \varsigma : \mathfrak{so}(3) \oplus \mathfrak{so}(3) \to \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ *is such that*

$$\begin{aligned} \varsigma_2 \circ \varsigma \cdot (\mathbf{E}_i, \mathbf{0}) &= (F_i, \mathbf{0}), \quad i = 1, 2, 3 \\ \varsigma_2 \circ \varsigma \cdot (\mathbf{0}, \mathbf{E}_i) &= (\mathbf{0}, F_i), \quad i = 1, 2, 3. \end{aligned}$$

These relations will prove useful in the investigation of control systems on $\mathsf{SO}(4)$.

4.1.3 REMARK. From this point on when we refer to the basis of $\mathfrak{so}(4)$ will mean the natural basis. We will be explicit when referring to the standard basis of $\mathfrak{so}(4)$.

In this chapter we will be considering the following types of optimal control problems. Let $\Sigma = (\mathsf{SO}(4), \Xi)$ be a left-invariant control affine system. An optimal control problem on $\mathsf{SO}(4)$ is given by

$$\begin{aligned} &\dot{g}(t) = g(t)\,\Xi(\mathbf{1}, u(t)), \quad g(\cdot) : [0, T] \to \mathsf{SO}(4),\ u(\cdot) : [0, T] \to \mathbb{R}^\ell \\ &g(0) = g_0, \quad g(T) = g_1, \quad g_0, g_1 \in \mathsf{SO}(4), \quad T > 0 \text{ fixed} \\ &\mathcal{J} = \int_0^T u(t)^\top Q u(t) dt \to \min. \end{aligned} \tag{4.1}$$

Here $Q$ is a positive definite $\ell \times \ell$ matrix. These systems are different from the ones considered on $\mathsf{SO}(3)$ as here we only consider quadratic cost functions, i.e., not quadratic affine cost functions (here $\mu = 0$). When considering optimal control problems on $\mathsf{SO}(4)$ we will also restrict ourselves to the case when the underlying control system is homogeneous. Under these restrictions we cannot consider a general classification of cost-equivalent systems, even when restricting ourselves to homogeneous control systems. However, our interest is primarily in the investigation of quadratic Hamilton-Poisson systems on $\mathfrak{so}(4)^*_-$. This investigation can be done without considering the cost equivalence of cost-extended systems on $\mathsf{SO}(4)$.

### Automorphisms

4.1.4 LEMMA. *The group of inner automorphisms of* $\mathfrak{so}(4)$ *is given by*

$$\mathsf{Int}(\mathfrak{so}(4)) = \left\{ \begin{bmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{bmatrix} : \psi_1, \psi_2 \in \mathsf{SO}(3) \right\}.$$

For convenience we will identify an inner automorphism $\psi = \begin{bmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{bmatrix}$ with the pair $(\psi_1, \psi_2)$.

4.1.5 PROPOSITION. *The group* $\mathsf{Aut}(\mathfrak{so}(4))$ *is generated by* $\mathsf{Int}(\mathfrak{so}(4))$ *and the swap automorphism* $\zeta = \begin{bmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{bmatrix}$. *Moreover, the group of automorphisms decomposes as a semi-direct product:*

$$\mathsf{Aut}(\mathfrak{so}(4)) = \mathsf{Int}(\mathfrak{so}(4)) \rtimes \{\mathbf{1}, \zeta\}.$$

PROOF. Let $M \in \mathsf{Aut}(\mathfrak{so}(4))$. We show that there exist $N_1, \ldots, N_k \in \mathsf{Int}(\mathfrak{so}(4)) \cup \zeta$ such that $N_1 \cdots N_k \cdot M = \mathbf{1}$. Write

$$M = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\ e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \end{bmatrix}.$$

There exists a rotation $\psi_1 \in \mathsf{SO}(3)$ such that $\psi_1 \cdot \begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix}^\top = \begin{bmatrix} a_1' & 0 & 0 \end{bmatrix}^\top$ with $a_1' \geq 0$. There exists another rotation $\psi_1'$, preserving $\begin{bmatrix} a_1' & 0 & 0 \end{bmatrix}^\top$, such that $\psi_1' \cdot \begin{bmatrix} a_2 & b_2 & c_2 \end{bmatrix}^\top = \begin{bmatrix} a_2' & b_2' & 0 \end{bmatrix}^\top$ with $b_2' \geq 0$. Therefore the top-left block of $(\psi_1', \mathbf{1}) \cdot (\psi_1, \mathbf{1}) \cdot M$ is

$$\begin{bmatrix} a_1' & a_2' & a_3' \\ 0 & b_2' & b_3' \\ 0 & 0 & c_3' \end{bmatrix}.$$

Similarly, by the application of an automorphism (preserving the top-left block), the entries $e_4$, $f_4$, $f_5$ can be made zero and the entries $d_4$, $e_5$ can be made nonnegative. Thus there exists an automorphism $N_1$ such that $N_1 \cdot M = M'$, where

$$M' = \begin{bmatrix} a_1' & a_2' & a_3' & a_4' & a_5' & a_6' \\ 0 & b_2' & b_3' & b_4' & b_5' & b_6' \\ 0 & 0 & c_3' & c_4' & c_5' & c_6' \\ d_1' & d_2' & d_3' & d_4' & d_5' & d_6' \\ e_1' & e_2' & e_3' & 0 & e_5' & e_6' \\ f_1' & f_2' & f_3' & 0 & 0 & f_6' \end{bmatrix}$$

and $a_1', b_2', d_4', e_5' \geq 0$.

As $M'$ is a Lie algebra automorphism, $M' \cdot [E_i, E_j] = [M' \cdot E_i, M' \cdot E_j]$, $i, j = 1, \ldots, 6$. Hence $a_2'$, $a_3'$, $b_3'$, $d_5'$, $d_6'$, $e_6'$ are all zero. Also $a_1' = b_2' c_3'$, $b_2' = a_1' c_3'$, $c_3' = a_1' b_2'$, $d_4' = e_5' f_6'$, $e_5' = d_4' f_6'$, and $f_6' = d_4' e_5'$. Consequently, the diagonal entries are either all zero or all one.

Suppose the diagonal entries are all one. Then

$$M' = \begin{bmatrix} 1 & 0 & 0 & a_4' & a_5' & a_6' \\ 0 & 1 & 0 & b_4' & b_5' & b_6' \\ 0 & 0 & 1 & c_4' & c_5' & c_6' \\ d_1' & d_2' & d_3' & 1 & 0 & 0 \\ e_1' & e_2' & e_3' & 0 & 1 & 0 \\ f_1' & f_2' & f_3' & 0 & 0 & 1. \end{bmatrix}.$$

Again, we impose the condition that $M'$ preserves the Lie bracket. Simple calculations show that $M' = \mathbf{1}$.

Suppose the diagonal entries of $M'$ are all zero. Then

$$\zeta \cdot M' = \begin{bmatrix} d_4' & d_5' & d_6' & 0 & 0 & 0 \\ e_4' & e_5' & e_6' & 0 & 0 & 0 \\ f_4' & f_5' & f_6' & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1' & a_2' & a_3' \\ 0 & 0 & 0 & b_1' & b_2' & b_3' \\ 0 & 0 & 0 & c_1' & c_2' & c_3'. \end{bmatrix}.$$

A similar argument shows that there exists an automorphism $N_2$ such that $N_2 \cdot \zeta \cdot M' = \mathbf{1}$.

□

4.1.6 PROPOSITION. *We have that* $\mathsf{dAut}(\mathsf{SO}(4)) = \mathsf{Aut}(\mathfrak{so}(4))$.

PROOF. Let $q : \mathsf{SU}(2) \times \mathsf{SU}(2) \to \mathsf{SO}(4)$ be the universal covering. It then follows that $\ker(q) = (\mathbf{1},\mathbf{1}) \cup (-\mathbf{1},-\mathbf{1})$. Then, according to proposition A.1.10, it follows that if $\psi \in \mathsf{Aut}(\mathfrak{so}(4))$, then $\psi \in \mathsf{dAut}(\mathsf{SO}(4))$ if and only if $\phi(\ker q) = \ker q$. Clearly, every inner automorphism $(\psi_1, \psi_2) \in \mathsf{dAut}(\mathsf{SO}(4))$. Thus we need only check that the swap automorphism $\zeta = \begin{bmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{bmatrix} \in \mathsf{dAut}(\mathsf{SO}(4))$. Now, if we consider the standard basis of $\mathsf{su}(2)$ and the natural basis of $\mathfrak{so}(4)$, then $T_{\mathbf{1}}q = \mathbf{1}$. Also, as $\mathsf{SU}(2) \times \mathsf{SU}(2)$ is simply connected, there exists a unique $\phi$ such that $T_{\mathbf{1}}\phi = (T_{\mathbf{1}}q)^{-1} \cdot \zeta \cdot T_{\mathbf{1}}q$. Let $(A, B) \in \mathfrak{su}(2) \times \mathfrak{su}(2)$ such that $\exp((A,B)) = (\exp(A), \exp(B)) = (\mathbf{1}, \mathbf{1})$. It then follows that

$$\begin{aligned} \phi((\mathbf{1},\mathbf{1})) = \phi(\exp((A,B))) &= \exp(T_1\phi \cdot (A,B)) \\ &= \exp((B,A)) \\ &= (\exp(B), \exp(A)) \\ &= (\mathbf{1},\mathbf{1}). \end{aligned}$$

Similarly, $\phi((-\mathbf{1},-\mathbf{1})) = (-\mathbf{1},-\mathbf{1})$, which implies $\phi(\ker q) = \ker q$. Hence, $\zeta \in \mathsf{dAut}(\mathsf{SO}(4))$. □

Let $(E_1^*, E_2^*, E_3^*, E_4^*, E_5^*, E_6^*)$ denote the dual of the natural basis. We shall write an element $p = \sum_{i=1}^6 p_i E_i^* \in \mathfrak{so}(4)^*$ as $\begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \end{bmatrix}$. The group of linear Poisson automorphisms (being the dual of the Lie algebra automorphisms) are given by

$$\mathsf{Aut}(\mathfrak{so}(4)^*) = \mathsf{Int}(\mathfrak{so}(4)) \rtimes \{\mathbf{1}, \zeta\}.$$

## Adjoint orbits

It is well known that there exists two functionally independent Casimir functions for the (minus) Lie-Poisson bracket on $\mathfrak{so}(4)^*_-$. In the natural basis the Casimir functions are given by

$$C_1(p) = p_1^2 + p_2^2 + p_3^2 \quad \text{and} \quad C_2(p) = p_4^2 + p_5^2 + p_6^2.$$

As $\mathfrak{so}(4)$ is semi-simple, we can identify the Lie algebra $\mathfrak{so}(4)$ and its dual space by using the Killing form. Let $X = \sum_{i=1}^6 x_i E_i$ and $Y = \sum_{i=1}^6 y_i E_i$. We shall use the pairing given by

$$\langle X, Y \rangle = -\tfrac{1}{2}\mathcal{K}_{\mathfrak{so}(4)}(X,Y) = \sum_{i=1}^{6} x_i y_i.$$

(Given our choice of basis the Killing form has the matrix $\mathcal{K}_{\mathfrak{so}(4)} = \operatorname{diag}(-2,-2,-2,-2,-2,-2)$.) Then, each curve $p(\cdot) \in \mathfrak{so}(4)^*$ is identified with a curve $p(\cdot) \in \mathfrak{so}(4)$ via the formula $\langle p(t), X\rangle = p(t)(X)$ for all $X \in \mathfrak{so}(4)^*$. (We use the same symbol, $p(t)$, for a curve in $\mathfrak{so}(4)$ and $\mathfrak{so}(4)^*$. It will be clear from the context which is intended.) Hence, there is a one-to-one correspondence between the adjoint and coadjoint orbits of $\mathsf{SO}(4)$. Given $p \in \mathfrak{so}(4)$, let $c_1 = C_1(p)$ and $c_2 = C_2(p)$. Generic adjoint (coadjoint) orbits are then exactly the level sets $\operatorname{Orb}(p) = (C_1 \times C_2)^{-1}(c_1, c_2)$, $c_1, c_2 \geq 0$. In this basis it can readily been seen that the adjoint orbits of $\mathsf{SO}(4)$ are given as follows. (The following result is a slight modification of the one given by Birtea et. al. [27].)

4.1.7 THEOREM. ([27]) *Denote by* $\mathbb{S}^2_r$ *the sphere in* $\mathbb{R}^3$ *of radius* $r$. *If* $c_1, c_2 > 0$, *then the adjoint orbit is given by* $\operatorname{Orb}(p) = \mathbb{S}^2_{\sqrt{c_1}} \times \mathbb{S}^2_{\sqrt{c_2}}$, *and hence is regular. If either* $c_1 = 0$ *or* $c_2 = 0$, *then the adjoint orbit is given by either* $\{0\} \times \mathbb{S}^2_{\sqrt{c_2}}$ *or* $\mathbb{S}^2_{\sqrt{c_1}} \times \{0\}$, *respectively, and so is singular. If* $c_1 = c_2 = 0$, *then the adjoint orbit is just the origin of* $\mathfrak{so}(4)$ *and so is singular.*

### Closed form solutions via $\mathsf{SU}(2) \times \mathsf{SU}(2)$

The results in this section follow closely that of Biggs and Holderbaum (see [19, 34, 36]). In this section we present an approach for obtaining the solutions $g(t) \in \mathsf{SO}(4)$ corresponding to the optimal control problem (4.1). Using the fact that $\mathsf{SU}(2) \times \mathsf{SU}(2)$ is a double cover of $\mathsf{SO}(4)$ we can decouple the system on $\mathsf{SO}(4)$ into two lower-dimensional systems. The solutions can then be computed for each of these lower-dimensional systems. The solutions of the decoupled systems can then be projected back to the original system on $\mathsf{SO}(4)$. We now show how this coupling can be achieved.

Using the pairing in the previous section, $\langle \cdot, \cdot \rangle = -\frac{1}{2}\mathcal{K}_{\mathfrak{so}(4)}(\cdot, \cdot)$, it follows that the extremal equations can be expressed in the dual form

$$\dot{p}(t) = [p(t), \mathbf{d}H(t)] \tag{4.2}$$

and the equations of motion on the group can be expressed using the equation

$$\frac{dg(t)}{dt} = g(t)\mathbf{d}H(t) = g(t)\Xi(\mathbf{1}, u(t)). \tag{4.3}$$

The equations (4.2) and (4.3) are what are known as the Lax-pair equations. These equations can be used to represent the equations for the Riemannian, sub-Riemannian, elastic and mechanical problems, where each problem differs by the appropriate left-invariant Hamiltonian $H$. On $\mathsf{SO}(4)$ it follows that (in our choice basis)

$$\mathbf{d}H_{\mathfrak{so}(4)} = \frac{1}{2}\begin{bmatrix} 0 & -\frac{\partial H}{\partial p_1} + \frac{\partial H}{\partial p_4} & -\frac{\partial H}{\partial p_2} + \frac{\partial H}{\partial p_5} & -\frac{\partial H}{\partial p_3} + \frac{\partial H}{\partial p_6} \\ \frac{\partial H}{\partial p_1} - \frac{\partial H}{\partial p_4} & 0 & -\frac{\partial H}{\partial p_3} - \frac{\partial H}{\partial p_6} & \frac{\partial H}{\partial p_2} + \frac{\partial H}{\partial p_5} \\ \frac{\partial H}{\partial p_2} - \frac{\partial H}{\partial p_5} & \frac{\partial H}{\partial p_3} + \frac{\partial H}{\partial p_6} & 0 & -\frac{\partial H}{\partial p_1} - \frac{\partial H}{\partial p_4} \\ \frac{\partial H}{\partial p_3} - \frac{\partial H}{\partial p_6} & -\frac{\partial H}{\partial p_2} - \frac{\partial H}{\partial p_5} & \frac{\partial H}{\partial p_1} + \frac{\partial H}{\partial p_4} & 0 \end{bmatrix}$$

$$p(t) = \frac{1}{2}\begin{bmatrix} 0 & -p_1 + p_4 & -p_2 + p_5 & -p_3 + p_6 \\ p_1 - p_4 & 0 & -p_3 - p_6 & p_2 + p_5 \\ p_2 - p_5 & p_3 + p_6 & 0 & -p_1 - p_4 \\ p_3 - p_6 & -p_2 - p_5 & p_1 + p_4 & 0 \end{bmatrix}$$

Recall that for $\mathfrak{su}(2)$ we have

$$\mathbf{d}H_{\mathfrak{su}(2)} = \frac{1}{2}\begin{bmatrix} i\frac{\partial H}{\partial p_1} & \frac{\partial H}{\partial p_2} + i\frac{\partial H}{\partial p_3} \\ -\frac{\partial H}{\partial p_2} + i\frac{\partial H}{\partial p_3} & -i\frac{\partial H}{\partial p_1} \end{bmatrix}, \qquad p(t) = \frac{1}{2}\begin{bmatrix} ip_1 & p_2 + ip_3 \\ -p_2 + ip_3 & -ip_1 \end{bmatrix}.$$

Then the system defined by (4.2) and (4.3) can be decoupled into a system on $\mathsf{SU}(2) \times \mathsf{SU}(2)$. Here $(g_1(t), g_2(t)) \in \mathsf{SU}(2) \times \mathsf{SU}(2)$ are the solutions of the following differential equations

$$\begin{aligned} \dot{g}_1(t) &= g_1(t)\mathbf{d}H_A, & \dot{p}(t) &= [p(t), \mathbf{d}H_A(t)] \\ \dot{g}_2(t) &= g_2(t)\mathbf{d}H_B, & \dot{p}(t) &= [p(t), \mathbf{d}H_B(t)] \end{aligned}$$

where

$$\mathbf{d}H_A = \frac{1}{2}\begin{bmatrix} i\frac{\partial H}{\partial p_1} & \frac{\partial H}{\partial p_2} + i\frac{\partial H}{\partial p_3} \\ -\frac{\partial H}{\partial p_2} + i\frac{\partial H}{\partial p_3} & -i\frac{\partial H}{\partial p_1} \end{bmatrix} \qquad \mathbf{d}H_B = \frac{1}{2}\begin{bmatrix} i\frac{\partial H}{\partial p_4} & \frac{\partial H}{\partial p_5} + i\frac{\partial H}{\partial p_6} \\ -\frac{\partial H}{\partial p_5} + i\frac{\partial H}{\partial p_6} & -i\frac{\partial H}{\partial p_4} \end{bmatrix}$$

$$p_A(t) = \frac{1}{2}\begin{bmatrix} ip_1 & p_2 + ip_3 \\ -p_2 + ip_3 & -ip_1 \end{bmatrix} \qquad p_B(t) = \frac{1}{2}\begin{bmatrix} ip_4 & p_5 + ip_6 \\ -p_5 + ip_6 & -ip_4 \end{bmatrix}.$$

The main difference between the work presented here and that of Biggs and Holderbaum [19], is that they identify the equations of motion on $\mathsf{SU}(2) \times \mathsf{SU}(2)$ (decoupled system) with the the equations of motion on $\mathsf{SO}(4)$ with respect to the standard basis on $\mathfrak{so}(4)$. This results in slightly more complex solutions for $\mathbf{d}H_A$ and $\mathbf{d}H_B$. (Their expression for $\mathbf{d}H_{\mathfrak{so}(4)}$, however, is simpler.) The choice of the natural basis is convenient here as the system is essentially already decoupled.

Consider the following set

$$S = \left\{ \frac{1}{2}\begin{bmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{bmatrix} \mid x_0, x_1, x_2, x_3 \in \mathbb{R} \right\}. \tag{4.4}$$

For any element $z \in \mathbb{R}^4$ we associate an element $Z \in S$ via the mapping:

$$\hat{z} = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \frac{1}{2}\begin{bmatrix} z_0 + iz_1 & z_2 + iz_3 \\ -z_2 + iz_3 & z_0 - iz_1 \end{bmatrix}.$$

The following result then holds.

4.1.8 THEOREM. ([35]) *The homomorphism* $\Phi : \mathsf{SU}(2) \times \mathsf{SU}(2) \to \mathsf{SO}(4)$ *is defined through the following equivalent group actions:*

$$g(t)\hat{z} = \hat{w}$$

*for* $g(t) \in \mathsf{SO}(4)$ *if and only if*

$$g_1(t) Z g_2(t)^{-1} = W$$

*where* $g_1(t), g_2(t) \in \mathsf{SU}(2)$.

Then, using the projection

$$g_1(t)Zg_2(t)^{-1} = W \mapsto \hat{w} \in \mathbb{R}^4$$

and the equivalence of the group actions, implies that the solution $g(t) \in \mathsf{SO}(4)$ can be constructed by associating each column of $g(t)$, denoted by $\hat{w}_i$, $i = 1, 2, 3, 4$, with each corresponding basis element of the orthonormal frame $\{e_1, e_2, e_3, e_4\}$ (the standard basis for $\mathbb{R}^4$). This is done using the projection $\hat{w}_i = g(t)e_i$, $i = 1, 2, 3, 4$, with each element of the orthonormal frame in $S$ (4.4). That is,

$$\frac{1}{2}g_1(t)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}g_2(t)^{-1} = W_1 \mapsto \hat{w}_1 \qquad \frac{1}{2}g_1(t)\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}g_2(t)^{-1} = W_2 \mapsto \hat{w}_2$$

$$\frac{1}{2}g_1(t)\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}g_2(t)^{-1} = W_3 \mapsto \hat{w}_3 \qquad \frac{1}{2}g_1(t)\begin{bmatrix}0 & i\\ i & 0\end{bmatrix}g_2(t)^{-1} = W_4 \mapsto \hat{w}_4$$

where

$$g(t) = \begin{bmatrix}\hat{w}_1 & \hat{w}_2 & \hat{w}_3 & \hat{w}_4\end{bmatrix}.$$

## 4.2 $\mathfrak{L}$-equivalence

### 4.2.1 Homogeneous systems

In this section we classify the homogeneous systems on $\mathsf{SO}(4)$. We may assume that $\Xi(\mathbf{1}, 0) = \mathbf{0}$, as any homogeneous system is equivalent to such a system by use of some reparametrization. We distinguish between the number $\ell$ of controls involved; this yields six types of systems. For each of these types we simplify an arbitrary system by successively applying automorphisms (as well as considering reparametrizations of the system). Finally, we verify that all the candidates for class representatives are distinct and non-equivalent. Families of representatives are typically parametrized by some vector $\boldsymbol{\alpha} = (\alpha_i)$ or some scalar $\beta$.

Any automorphism of $\mathfrak{so}(4)$ preserves the dot product $A \bullet B = \sum_{i=1}^{6} a_i b_i$. (Here $A = \sum_{i=1}^{6} a_i E_i$ and $B = \sum_{i=1}^{6} b_i E_i$.) Let $\Gamma^{\perp}$ denote the orthogonal complement of a subspace $\Gamma \subset \mathfrak{so}(4)$.

4.2.1 Lemma. *Suppose* $\Gamma, \widetilde{\Gamma}$ *are subspaces of* $\mathfrak{so}(4)$ *and* $\psi \in \mathsf{Aut}(\mathfrak{so}(4))$. *Then* $\psi \cdot \Gamma = \widetilde{\Gamma}$ *if and only if* $\psi \cdot \Gamma^{\perp} = \widetilde{\Gamma}^{\perp}$.

Proof. As $\psi$ is an isomorphism we only need show that $\psi \cdot \Gamma^{\perp} \subset \widetilde{\Gamma}^{\perp}$. Let $\psi \in \mathsf{Aut}(\mathfrak{so}(4))$, $A \in \Gamma$, and $B \in \Gamma^{\perp}$. Then, as these mappings preserve the inner product on $\mathfrak{so}(3) \times \mathfrak{so}(3)$, it follows that $0 = A \bullet B = \psi \cdot A \bullet \psi \cdot B$. Now $\psi \cdot A \in \widetilde{\Gamma}$ and thus by definition $\psi \cdot B \in \widetilde{\Gamma}^{\perp}$. Hence $\psi \cdot \Gamma^{\perp} \subset \widetilde{\Gamma}^{\perp}$. The argument for the converse is almost identical.

□

The classification of the $(6 - \ell)$-input systems therefore follows from the classification of the $\ell$-input systems. Hence, we need only classify the single-input, two-input, and three-input systems. The results for the four-input and five-input systems then follow as corollaries. (The classification for the six-input systems is trivial.)

When convenient, an $\ell$-input homogeneous system $\Sigma \,:\, u_1 \sum_{i=1}^{6} b_1^i E_i + \cdots + u_\ell \sum_{i=1}^{6} b_\ell^i E_i$ will be written (in matrix form) as

$$\Sigma\,:\,\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} b_1^1 & \dots & b_\ell^1 \\ \vdots & & \vdots \\ b_1^6 & \dots & b_\ell^6 \end{bmatrix}.$$

Here $M_1,\, M_2 \in \mathbb{R}^{3\times\ell}$. The evaluation $\psi \cdot \Xi(\mathbf{1}, u)$ then becomes a matrix multiplication. Accordingly, two $\ell$-input homogeneous systems $\Sigma\,:\,\begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ and $\Sigma'\,:\,\begin{bmatrix} M_1' \\ M_2' \end{bmatrix}$ are equivalent if and only if there exist an automorphism $\psi \in \mathsf{Aut}\,(\mathfrak{so}\,(4))$ and $K \in \mathsf{GL}\,(\ell, \mathbb{R})$ such that

$$\psi \cdot \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} K = \begin{bmatrix} M_1' \\ M_2' \end{bmatrix}.$$

($K$ corresponds to a reparametrization $\Xi(\mathbf{1}, Ku)$ of the system $\Sigma$.) More precisely, $\Sigma$ and $\Sigma'$ are equivalent if and only if there exist $R_1, R_2 \in \mathsf{SO}\,(3)$ and $K \in \mathsf{GL}\,(\ell, \mathbb{R})$ such that

$$\begin{array}{cl} & \big(R_1 M_1 K = M_1' \quad\text{and}\quad R_2 M_2 K = M_2'\big) \\ \text{or} & \big(R_1 M_2 K = M_1' \quad\text{and}\quad R_2 M_1 K = M_2'\big)\,. \end{array}$$

The *singular value decomposition* (SVD) (see e.g., [44]) turns out to be useful in classifying systems. For any matrix $M \in \mathbb{R}^{m\times n}$ of rank $r$, there exist orthogonal matrices $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$ and a diagonal matrix $D \in \mathbb{R}^{r\times r} = \operatorname{diag}(\sigma_1, \dots, \sigma_r)$ such that $M = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} V^\top$ with $\sigma_1 \geq \dots \geq \sigma_r > 0$. Specialised forms of the SVD (stated as lemmas) will be used in classifying the two-input and three-input homogeneous systems.

For each theorem and corollary, we include a remark about which representatives are controllable (i.e., have full rank). The method of determining which systems are controllable is dealt with in appendix E.

4.2.2 THEOREM. *Any single-input homogeneous system is equivalent to* $\Sigma_\beta^{(1,0)}\,:\,u(E_1 + \beta E_4)$ *for some* $0 \leq \beta \leq 1$*. Here* $\beta$ *parametrizes a family of class representatives, each different value corresponding to a distinct non-equivalent representative.*

4.2.3 REMARK. Clearly, no single-input homogeneous system is controllable.

PROOF. Let $\Sigma\,:\,\begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ be a single-input system. (Here $M_1, M_2 \in \mathbb{R}^{3\times 1}$.) We may assume that $M_1 \neq \mathbf{0}$. (If not, consider $\Sigma\,:\,\zeta \cdot \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$.) There exist $R_1, R_2 \in \mathsf{SO}\,(3)$ such that

$$R_1\, M_1\, \frac{1}{\|M_1\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad\text{and}\qquad R_2\, M_2\, \frac{1}{\|M_1\|} = \begin{bmatrix} \frac{\|M_2\|}{\|M_1\|} \\ 0 \\ 0 \end{bmatrix}.$$

(Here $\|\cdot\|$ denotes the usual Euclidean norm) . Thus $\Sigma$ is equivalent to $\Sigma'\,:\,u_1(E_1 + \frac{\|M_2\|}{\|M_1\|} E_4)$.

If $\frac{\|M_2\|}{\|M_1\|} > 1$, then we have $\zeta \cdot \left\langle E_1 + \frac{\|M_2\|}{\|M_1\|} E_4 \right\rangle = \left\langle E_1 + \frac{\|M_1\|}{\|M_2\|} E_4 \right\rangle$, and so $\Sigma$ is equivalent to $\Sigma'' : u_1(E_1 + \frac{\|M_1\|}{\|M_2\|} E_4)$. Hence $\Sigma$ is equivalent to $\Sigma_\beta^{(1,0)}$ for some $0 \leq \beta \leq 1$.

Suppose $\Sigma_\beta^{(1,0)}$ and $\Sigma_{\beta'}^{(1,0)}$ are equivalent. Then there exist $R_1, R_2 \in \mathsf{SO}(3)$ and $k \neq 0$ such that

$$\left( R_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} k = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad R_2 \begin{bmatrix} \beta \\ 0 \\ 0 \end{bmatrix} k = \begin{bmatrix} \beta' \\ 0 \\ 0 \end{bmatrix} \right)$$

$$\text{or} \qquad \left( R_1 \begin{bmatrix} \beta \\ 0 \\ 0 \end{bmatrix} k = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad R_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} k = \begin{bmatrix} \beta' \\ 0 \\ 0 \end{bmatrix} \right).$$

Therefore $|\beta| = |\beta'|$ or $|\beta\beta'| = 1$. Thus, as $0 \leq \beta, \beta' \leq 1$, we get $\beta = \beta'$.

□

4.2.4 COROLLARY. *Any five-input homogeneous system is equivalent to* $\Sigma_\beta^{(5,0)} : u_1(E_4 - \beta E_1) + u_2 E_2 + u_3 E_3 + u_4 E_5 + u_5 E_6$ *for some* $0 \leq \beta \leq 1$. *Here* $\beta$ *parametrizes a family of class representatives, each different value corresponding to a distinct non-equivalent representative.*

4.2.5 REMARK. Every five-input homogeneous system is controllable.

The following lemma will assist us in the classification of two-input systems. It is just a reformulation of the singular value decomposition (SVD).

4.2.6 LEMMA. *For any* $M \in \mathbb{R}^{3\times 2}$ *there exist orthogonal matrices* $R_1 \in \mathsf{SO}(3)$ *and* $R_2 \in \mathsf{O}(2)$ *such that* $R_1 \, M \, R_2 = \begin{bmatrix} D \\ 0 \;\; 0 \end{bmatrix}$, *where* $D = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$ *and* $a_1 \geq a_2 \geq 0$. *If* $R_1 \begin{bmatrix} D \\ 0 \;\; 0 \end{bmatrix} R_2 = \begin{bmatrix} D' \\ 0 \;\; 0 \end{bmatrix}$ *for some* $R_1 \in \mathsf{SO}(3)$ *and* $R_2 \in \mathsf{O}(2)$, *then* $D = D'$ *(provided that* $D$ *and* $D'$ *are diagonal matrices such that* $a_1 \geq a_2 \geq 0$ *and* $a_1' \geq a_2' \geq 0$*).*

4.2.7 THEOREM. *Any two-input homogeneous system is equivalent to exactly one of the systems*

$$\begin{aligned} \Sigma_1^{(2,0)} &: u_1 E_1 + u_2 E_4 \\ \Sigma_{2,\boldsymbol{\alpha}}^{(2,0)} &: u_1(E_1 + \alpha_1 E_4) + u_2(E_2 + \alpha_2 E_5) \end{aligned}$$

*for some* $\alpha_1, \alpha_2 \in \mathbb{R}$, *where* $(0 = \alpha_2 \leq \alpha_1) \vee (1 \leq \frac{1}{\alpha_2} \leq \alpha_1) \vee (0 < \alpha_2 \leq \alpha_1 < 1)$. *Here* $\boldsymbol{\alpha}$ *parametrizes a family of class representatives, each different value corresponding to a distinct non-equivalent representative.*

4.2.8 REMARK. $\Sigma_1^{(2,0)}$ is not controllable. $\Sigma_{2,\boldsymbol{\alpha}}^{(2,0)}$ is not controllable exactly when $\alpha_2 = 0$ or $\alpha_1 = \alpha_2 = 1$.

PROOF. Let $\Sigma : \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ be a two-input system. (Here $M_1, M_2 \in \mathbb{R}^{3\times 2}$.) Now either $\operatorname{rank}(M_1) = \operatorname{rank}(M_2) = 1$ or $\max\{\operatorname{rank}(\mathrm{M}_1), \operatorname{rank}(\mathrm{M}_2)\} = 2$. Suppose $\operatorname{rank}(\mathrm{M}_1) = \operatorname{rank}(\mathrm{M}_2) = 1$.

Then there exists $K \in \mathsf{GL}\,(2,\mathbb{R})$ such that

$$M_1K = \begin{bmatrix} b_1 & 0 \\ b_2 & 0 \\ b_3 & 0 \end{bmatrix} \qquad \text{and} \qquad M_2K = \begin{bmatrix} 0 & b_4 \\ 0 & b_5 \\ 0 & b_6 \end{bmatrix}.$$

Hence there exists $R_1, R_2 \in \mathsf{SO}\,(3)$ such that

$$\begin{aligned} R_1 \begin{bmatrix} b_1 & 0 \\ b_2 & 0 \\ b_3 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{b_1^2+b_2^2+b_3^2}} & 0 \\ 0 & \frac{1}{\sqrt{b_4^2+b_5^2+b_6^2}} \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \text{and} \qquad R_2 \begin{bmatrix} 0 & b_4 \\ 0 & b_5 \\ 0 & b_6 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{b_1^2+b_2^2+b_3^2}} & 0 \\ 0 & \frac{1}{\sqrt{b_4^2+b_5^2+b_6^2}} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Therefore $\Sigma$ is equivalent to $\Sigma_1^{(2,0)}$.

On the other hand, suppose $\operatorname{rank}(\mathrm{M}_1) = 2$ or $\operatorname{rank}(\mathrm{M}_2) = 2$. We may assume $\operatorname{rank}(\mathrm{M}_1) = 2$. (If not, consider $\Sigma : \zeta \cdot \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$.) There exists $R_1 \in \mathsf{SO}\,(3)$ such that $R_1M_1 = \begin{bmatrix} M_1' \\ 0 \;\; 0 \end{bmatrix}$. Hence, there exists $K \in \mathsf{GL}\,(2,\mathbb{R})$ such that $R_1\,M_1\,K = I_{2,0}$, where $I_{2,0} = \begin{bmatrix} I_2 \\ 0 \;\; 0 \end{bmatrix}$. Thus $\Sigma$ is equivalent to $\Sigma' : \begin{bmatrix} I_{2,0} \\ M_2' \end{bmatrix}$. By lemma 4.2.6, there exist $R_2 \in \mathsf{SO}\,(3)$ and $K \in \mathsf{O}\,(2)$ such that

$$\begin{bmatrix} K^{-1} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \det K \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad R_2\,M_2'\,K = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \\ 0 & 0 \end{bmatrix}$$

for some $\alpha_1 \geq \alpha_2 \geq 0$. If $\alpha_2 = 0$ or $0 \leq \alpha_2 \leq \alpha_1 < 1$, then $\Sigma$ is equivalent to $\Sigma_{2,\boldsymbol{\alpha}}^{(2,0)}$. Suppose $1 < \alpha_2 \leq \alpha_1$. Then

$$\begin{aligned} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\alpha_1} \\ \frac{1}{\alpha_2} & 0 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \text{and} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\alpha_1} \\ \frac{1}{\alpha_2} & 0 \end{bmatrix} &= \begin{bmatrix} \frac{1}{\alpha_2} & 0 \\ 0 & \frac{1}{\alpha_1} \\ 0 & 0 \end{bmatrix} \end{aligned}$$

with $0 < \frac{1}{\alpha_1} \leq \frac{1}{\alpha_2} < 1$. Thus $\Sigma$ is equivalent to $\Sigma_{2,\boldsymbol{\alpha}'}^{(2,0)}$ for some $0 < \alpha_2' \leq \alpha_1' < 1$. Suppose $\alpha_2 \leq 1 \leq \alpha_1$. If $\frac{1}{\alpha_2} \leq \alpha_1$, then we are done. If $\frac{1}{\alpha_2} > \alpha_1$, then $\Sigma$ is likewise equivalent to $\Sigma_{2,\boldsymbol{\alpha}'}^{(2,0)}$ for some $1 \leq \frac{1}{\alpha_2'} \leq \alpha_1'$.

We now verify that none of the class representatives are equivalent. As the traces of $\Sigma_1^{(2,0)}$ and $\Sigma_{2,\boldsymbol{\alpha}}^{(2,0)}$, respectively, do not generate the same subalgebra (for any $\alpha_1, \alpha_2 \in \mathbb{R}$), they cannot be equivalent. We claim that $\Sigma_{2,\boldsymbol{\alpha}}^{(2,0)}$ and $\Sigma_{2,\boldsymbol{\alpha}'}^{(2,0)}$ are equivalent only if $\boldsymbol{\alpha} = \boldsymbol{\alpha}'$. Indeed, assume there exist $R_1, R_2 \in \mathsf{SO}(3)$ and $K \in \mathsf{GL}(2,\mathbb{R})$ such that

$$R_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad R_2 \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \\ 0 & 0 \end{bmatrix} K = \begin{bmatrix} \alpha_1' & 0 \\ 0 & \alpha_2' \\ 0 & 0 \end{bmatrix}.$$

Then $K \in \mathsf{O}(2)$ and so, by lemma 4.2.6, it follows that $\alpha_1 = \alpha_1'$ and $\alpha_2 = \alpha_2'$. On the other hand, assume there exist $R_1, R_2 \in \mathsf{SO}(3)$ and $K \in \mathsf{GL}(2,\mathbb{R})$ such that

$$R_1 \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \\ 0 & 0 \end{bmatrix} K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad R_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} K = \begin{bmatrix} \alpha_1' & 0 \\ 0 & \alpha_2' \\ 0 & 0 \end{bmatrix}.$$

Then $\alpha_2 \neq 0$ and $\alpha_2' \neq 0$. Hence, we need only consider the cases: (i) $1 \leq \frac{1}{\alpha_2} \leq \alpha_1$ and $0 < \alpha_2' \leq \alpha_1' < 1$, (ii) $0 < \alpha_2 \leq \alpha_1 < 1$ and $0 < \alpha_2' \leq \alpha_1' < 1$, (iii) $1 \leq \frac{1}{\alpha_2} \leq \alpha_1$ and $1 \leq \frac{1}{\alpha_2'} \leq \alpha_1'$. Assume (i) holds. It follows that $R_1 = \begin{bmatrix} S_1 & 0 \\ 0 & \det S_1 \end{bmatrix}$ and $R_2 = \begin{bmatrix} S_2 & 0 \\ 0 & \det S_2 \end{bmatrix}$ for some $S_1, S_2 \in \mathsf{O}(2)$. Thus $K = \begin{bmatrix} \frac{1}{\alpha_1} & 0 \\ 0 & \frac{1}{\alpha_2} \end{bmatrix} S_1^{-1}$ and so $S_2 \begin{bmatrix} \frac{1}{\alpha_1} & 0 \\ 0 & \frac{1}{\alpha_2} \end{bmatrix} S_1^{-1} = \begin{bmatrix} \alpha_1' & 0 \\ 0 & \alpha_2' \end{bmatrix}$. By applying the mapping $A \mapsto AA^\top$, we get $S_2 \begin{bmatrix} \frac{1}{\alpha_1^2} & 0 \\ 0 & \frac{1}{\alpha_2^2} \end{bmatrix} S_2^\top = \begin{bmatrix} {\alpha_1'}^2 & 0 \\ 0 & {\alpha_2'}^2 \end{bmatrix}$. As $\frac{1}{\alpha_2} \geq \frac{1}{\alpha_1} \geq 0$ and $\alpha_1' \geq \alpha_2' \geq 0$, it follows that $\alpha_1^2 {\alpha_2'}^2 = 1$ and ${\alpha_1'}^2 \alpha_2^2 = 1$. Hence $\alpha_1' \geq 1$, a contradiction. Similarly, if (ii) or (iii) hold, then we arrive at a contradiction.

$\square$

4.2.9 COROLLARY. *Any four-input homogeneous system is equivalent to exactly one of the systems*

$$\begin{aligned} \Sigma_1^{(4,0)} &: u_1 E_2 + u_2 E_3 + u_3 E_5 + u_4 E_6 \\ \Sigma_{2,\boldsymbol{\alpha}}^{(4,0)} &: u_1(E_4 - \alpha_1 E_1) + u_2(E_5 - \alpha_2 E_2) + u_3 E_3 + u_4 E_6 \end{aligned}$$

*for some* $\alpha_1, \alpha_2 \in \mathbb{R}$, *where* $(0 = \alpha_2 \leq \alpha_1) \vee (1 \leq \frac{1}{\alpha_2} \leq \alpha_1) \vee (0 < \alpha_2 \leq \alpha_1 < 1)$. *Here* $\boldsymbol{\alpha}$ *parametrizes a family of class representatives, each different value corresponding to a distinct non-equivalent representative.*

4.2.10 REMARK. $\Sigma_1^{(4,0)}$ is controllable. $\Sigma_{2,\boldsymbol{\alpha}}^{(4,0)}$ is not controllable exactly when $\alpha_1 = \alpha_2 = 0$.

The following lemma will assist us in the classification of three-input systems. It is just a reformulation of the SVD.

4.2.11 LEMMA. *For any* $M \in \mathbb{R}^{3\times 3}$ *there exist* $R_1, R_2 \in \mathsf{SO}(3)$ *such that* $R_1 M R_2 = \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3)$, *where* $\alpha_1 \geq \alpha_2 \geq |\alpha_3| \geq 0$. *Moreover, if* $\operatorname{diag}(\alpha_1, \alpha_2, \alpha_3)$ *and* $\operatorname{diag}(\alpha_1', \alpha_2', \alpha_3')$ *are two such matrices and*

$$R_1 \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3) R_2 = \operatorname{diag}(\alpha_1', \alpha_2', \alpha_3')$$

*for some* $R_1, R_2 \in \mathsf{SO}(3)$, *then* $\alpha_1 = \alpha_1'$, $\alpha_2 = \alpha_2'$, *and* $\alpha_3 = \alpha_3'$.

4.2.12 THEOREM. *Any three-input homogeneous system is equivalent to exactly one of the systems*

$$\begin{aligned}\Sigma^{(3,0)}_{1,\beta} &: \ u_1(E_1+\beta E_4)+u_2E_2+u_3E_6\\ \Sigma^{(3,0)}_{2,\boldsymbol{\alpha}} &: \ u_1(E_1+\alpha_1E_4)+u_2(E_2+\alpha_2E_5)+u_3(E_3+\alpha_3E_6)\end{aligned}$$

*for some* $\alpha_1,\alpha_2,\alpha_3,\beta\in\mathbb{R}$, *where* $0\le\beta\le 1$ *and* $(0=\alpha_3\le\alpha_2\le\alpha_1)\ \vee\ (0<|\alpha_3|\le\alpha_2<1\wedge\alpha_2\le\alpha_1)\vee(\alpha_2=1\le\frac{1}{|\alpha_3|}\le\alpha_1)$. *Here* $\boldsymbol{\alpha}$ *and* $\beta$ *parametrize families of class representatives, each different value corresponding to a distinct non-equivalent representative.*

4.2.13 REMARK. $\Sigma^{(3,0)}_{1,\beta}$ is controllable exactly when $\beta>0$. $\Sigma^{(3,0)}_{2,\boldsymbol{\alpha}}$ is not controllable exactly when $\alpha_1=\alpha_2=\alpha_3=1$ or $\alpha_2=0$.

PROOF. Let $\Sigma:\begin{bmatrix}M_1\\M_2\end{bmatrix}$ be a three-input system. (Here $M_1,M_2\in\mathbb{R}^{3\times3}$.)

Clearly either $\max\{\mathrm{rank}\,(\mathrm{M}_1),\mathrm{rank}\,(\mathrm{M}_2)\}=3$ or $\max\{\mathrm{rank}\,(\mathrm{M}_1),\mathrm{rank}\,(\mathrm{M}_2)\}=2$. Suppose, $\mathrm{rank}\,(\mathrm{M}_1)=3$ or $\mathrm{rank}\,(\mathrm{M}_2)=3$. We may assume $\mathrm{rank}\,(\mathrm{M}_1)=3$. (If not, consider $\Sigma:\ \zeta\cdot\begin{bmatrix}M_1\\M_2\end{bmatrix}$.) Then there exists $K\in\mathsf{GL}\,(3,\mathbb{R})$ such that $M_1K=I_3$. Thus $\Sigma$ is equivalent to $\Sigma':\begin{bmatrix}I_3\\M_2'\end{bmatrix}$. By lemma 4.2.11, there exist $R_2,K\in\mathsf{SO}\,(3)$ such that $R_2M_2'K=\mathrm{diag}\,(\alpha_1,\alpha_2,\alpha_3)$ for some $\alpha_1\ge\alpha_2\ge|\alpha_3|\ge0$.

If $\alpha_3=0$ or $|\alpha_3|\le\alpha_2<1$ or $1=\alpha_2\le\frac{1}{|\alpha_3|}\le\alpha_1$, then we are done. Suppose $1<|\alpha_3|\le\alpha_2\le\alpha_1$ or $0<|\alpha_3|<1<\alpha_2\le\alpha_1$ . If $\alpha_3>0$, then

$$\begin{bmatrix}0&0&1\\0&1&0\\-1&0&0\end{bmatrix}\begin{bmatrix}\alpha_1&0&0\\0&\alpha_2&0\\0&0&\alpha_3\end{bmatrix}\begin{bmatrix}0&0&-\frac{1}{\alpha_1}\\0&\frac{1}{\alpha_2}&0\\\frac{1}{\alpha_3}&0&0\end{bmatrix}=I_3$$

$$\text{and}\qquad\begin{bmatrix}0&0&1\\0&1&0\\-1&0&0\end{bmatrix}\begin{bmatrix}0&0&-\frac{1}{\alpha_1}\\0&\frac{1}{\alpha_2}&0\\\frac{1}{\alpha_3}&0&0\end{bmatrix}=\begin{bmatrix}\frac{1}{\alpha_3}&0&0\\0&\frac{1}{\alpha_2}&0\\0&0&\frac{1}{\alpha_1}\end{bmatrix}.$$

If $\alpha_3<0$, then

$$\begin{bmatrix}0&0&-1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}\alpha_1&0&0\\0&\alpha_2&0\\0&0&\alpha_3\end{bmatrix}\begin{bmatrix}0&0&\frac{1}{\alpha_1}\\0&\frac{1}{\alpha_2}&0\\-\frac{1}{\alpha_3}&0&0\end{bmatrix}=I_3$$

$$\text{and}\qquad\begin{bmatrix}0&0&1\\0&1&0\\-1&0&0\end{bmatrix}\begin{bmatrix}0&0&\frac{1}{\alpha_1}\\0&\frac{1}{\alpha_2}&0\\-\frac{1}{\alpha_3}&0&0\end{bmatrix}=\begin{bmatrix}-\frac{1}{\alpha_3}&0&0\\0&\frac{1}{\alpha_2}&0\\0&0&-\frac{1}{\alpha_1}\end{bmatrix}.$$

In both cases $0<\frac{1}{\alpha_1}\le\frac{1}{\alpha_2}\le\frac{1}{|\alpha_3|}$. Thus $\Sigma$ is equivalent to some system $\Sigma^{(3,0)}_{2,\boldsymbol{\alpha}'}$ with $0<|\alpha_3'|\le\alpha_2'<1$ and $\alpha_2'\le\alpha_1'$. Likewise, if $\frac{1}{|\alpha_3|}\ge\alpha_1\ge\alpha_2=1$, then $\Sigma$ is equivalent to some system $\Sigma^{(3,0)}_{2,\boldsymbol{\alpha}'}$ with $1=\alpha_2'\le\frac{1}{|\alpha_3'|}\le\alpha_1'$.

On the other hand, suppose $\operatorname{rank}(\mathrm{M}_1) = 2$ and $\operatorname{rank}(\mathrm{M}_2) \in \{1, 2\}$. Again, a simple argument shows that $\Sigma$ is equivalent to some system $\Sigma' : \begin{bmatrix} I_{2,0} \\ M_1' \end{bmatrix}$, where $I_{2,0} = \begin{bmatrix} I_2 & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}$. If $\operatorname{rank}(M_1') = 1$, it is easy to show that $\Sigma$ is equivalent to $\Sigma_{1,0}^{(3,0)}$. Assume that $\operatorname{rank}(M_1') = 2$. Then there exist $R_1$, $R_2 \in \mathsf{SO}(3)$ and $K \in \mathsf{GL}(3, \mathbb{R})$ such that

$$R_1 I_{2,0} K = I_{2,0} \qquad \text{and} \qquad R_2 M_1' K = \begin{bmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

By the SVD there exist $S_1, S_2 \in \mathsf{O}(2)$ such that $S_2 \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} S_1 = \operatorname{diag}(\beta, 0)$ for some $\beta \geq 0$. Let $K' = \begin{bmatrix} S_1 & \mathbf{0} \\ \mathbf{0} & \det S_1 \end{bmatrix} \in \mathsf{SO}(3)$ and $R_2' = \begin{bmatrix} S_2 & \mathbf{0} \\ \mathbf{0} & \det S_2 \end{bmatrix} \in \mathsf{SO}(3)$. Now

$$(K')^{-1} I_{2,0} K' = I_{2,0} \qquad \text{and} \qquad R_2' \begin{bmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & 0 \\ 0 & 0 & 1 \end{bmatrix} K' = \begin{bmatrix} \beta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If $\beta \leq 1$, then we are done (i.e., $\Sigma$ is equivalent to $\Sigma_{1,\beta}^{(3,0)}$). Suppose that $\beta > 1$. Then $\zeta \cdot \langle E_1 + \beta E_4, E_2, E_6 \rangle = \left\langle \frac{1}{\beta} E_4 + E_1, E_5, E_3 \right\rangle$. It is a simple matter to show that there exists an automorphism $\psi$ such that $\psi \cdot \left\langle \frac{1}{\beta} E_4 + E_1, E_5, E_3 \right\rangle = \left\langle E_1 + \frac{1}{\beta} E_4, E_2, E_6 \right\rangle$. Thus $\Sigma$ is equivalent to $\Sigma_{1,\beta'}^{(3,0)}$ for some $0 \leq \beta' \leq 1$.

We now verify that none of these class representatives are equivalent. As the traces of $\Sigma_{1,\beta}^{(3,0)}$ and $\Sigma_{2,\boldsymbol{\alpha}}^{(3,0)}$, respectively, do not generate the same subalgebra (for any $\beta, \alpha_1, \alpha_2 \in \mathbb{R}$), they cannot be equivalent. Suppose two systems $\Sigma_{2,\boldsymbol{\alpha}}^{(3,0)}$ and $\Sigma_{2,\boldsymbol{\alpha}'}^{(3,0)}$, with $\alpha_1 \geq \alpha_2 \geq |\alpha_3| \geq 0$ and $\alpha_1' \geq \alpha_2' \geq |\alpha_3'| \geq 0$, are equivalent. We claim that $\boldsymbol{\alpha} = \boldsymbol{\alpha}'$. Indeed, assume there exist $R_1, R_2 \in \mathsf{SO}(3)$ and $K \in \mathsf{GL}(3, \mathbb{R})$ such that $R_1 I_3 K = I_3$ and $R_2 \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3) K = \operatorname{diag}(\alpha_1', \alpha_2', \alpha_3')$. Then, by lemma 4.2.11, it follows that $\boldsymbol{\alpha} = \boldsymbol{\alpha}'$. On the other hand, assume there exist $R_1, R_2 \in \mathsf{SO}(3)$ and $K \in \mathsf{GL}(3, \mathbb{R})$ such that $R_1 \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3) K = I_3$ and $R_2 I_3 K = \operatorname{diag}(\alpha_1', \alpha_2', \alpha_3')$. Then $\alpha_1^2 {\alpha_3'}^2 = 1$, $\alpha_2^2 {\alpha_2'}^2 = 1$ and $\alpha_3^2 {\alpha_1'}^2 = 1$. Clearly, $\alpha_3, \alpha_3' \neq 0$. Three possibilities remain, either (i) $0 < |\alpha_3| \leq \alpha_2 < 1$ and $0 < |\alpha_3'| \leq \alpha_2' < 1$, or (ii) $0 < |\alpha_3| \leq \alpha_2 < 1$ and $0 < |\alpha_3'| \leq \alpha_2' < 1 \wedge \alpha_2' \leq \alpha_1'$, or (iii) $0 < |\alpha_3| \leq \alpha_2 < 1 \wedge \alpha_2 \leq \alpha_1$ and $0 < |\alpha_3'| \leq \alpha_2' < 1 \wedge \alpha_2' \leq \alpha_1'$. Again (as in theorem 4.2.7), each case leads to a contradiction.

$\square$

4.2.14 REMARK. There is only one six-dimensional affine subspace of $\mathfrak{so}(4)$, namely $\mathfrak{so}(4)$. Therefore any six-input system is equivalent to the system $\Sigma^{(6,0)} : u_1 E_1 + u_2 E_2 + u_3 E_3 + u_4 E_4 + u_5 E_5 + u_6 E_6$. Clearly, this system is controllable.

### 4.2.2 Inhomogeneous systems

MATHEMATICA. *In this section we find it simpler to verify that the representatives obtained are distinct and nonequivalent using Mathematica. These results can be found in the following Mathematica file:*

*Thesis Mathematica\SO(4)\Lequivalence\InHomSys.nb*

We now proceed to the classification of the inhomogeneous systems on $\mathsf{SO}(4)$. This classification is, in part, based on our classification of homogeneous systems. As before, we distinguish between the number $\ell$ of controls involved; this yields five types of systems. (Clearly there are no six-input inhomogeneous systems.) Suppose $\Sigma: A + u_1B_1 + \cdots + u_\ell B_\ell$ is an inhomogeneous system. Then the corresponding homogeneous system $\widetilde{\Sigma} : u_1B_1 + \cdots + u_\ell B_\ell$ is equivalent to exactly one homogeneous class representative $\Sigma^0$. Consequently, $\Sigma$ is equivalent to a system $\Sigma'$ with parametrization map $\Xi'(\mathbf{1}, u) = A' + \Xi^0(\mathbf{1}, u)$. Such an (arbitrary) system is then further simplified by applying automorphisms preserving the trace $\Gamma^0$ of $\Sigma^0$. Accordingly, for each homogeneous class representative $\Sigma^0$, representatives for the associated class of inhomogeneous systems are identified. We will, in addition, use vectors $\boldsymbol{\varepsilon} = (\varepsilon_i)$ to parametrize class representatives.

Again, it is convenient to write the condition of equivalence in matrix form. An $\ell$-input inhomogeneous system specified by $\Sigma : \sum_{i=1}^6 a^iE_i + u_1 \sum_{i=1}^6 b_1^iE_i + \cdots + u_\ell \sum_{i=1}^6 b_\ell^iE_i$ will be written (in matrix form) as

$$\Sigma : \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} a^1 & b_1^1 & \dots & b_\ell^1 \\ \vdots & \vdots & \ddots & \vdots \\ a^6 & b_1^6 & \dots & b_\ell^6 \end{bmatrix}.$$

Here $M_1, M_2 \in \mathbb{R}^{3\times(\ell+1)}$. Two $\ell$-input inhomogeneous systems $\Sigma : \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ and $\Sigma' : \begin{bmatrix} M_1' \\ M_2' \end{bmatrix}$ are equivalent if and only if there exist an automorphism $\psi \in \mathsf{Aut}(\mathfrak{so}(4))$ and $K \in \mathsf{Aff}(\mathbb{R}^\ell)$ such that

$$\psi \cdot \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} K = \begin{bmatrix} M_1' \\ M_2' \end{bmatrix}.$$

For an inhomogeneous system $\Sigma : A + u_1B_1 + \cdots + u_\ell B_\ell$, with $A = \sum_{i=1}^6 \varepsilon_iE_i$, it follows that $\sum_{i=1}^6 \varepsilon_i^2 \neq 0$. We omit this condition in the statements of the theorems throughout this section. A proof sketch is provided for theorem 4.2.15 to elucidate the approach used in the classification of inhomogeneous systems. More details are provided in the proof of theorems 4.2.17, 4.2.21, and 4.2.23. The proof of theorem 4.2.19 is similar to the other cases, however there are a much larger number of subcases. We shall therefore omit the proof of theorem 4.2.19.

4.2.15 THEOREM. *Every single-input inhomogeneous system is equivalent to exactly one of the systems*

$\Sigma_{\beta\boldsymbol{\varepsilon}}^{(1,1)} : A + u_1(E_1 + \beta E_4)$ $\qquad 0 \leq \beta \leq 1$

*(i)* $A = \varepsilon_2E_2 + \varepsilon_4E_4$ *if* $\beta = 0$ $\qquad \varepsilon_2, \varepsilon_4 \geq 0$

*(ii)* $A = \varepsilon_2E_2 + \varepsilon_4E_4 + \varepsilon_5E_5$ *if* $0 < \beta \leq 1$ $\qquad (\beta = 1 \Rightarrow \varepsilon_2 \geq \varepsilon_5)$, $\varepsilon_2, \varepsilon_4, \varepsilon_5 \geq 0$.

*Here $\beta$ and $\varepsilon$ parametrize a family of class representatives, each different value corresponding to a distinct non-equivalent representative.*

4.2.16 REMARK. If $\beta = 0$, then $\Sigma^{(1,1)}_{\beta\varepsilon}$ is not controllable. If $\beta > 0$, then $\Sigma^{(1,1)}_{\beta\varepsilon}$ is not controllable exactly when $\varepsilon_2 = 0 \vee \varepsilon_5 = 0 \vee (\varepsilon_2 = \varepsilon_5 \wedge \varepsilon_4 = 0 \wedge \beta = 1)$.

PROOF. Let $\Sigma : A + u_1 B_1$ be a single-input system. Then, by theorem 4.2.2, $\Sigma$ is equivalent to a system $\widehat{\Sigma} : \sum_{i=2}^{6} \varepsilon_i E_i + u_1(E_1 + \beta E_4)$ for some $0 \leq \beta \leq 1$. Suppose $\beta > 0$. Now

$$R_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} k = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad R_2 \begin{bmatrix} \beta \\ 0 \\ 0 \end{bmatrix} k = \begin{bmatrix} \beta \\ 0 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad R_1, R_2 \in \mathsf{SO}\,(3)$$

exactly when $k = \det S_1 = \det S_2$, $R_1 = \begin{bmatrix} \det S_1 & 0 \\ 0 & S_1 \end{bmatrix}$, $R_2 = \begin{bmatrix} \det S_2 & 0 \\ 0 & S_2 \end{bmatrix}$, and $S_1, S_2 \in \mathsf{O}\,(2)$. Accordingly, there exist $S_1, S_2 \in \mathsf{O}\,(2)$ such that

$$\begin{bmatrix} \det S_1 & 0 \\ 0 & S_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \varepsilon_2 & 0 \\ \varepsilon_3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \det S_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \varepsilon_2' & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{and} \qquad \begin{bmatrix} \det S_2 & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} \varepsilon_4 & \beta \\ \varepsilon_5 & 0 \\ \varepsilon_6 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \det S_1 \end{bmatrix} = \begin{bmatrix} \varepsilon_4' & \beta \\ \varepsilon_5' & 0 \\ 0 & 0 \end{bmatrix}$$

for some $\varepsilon_2', \varepsilon_4', \varepsilon_5' \geq 0$. Therefore $\Sigma$ is equivalent to the system $\Sigma' : \varepsilon_2' E_2 + \varepsilon_4' E_4 + \varepsilon_5' E_5 + u_1(E_1 + \beta E_4)$. Moreover, if $\beta = 1$, then $\Sigma$ can be shown to be equivalent to a system $\Sigma'' : \varepsilon_2'' E_2 + \varepsilon_5'' E_5 + u_1(E_1 + E_4)$ for some $\varepsilon_2'' \geq \varepsilon_5'' \geq 0$ using a swap automorphism $\zeta \cdot \psi = \begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix}$. Here $D = \operatorname{diag}(-1, 1, -1)$. Likewise, if $\beta = 0$, it follows that $\Sigma$ is equivalent to a system $\Sigma' : \varepsilon_2' E_2 + \varepsilon_4' E_4 + u_1 E_1$ for some $\varepsilon_2', \varepsilon_4' \geq 0$. (Again, as in the homogeneous case, one verifies that all the systems obtained are distinct and non-equivalent.) ☐

4.2.17 THEOREM. *Every two-input inhomogeneous system is equivalent to exactly one of the systems*

*(1)* $\Sigma^{(2,1)}_{1,\varepsilon} : \ \varepsilon_2 E_2 + \varepsilon_5 E_5 + u_1 E_1 + u_2 E_4$ $\qquad \varepsilon_2 \geq \varepsilon_5 \geq 0$

*(2)* $\Sigma^{(2,1)}_{2,\alpha\varepsilon} : \ A + u_1(E_1 + \alpha_1 E_4) + u_2(E_2 + \alpha_2 E_5)$

$(\alpha_1 \geq \alpha_2 = 0) \vee (1 \leq \frac{1}{\alpha_2} \leq \alpha_1) \vee (0 < \alpha_2 \leq \alpha_1 < 1)$

- *(i)* $A = \varepsilon_3 E_3 + \varepsilon_4 E_4$ *if* $\alpha_1 = \alpha_2 = 0$ $\qquad \varepsilon_3, \varepsilon_4 \geq 0$
- *(ii)* $A = \varepsilon_3 E_3 + \varepsilon_4 E_4 + \varepsilon_6 E_6$ *if* $\alpha_1 = \alpha_2 > 0$
  $(\varepsilon_3 = 0 \Rightarrow \varepsilon_6 \geq 0) \wedge (\alpha_1 = 1 \Rightarrow \varepsilon_4 \geq |\varepsilon_6| \geq 0)$, $\varepsilon_6 \in \mathbb{R}$, $\varepsilon_3, \varepsilon_4 \geq 0$
- *(iii)* $A = \varepsilon_3 E_3 + \varepsilon_4 E_4 + \varepsilon_5 E_5$ *if* $\alpha_1 > \alpha_2 = 0$ $\qquad \varepsilon_3, \varepsilon_4, \varepsilon_5 \geq 0$
- *(iv)* $A = \varepsilon_3 E_3 + \varepsilon_4 E_4 + \varepsilon_5 E_5 + \varepsilon_6 E_6$ *if* $\alpha_1 > \alpha_2 > 0$
  $(\varepsilon_3, \varepsilon_4 > 0) \vee (\varepsilon_3 > 0 \wedge \varepsilon_5 \geq 0) \vee (\varepsilon_4, \varepsilon_5 \geq 0) \vee (\varepsilon_5, \varepsilon_6 \geq 0)$, $\varepsilon_5, \varepsilon_6 \in \mathbb{R}$, $\varepsilon_3, \varepsilon_4 \geq 0$.

*Here $\boldsymbol{\alpha}$ and $\boldsymbol{\varepsilon}$ parametrize families of class representatives, each different value corresponding to a distinct non-equivalent representative.*

4.2.18 REMARK. $\Sigma^{(2,1)}_{1,\boldsymbol{\varepsilon}}$ is controllable exactly when $\varepsilon_5 \neq 0$. $\Sigma^{(2,1)}_{2,\boldsymbol{\alpha\varepsilon}}$ is not controllable exactly when $\alpha_2 = 0 \wedge (\alpha_1 = 0 \vee \varepsilon_5 = \varepsilon_6 = 0)$ or $\alpha_1 = \alpha_2 = 1 \wedge \varepsilon_4 = 0 \wedge \varepsilon_3 = \varepsilon_6$.

PROOF. Let $\Sigma : A + u_1 B_1 + u_2 B_2$ be a two-input system. Then, by theorem 4.2.7, $\Sigma$ is equivalent either to $\widehat{\Sigma}_1 : \sum_{i=1}^{6} \varepsilon_i E_i + u_1 E_1 + u_2 E_4$ or $\widehat{\Sigma}_2 : \sum_{i=3}^{6} \varepsilon_i E_i + u_1(E_1 + \alpha_1 E_4) + u_2(E_2 + \alpha_2 E_5)$. It is easy to show that $\widehat{\Sigma}_1$ is equivalent to $\Sigma^{(2,1)}_{1,\boldsymbol{\varepsilon}}$. Suppose $\Sigma$ is equivalent to $\widehat{\Sigma}_2$ and $\alpha_1 > \alpha_2 > 0$. Now

$$R_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad R_2 \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \\ 0 & 0 \end{bmatrix} N = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \\ 0 & 0 \end{bmatrix}$$

$R_1, R_2 \in \mathsf{SO}(3)$, and $N \in \mathsf{GL}(2, \mathbb{R})$ exactly when $N = S$, $R_1 = R_2 = \begin{bmatrix} S & 0 \\ 0 & \det S \end{bmatrix}$, and $S = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$, $\sigma_1, \sigma_2 \in \{-1, 1\}$. Accordingly, (a tedious but straightforward computation shows that) there exists $\sigma_1, \sigma_2 \in \{-1, 1\}$ such that

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_1\sigma_2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sigma_1 & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon_3' & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_1\sigma_2 \end{bmatrix} \begin{bmatrix} \varepsilon_4 & \alpha_1 & 0 \\ \varepsilon_5 & 0 & \alpha_2 \\ \varepsilon_6 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sigma_1 & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} \varepsilon_4' & \alpha_1 & 0 \\ \varepsilon_5' & 0 & \alpha_2 \\ \varepsilon_6' & 0 & 0 \end{bmatrix}$$

where $(\varepsilon_3', \varepsilon_4' \geq 0) \wedge ((\varepsilon_3' = 0 \vee \varepsilon_4' = 0) \Rightarrow \varepsilon_5' \geq 0) \wedge (\varepsilon_3' = \varepsilon_4' = 0 \Rightarrow (\varepsilon_5', \varepsilon_6' \geq 0)) \wedge (\varepsilon_3' = \varepsilon_5' = 0 \Rightarrow \varepsilon_6' \geq 0)$. These conditions are equivalent to those given in the theorem.

On the other hand, suppose $\Sigma$ is equivalent to $\widehat{\Sigma}_2$ and $\alpha_1 = \alpha_2 > 0$. Then

$$R_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad R_2 \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \\ 0 & 0 \end{bmatrix} N = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \\ 0 & 0 \end{bmatrix}$$

$R_1, R_2 \in \mathsf{SO}(3)$, and $N \in \mathsf{GL}(2, \mathbb{R})$ exactly when $N = S^\top$, $R_1 = R_2 = \begin{bmatrix} S & 0 \\ 0 & \det S \end{bmatrix}$, and $S \in \mathsf{O}(2)$. Therefore there exists $S \in \mathsf{O}(2)$ such that

$$\begin{bmatrix} S & 0 \\ 0 & \det S \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & S^\top \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon_3' & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} S & 0 \\ 0 & \det S \end{bmatrix} \begin{bmatrix} \varepsilon_4 & \alpha_1 & 0 \\ \varepsilon_5 & 0 & \alpha_1 \\ \varepsilon_6 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & S^\top \end{bmatrix} = \begin{bmatrix} \varepsilon_4' & \alpha_1 & 0 \\ 0 & 0 & \alpha_1 \\ \varepsilon_6' & 0 & 0 \end{bmatrix}$$

where $\varepsilon_3', \varepsilon_4' \geq 0$ and $\varepsilon_3' = 0 \Rightarrow \varepsilon_6' \geq 0$. If $\alpha_1 = 1$, then we can apply a swap automorphism to interchange the values of $\varepsilon_4$ and $\varepsilon_6$. Thus, in this case, we can assume that $\varepsilon_4 \geq \varepsilon_6$.

The (families of) equivalence representatives 2(i) and 2(iii) are obtained similarly. (Again one verifies that all the systems obtained are distinct and non-equivalent.) ☐

4.2.19 THEOREM. *Every three-input inhomogeneous system is equivalent to exactly one of the systems*

*(1)* $\Sigma^{(3,1)}_{1,\beta\boldsymbol{\varepsilon}} : A + u_1(E_1 + \beta E_4) + u_2 E_2 + u_3 E_6 \qquad 0 \leq \beta \leq 1$

- *(i)* $A = \varepsilon_3 E_3 + \varepsilon_4 E_4$ *if* $\beta = 0 \qquad \varepsilon_3, \varepsilon_4 \geq 0$
- *(ii)* $A = \varepsilon_3 E_3 + \varepsilon_4 E_4 + \varepsilon_5 E_5$ *if* $0 < \beta \leq 1$
  $(\beta = 1 \Rightarrow \varepsilon_3 \geq \varepsilon_5),\ \varepsilon_3, \varepsilon_4, \varepsilon_5 \geq 0$

*(2)* $\Sigma^{(3,1)}_{2,\boldsymbol{\alpha\varepsilon}} : A + u_1(E_1 + \alpha_1 E_4) + u_2(E_2 + \alpha_2 E_5) + u_3(E_3 + \alpha_3 E_6)$
$(0 = \alpha_3 \leq \alpha_2 \leq \alpha_1) \ \vee\ (0 < |\alpha_3| \leq \alpha_2 < 1 \ \wedge\ \alpha_2 \leq \alpha_1) \ \vee\ (\alpha_2 = 1 \leq \frac{1}{|\alpha_3|} \leq \alpha_1)$

- *(i)* $A = \varepsilon_4 E_4$ *if* $\alpha_1 = \alpha_2 = |\alpha_3| \qquad \varepsilon_4 \geq 0$
- *(ii)* $A = \varepsilon_4 E_4 + \varepsilon_5 E_5$ *if* $\alpha_1 > \alpha_2 = |\alpha_3| \qquad \varepsilon_4, \varepsilon_5 \geq 0$
- *(iii)* $A = \varepsilon_4 E_4 + \varepsilon_6 E_6$ *if* $\alpha_1 = \alpha_2 > |\alpha_3| \qquad \varepsilon_4, \varepsilon_6 \geq 0$
- *(iv)* $A = \varepsilon_4 E_4 + \varepsilon_5 E_5 + \varepsilon_6 E_6$ *if* $\alpha_1 > \alpha_2 > |\alpha_3|$
  $((\varepsilon_4 = 0 \vee \varepsilon_5 = 0) \Rightarrow \varepsilon_6 \geq 0) \wedge ((\varepsilon_6 > 0 \wedge \alpha_2 = 1 \wedge \alpha_1 = \frac{1}{|\alpha_3|}) \Rightarrow \varepsilon_4 \geq \varepsilon_6 \geq 0),\ \varepsilon_6 \in \mathbb{R},\ \varepsilon_4, \varepsilon_5 \geq 0.$

*Here* $\boldsymbol{\alpha}$, $\beta$ *and* $\boldsymbol{\varepsilon}$ *parametrize families of class representatives, each different value corresponding to a distinct non-equivalent representative.*

4.2.20 REMARK. $\Sigma^{(3,1)}_{1,\beta\boldsymbol{\varepsilon}}$ is controllable exactly when $\beta \neq 0$ or $\varepsilon_4 \neq 0$. $\Sigma^{(3,1)}_{2,\boldsymbol{\alpha\varepsilon}}$ is not controllable exactly when $\alpha_2 = 0 \wedge (\alpha_1 = 0 \vee \varepsilon_5 = 0)$.

PROOF. These representatives are obtained using the same ideas as theorems 4.2.17 and 4.2.21. However, in this case there is just a larger number of subcases. As we do not want to repeat similar calculations unnecessarily, we omit the details on how we obtained these representatives. The verification that each of these representatives are distinct and non-equivalent is included in the Mathematica file previously mentioned for this section. ☐

4.2.21 THEOREM. *Every four-input inhomogeneous system is equivalent to exactly one of the systems*

*(1)* $\Sigma^{(4,1)}_{1,\boldsymbol{\varepsilon}} : \varepsilon_1 E_1 + \varepsilon_4 E_4 + u_1 E_2 + u_2 E_3 + u_3 E_5 + u_4 E_6 \qquad \varepsilon_1 \geq \varepsilon_4 \geq 0$

*(2)* $\Sigma^{(4,1)}_{2,\boldsymbol{\alpha\varepsilon}} : A + u_1(E_4 - \alpha_1 E_1) + u_2(E_5 - \alpha_2 E_2) + u_3 E_3 + u_4 E_6$
$(\alpha_1 \geq \alpha_2 = 0) \vee (1 \leq \frac{1}{\alpha_2} \leq \alpha_1) \vee (0 < \alpha_2 \leq \alpha_1 < 1)$

- *(i)* $A = \varepsilon_1 E_1$ *if* $\alpha_1 = \alpha_2 \qquad \varepsilon_1 \geq 0$
- *(ii)* $A = \varepsilon_1 E_1 + \varepsilon_2 E_2$ *if* $\alpha_1 > \alpha_2 \qquad \varepsilon_1, \varepsilon_2 \geq 0.$

*Here* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\varepsilon}$ *parametrize families of class representatives, each different value corresponding to a distinct non-equivalent representative.*

4.2.22 REMARK. Every four-input inhomogeneous system is controllable.

PROOF. Let $\Sigma : A + \sum_{i=1}^{4} u_i B_i$ be a four-input system. Then, by corollary 4.2.9, $\Sigma$ is equivalent either to

$$\hat{\Sigma}_1 : \varepsilon_1 E_1 + \varepsilon_4 E_4 + u_1 E_2 + u_2 E_3 + u_3 E_5 + u_4 E_6$$

or

$$\hat{\Sigma}_2 : \varepsilon_1 E_1 + \varepsilon_2 E_2 + u_1(E_4 - \alpha_1 E_1) + u_2(E_5 - \alpha_2 E_2) + u_3 E_3 + u_4 E_6.$$

Consider the system $\hat{\Sigma}_1$. Let $R_1 = \operatorname{diag}(\sigma_1\sigma_2, \sigma_1, \sigma_2)$, $R_2 = \operatorname{diag}(\sigma_3\sigma_4, \sigma_3, \sigma_4)$ , and $K = \operatorname{diag}(1, \sigma_1, \sigma_2, \sigma_3, \sigma_4)$, for $\sigma_i \in \{-1, 1\}$, $i = 1, 2, 3, 4$. (Note that for any such choice of the $\sigma_i$'s, $R_1, R_2 \in \mathsf{SO}(3)$.) Then

$$\begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} \varepsilon_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \varepsilon_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} K = \begin{bmatrix} \varepsilon_1\sigma_1\sigma_2 & 0 & 0 & 0 & 0 \\ 0 & \sigma_1^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_2^2 & 0 & 0 \\ \varepsilon_4\sigma_3\sigma_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_3^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_4^2 \end{bmatrix}.$$

That is, we can choose $\sigma_i$, $i = 1, 2, 3, 4$, such that $\varepsilon_1, \varepsilon_4 \geq 0$. If $\varepsilon_4 > \varepsilon_1$, applying the swap automorphism $\zeta$ along with the reparametrization

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

allows us to swap $\varepsilon_1$ and $\varepsilon_4$. Thus $\hat{\Sigma}_1$ is equivalent to one of the systems $\Sigma_{1,\varepsilon}^{(4,1)}$. It is then straightforward to verify that each choice of $\varepsilon_1 \geq \varepsilon_4 \geq 0$ leads to a distinct non-equivalent representative.

Now consider the system $\hat{\Sigma}_2$. Assume $\alpha_1 > \alpha_2$. Let $R_1 = R_2 = \operatorname{diag}(\sigma_1, \sigma_2, \sigma_1\sigma_2)$ and $K = \operatorname{diag}(1, \sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_1\sigma_2)$, for $\sigma_1, \sigma_2 \in \{-1, 1\}$. In a very similar way as above, we can then choose suitable values of $\sigma_1$ and $\sigma_2$ such that any system $\hat{\Sigma}_2$ is equivalent to a system $\Sigma_{2,\alpha\varepsilon}^{(4,1)}$ with $\varepsilon_1, \varepsilon_2 \geq 0$. On the other hand, assume $\alpha_1 = \alpha_2$. Let

$$R_1 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 & 0 \\ 0 & \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Clearly, $R_1 \in \mathsf{SO}(3)$ and $\det(K) = 1$. It follows that

$$\begin{bmatrix} R_1 & 0 \\ 0 & R_1 \end{bmatrix} \begin{bmatrix} \varepsilon_1 & -\alpha_1 & 0 & 0 & 0 \\ \varepsilon_2 & 0 & -\alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} K = \begin{bmatrix} \varepsilon_1\cos\theta + \varepsilon_2\sin\theta & -\alpha_1 & 0 & 0 & 0 \\ \varepsilon_2\cos\theta - \varepsilon_1\sin\theta & 0 & -\alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, there exists a $\theta \in \mathbb{R}$ such that $\varepsilon_2 \cos\theta - \varepsilon_1 \sin\theta = 0$. Hence, $\hat{\Sigma}_2$ is equivalent to a system $\Sigma^{(4,1)}_{2,\alpha\varepsilon'}$ (with $\alpha_1 = \alpha_2$) for some $\epsilon_1' > 0$. Again it is straightforward to verify that each choice of $\varepsilon_1 > 0$ leads to a distinct non-equivalent representative.

□

4.2.23 THEOREM. *Every five-input inhomogeneous system is equivalent to exactly one of the systems*

$$\Sigma^{(5,1)}_{\beta\varepsilon} : \varepsilon_1 E_1 + u_1(E_4 - \beta E_1) + u_2 E_2 + u_3 E_3 + u_4 E_5 + u_5 E_6 \qquad 0 \leq \beta \leq 1,\ \varepsilon_1 \geq 0.$$

*Here $\beta$ and $\varepsilon$ parametrize families of class representatives, each different value corresponding to a distinct non-equivalent representative.*

4.2.24 REMARK. Clearly, as every five-input homogeneous system is controllable, so is every five-input inhomogeneous system.

PROOF. The proof of this theorem is very to similar to that of theorem 4.2.21. Clearly any five-input inhomogeneous system is equivalent to a system

$$\hat{\Sigma} : \varepsilon_1 E_1 + u_1(E_4 - \beta E_1) + u_2 E_2 + u_3 E_3 + u_4 E_5 + u_5 E_6$$

for some $\varepsilon_1 \neq 0$. Using a similar automorphism as in theorem 4.2.21 we can always make the sign of $\varepsilon_1$ positive. It is then straightforward to verify that each choice of $\varepsilon_1 > 0$ leads to a distinct non-equivalent representative.

□

As a simple by-product of the classification of homogeneous systems, we recover a classification of subalgebras of $\mathfrak{so}(4)$. (Two subalgebras $\mathfrak{a}_1, \mathfrak{a}_2 \subset \mathfrak{so}(4)$ are equivalent if there exists $\psi \in \mathsf{Aut}(\mathfrak{so}(4))$ such that $\psi \cdot \mathfrak{a}_1 = \mathfrak{a}_2$). Any (non-trivial) subalgebra of $\mathfrak{so}(4)$ is equivalent to exactly one of the following subalgebras

$$\begin{aligned}
\mathfrak{a}^{(1)}_{\alpha} &= \langle E_1 + \alpha E_4 \rangle & &= \varsigma \cdot \langle (\mathbf{E}_1, \alpha \mathbf{E}_1) \rangle \\
\mathfrak{a}^{(2)} &= \langle E_1, E_4 \rangle & &= \varsigma \cdot \langle \mathbf{E}_1 \rangle \oplus \langle \mathbf{E}_1 \rangle \\
\mathfrak{a}^{(3)}_1 &= \langle E_1, E_2, E_3 \rangle & &= \varsigma \cdot \mathfrak{so}(3) \oplus \{\mathbf{0}\} \\
\mathfrak{a}^{(3)}_2 &= \langle E_1 + E_4, E_2 + E_5, E_4 + E_6 \rangle & &= \varsigma \cdot \{(A, A) \,:\, A \in \mathfrak{so}(3)\} \\
\mathfrak{a}^{(4)} &= \langle E_1, E_2, E_3, E_4 \rangle & &= \varsigma \cdot \mathfrak{so}(3) \oplus \langle \mathbf{E}_1 \rangle .
\end{aligned}$$

Here $0 \leq \alpha \leq 1$ parametrizes a family of nonequivalent class representatives. (Only $\mathfrak{a}^{(3)}_1$ is an ideal.)

| **Single-input** | | |
|---|---|---|
| | $\Xi^0(\mathbf{1},u)$ | $A$ |
| $\Sigma^{(1,1)}_{\beta\varepsilon}$ | $\begin{bmatrix}1\\0\\0\\\beta\\0\\0\end{bmatrix}$ | $\begin{bmatrix}0\\\varepsilon_2\\0\\\varepsilon_4\\0\\0\end{bmatrix}$ $\beta=0$ <br> $\begin{bmatrix}0\\\varepsilon_2\\0\\\varepsilon_4\\\varepsilon_5\\0\end{bmatrix}$ $0<\beta\leq 1$ |

| **Two-input** | | |
|---|---|---|
| | $\Xi^0(\mathbf{1},u)$ | $A$ |
| $\Sigma^{(2,1)}_{1,\varepsilon}$ | $\begin{bmatrix}1&0\\0&0\\0&0\\0&1\\0&0\\0&0\end{bmatrix}$ | $\begin{bmatrix}0\\\varepsilon_2\\0\\0\\\varepsilon_5\\0\end{bmatrix}$ |
| $\Sigma^{(2,1)}_{2,\alpha\varepsilon}$ | $\begin{bmatrix}1&0\\0&1\\0&0\\\alpha_1&0\\0&\alpha_2\\0&0\end{bmatrix}$ | $\begin{bmatrix}0\\0\\\varepsilon_3\\\varepsilon_4\\0\\0\end{bmatrix}$ $\alpha_1=\alpha_2=0$ <br> $\begin{bmatrix}0\\0\\\varepsilon_3\\\varepsilon_4\\0\\\varepsilon_6\end{bmatrix}$ $\alpha_1=\alpha_2>0$ <br> $\begin{bmatrix}0\\0\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\end{bmatrix}$ $\alpha_1>\alpha_2=0$ <br> $\begin{bmatrix}0\\0\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\end{bmatrix}$ $\alpha_1>\alpha_2>0$ |

| **Three-input** | | |
|---|---|---|
| | $\Xi^0(\mathbf{1},u)$ | $A$ |
| $\Sigma^{(3,1)}_{1,\alpha\varepsilon}$ | $\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\\\beta&0&0\\0&0&0\\0&0&1\end{bmatrix}$ | $\begin{bmatrix}0\\0\\\varepsilon_3\\\varepsilon_4\\0\\0\end{bmatrix}$ $\beta=0$ <br> $\begin{bmatrix}0\\0\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\0\end{bmatrix}$ $0<\beta\leq 1$ |
| $\Sigma^{(3,1)}_{2,\alpha\varepsilon}$ | $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\\alpha_1&0&0\\0&\alpha_2&0\\0&0&\alpha_3\end{bmatrix}$ | $\begin{bmatrix}0\\0\\0\\\varepsilon_4\\0\\0\end{bmatrix}$ $\alpha_1=\alpha_2=\lvert\alpha_3\rvert$ <br> $\begin{bmatrix}0\\0\\0\\\varepsilon_4\\\varepsilon_5\\0\end{bmatrix}$ $\alpha_1>\alpha_2=\lvert\alpha_3\rvert$ <br> $\begin{bmatrix}0\\0\\0\\\varepsilon_4\\0\\\varepsilon_6\end{bmatrix}$ $\alpha_1=\alpha_2>\lvert\alpha_3\rvert$ <br> $\begin{bmatrix}0\\0\\0\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\end{bmatrix}$ $\alpha_1>\alpha_2>\lvert\alpha_3\rvert$ |

Table 4.3: Classification of systems on $\mathsf{SO}\,(4)$ in matrix form (the homogeneous systems correspond to $A=0$)

| **Four-input** | | |
|---|---|---|
| | $\Xi^0(\mathbf{1},u)$ | $A$ |
| $\Sigma^{(4,1)}_{1,\varepsilon}$ | $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} \varepsilon_1 \\ 0 \\ 0 \\ \varepsilon_4 \\ 0 \\ 0 \end{bmatrix}$ |
| $\Sigma^{(4,1)}_{2,\alpha\varepsilon}$ | $\begin{bmatrix} -\alpha_1 & 0 & 0 & 0 \\ 0 & -\alpha_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ | $\underset{\alpha_1=\alpha_2}{\begin{bmatrix} \varepsilon_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$ $\underset{\alpha_1>\alpha_2}{\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$ |

| **Five-input** | | |
|---|---|---|
| | $\Xi^0(\mathbf{1},u)$ | $A$ |
| $\Sigma^{(5,1)}_{\beta\varepsilon}$ | $\begin{bmatrix} -\beta & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} \varepsilon_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ |

| **Six-input** | |
|---|---|
| | $\Xi^0(\mathbf{1},u)$ |
| $\Sigma^{(6,0)}_{\beta\varepsilon}$ | $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ |

$$\Sigma \ : \ A + \Xi^0(\mathbf{1},u) \qquad \Xi^0(\mathbf{1},u) = u_1B_1 + \cdots + u_\ell B_\ell \longleftrightarrow [B_1 \quad \cdots \quad B_\ell]$$

Table 4.4: Classification of systems on $\mathsf{SO}\,(4)$ in matrix form (the homogeneous systems correspond to $A = 0$), cont.

## 4.3 Linear equivalence of Hamilton-Poisson systems on $\mathfrak{so}(4)^*_-$

In this section we present some results on the classification of quadratic homogeneous Hamilton-Poisson systems on $\mathfrak{so}(4)$, under linear equivalence. We were unable to obtain a complete classification in this case, unlike for $\mathfrak{so}(3)$. This is primarily due to the following. In general, we need to find a linear isomorphism $\psi : \mathfrak{so}(4)^* \to \mathfrak{so}(4)^*$ such that $T\psi \cdot \vec{G} = \vec{H} \circ \psi$. In this case, however, $\psi \in \mathbb{R}^{6\times 6}$ and even with MATHEMATICA the number of equations that need to be solved are just too many to manage. We are able to obtain a partial classification using the properties of lemma 2.2.24.

4.3.1 LEMMA. *Every homogeneous quadratic Hamilton-Poisson system* $H_Q(p) = p\,Q\,p^\top$ *(where* $Q$ *is a* $6\times 6$ *positive semi-definite matrix) is equivalent to one of the systems*

$$H_{\boldsymbol{\alpha}\boldsymbol{\beta}X}(p) = p\begin{bmatrix} D_{\boldsymbol{\alpha}} & X^\top \\ X & D_{\boldsymbol{\beta}} \end{bmatrix} p^\top, \qquad D_{\boldsymbol{\gamma}} = \operatorname{diag}(\gamma_1, \gamma_2, 0),\ \gamma_1 \geq \gamma_2 \geq 0, \gamma = \alpha, \beta,\ X \in \mathbb{R}^{3\times 3}.$$

PROOF. Let $H_Q(p) = pQp^\top$ be a homogeneous quadratic Hamilton-Poisson system on $\mathfrak{so}(4)^*$. We write $Q$ as

$$Q = \begin{bmatrix} X_1 & X_2^\top \\ X_2 & X_3 \end{bmatrix}$$

where $X_1, X_3 \in \operatorname{Sym}(3)$ and $X_2 \in \mathbb{R}^{3\times 3}$. Then for $\psi = (\psi_1, \psi_2) \in \mathsf{Aut}(\mathfrak{so}(4)^*)$ it follows that

$$(H \circ \psi)(p) = p\begin{bmatrix} \psi_1 X_1 \psi_1^\top & \psi_1 X_2^\top \psi_2^\top \\ \psi_2 X_2 \psi_1^\top & \psi_2 X_3 \psi_2^\top \end{bmatrix} p^\top.$$

Now, as $X_1$ and $X_3$ are symmetric it follows that they can be diagonalized by orthogonal matrices. Thus there exists $\psi_1, \psi_2 \in \mathsf{SO}(3)$ such that

$$(H \circ \psi)(p) = p\begin{bmatrix} \operatorname{diag}(\alpha_1', \alpha_2', \alpha_3') & Y^\top \\ Y & \operatorname{diag}(\beta_1', \beta_2', \beta_3') \end{bmatrix} p^\top, \quad \text{where}$$

where $Y = \psi_2 X_2 \psi_1^\top$, $\alpha_1 \geq \alpha_2 \geq 0$, and $\beta_1 \geq \beta_2 \geq 0$. It then follows that

$$(H \circ \psi)(p) - \alpha_3' C_1(p) - \beta_3' C_2(p) = p\begin{bmatrix} D_{\boldsymbol{\alpha}} & Y^\top \\ Y & D_{\boldsymbol{\beta}} \end{bmatrix} p^\top.$$

Here $D_{\boldsymbol{\gamma}} = \operatorname{diag}(\gamma_1, \gamma_2, 0)$, $\gamma_1 \geq \gamma_2 \geq 0$, $\gamma = \alpha, \beta$. ☐

Throughout the remainder of the thesis we consider several special cases of the Hamilton-Poisson system $H_{\boldsymbol{\alpha}\boldsymbol{\beta}X}$.

**Case 1:** Assume $X = \mathbf{0}$ and $\boldsymbol{\alpha}, \boldsymbol{\beta} \neq \mathbf{0}$.

4.3.2 PROPOSITION. *If* $X = \mathbf{0}$ *and* $\boldsymbol{\alpha}, \boldsymbol{\beta} \neq 0$*, then* $H_{\boldsymbol{\alpha}\boldsymbol{\beta}}$ *is equivalent to one of the systems*

$$H_{D,0}(p) = p\begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} p^\top$$

*where* $D_1, D_2 \in \{\mathbf{0}, \operatorname{diag}(\frac{1}{2}, 0, 0), \operatorname{diag}(1, \frac{1}{2}, 0)\}$. *This system is just the product of two of the homogeneous systems obtained in the classification for* $\mathfrak{so}(3)^*$.

PROOF. Let $H_{\boldsymbol{\alpha\beta}}$ (with $X = \mathbf{0}$) be a quadratic Hamilton-Poisson system on $\mathfrak{so}(4)^*$. Assume that $\alpha_1 > \alpha_2 > 0$ and $\beta_1 > \beta_2 > 0$. Then $\vec{H}_{\boldsymbol{\alpha\beta}}$ and $\vec{H}_{D,0}$, with $D_1 = D_2 = \mathrm{diag}\,(1, \frac{1}{2}, 0)$, are compatible with the linear isomorphism

$$p \mapsto p \begin{bmatrix} \psi_{\boldsymbol{\alpha}} & \mathbf{0} \\ \mathbf{0} & \psi_{\boldsymbol{\beta}} \end{bmatrix}$$

where $\psi_{\boldsymbol{\gamma}} = \sqrt{2}\,\mathrm{diag}\,(\sqrt{\gamma_1(\gamma_1 - \gamma_2)}, \sqrt{2}\sqrt{(\gamma_1 - \gamma_2)\gamma_2}, \sqrt{\gamma_1\gamma_2})$, $\gamma = \alpha, \beta$. On the other hand, assume $\alpha_2 = \beta_2 = 0$. Then $\vec{H}_{\boldsymbol{\alpha\beta}}$ and $\vec{H}_{D,0}$, with $D_1 = D_2 = \mathrm{diag}\,(1, 0, 0)$, are compatible with the linear isomorphism $p \mapsto p\,\mathrm{diag}\,(\alpha_1, 1, 1, \beta_1, 1, 1)$. The remaining cases are just combinations of these of these linear maps.

□

**Case 2:** Assume $\boldsymbol{\alpha} = \boldsymbol{\beta} = \mathbf{0}$.

4.3.3 PROPOSITION. *If $\boldsymbol{\alpha} = \boldsymbol{\beta} = 0$, then $H_X$ is equivalent to one of the systems*

$$H_{0,D}(p) = \frac{1}{2} p \begin{bmatrix} 0 & D_{\boldsymbol{\alpha}} \\ D_{\boldsymbol{\alpha}} & 0 \end{bmatrix} p^\top$$

*where $D = \mathrm{diag}\,(1, \alpha_1, \alpha_2)$, with $0 \leq |\alpha_2| \leq \alpha_1 \leq 1$.*

PROOF. For any $X \in \mathbb{R}^{3\times 3}$, it follows from lemma 4.2.11 that there exist $R_1, R_2 \in \mathsf{SO}(3)$ such that $R_1 X R_2 = \mathrm{diag}\,(\beta_1, \beta_2, \beta_3)$, where $\beta_1 \geq \beta_2 \geq |\beta_3| \geq 0$. (We shall assume that $\beta_1 \neq 0$, as otherwise we just have the trivial system $H = \mathbf{0}$.) Thus let $\psi = (R_2^\top, R_1) \in \mathsf{Aut}(\mathfrak{so}(4)^*)$. Then

$$\begin{aligned} (H_X \circ \psi)(p) &= p \begin{bmatrix} 0 & R_2^\top X^\top R_1^\top \\ R_1 X R_2 & 0 \end{bmatrix} p^\top \\ &= p \begin{bmatrix} 0 & \mathrm{diag}\,(\beta_1, \beta_2, \beta_3) \\ \mathrm{diag}\,(\beta_1, \beta_2, \beta_3) & 0 \end{bmatrix} p^\top. \end{aligned}$$

It then follows that

$$H_{0,D}(p) = \frac{1}{2\beta_1}(H_X \circ \psi)(p) = \frac{1}{2} p \begin{bmatrix} 0 & D_{\boldsymbol{\alpha}} \\ D_{\boldsymbol{\alpha}} & 0 \end{bmatrix} p^\top$$

where $1 \geq \alpha_1 \geq |\alpha_2| \geq 0$. Here $\alpha_1 = \frac{\beta_2}{\beta_1}$ and $\alpha_2 = \frac{\beta_3}{\beta_1}$. Note that these two operations correspond to $(\mathfrak{C}1)$ and $(\mathfrak{C}2)$ of lemma 2.2.24. Thus any system $H_X$ (for $\boldsymbol{\alpha} = \boldsymbol{\beta} = 0$) is equivalent to the system $H_{0,D}$ (for some $1 \geq \alpha_1 \geq |\alpha_2| \geq 0$).

□

**Case 3:** Assume either $\boldsymbol{\alpha} = \mathbf{0}$ or $\boldsymbol{\beta} = \mathbf{0}$.

4.3.4 PROPOSITION. *If either $\boldsymbol{\alpha} = \mathbf{0}$ or $\boldsymbol{\beta} = \mathbf{0}$, then $H_{\boldsymbol{\alpha\beta}X}$ is equivalent to one of the systems*

$$H_{D,U}(p) = p \begin{bmatrix} D_{\boldsymbol{\alpha}} & U^\top \\ U & \mathbf{0} \end{bmatrix} p^\top, \quad \textit{where} \quad U = \begin{bmatrix} y_1 & y_2 & y_3 \\ 0 & y_4 & y_5 \\ 0 & 0 & y_6 \end{bmatrix}, \; D = \mathrm{diag}\,(\alpha_1, \alpha_2, 0)$$

*for some $y_1, \ldots, y_6 \in \mathbb{R}$, where $y_1, y_2 \geq 0$, and $\alpha_1 \geq \alpha_2 \geq 0$.*

PROOF. Assume either $\boldsymbol{\alpha} = \mathbf{0}$ or $\boldsymbol{\beta} = \mathbf{0}$. In fact we can assume that $\boldsymbol{\beta} = \mathbf{0}$ as we can use the swap automorphism to swap $D_{\boldsymbol{\alpha}}$ and $D_{\boldsymbol{\beta}}$. It then follows that there exists an automorphism $\psi = (\mathbf{1}_3, \psi_1) \in \mathsf{Aut}(\mathfrak{so}(4)^*)$ that

$$\begin{aligned} H_{D,U}(p) = (H' \circ \psi)(p) &= p \begin{bmatrix} D_{\boldsymbol{\alpha}} & X^\top \psi_1^\top \\ \psi_1 X & \mathbf{0} \end{bmatrix} p^\top \\ &= p \begin{bmatrix} D_{\boldsymbol{\alpha}} & U^\top \\ U & \mathbf{0} \end{bmatrix} p^\top . \end{aligned}$$

Here $U$ is as an upper-triangular matrix, as in the statement.

□

# Chapter 5

# Homogeneous systems on $\mathfrak{so}(4)^*$

In this chapter we investigate the homogeneous quadratic Hamilton-Poisson systems obtained in case 1 and case 2 of section 4.3. We provide an analysis of the stability nature of the equilibrium states for each system.

Unlike for $\mathfrak{so}(3)^*$ it is possible that there exist extra integrals of motion for a Hamilton-Poisson system on $\mathfrak{so}(4)^*_-$. In fact, such a system on $\mathfrak{so}(4)^*_-$ is integrable only if we can find at least one additional integral of motion (i.e., apart from the Casimirs and Hamiltonian) which is functionally independent (see appendix D). We therefore only consider Hamilton-Poisson systems on $\mathfrak{so}(4)^*_-$ for which we can find such an additional integral of motion.

## 5.1 Decomposable systems

Recall that, in the natural basis, we have $\mathfrak{so}(4)^* = \mathfrak{so}(3)^* \oplus \mathfrak{so}(3)^*$. We shall say that a (quadratic) Hamilton-Poisson system $(\mathfrak{so}(4)^*_-, H)$ decomposes into two Hamilton-Poisson systems $(\mathfrak{so}(3)^*_-, H_1)$ and $(\mathfrak{so}(3)^*_-, H_2)$ if

$$H(p) = \tilde{H}_1(x, y) + \tilde{H}_2(x, y), \qquad \text{for all} \qquad p = (x, y) \in \mathfrak{so}(4)^*$$

where $\tilde{H}_1(x, y) := H_1(x)$ and $\tilde{H}_2(x, y) := H_2(y)$. A similar approach concerning the derivation of explicit expression on $\mathfrak{so}(4)^*_-$

We define the two canonical projections $\pi_1, \pi_2 : \mathfrak{so}(4)^* \to \mathfrak{so}(3)^*$, for $p = \sum_{i=1}^{6} p_i E_i^*$, by

$$\pi_1 : p \mapsto (p_1, p_2, p_3) \qquad \text{and} \qquad \pi_2 : p \mapsto (p_4, p_5, p_6).$$

Let $\mathrm{ad}^*_X$ denote the dual of the linear map $\mathrm{ad}_X : \mathfrak{so}(4)^* \to \mathfrak{so}(4)^*$, where $\mathrm{ad}_X : Y \mapsto [X, Y]$. We now consider the Hamiltonian vector field associated to $H$, i.e., for any $p = (x, y) \in \mathfrak{so}(4)^*$,

$$\begin{aligned}\vec{H}(p) = \mathrm{ad}^*_{\mathbf{d}H(p)}(p) &= \mathrm{ad}^*_{d\tilde{H}_1(x,y)}(x, y) + \mathrm{ad}^*_{d\tilde{H}_2(x,y)}(x, y) \\ &= \mathrm{ad}^*_{d\tilde{H}_1(x,0)}(x, y) + \mathrm{ad}^*_{d\tilde{H}_2(0,y)}(x, y).\end{aligned}$$

It now follows that $\mathrm{ad}^*_{\mathbf{d}H_1(x,0)}(x, y) = \mathrm{ad}^*_{\mathbf{d}H_1(x,0)}(x, 0)$. Let $A = \sum_{i=1}^{6} a_i E_i \in \mathfrak{so}(4)^*$.

Then

$$\begin{aligned}\left\langle \mathrm{ad}^*_{\mathbf{d}H_1(x,0)}(x,y), A\right\rangle &= \left\langle p, \mathrm{ad}_{\mathbf{d}H_1(x,0)}(A)\right\rangle \\ &= \left\langle p, [\mathbf{d}H_1(x,0), A]\right\rangle \\ &= \left\langle p, \left[\mathbf{d}H_1(x,0), \sum_{i=1}^{3} a_i E_i\right]\right\rangle \qquad \text{as } [E_i, E_j] = 0,\ i \in \{1,2,3\},\ j \in \{4,5,6\} \\ &= \left\langle (x,0), \left[\mathbf{d}H_1(x,0), \sum_{i=1}^{3} a_i E_i\right]\right\rangle\end{aligned}$$

as $\left[\mathbf{d}H_1(x,0), \sum_{i=1}^{3} a_i E_i\right] \in \langle E_1, E_2, E_3\rangle$. (Similarly, $\mathrm{ad}^*_{\mathbf{d}H_2(0,y)}(x,y) = \mathrm{ad}^*_{\mathbf{d}H_2(0,y)}(0,y)$.) Therefore,

$$\begin{aligned}\vec{H}(p) &= \mathrm{ad}^*_{d\tilde{H}_1(x,0)}(x,0) + \mathrm{ad}^*_{d\tilde{H}_2(0,y)}(0,y) \\ &= \vec{\tilde{H}}_1(x,0) + \vec{\tilde{H}}_2(0,y) \\ &= \left(\vec{H}_1(x), \vec{H}_2(y)\right).\end{aligned}$$

The following proposition thus immediately follows.

5.1.1 Proposition. *We have that, $p(\cdot)$ is an integral curve of the Hamilton-Poisson system $(\mathfrak{so}(4)^*_-, H)$ if and only if $\pi_1(p(\cdot))$ and $\pi_2(p(\cdot))$ are integral curves of the Hamilton-Poisson systems $(\mathfrak{so}(3)^*_-, H_1)$ and $(\mathfrak{so}(3)^*_-, H_2)$, respectively.*

5.1.2 Corollary. *We have that, $\mathsf{e}^{\mu} = (\mathsf{e}_1^{\mu_1}, \mathsf{e}_2^{\mu_2})$ is an equilibrium state of the system $H$ if and only if $\mathsf{e}_1^{\mu_1}$ and $\mathsf{e}_2^{\mu_2}$ are equilibrium states of the systems $H_1$ and $H_2$, respectively.*

5.1.3 Theorem. *An equilibrium state $\mathsf{e}^{\mu} = (\mathsf{e}_1^{\mu_1}, \mathsf{e}_2^{\mu_2})$ of the system $H = \tilde{H}_1 + \tilde{H}_2$ is stable if and only if $\mathsf{e}_1^{\mu_1}$ and $\mathsf{e}_2^{\mu_2}$ are stable equilibrium states of the systems $H_1$ and $H_2$, respectively.*

Proof. Assume $\mathsf{e}^{\mu} = (\mathsf{e}_1^{\mu_1}, \mathsf{e}_2^{\mu_2})$ is a stable equilibrium state of $H$. Let $U_1$ and $U_2$ be any pair of open neighbourhoods of $\mathsf{e}_1^{\mu_1}$ and $\mathsf{e}_2^{\mu_2}$, respectively. Then $\pi_1^{-1}(U_1) \cap \pi_2^{-1}(U_2) = (U_1, U_2)$ is an open neighbourhood of $\mathsf{e}^{\mu}$. Thus there exists an open neighbourhood $V \subset (U_1, U_2)$ such that, if $p(0) \in V$, then $p(t) \in (U_1, U_2)$ for all $t > 0$. Now, $V_1 = \pi_1(V)$ and $V_2 = \pi_2(V)$ will be open sets containing $\mathsf{e}_1^{\mu_1}$ and $\mathsf{e}_2^{\mu_2}$, respectively. (The projections are open maps.) Then, there exist open neighbourhoods $W_1 \subset V_1$ and $W_2 \subset V_2$ of $\mathsf{e}_1^{\mu_1}$ and $\mathsf{e}_2^{\mu_2}$, respectively, such that $\pi_1^{-1}(W_1) \cap \pi_2^{-1}(W_2) = (W_1, W_2) \subset V$. If $x(\cdot)$ and $y(\cdot)$ are any trajectories such that $x(0) \in W_1$ and $y(0) \in W_2$, then $p(0) = (x(0), y(0)) \in V$. Now as any trajectory, such that $p(0) \in V$, satisfies $p(t) \in (U_1, U_2)$ for all $t > 0$, it follows that $\pi_1(p(t)) \in \pi_1((U_1, U_2)) = U_1$ and $\pi_2(p(t)) \in \pi_2((U_1, U_2)) = U_2$ for all $t > 0$. Thus for any open neighbourhoods $U_1$ and $U_2$ of $\mathsf{e}_1^{\mu_1}$ and $\mathsf{e}_2^{\mu_2}$, respectively, there exist open neighbourhoods $W_1$ and $W_2$ containing these respective equilibrium states, such that any trajectory starting in $W_1$, respectively $W_2$, never leaves $U_1$, respectively $U_2$. Hence, the equilibrium states $\mathsf{e}_1^{\mu_1}$ and $\mathsf{e}_2^{\mu_2}$ are stable.

On the other hand, assume $\mathsf{e}_1^{\mu_1}$ and $\mathsf{e}_2^{\mu_2}$ are both stable equilibrium states of $H_1$ and $H_2$, respectively. For every open neighbourhood $\tilde{U} \subset \mathfrak{so}(4)^*$ of $\mathsf{e}^{\mu}$, there exist open neighbourhoods

$U_1$ and $U_2$ of $\mathsf{e}_1^{\mu_1}$ and $\mathsf{e}_2^{\mu_2}$, respectively, such that $\pi_1^{-1}(U_1) \cap \pi_2^{-1}(U_2) = (U_1, U_2) \subset \tilde{U}$ is an open set containing $\mathsf{e}^{\mu}$. For each such $U_1, U_2$ there exist $V_1 \subset U_1, V_2 \subset U_2$ such that any trajectory starting in $V_1$, respectively $V_2$, will never leave $U_1$, respectively $U_2$. Let $V = \pi_1^{-1}(V_1) \cap \pi_2^{-1}(V_2) = (V_1, V_2)$. Then, as every trajectory is of the form $p(\cdot) = (\pi_1(p(\cdot)), \pi_2(p(\cdot)))$, if $p(0) \in (V_1, V_2)$, then $p(t) \in (U_1, U_2) \subset \tilde{U}$ for all $t > 0$. Hence, $\mathsf{e}^{\mu}$ is stable.

□

The following results now follow immediately from our investigation of homogeneous Hamilton-Poisson systems on $\mathfrak{so}(3)^*$, see the systems $H^1(p) = \frac{1}{2}p_1^2$ and $H^2(p) = p_1^2 + \frac{1}{2}p_2^2$ of section 3.3.2.

**Subcase 1.1:** $D_1 = D_2 = \operatorname{diag}(1, \frac{1}{2}, 0)$

The first subcase we consider is when $D_1 = D_2 = \operatorname{diag}(1, \frac{1}{2}, 0)$, i.e., we are considering the system specified by the Hamiltonian $H^1_{D,0}(p) = p_1^2 + \frac{1}{2}p_2^2 + p_4^2 + \frac{1}{2}p_5^2$.

The equilibrium states of this system are given as follows:

$$\begin{array}{lll}
\mathsf{e}_1^{\boldsymbol{\mu}} = (0, 0, \mu_3, 0, 0, \mu_6) & \mathsf{e}_2^{\boldsymbol{\mu\nu}} = (0, 0, \mu_3, 0, \nu_5, 0) & \mathsf{e}_3^{\boldsymbol{\mu\nu}} = (0, 0, \mu_3, \nu_4, 0, 0) \\
\mathsf{e}_4^{\boldsymbol{\mu\nu}} = (0, \nu_2, 0, 0, 0, \mu_6) & \mathsf{e}_5^{\boldsymbol{\nu}} = (0, \nu_2, 0, 0, \nu_5, 0) & \mathsf{e}_6^{\boldsymbol{\nu}} = (0, \nu_2, 0, \nu_4, 0, 0) \\
\mathsf{e}_7^{\boldsymbol{\mu\nu}} = (\nu_1, 0, 0, 0, 0, \mu_6) & \mathsf{e}_8^{\boldsymbol{\nu}} = (\nu_1, 0, 0, 0, \nu_5, 0) & \mathsf{e}_9^{\boldsymbol{\nu}} = (\nu_1, 0, 0, \nu_4, 0, 0).
\end{array}$$

Here we assume that $e_i^{\boldsymbol{\mu}} \neq \mathbf{0}$, $i = 1, \ldots, 9$. (This is as the origin is always a stable equilibrium state for any Hamilton-Poisson system on $\mathfrak{so}(4)^*_-$.)

5.1.4 Proposition. *The equilibrium states have the following behaviour:*

*(i) The states* $\mathsf{e}_1^{\boldsymbol{\mu}}$, $\mathsf{e}_3^{\boldsymbol{\mu\nu}}$, *and* $\mathsf{e}_7^{\boldsymbol{\mu\nu}}$ *are stable.*

*(ii) The states* $\mathsf{e}_2^{\boldsymbol{\mu\nu}}$ $\mathsf{e}_4^{\boldsymbol{\mu\nu}}$, $\mathsf{e}_5^{\boldsymbol{\nu}}$, $\mathsf{e}_6^{\boldsymbol{\mu}}$, $\mathsf{e}_8^{\boldsymbol{\nu}}$, *and* $\mathsf{e}_9^{\boldsymbol{\nu}}$ *are unstable.*

**Subcase 1.2:** $D_1 = \operatorname{diag}(1, 0, 0)$ **and** $= D_2 = \operatorname{diag}(1, \frac{1}{2}, 0)$

Now, consider the case when $D_1 = \operatorname{diag}(1, 0, 0)$ and $D_2 = \operatorname{diag}(1, \frac{1}{2}, 0)$, i.e., we are considering the system specified by the Hamiltonian $H^2_{D,0}(p) = p_1^2 + p_4^2 + \frac{1}{2}p_5^2$.

The equilibrium states of this system are given as follows:

$$\begin{array}{lll}
\mathsf{e}_1^{\boldsymbol{\mu}} = (0, \mu_2, \mu_3, 0, 0, \mu_6) & \mathsf{e}_2^{\boldsymbol{\mu}} = (0, \mu_2, \mu_3, 0, \mu_5, 0) & \mathsf{e}_3^{\boldsymbol{\mu}} = (0, \mu_2, \mu_3, \mu_4, 0, 0) \\
\mathsf{e}_4^{\boldsymbol{\mu}} = (\mu_1, 0, 0, 0, 0, \mu_6) & \mathsf{e}_5^{\boldsymbol{\mu\nu}} = (\mu_1, 0, 0, 0, \nu_5, 0) & \mathsf{e}_6^{\boldsymbol{\mu\nu}} = (\mu_1, 0, 0, \nu_4, 0, 0).
\end{array}$$

Here we assume that $e_i^{\boldsymbol{\mu}} \neq \mathbf{0}$, $i = 1, \ldots, 6$.

5.1.5 Theorem. *The equilibrium states have the following behaviour:*

*(i) The states* $\mathsf{e}_4^{\boldsymbol{\mu}}$ *and* $\mathsf{e}_6^{\boldsymbol{\mu}}$ *are stable.*

*(ii) The states* $\mathsf{e}_1^{\boldsymbol{\mu}}$, $\mathsf{e}_2^{\boldsymbol{\mu}}$, $\mathsf{e}_3^{\boldsymbol{\mu}}$, *and* $\mathsf{e}_5^{\boldsymbol{\mu}}$ *are unstable.*

**Subcase 1.3:** $D_1 = D_2 = \operatorname{diag}(1,0,0)$

We now consider the case when $D_1 = D_2 = \operatorname{diag}(1,0,0)$, i.e., we are considering the system specified by the Hamiltonian $H^3_{D,0}(p) = p_1^2 + p_4^2$.

The equilibrium states of this system are given as follows

$$\mathsf{e}_1^{\boldsymbol{\mu}} = (0, \mu_2, \mu_3, 0, \mu_5, \mu_6) \qquad \mathsf{e}_2^{\boldsymbol{\mu}} = (0, \mu_2, \mu_3, \mu_4, 0, 0)$$
$$\mathsf{e}_3^{\boldsymbol{\mu}} = (\mu_1, 0, 0, 0, \mu_5, \mu_6) \qquad \mathsf{e}_4^{\boldsymbol{\mu}} = (\mu_1, 0, 0, \mu_4, 0, 0).$$

Here we assume that $e_i^{\boldsymbol{\mu}} \neq \mathbf{0},\ i = 1, \ldots, 4$.

5.1.6 THEOREM. *The equilibrium states have the following behaviour:*

*(i) The states* $\mathsf{e}_4^{\boldsymbol{\mu}}$ *are stable.*

*(ii) The states* $\mathsf{e}_1^{\boldsymbol{\mu}}$, $\mathsf{e}_2^{\boldsymbol{\mu}}$, *and* $\mathsf{e}_3^{\boldsymbol{\mu}}$ *are unstable.*

## 5.2 Indecomposable systems

In this section we investigate the Hamilton-Poisson systems $H_{0,D}(p) = \frac{1}{2}p \begin{bmatrix} 0 & D_{\boldsymbol{\alpha}} \\ D_{\boldsymbol{\alpha}} & 0 \end{bmatrix} p^\top$, where $D = \operatorname{diag}(1, \alpha_1, \alpha_2)$ with $0 < |\alpha_2| \leq \alpha_1 \leq 1$. In particular, we investigate the stability nature of the equilibria for several subclasses of these systems. In the case $\alpha_2 = \alpha_1 = 1$, we integrate the associated Hamiltonian vector field in terms of elementary functions. It is known that not all quadratic Hamilton-Poisson systems on $\mathfrak{so}(4)^*_-$ are completely integrable. We restrict ourselves to several cases where we have found an additional first integral. Examples of deriving the explicit expression on $\mathfrak{so}(4)^*_-$ (when such a first integral exists) can be found in the paper [18].

In general, the use of Kowalevski exponents can prove useful in determining which systems possess such an additional first integral, see e.g., [57]. We, however, do not provide the details of calculating these exponents here as they fall outside the scope of this thesis. For those systems that we investigate we will provide the explicit additional first integral.

5.2.1 REMARK. In this section $\mu_i \in \mathbb{R},\ i = 1, \ldots, 6$, and $\nu_i \neq 0,\ i = 1, \ldots, 6$. Also, we let $\{e_1, \ldots e_6\} \in \mathbb{R}^6$ denote the standard ordered basis of $\mathbb{R}^6$.

MATHEMATICA. *The details of the calculations for the stability analysis of the systems in this section can be found in the Mathematica file:*

*Thesis Mathematica\SO(4)\Stability\StabilitySO(4).nb*

### 5.2.1 Subcase 2.1: $0 < \alpha_2 < \alpha_1 < 1$

The first subcase we consider is $0 < \alpha_2 < \alpha_1 < 1$, i.e., we are considering the system specified by the Hamiltonian $H^1_{0,D}(p) = p_1p_4 + \alpha_1 p_2 p_5 + \alpha_2 p_3 p_6$. In this case, there exists a functionally independent first integral given by

$$I(p) = \left(1-\alpha_1^2\right)\left(p_2^2 + 2\alpha_2 p_2 p_5 + p_5^2\right) + \left(1-\alpha_2^2\right)\left(p_3^2 + 2\alpha_1 p_3 p_6 + p_6^2\right)$$

Indeed, one can verify that $\{H^1_{0,D}, I\} = 0$. The Jacobian matrix of $I$, $H^1_{0,D}$, $C_1$, $C_2$ is given by

$$\begin{bmatrix} 0 & 2\gamma_1 (p_2 + p_5\alpha_2) & 2(p_3 + p_6\alpha_1)\gamma_2 & 0 & 2\gamma_1 (p_5 + p_2\alpha_2) & 2(p_6 + p_3\alpha_1)\gamma_2 \\ p_4 & p_5\alpha_1 & p_6\alpha_2 & p_1 & p_2\alpha_1 & p_3\alpha_2 \\ 2p_1 & 2p_2 & 2p_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2p_4 & 2p_5 & 2p_6 \end{bmatrix}$$

which has rank 4 for a generic point $p \in \mathfrak{so}(4)^*$ (i.e., these functions are functionally independent). Here $\gamma_1 = 1 - \alpha_1^2$ and $\gamma_2 = 1 - \alpha_2^2$. This system has equations of motion

$$\begin{cases} \dot p_1 = \alpha_2 p_2 p_6 - \alpha_1 p_3 p_5 \\ \dot p_2 = p_3 p_4 - \alpha_2 p_1 p_6 \\ \dot p_3 = \alpha_1 p_1 p_5 - p_2 p_4 \end{cases} \qquad \begin{cases} \dot p_4 = \alpha_2 p_3 p_5 - \alpha_1 p_2 p_6 \\ \dot p_5 = p_1 p_6 - \alpha_2 p_3 p_4 \\ \dot p_6 = \alpha_1 p_2 p_4 - p_1 p_5. \end{cases}$$

The equilibrium states of this system are given as follows:

$$\begin{array}{lll} \mathsf{e}_1^{\boldsymbol{\mu}} = (0,0,0,\mu_4,\mu_5,\mu_6) & \mathsf{e}_2^{\boldsymbol{\nu}} = (0,0,\nu_3,0,0,\nu_6) & \mathsf{e}_3^{\boldsymbol{\nu}} = (0,\nu_2,0,0,\nu_5,0) \\ \mathsf{e}_4^{\boldsymbol{\nu}} = (\nu_1,0,0,\nu_4,0,0) & \mathsf{e}_5^{\boldsymbol{\mu}} = (\mu_1,\mu_2,\mu_3,0,0,0). & \end{array}$$

We were unable to determine the nature of the equilibria for every equilibrium state. We only state those equilibria and conditions for which we could determine their stability nature. For the omitted equilibrium states both spectral and energy methods provided no insight into their stability nature.

5.2.2 Theorem. *The equilibrium states have the following behaviour:*

*(i) The states $\mathsf{e}_1^{\boldsymbol{\mu}}$ are stable if either $\mu_4 = \mu_5 = 0$, or $\mu_4 = \mu_6 = 0$, or $\mu_5 = \mu_6 = 0$.*

*(ii) Consider the states $\mathsf{e}_2^{\boldsymbol{\nu}}$.*

  *(1) The states $\mathsf{e}_2^{\boldsymbol{\nu}}$ are stable for either $\nu_3, \nu_6 > 0$ or $\nu_3, \nu_6 < 0$.*

  *(2) If either $\nu_3 = -\nu_6$, or $\alpha_1\nu_3 = -\nu_6$, or $\nu_3 = -\alpha_1\nu_6$, then the states are unstable.*

  *(3) If $\alpha_2 \geq \eta$, $\operatorname{sgn}(\nu_3) \neq \operatorname{sgn}(\nu_6)$, and min$\left(-\frac{\nu_3}{\nu_6}, -\frac{\nu_6}{\nu_3}\right) < \alpha_1 < -\frac{2\nu_3\nu_6}{\nu_3^2+\nu_6^2}$, then the states are unstable.*

  *(4) If $\alpha_2 > \eta$, $\operatorname{sgn}(\nu_3) \neq \operatorname{sgn}(\nu_6)$, and $\alpha_1 > -\frac{2\nu_3\nu_6}{\nu_3^2+\nu_6^2}$, then the states are stable.*

*Here $\eta = 2\sqrt{-\frac{\nu_3\nu_6(\alpha_1\nu_3+\nu_6)(\nu_3+\alpha_1\nu_6)}{(\nu_3^2-\nu_6^2)^2}}$. (If $\alpha_2 < \eta$, then the states are unknown.)*

*(iii) The states $\mathsf{e}_3^{\boldsymbol{\nu}}$ are unstable.*

*(iv) The states $\mathsf{e}_4^{\boldsymbol{\nu}}$ are stable.*

*(v) The states $\mathsf{e}_5^{\boldsymbol{\mu}}$ are stable if either $\mu_1 = \mu_2 = 0$, or $\mu_1 = \mu_3 = 0$, or $\mu_2 = \mu_3 = 0$.*

PROOF. Let $H_\lambda(p) = \lambda_1 H^1_{0,D}(p) + \lambda_2 C_1(p) + \lambda_3 C_2(p) + \lambda_4 I(p)$.

(i) First, assume $\mu_4 = \mu_5 = 0$. Let $\lambda_1 = 2\alpha_1\alpha_2$, $\lambda_2 = 1$, $\lambda_3 = \alpha_2^2$, and $\lambda_4 = -\frac{\alpha_2^2}{1-\alpha_2^2}$. We have $\mathbf{d}H_\lambda(\mathsf{e}_1^{\boldsymbol{\mu}}) = 0$ and that the Hessian

$$\mathbf{d}^2 H_\lambda(\mathsf{e}_2^{\boldsymbol{\mu}})_{|W\times W} = \begin{bmatrix} 2 & 0 & 2\alpha_1\alpha_2 & 0 \\ 0 & \frac{2-2\left(2-\alpha_1^2\right)\alpha_2^2}{1-\alpha_2^2} & 0 & \frac{2\alpha_2\left(\alpha_1^2-\alpha_2^2\right)}{1-\alpha_2^2} \\ 2\alpha_1\alpha_2 & 0 & 2\alpha_2^2 & 0 \\ 0 & \frac{2\alpha_2\left(\alpha_1^2-\alpha_2^2\right)}{1-\alpha_2^2} & 0 & \frac{2\alpha_2^2\left(\alpha_1^2-\alpha_2^2\right)}{1-\alpha_2^2} \end{bmatrix}$$

where $W = \operatorname{span}\{e_1, e_2, e_4, e_5\}$, is (positive) definite. Thus, by the generalized energy-Casimir method, the states $\mathsf{e}_1^{\boldsymbol{\mu}}$, $\mu_4 = \mu_5 = 0$ are stable.

Next, assume $\mu_4 = \mu_6 = 0$. Let $\lambda_1 = -\frac{2\alpha_2}{\alpha_1}$, $\lambda_2 = -\alpha_2^2$, $\lambda_3 = -1$, and $\lambda_4 = \frac{1}{1-\alpha_1^2}$. We have $\mathbf{d}H_\lambda(\mathsf{e}_1^{\boldsymbol{\mu}}) = 0$ and that the Hessian

$$\mathbf{d}^2 H_\lambda(\mathsf{e}_2^{\boldsymbol{\mu}})_{|W\times W} = \begin{bmatrix} -2\alpha_2^2 & 0 & -\frac{2\alpha_2}{\alpha_1} & 0 \\ 0 & \frac{2-2\left(2-\alpha_1^2\right)\alpha_2^2}{1-\alpha_1^2} & 0 & \frac{2(\alpha_1^2-\alpha_2^2)(\alpha_1+\alpha_2)}{\alpha_1(1-\alpha_1^2)} \\ -\frac{2\alpha_2}{\alpha_1} & 0 & -2 & 0 \\ 0 & \frac{2(\alpha_1^2-\alpha_2^2)(\alpha_1+\alpha_2)}{\alpha_1(1-\alpha_1^2)} & 0 & \frac{2(\alpha_1^2-\alpha_2^2)(\alpha_1+\alpha_2)}{1-\alpha_1^2} \end{bmatrix}$$

where $W = \operatorname{span}\{e_1, e_3, e_4, e_6\}$, is (negative) definite. Thus, by the generalized energy-Casimir method, the states $\mathsf{e}_1^{\boldsymbol{\mu}}$, $\mu_4 = \mu_6 = 0$ are stable.

Lastly, assume $\mu_5 = \mu_6 = 0$. Let $\lambda_1 = \lambda_3 = 0$, $\lambda_2 = 1$, and $\lambda_4 = \frac{1}{1-\alpha_1^2}$. We have $\mathbf{d}H_\lambda(\mathsf{e}_1^{\boldsymbol{\mu}}) = 0$ and that the Hessian

$$\mathbf{d}^2 H_\lambda(\mathsf{e}_2^{\boldsymbol{\mu}})_{|W\times W} = \begin{bmatrix} 4 & 0 & 2\alpha_2 & 0 \\ 0 & \frac{2\left(2-\alpha_1^2-\alpha_2^2\right)}{1-\alpha_1^2} & 0 & \frac{2\alpha_1\left(1-\alpha_2^2\right)}{1-\alpha_1^2} \\ 2\alpha_2 & 0 & 2 & 0 \\ 0 & \frac{2\alpha_1\left(1-\alpha_2^2\right)}{1-\alpha_1^2} & 0 & \frac{2\left(1-\alpha_2^2\right)}{1-\alpha_1^2} \end{bmatrix}$$

where $W = \operatorname{span}\{e_2, e_3, e_5, e_6\}$, is (positive) definite. Thus, by the generalized energy-Casimir method, the states $\mathsf{e}_1^{\boldsymbol{\mu}}$, $\mu_5 = \mu_6 = 0$ are stable.

(ii) We consider the states $\mathsf{e}_2^{\boldsymbol{\nu}}$.

(1) Assume either $\nu_3, \nu_6 > 0$ or $\nu_3, \nu_6 < 0$. Let $\lambda_1 = 2\alpha_2(1-\alpha_1^2)$, $\lambda_2 = \alpha_1\left(1-\alpha_2^2\right) + \frac{\nu_6}{\nu_3}\left(\alpha_1^2-\alpha_2^2\right)$, $\lambda_3 = \alpha_1\left(1-\alpha_2^2\right) + \frac{\nu_3}{\nu_6}\left(\alpha_1^2-\alpha_2^2\right)$, and $\lambda_4 = -\alpha_1$. We have $\mathbf{d}H_\lambda(\mathsf{e}_2^{\boldsymbol{\nu}}) = 0$ and that the Hessian

$$\mathbf{d}^2 H_\lambda(\mathsf{e}_2^{\boldsymbol{\nu}})_{|W\times W} = \begin{bmatrix} \frac{2\left(\alpha_1\left(\nu_3-\alpha_2^2\nu_3\right)+\nu_6\gamma\right)}{\nu_3} & 0 & 2\left(1-\alpha_1^2\right)\alpha_2 & 0 \\ 0 & \frac{2\gamma(\alpha_1\nu_3+\nu_6)}{\nu_3} & 0 & 0 \\ 2\left(1-\alpha_1^2\right)\alpha_2 & 0 & \frac{2\left(\nu_3\gamma+\alpha_1\left(\nu_6-\alpha_2^2\nu_6\right)\right)}{\nu_6} & 0 \\ 0 & 0 & 0 & \frac{2\gamma(\nu_3+\alpha_1\nu_6)}{\nu_6} \end{bmatrix}$$

where $W = \operatorname{span}\{e_1, e_2, e_4, e_5\}$, is (positive) definite. Here $\gamma = \alpha_1^2 - \alpha_2^2$. Thus, by the generalized energy-Casimir method, the states $\mathsf{e}_2^{\boldsymbol{\nu}}$, for either $\nu_3, \nu_6 > 0$ or $\nu_3, \nu_6 < 0$, are stable.

In the following subcases (1) – (3) we assume $\nu_3$ and $\nu_6$ have opposite signs.

(2) The linearization of the system at $\mathsf{e}_2^{\boldsymbol{\nu}}$ when either $\nu_3 = -\nu_6$, or $\alpha_1\nu_3 = -\nu_6$, or $\nu_3 = -\alpha_1\nu_6$, has respective eigenvalues

$$\begin{aligned}
&\lambda_{1,2} = 0 \qquad && \lambda_{3,4} = \pm\nu_3\sqrt{(\alpha_1 - \alpha_2)(1 + \alpha_2)} \qquad && \lambda_{5,6} = \pm\nu_3\sqrt{(1 - \alpha_2)(\alpha_1 + \alpha_2)}\\
&\lambda_{1,2} = 0 && \lambda_{3,4} = \pm\nu_3\sqrt{(\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2)} && \lambda_{5,6} = \pm\nu_3\alpha_1\sqrt{1 - \alpha_2^2}\\
&\lambda_{1,2} = 0 && \lambda_{3,4} = \pm\nu_6\sqrt{(\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2)} && \lambda_{5,6} = \pm\nu_6\alpha_1\sqrt{1 - \alpha_2^2}.
\end{aligned}$$

Clearly, as $0 < \alpha_2 < \alpha_1 < 1$, at least one of the eigenvalues in each of these cases will have a positive real part. Hence the states $\mathsf{e}_2^{\boldsymbol{\nu}}$ for $\nu_3 = -\nu_6$, $\alpha_1\nu_3 = -\nu_6$, or $\nu_3 = -\alpha_1\nu_6$ are unstable.

In the following subcases we now assume that $\nu_3 \neq -\nu_6$, $\alpha_1\nu_3 \neq -\nu_6$, and $\nu_3 \neq -\alpha_1\nu_6$. Also, let $\eta = 2\sqrt{-\frac{\nu_3\nu_6(\alpha_1\nu_3+\nu_6)(\nu_3+\alpha_1\nu_6)}{(\nu_3^2-\nu_6^2)^2}}$.

(3) The linearization of the system at $\mathsf{e}_2^{\boldsymbol{\nu}}$ has eigenvalues $\lambda_{1,2} = 0$, $\lambda_{3,4} = \pm\sqrt{\gamma_1 - \gamma_2}$, and $\lambda_{5,6} = \pm\sqrt{\gamma_1 + \gamma_2}$, where

$$\begin{aligned}
\gamma_1 &= -2\alpha_1\nu_3\nu_6 - \alpha_2^2\left(\nu_3^2 + \nu_6^2\right)\\
\gamma_2 &= \alpha_2\sqrt{4\nu_3\nu_6\left(\alpha_1\nu_3 + \nu_6\right)\left(\nu_3 + \alpha_1\nu_6\right) + \alpha_2^2\left(\nu_3^2 - \nu_6^2\right)^{2}}.
\end{aligned}$$

Note that if $\alpha_2 < \eta$, then $\gamma_2$ will be imaginary. Thus at least $\alpha_2 \geq \eta$ for one of the eigenvalues to have a real part. Assume, $\nu_3 < 0$ and $\nu_6 > 0$. It then follows that $\gamma_1 - \gamma_2 > 0$, if $-\nu_3 > \nu_6$ and $-\frac{\nu_6}{\nu_3} < \alpha_1 < -\frac{2\nu_3\nu_6}{\nu_3^2+\nu_6^2}$, or if $-\nu_3 < \nu_6$ and $-\frac{\nu_3}{\nu_6} < \alpha_1 < -\frac{2\nu_3\nu_6}{\nu_3^2+\nu_6^2}$. On the other hand, assume $\nu_3 > 0$ and $\nu_6 < 0$. It then follows that $\gamma_1 - \gamma_2 > 0$, if $\nu_3 < -\nu_6$ and $-\frac{\nu_3}{\nu_6} < \alpha_1 < -\frac{2\nu_3\nu_6}{\nu_3^2+\nu_6^2}$, or if $\nu_3 > -\nu_6$ and $-\frac{\nu_6}{\nu_3} < \alpha_1 < -\frac{2\nu_3\nu_6}{\nu_3^2+\nu_6^2}$. Combining all these conditions together gives us that the states $\mathsf{e}_2^{\boldsymbol{\nu}}$, $\min\left(-\frac{\nu_3}{\nu_6}, -\frac{\nu_6}{\nu_3}\right) < \alpha_1 < -\frac{2\nu_3\nu_6}{\nu_3^2+\nu_6^2}$ are unstable.

(4) Let $\lambda_1 = \frac{4}{\alpha_2}\left(1 - \alpha_2^2\right)$, $\lambda_2 = \frac{\left(1-\alpha_2^2\right)(\nu_3-\nu_6)(\nu_3+\nu_6)}{\nu_3(\alpha_1\nu_3+\nu_6)}$, $\lambda_3 = -\frac{\left(1-\alpha_2^2\right)(\nu_3-\nu_6)(\nu_3+\nu_6)}{\nu_6(\nu_3+\alpha_1\nu_6)}$, and $\lambda_4 = -\frac{\nu_3^2+2\alpha_1\nu_3\nu_6+\nu_6^2}{(\alpha_1\nu_3+\nu_6)(\nu_3+\alpha_1\nu_6)}$. We have $\mathbf{d}H_\lambda(\mathsf{e}_2^{\boldsymbol{\nu}}) = 0$ and that the Hessian $\mathbf{d}^2H_\lambda(\mathsf{e}_2^{\boldsymbol{\nu}})_{|W\times W}$, where $W = \operatorname{span}\{e_1, e_2, e_4, e_5\}$, is (positive) definite when $\alpha_1 > -\frac{2\nu_3\nu_6}{\nu_3^2+\nu_6^2}$ and $\alpha_2 > \eta$. (The expression for $\mathbf{d}^2H_\lambda(\mathsf{e}_2^{\boldsymbol{\nu}})_{|W\times W}$ is omitted in this case as the expression is rather large; the expression can be found in the Mathematica code given for this section.) Thus, by the generalized energy-Casimir method, the states $\mathsf{e}_2^{\boldsymbol{\nu}}$, $\operatorname{sgn}(\nu_3) \neq \operatorname{sgn}(\nu_6)$, $\alpha_1 > -\frac{2\nu_3\nu_6}{\nu_3^2+\nu_6^2}$, $\alpha_2 > \eta$ are stable.

(iii) The linearization of the system at $\mathsf{e}_3^{\boldsymbol{\nu}}$ has eigenvalues $\lambda_1 = \lambda_2 = 0$, $\lambda_{3,4} = \pm\sqrt{\frac{\gamma_1-\gamma_2}{2}}$, and $\lambda_{5,6} = \pm\sqrt{\frac{\gamma_1+\gamma_2}{2}}$, where

$$\begin{aligned}
\gamma_1 &= -4\alpha_2\nu_2\nu_5 - 2\alpha_1^2\left(\nu_2^2 + \nu_5^2\right)\\
\gamma_2 &= 2\alpha_1\sqrt{4\nu_2\nu_5\left(\alpha_2\nu_2 + \nu_5\right)\left(\nu_2 + \alpha_2\nu_5\right) + \alpha_1^2\left(\nu_2^2 - \nu_5^2\right)^2}.
\end{aligned}$$

It is straightforward to determine that the eigenvalue $\lambda_5 = \sqrt{\frac{\gamma_1+\gamma_2}{2}} > 0$ for all $\nu_2, \nu_5$. Hence the states $\mathsf{e}_3^{\boldsymbol{\nu}}$ are unstable.

(iv) Let $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and $\lambda_4 = 1$. We have $\mathbf{d}H_\lambda(\mathsf{e}_4^{\boldsymbol{\nu}}) = 0$ and that the Hessian

$$\mathbf{d}^2 H_\lambda(\mathsf{e}_4^{\boldsymbol{\nu}})_{|W\times W} = \begin{bmatrix} 2(1-\alpha_1^2) & 0 & 2\left(1-\alpha_1^2\right)\alpha_2 & 0 \\ 0 & 2(1-\alpha_2^2) & 0 & 2\alpha_1\left(1-\alpha_2^2\right) \\ 2\left(1-\alpha_1^2\right)\alpha_2 & 0 & 2(1-\alpha_1^2) & 0 \\ 0 & 2\alpha_1\left(1-\alpha_2^2\right) & 0 & 2(1-\alpha_2^2) \end{bmatrix}$$

where $W = \operatorname{span}\{e_2, e_3, e_5, e_6\}$, is definite. Thus, by the generalized energy-Casimir method, the states $\mathsf{e}_4^{\boldsymbol{\nu}}$ are stable.

(v) First, assume $\mu_1 = \mu_2 = 0$. Let $\lambda_1 = 2\alpha_1\alpha_2$, $\lambda_2 = \alpha_2^2$, $\lambda_3 = 1$, and $\lambda_4 = -\frac{\alpha_2^2}{1-\alpha_2^2}$. We have $\mathbf{d}H_\lambda(\mathsf{e}_5^{\boldsymbol{\mu}}) = 0$ and that the Hessian

$$\mathbf{d}^2 H_\lambda(\mathsf{e}_2^{\boldsymbol{\mu}})_{|W\times W} = \begin{bmatrix} 2\alpha_2^2 & 0 & 2\alpha_1\alpha_2 & 0 \\ 0 & \frac{2\alpha_2^2\left(\alpha_1^2-\alpha_2^2\right)}{1-\alpha_2^2} & 0 & \frac{2\alpha_2\left(\alpha_1^2-\alpha_2^2\right)}{1-\alpha_2^2} \\ 2\alpha_1\alpha_2 & 0 & 2 & 0 \\ 0 & \frac{2\alpha_2\left(\alpha_1^2-\alpha_2^2\right)}{1-\alpha_2^2} & 0 & \frac{2-2\left(2-\alpha_1^2\right)\alpha_2^2}{1-\alpha_2^2} \end{bmatrix}$$

where $W = \operatorname{span}\{e_1, e_2, e_4, e_5\}$, is (positive) definite. Thus, by the generalized energy-Casimir method, the states $\mathsf{e}_5^{\boldsymbol{\mu}}$, $\mu_1 = \mu_2 = 0$ are stable.

Next, assume $\mu_1 = \mu_3 = 0$. Let $\lambda_1 = -2\alpha_2$, $\lambda_2 = -\alpha_1$, $\lambda_3 = 1$, and $\lambda_4 = \frac{\alpha_1}{1-\alpha_1^2}$. We have $\mathbf{d}H_\lambda(\mathsf{e}_5^{\boldsymbol{\mu}}) = 0$ and that the Hessian

$$\mathbf{d}^2 H_\lambda(\mathsf{e}_2^{\boldsymbol{\mu}})_{|W\times W} = \begin{bmatrix} -2\alpha_1 & 0 & -2\alpha_2 & 0 \\ 0 & \frac{2\alpha_1(\alpha_1^2-\alpha_2^2)}{1-\alpha_1^2} & 0 & \frac{2(\alpha_1^2-\alpha_2^2)}{1-\alpha_1^2} \\ -2\alpha_2 & 0 & 2 & 0 \\ 0 & \frac{2(\alpha_1^2-\alpha_2^2)}{1-\alpha_1^2} & 0 & \frac{2\left(1+\alpha_1\left(1-\alpha_1-\alpha_2^2\right)\right)}{1-\alpha_1^2} \end{bmatrix}$$

where $W = \operatorname{span}\{e_1, e_3, e_4, e_6\}$, is (negative) definite. Thus, by the generalized energy-Casimir method, the states $\mathsf{e}_5^{\boldsymbol{\mu}}$, $\mu_1 = \mu_3 = 0$ are stable.

Lastly, assume $\mu_2 = \mu_3 = 0$. Let $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 2\lambda_4 = 1$. We have $\mathbf{d}H_\lambda(\mathsf{e}_5^{\boldsymbol{\mu}}) = 0$ and that the Hessian

$$\mathbf{d}^2 H_\lambda(\mathsf{e}_2^{\boldsymbol{\mu}})_{|W\times W} = \begin{bmatrix} 1-\alpha_1^2 & 0 & \alpha_2(1-\alpha_1^2) & 0 \\ 0 & 1-\alpha_2^2 & 0 & \alpha_1(1-\alpha_2^2) \\ \alpha_2(1-\alpha_1^2) & 0 & 3-\alpha_1^2 & 0 \\ 0 & \alpha_1(1-\alpha_2^2) & 0 & 3-\alpha_2^2 \end{bmatrix}$$

where $W = \operatorname{span}\{e_2, e_3, e_5, e_6\}$, is (positive) definite. Thus, by the generalized energy-Casimir method, the states $\mathsf{e}_5^{\boldsymbol{\mu}}$, $\mu_2 = \mu_3 = 0$ are stable.

$\square$

### 5.2.2 Subcase 2.2: $\alpha_1 = \alpha_2 = 1$

Next, we consider the case when $\alpha_1 = \alpha_2 = 1$, i.e., we are considering the system specified by the Hamiltonian $H^1_{0,D}(p) = p_1p_4 + p_2p_5 + p_3p_6$. In this case, there exists two functionally independent first integrals given by

$$I_1(p) = p_1 + p_4$$
$$I_2(p) = p_2 + p_5$$

Indeed, one can easily verify that $\{H^1_{0,D}, I_i\} = 0\,,\ i = 1, 2$. The Jacobian matrix of $H^1_{0,D}$, $I_1$, $I_2$, $C_1$, $C_2$ is given by

$$\begin{bmatrix} p_4 & p_5 & p_6 & p_1 & p_2 & p_3 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2p_1 & 2p_2 & 2p_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2p_4 & 2p_5 & 2p_6 \end{bmatrix}$$

which has rank 5 (i.e., these functions are functionally independent). Note that although $I_1(p)$ and $I_2(p)$ are not in involution, i.e., $\{I_1, I_2\} \neq 0$, they can still both be used in the study of the stability of this system.

This system has equations of motion

$$\begin{cases} \dot p_1 = p_2p_6 - p_3p_5 \\ \dot p_2 = p_3p_4 - p_1p_6 \\ \dot p_3 = p_1p_5 - p_2p_4 \end{cases} \qquad \begin{cases} \dot p_4 = p_3p_5 - p_2p_6 \\ \dot p_5 = p_1p_6 - p_3p_4 \\ \dot p_6 = p_2p_4 - p_1p_5. \end{cases}$$

The equilibrium states of this system are given as follows:

$$\mathsf{e}_1^{\mu\nu} = (\mu_1, \nu_2, \mu_3, \frac{\mu_1\mu_5}{\nu_2}, \mu_5, \frac{\mu_3\mu_5}{\nu_2}) \qquad \mathsf{e}_2^{\mu} = (0, 0, 0, \mu_4, \mu_5, \mu_6)$$
$$\mathsf{e}_3^{\mu\nu} = (\nu_1, 0, \mu_3, \mu_4, 0, \frac{\mu_3\mu_4}{\nu_1}) \qquad \mathsf{e}_4^{\mu} = (0, 0, \mu_3, 0, 0, \mu_6).$$

Here we assume that $e_i^{\mu\nu} \neq \mathbf{0}$, $i = 1, \ldots, 4$.

5.2.3 Theorem. *The equilibrium states have the following behaviour:*

*(i) The states* $\mathsf{e}_1^{\mu\nu}$ *are stable for* $\mu_2 + \mu_5 \neq 0$.

*(ii) The states* $\mathsf{e}_2^{\mu}$ *are stable.*

*(iii) The states* $\mathsf{e}_3^{\mu\nu}$ *are stable for* $\mu_1 + \mu_4 \neq 0$.

*(iv) The states* $\mathsf{e}_4^{\mu}$ *are stable for* $\mu_3 + \mu_6 \neq 0$.

Proof. Let $H_\lambda(p) = \lambda_1 H^1_{0,D}(p) + \lambda_2 C_1(p) + \lambda_3 C_2(p) + \lambda_4 I_1(p) + \lambda_5 I_2(p)$.

(i) First, assume $\mu_3, \mu_5 \neq 0$. Let $\lambda_1 = -\sigma = -\mathrm{sgn}(\nu_2\mu_5)$, $\lambda_2 = \frac{\sigma\mu_5}{2\nu_2}$, $\lambda_3 = \frac{\sigma\nu_2}{2\mu_5}$, and $\lambda_4 = \lambda_5 = 0$.

We have $\mathbf{d}H_\lambda(\mathsf{e}_1^{\boldsymbol{\mu\nu}}) = 0$ and that the Hessian

$$\mathbf{d}^2H_\lambda(\mathsf{e}_1^{\boldsymbol{\mu\nu}})_{|W\times W} = \begin{bmatrix} \frac{\sigma\left(\mu_1^2+\mu_3^2\right)(\nu_2+\mu_5)^2}{\nu_2\mu_3^2\mu_5} & \frac{\sigma\mu_1(\nu_2+\mu_5)^2}{\mu_3^2\mu_5} \\ \frac{\sigma\mu_1(\nu_2+\mu_5)^2}{\mu_3^2\mu_5} & \frac{\sigma\left(\nu_2^2+\mu_3^2\right)(\nu_2+\mu_5)^2}{\nu_2\mu_3^2\mu_5} \end{bmatrix}$$

where $W = \operatorname{span}\left\{(e_1-e_4) - \frac{\mu_1}{\mu_3}(e_3-e_6), (e_2-e_5) - \frac{\nu_2}{\mu_3}(e_3-e_6)\right\}$, is definite when $\nu_2+\mu_5 \neq 0$. Thus, by the generalized energy-Casimir method, the states $\mathsf{e}_1^{\boldsymbol{\mu\nu}}$, $\mu_3\mu_5 \neq 0$ and $\nu_2+\mu_5 \neq 0$, are stable. When $\nu_2+\mu_5 = 0$ both spectral and energy-Casimir results provide no useful insight.

Now, assume $\mu_3 = 0$ and $\mu_5 \neq 0$. Let $\lambda_2 = \frac{1}{2}\left(\lambda_1 - \frac{(\lambda_1-2\lambda_3)\mu_5}{\nu_2}\right)$, $\lambda_4 = -\mu_1\left(\lambda_1 + \frac{2\lambda_3\mu_5}{\nu_2}\right)$, and $\lambda_5 = -\lambda_1\nu_2 - 2\lambda_3\mu_5$. We have $\mathbf{d}H_\lambda(\mathsf{e}_1^{\boldsymbol{\mu\nu}}) = 0$ and that the Hessian

$$\mathbf{d}^2H_\lambda(\mathsf{e}_1^{\boldsymbol{\mu\nu}})_{|W\times W} = \begin{bmatrix} -\frac{(\lambda_1-2\lambda_3)\left(\mu_1^2+\nu_2^2\right)(\nu_2+\mu_5)}{\nu_2^3} & 0 & 0 \\ 0 & \lambda_1 - \frac{(\lambda_1-2\lambda_3)\mu_5}{\nu_2} & \lambda_1 \\ 0 & \lambda_1 & 2\lambda_3 \end{bmatrix}.$$

where $W = \operatorname{span}\left\{(e_1-e_4) - \frac{\mu_1}{\nu_2}(e_2-e_5), e_3, e_6\right\}$. Now, for $\nu_2, \mu_5 > 0$ or $\nu_2, \mu_5 < 0$, if we choose $\lambda_1 = 1$ and $\lambda_3 = 2$, then the Hessian is (positive) definite. If either ($\nu_2 > 0$, $\mu_5 < 0$, and $\nu_2+\mu_5 > 0$) or ($\nu_2 < 0$, $\mu_5 > 0$, and $\nu_2+\mu_5 < 0$), then there exist $\lambda_1, \lambda_3 > 0$ with $-2\lambda_3\frac{\mu_5}{\nu_2} < \lambda_1 < 2\lambda_3$, such that the Hessian is (positive) definite. On the other hand, if either ($\nu_2 > 0$, $\mu_5 < 0$, and $\nu_2+\mu_5 < 0$) or ($\nu_2 < 0$, $\mu_5 > 0$, and $\nu_2+\mu_5 > 0$), then there exist $\lambda_1, \lambda_3 > 0$ with $2\lambda_3 < \lambda_1 < -2\lambda_3\frac{\mu_5}{\nu_2}$, such that the Hessian is (positive) definite.

Thus, by the generalized energy-Casimir method, the states $\mathsf{e}_1^{\boldsymbol{\mu\nu}}$ ($\mu_3 = 0$, $\mu_5 \neq 0$, and $\nu_2+\mu_5 \neq 0$) are stable. A very similar argument shows that $\mathsf{e}_1^{\boldsymbol{\mu\nu}}$ is stable whenever $\mu_3 \neq 0$, $\mu_5 = 0$ or $\mu_3 = \mu_5 = 0$.

(ii) Assume $\mu_6 \neq 0$. Let $\lambda_1 = \lambda_3 = \lambda_4 = \lambda_5 = 0$ and $\lambda_2 = \frac{1}{2}$. We have $\mathbf{d}H_\lambda(\mathsf{e}_2^{\boldsymbol{\mu}}) = 0$ and that the Hessian

$$\mathbf{d}^2H_\lambda(\mathsf{e}_2^{\boldsymbol{\mu}})_{|W\times W} = \begin{bmatrix} 1+\frac{\mu_4^2}{\mu_6^2} & \frac{\mu_4\mu_5}{\mu_6^2} \\ \frac{\mu_4\mu_5}{\mu_6^2} & 1+\frac{\mu_5^2}{\mu_6^2} \end{bmatrix}$$

where $W = \operatorname{span}\left\{(e_1-e_4) - \frac{\mu_4}{\mu_6}(e_3-e_6), (e_2-e_5) - \frac{\mu_5}{\mu_6}(e_3-e_6)\right\}$, is (positive) definite. Thus, by the generalized energy-Casimir method, the states $\mathsf{e}_2^{\boldsymbol{\mu}}$, $\mu_6 \neq 0$ are stable.

On the other hand, assume $\mu_6 = 0$. Let $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \frac{1}{2}$, $\lambda_4 = -\mu_4$, and $\lambda_5 = -\mu_5$. We have $\mathbf{d}H_\lambda(\mathsf{e}_2^{\boldsymbol{\mu}}) = 0$ and that the Hessian

$$\mathbf{d}^2H_\lambda(\mathsf{e}_2^{\boldsymbol{\mu}})_{|W\times W} = \begin{bmatrix} \mu_4^2+\mu_5^2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

where $W = \operatorname{span}\{\mu_5(e_1-e_4) - \mu_4(e_2-e_5), e_3, e_6\}$, is (positive) definite. Thus, by the generalized energy-Casimir method, the states $\mathsf{e}_2^{\boldsymbol{\mu}}$, $\mu_6 = 0$ are stable.

(iii) Assume $\mu_4 \neq 0$. Let $\lambda_1 = -\sigma = -\operatorname{sgn}(\nu_1\mu_4)$, $\lambda_2 = \frac{\sigma\mu_4}{2\nu_1}$, $\lambda_3 = \frac{\sigma\nu_1}{2\mu_4}$, and $\lambda_4 = \lambda_5 = 0$. We have $\mathbf{d}H_\lambda(\mathsf{e}_3^{\boldsymbol{\mu\nu}}) = 0$ and that the Hessian

$$\mathbf{d}^2H_\lambda(\mathsf{e}_3^{\boldsymbol{\mu\nu}})_{|W\times W} = \begin{bmatrix} \frac{\sigma\left(\nu_1^2+\mu_3^2\right)(\nu_1+\mu_4)^2}{\nu_1\mu_4} & 0 \\ 0 & \frac{\sigma(\nu_1+\mu_4)^2}{\nu_1\mu_4} \end{bmatrix}$$

where $W = \operatorname{span}\{\mu_3(e_1-e_4) - \nu_1(e_3-e_6), e_2-e_5\}$, is definite when $\nu_1+\mu_4 \neq 0$. Thus, by the generalized energy-Casimir method, the states $\mathsf{e}_3^{\boldsymbol{\mu\nu}}$, $\mu_4 \neq 0$ and $\nu_1+\mu_4 \neq 0$, are stable. When $\nu_1+\mu_4 = 0$ both spectral and energy-Casimir results provide no useful insight.

On the other hand, assume, $\mu_4 = 0$. Let $\lambda_1 = \lambda_2 = \lambda_4 = \lambda_5 = 0$ and $\lambda_3 = \frac{1}{2}$. We have $\mathbf{d}H_\lambda(\mathsf{e}_3^{\boldsymbol{\mu\nu}}) = 0$ and that the Hessian

$$\mathbf{d}^2H_\lambda(\mathsf{e}_3^{\boldsymbol{\mu\nu}})_{|W\times W} = \begin{bmatrix} \nu_1^2+\mu_3^2 & 0 \\ 0 & 1 \end{bmatrix}$$

where $W = \operatorname{span}\{\mu_3(e_1-e_4) - \nu_1(e_3-e_6), e_2-e_5\}$, is definite. Thus, by the generalized energy-Casimir method, the states $\mathsf{e}_3^{\boldsymbol{\mu\nu}}$, $\mu_4 = 0$, are stable.

(iv) Assume $\mu_3, \mu_6 \neq 0$. Let $\lambda_1 = -\sigma = -\operatorname{sgn}(\mu_3\mu_6)$, $\lambda_2 = \frac{\sigma\mu_6}{2\mu_3}$, $\lambda_3 = \frac{\sigma\mu_3}{2\mu_6}$, and $\lambda_4 = \lambda_5 = 0$. We have $\mathbf{d}H_\lambda(\mathsf{e}_4^{\boldsymbol{\mu}}) = 0$ and that the Hessian

$$\mathbf{d}^2H_\lambda(\mathsf{e}_4^{\boldsymbol{\mu}})_{|W\times W} = \begin{bmatrix} \frac{(\mu 3+\mu 6)^2\sigma}{\mu 3\mu 6} & 0 \\ 0 & \frac{(\mu 3+\mu 6)^2\sigma}{\mu 3\mu 6} \end{bmatrix}$$

where $W = \operatorname{span}\{e_1-e_4, e_2-e_5\}$, is definite when $\mu_3+\mu_6 \neq 0$. Thus, by the generalized energy-Casimir method, the states $\mathsf{e}_4^{\boldsymbol{\mu}}$, $\mu_3, \mu_6 \neq 0$ and $\mu_3+\mu_6 \neq 0$, are stable. When $\mu_3+\mu_6 = 0$ both spectral and energy-Casimir results provide no useful insight.

Now, assume $\mu_3 = 0$ and $\mu_6 \neq 0$. Let $\lambda_1 = \lambda_3 = \lambda_4 = \lambda_5 = 0$ and $\lambda_2 = \frac{1}{2}$. We have $\mathbf{d}H_\lambda(\mathsf{e}_4^{\boldsymbol{\mu}}) = 0$ and that the Hessian

$$\mathbf{d}^2H_\lambda(\mathsf{e}_4^{\boldsymbol{\mu}})_{|W\times W} = \operatorname{diag}(1,1)$$

where $W = \operatorname{span}\{e_1, e_2\}$, is definite. Thus, by the generalized energy-Casimir method, the states $\mathsf{e}_4^{\boldsymbol{\mu}}$, $\mu_3 = 0, \mu_6 \neq 0$, are stable. A very similar argument shows that $\mathsf{e}_4^{\boldsymbol{\mu}}$ is stable whenever $\mu_3 \neq 0$ and $\mu_6 = 0$.

□

MATHEMATICA. *The verification of the integration for this system can be found in the Mathematica file:*

*Thesis Mathematica\SO(4)\Integration\SimpleHOD.nb*

5.2.4 THEOREM. *Let $p(\cdot)$ be an integral curve of the system $H^1_{0,D}$ through $p(0)$. Let $h_0 = H^1_{0,D}(p(0))$, $c_{10} = C_1(p(0))$, $c_{20} = C_2(p(0))$, $d_{14} = p_1(0)+p_4(0)$, $d_{25} = p_2(0)+p_5(0)$, and $d_{36} = p_3(0)+p_6(0)$.*

*Then there exists* $t_0 \in \mathbb{R}$ *such that* $p(t) = \bar{p}(t + t_0)$, *where*

$$\begin{aligned}
\bar{p}_1(t) &= \frac{1}{\Omega^2}\left(d_{14}\left(c_{10} + h_0\right) + \gamma_{14}\,\Omega\,\sin(\Omega\,t) + \cos(\Omega\,t)\eta_1\right)\\
\bar{p}_2(t) &= \frac{1}{\Omega^2}\left(d_{25}\left(c_{10} + h_0\right) + \gamma_{25}\,\Omega\,\sin(\Omega\,t) + \cos(\Omega\,t)\eta_2\right)\\
\bar{p}_3(t) &= \frac{1}{\Omega^2}\left(d_{36}\left(c_{10} + h_0\right) + \gamma_{36}\,\Omega\,\sin(\Omega\,t) + \cos(\Omega\,t)\eta_3\right)\\
\bar{p}_4(t) &= \frac{1}{\Omega^2}\left(d_{14}\left(c_{20} + h_0\right) - \gamma_{14}\,\Omega\,\sin(\Omega\,t) - \cos(\Omega\,t)\eta_1\right)\\
\bar{p}_5(t) &= \frac{1}{\Omega^2}\left(d_{25}\left(c_{20} + h_0\right) - \gamma_{25}\,\Omega\,\sin(\Omega\,t) - \cos(\Omega\,t)\eta_2\right)\\
\bar{p}_6(t) &= \frac{1}{\Omega^2}\left(d_{36}\left(c_{20} + h_0\right) - \gamma_{36}\,\Omega\,\sin(\Omega\,t) - \cos(\Omega\,t)\eta_3\right).
\end{aligned}$$

*Here* $\Omega = \sqrt{d_{14}^2 + d_{25}^2 + d_{36}^2}$, $\eta_i = p_i(0)\Omega^2 - d_{i0}(c_{10} + h_0)$, $i = 1, 2, 3$, *and*

$$\gamma_{14} = p_2(0)p_6(0) - p_3(0)p_5(0), \quad \gamma_{25} = p_3(0)p_4(0) - p_1(0)p_6(0), \quad \gamma_{36} = p_1(0)p_5(0) - p_2(0)p_4(0).$$

PROOF. In this case one can immediately notice that $\dot{p}_i(t) = -\dot{p}_{i+3}(t)$, $i = 1, 2, 3$, for all $t \in \mathbb{R}$. Therefore $p_4(t) = -p_1(t) + d_{14}$, $p_5(t) = -p_2(t) + d_{25}$, and $p_6(t) = -p_3(t) + d_{36}$ for some constants $d_{14}, d_{25}, d_{36}$. Substituting these conditions back into the original equations of motion we obtain the linear system of ordinary differential equations

$$\begin{cases} \dot{p}_1 = d_{36}p_2 - d_{25}p_3 \\ \dot{p}_2 = d_{14}p_3 - d_{36}p_1 \\ \dot{p}_3 = d_{25}p_1 - d_{14}p_2 \end{cases} \qquad \begin{cases} \dot{p}_4 = d_{36}p_5 - d_{25}p_6 \\ \dot{p}_5 = d_{14}p_6 - d_{36}p_4 \\ \dot{p}_6 = d_{25}p_4 - d_{14}p_5. \end{cases}$$

This system can be written in matrix form as

$$\dot{p}(t) = \begin{bmatrix} B_1 & 0 \\ 0 & B_1 \end{bmatrix} p(t), \quad \text{where} \quad B_1 = \begin{bmatrix} 0 & d_{36} & -d_{25} \\ -d_{36} & 0 & d_{14} \\ d_{25} & -d_{14} & 0 \end{bmatrix}.$$

Here $B_1 \in \mathfrak{so}(3)$ and this system can be solved directly using the matrix exponential. Therefore,

$$p(t) = \exp\left(t\begin{bmatrix} B_1 & 0 \\ 0 & B_1 \end{bmatrix}\right) p(0) = \begin{bmatrix} \exp(tB_1) & 0 \\ 0 & \exp(tB_1) \end{bmatrix} p(0)$$

where $\exp(tB_1)$ is given by

$$\frac{1}{\Omega^2}\begin{bmatrix} d_{14}^2 + \cos(\Omega\,t)\left(d_{25}^2 + d_{36}^2\right) & f(t)d_{14}d_{25} + \Omega\sin(\Omega\,t)d_{36} & -\Omega\sin(\Omega\,t)d_{25} + f(t)d_{14}d_{36} \\ f(t)d_{14}d_{25} - \Omega\sin(\Omega\,t)d_{36} & d_{25}^2 + \cos(\Omega\,t)\left(d_{14}^2 + d_{36}^2\right) & \Omega\sin(\Omega\,t)d_{14} + f(t)d_{25}d_{36} \\ \Omega\sin(\Omega\,t)d_{25} + f(t)d_{14}d_{36} & f(t)d_{25}d_{36} - \Omega\sin(\Omega\,t)d_{14} & \cos(\Omega\,t)\left(d_{14}^2 + d_{25}^2\right) + d_{36}^2 \end{bmatrix}.$$

Here $f(t) = 1 - \cos(\Omega\,t)$ and $\Omega = \sqrt{d_{14}^2 + d_{25}^2 + d_{36}}$. It is straightforward to verify that $H^1_{0,D}(p(t)) = h_0$, $C_1(p(t)) = c_{10}$, $C_2(p(t)) = c_{20}$, $p_1(t) + p_4(t) = d_{14}$, $p_2(t) + p_5(t) = d_{25}$, and $p_3(t) + p_6(t) = d_{36}$, for all $t \in \mathbb{R}$. Making several simplifications the expression for $p(t)$ can then be found to be given as in the statement of the theorem. $\square$

### 5.2.3 Subcase 2.3: $0 < \alpha_1 = \alpha_2 < 1$

Now, we consider the case when $0 < \alpha_1 = \alpha_2 < 1$, i.e., we are considering the system specified by the Hamiltonian $H^1_{0,D}(p) = p_1p_4 + \alpha_1p_2p_5 + \alpha_1p_3p_6$. In this case, there exists a functionally independent first integral given by

$$I_1(p) = p_1 + p_4.$$

Indeed, one can easily verify that $\{H^1_{0,D}, I\} = 0$. The Jacobian matrix of $H^1_{0,D}$, $I$, $C_1$, $C_2$ is given by

$$\begin{bmatrix} p_4 & p_5\alpha_1 & p_6\alpha_1 & p_1 & p_2\alpha_1 & p_3\alpha_1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 2p_1 & 2p_2 & 2p_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2p_4 & 2p_5 & 2p_6 \end{bmatrix}$$

which has rank 4 (i.e., these functions are functionally independent).

This system has equations of motion

$$\begin{cases} \dot p_1 = \alpha_1(p_2p_6 - p_3p_5) \\ \dot p_2 = p_3p_4 - \alpha_1p_1p_6 \\ \dot p_3 = \alpha_1p_1p_5 - p_2p_4 \end{cases} \qquad \begin{cases} \dot p_4 = \alpha_1(p_3p_5 - p_2p_6) \\ \dot p_5 = p_1p_6 - \alpha_1p_3p_4 \\ \dot p_6 = \alpha_1p_2p_4 - p_1p_5. \end{cases}$$

The equilibrium states of this system are given as follows:

$$\mathsf{e}_1^{\mu} = (0,0,0,\mu_4,\mu_5,\mu_6) \qquad \mathsf{e}_2^{\mu} = (\mu_1,\mu_2,\mu_3,0,0,0) \qquad \mathsf{e}_3^{\mu\nu} = (0,\nu_2,\mu_3,0,\nu_5,\frac{\mu_3\nu_5}{\nu_2})$$

$$\mathsf{e}_4^{\nu} = (0,0,\nu_3,0,0,\nu_6) \qquad \mathsf{e}_5^{\nu} = (\nu_1,0,0,\nu_4,0,0)$$

Here we assume that $e_i^{\mu\nu} \neq \mathbf{0}$, $i = 1,\ldots,6$.

We were unable to determine the nature of the equilibria for every equilibrium state. We only state those equilibria and conditions for which we could determine their stability nature. For the omitted equilibrium states both spectral and energy methods provided no insight into their stability nature.

5.2.5 THEOREM. *The equilibrium states have the following behaviour:*

*(i) The equilibrium states* $\mathsf{e}_1^{\mu}$ *are stable if* $\mu_5 = \mu_6 = 0$.

*(ii) The equilibrium states* $\mathsf{e}_2^{\mu}$ *are stable if* $\mu_2 = \mu_3 = 0$.

*(iii) The equilibrium states* $\mathsf{e}_3^{\mu\nu}$ *are unstable if* $\operatorname{sgn}(\nu_2) \neq \operatorname{sgn}(\nu_5)$ *and* $0 < \alpha_1 < -\frac{2\mu_2\mu_5}{\mu_2^2+\mu_5^2}$.

*(iv) The equilibrium states* $\mathsf{e}_4^{\mu}$ *are unstable if* $\operatorname{sgn}(\nu_3) \neq \operatorname{sgn}(\nu_6)$ *and* $0 < \alpha_1 < -\frac{2\mu_3\mu_6}{\mu_3^2+\mu_6^2}$.

*(v) The equilibrium states* $\mathsf{e}_5^{\nu}$ *are stable.*

PROOF. Let $H_\lambda(p) = \lambda_1 H^1_{0,D}(p) + \lambda_2 C_1(p) + \lambda_3 C_2(p) + \lambda_4 I_1(p)$.

(i) Assume $\mu_5 = \mu_6 = 0$. Let $\lambda_1 = 1$, $\lambda_3 = \frac{1}{2}$, $\lambda_2 = \alpha_1^2$, and $\lambda_4 = -\mu_4$. We have $\mathbf{d}H_\lambda(\mathsf{e}_1^{\boldsymbol{\mu}}) = 0$ and that the Hessian

$$\mathbf{d}^2 H_\lambda(\mathsf{e}_1^{\boldsymbol{\mu}})_{|W\times W} = \begin{bmatrix} 4\alpha_1^2 & 0 & 2\alpha_1 & 0 \\ 0 & 4\alpha_1^2 & 0 & 2\alpha_1 \\ 2\alpha_1 & 0 & 2 & 0 \\ 0 & 2\alpha_1 & 0 & 2 \end{bmatrix}$$

where $W = \operatorname{span}\{e_2, e_3, e_5, e_6\}$, is positive definite for all $\mu_4 \neq 0$. Thus, by the generalized energy-Casimir method, the states $\mathsf{e}_1^{\boldsymbol{\mu}}$, $\mu_4 \neq 0$ are stable. (When $\mu_5 \neq 0$ or $\mu_6 \neq 0$ both spectral and energy-Casimir results provide no useful insight.)

(ii) Assume $\mu_2 = \mu_3 = 0$. Let $\lambda_1 = 1$, $\lambda_2 = \frac{1}{2}$, $\lambda_3 = \alpha_1^2$, and $\lambda_4 = -\mu_1$. We have $\mathbf{d}H_\lambda(\mathsf{e}_2^{\boldsymbol{\mu}}) = 0$ and that the Hessian

$$\mathbf{d}^2 H_\lambda(\mathsf{e}_2^{\boldsymbol{\mu}})_{|W\times W} = \begin{bmatrix} 2 & 0 & 2\alpha_1 & 0 \\ 0 & 2 & 0 & 2\alpha_1 \\ 2\alpha_1 & 0 & 4\alpha_1^2 & 0 \\ 0 & 2\alpha_1 & 0 & 4\alpha_1^2 \end{bmatrix}$$

where $W = \operatorname{span}\{e_2, e_3, e_5, e_6\}$, is positive definite for all $\mu_1 \neq 0$. Thus, by the generalized energy-Casimir method, the states $\mathsf{e}_2^{\boldsymbol{\mu}}$, $\mu_1 \neq 0$ are stable.

(iii) The linearization of the system at $\mathsf{e}_3^{\boldsymbol{\mu\nu}}$ has eigenvalues $\lambda_{1,2,3,4} = 0$ and

$$\lambda_{5,6} = \pm i \frac{\sqrt{\alpha_1(\nu_2^2+\mu_3^2)(2\nu_2\nu_5+\alpha_1(\nu_2^2+\nu_5^2))}}{\nu_2}.$$

Thus, either $\lambda_5$ or $\lambda_6$ will be positive when $2\nu_2\nu_5 + \alpha_1\left(\nu_2^2 + \nu_5^2\right) < 0$. This is equivalent to the condition $0 < \alpha_1 < -\frac{2\nu_2\nu_5}{\nu_2^2+\nu_5^2}$. Thus $\mathsf{e}_3^{\boldsymbol{\mu\nu}}$ is unstable if $\operatorname{sgn}(\nu_2) \neq \operatorname{sgn}(\nu_5)$ and $0 < \alpha_1 < -\frac{2\nu_2\nu_5}{\nu_2^2+\nu_5^2}$.

(iv) The linearization of the system at $\mathsf{e}_4^{\boldsymbol{\nu}}$ has eigenvalues $\lambda_{1,2,3,4} = 0$ and

$$\lambda_{5,6} = \pm\frac{i}{\nu_2}\sqrt{\alpha_1\left(2\nu_3\nu_6 + \alpha_1\left(\nu_3^2 + \nu_6^2\right)\right)}.$$

Thus, either $\lambda_5$ or $\lambda_6$ will be positive when $0 < \alpha_1 < -\frac{2\nu_3\nu_6}{\nu_3^2+\nu_6^2}$. Thus $\mathsf{e}_4^{\boldsymbol{\nu}}$ is unstable if $\operatorname{sgn}(\nu_3) \neq \operatorname{sgn}(\nu_6)$ and $0 < \alpha_1 < -\frac{2\nu_3\nu_6}{\nu_3^2+\nu_6^2}$.

(v) Let $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = 1$, and $\lambda_4 = -2\left(\nu_1 + \nu_4\right)$. We have $\mathbf{d}H_\lambda(\mathsf{e}_5^{\boldsymbol{\nu}}) = 0$ and that the Hessian

$$\mathbf{d}^2 H_\lambda(\mathsf{e}_5^{\boldsymbol{\nu}})_{|W\times W} = \begin{bmatrix} 2 & 0 & 2\alpha_1 & 0 \\ 0 & 2 & 0 & 2\alpha_1 \\ 2\alpha_1 & 0 & 2 & 0 \\ 0 & 2\alpha_1 & 0 & 2 \end{bmatrix}$$

where $W = \operatorname{span}\{e_2, e_3, e_5, e_6\}$, is positive definite for all $\nu_1, \nu_4$ (as $0 < \alpha_1 < 1$). Thus, by the generalized energy-Casimir method, the states $\mathsf{e}_5^{\boldsymbol{\nu}}$ are stable.

□

# Chapter 6

# Some illustrative examples of optimal control problems

## 6.1 Path planning on $\mathsf{SO}(3)$

In this section we provide several examples of optimal control problems related to our classification of cost-extended systems on $\mathsf{SO}(3)$. In each case we find closed-form solutions for the extremal curves on the sphere $\mathbb{S}^2$, obtained by projecting the extremal curves on $\mathsf{SO}(3)$.

The examples we consider show how our classifications of Hamilton-Poisson systems, under affine equivalence, and of cost-extended systems, under cost-equivalence, can be used to investigate optimal control problems on the group $\mathsf{SO}(3)$. We provide an example corresponding to each number of possible inputs. Specifically, we consider a single-input inhomogeneous, two-input homogeneous, two-input inhomogeneous, and three input (homogeneous) system. (Recall that a single-input homogeneous system cannot be controllable.)

In general, an optimal control problem of interest will not correspond to a representative of our classification of cost-extended systems. Thus, we require a way of transforming the extremal curves of our representative cost-extended system to the system of interest. In this regard, our investigation of the optimal control problems corresponding to two-input homogeneous systems are of particular interest. In this case, we consider two systems which are cost-equivalent and show how the expressions for the extremal curves on $\mathsf{SO}(3)$ for the one system can be obtained from the other; using a Lie group isomorphism. We show that the construction of the appropriate Lie group isomorphism is far simpler than integrating the system independently.

A large part of the investigation of optimal control problems on Lie groups reduces to the investigation of an associated Hamilton-Poisson system and the integral curves of its Hamiltonian vector field. However, the final step of obtaining expressions for the extremal curves on the group still requires a further step of integration. In our case, for obtaining expressions for the Euler angles. Our examples indicate the level of complexity that can be involved in this step, which turns out to be nontrivial; even for rather simple representative systems.

We do note that there are a number of cost-extended systems (optimal control problems) we do not consider in this thesis. Clearly, any system whose associated Hamilton-Poisson system is equivalent to one of the systems of type II cannot be investigated. This is because we were unable to obtain expressions for the integral curves of such systems due to computational complexities.

We also do not consider any system whose associated Hamilton-Poisson system is equivalent to one of the systems $H^2_{1,\alpha}$ or $H^2_{2,\alpha}$. These systems divide into a large number of subcases; some of which have rather complex expressions for their integral curves. To obtain the extremal trajectories on the group requires the integration of these expressions which we were not always able to perform.

Each example is presented following the approach outlined in section 3.5. In each case, we find the associated Hamilton-Poisson system, $H$, as well as the representative of theorem 3.3.1 it is equivalent to. We then use the expressions of the integral curves of $\vec{H}$ to obtain the optimal controls and to solve for the Euler angles. Using the expressions for the Euler angles we then obtain the explicit expressions for the extremal trajectories on the sphere $\mathbb{S}^2$. In each case we also plot these trajectories on $\mathbb{S}^2$ for some specific values.

### 6.1.1 The system $(\Sigma^{(2,0)}, \chi^2_\eta)$

MATHEMATICA. *The details of the calculations for this system are contained in the Mathematica file:*

*Thesis Mathematica\SO(3)\Applications\Sys20.nb*

We begin by considering the cost-extended system $(\Sigma^{(2,0)}, \chi^2_\eta)$. In particular, we consider the case when $\eta = 0$. Thus, we are considering the optimal control problem given by

$$\begin{aligned} \dot{g}(t) &= g(t)\ (u_1(t)E_2 + u_2(t)E_3)\,, \quad g(\cdot) : [0,T] \to \mathsf{SO}(3),\ u(\cdot) : [0,T] \to \mathbb{R}^2 \\ g(0) &= g_0, \quad g(T) = g_1, \quad g_0, g_1 \in \mathsf{SO}(3), \quad T > 0 \text{ fixed} \\ \mathcal{J} &= \int_0^T (u_1(t)^2 + u_2(t)^2)dt \to \min. \end{aligned} \tag{6.1}$$

The associated quadratic Hamilton-Poisson system is $H_2^{(2,0)}(p) = \frac{1}{2}(p_2^2 + p_3^2)$, and the optimal controls associated to this system are given by $(u_1(t), u_2(t)) = (p_2(t), p_3(t))$.

It follows that the system $H_2^{(2,0)}(p)$ is linearly equivalent to the system $H^1(\tilde{p}) = \frac{1}{2}\tilde{p}_1^2$. Indeed,

$$\psi : \tilde{p} \mapsto \tilde{p} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

is a linear isomorphism such that $T\psi \cdot \vec{H}^1(\tilde{p}) = (\vec{H}_2^{(2,0)} \circ \psi)(\tilde{p})$. The original integral curves for the system $H^1(\tilde{p})$ can be expressed as follows (see theorem 3.3.5):

$$\begin{aligned} \bar{p}_1(t) &= \sigma\sqrt{2\tilde{h}_0} \\ \bar{p}_2(t) &= \sigma\sqrt{\tilde{c}_0 - 2\tilde{h}_0}\ \sin(\sqrt{2\tilde{h}_0}\, t) \\ \bar{p}_3(t) &= \sqrt{\tilde{c}_0 - 2\tilde{h}_0}\ \cos(\sqrt{2\tilde{h}_0}\, t). \end{aligned}$$

Here $\tilde{h}_0 = H^1(\tilde{p}(0))$ and $\tilde{c}_0 = C(\tilde{p}(0))$. Let $h_0 = H_2^{(2,0)}(\psi(\tilde{p}(0)))$ and $c_0 = C(\psi(\tilde{p}(0)))$. We then obtain the relations $c_0 = \tilde{c}_0$ and $\tilde{h}_0 = \frac{1}{2}c_0 - h_0$. The integral curves of the system $H_2^{(2,0)}$ are obtained by applying $\psi$ to the integral curves of the system $H^1$. Explicitly, we get

$$\begin{aligned}
p_1(t) &= -\sigma\sqrt{c_0 - 2h_0} \\
p_2(t) &= -\sigma\sqrt{2h_0}\, \sin(t\sqrt{c_0 - 2h_0}) \\
p_3(t) &= -\sqrt{2h_0}\, \cos(t\sqrt{c_0 - 2h_0}).
\end{aligned}$$

For the remainder of this example we shall for convenience assume that $\sigma = -1$. Using the Euler angles $\phi_1, \phi_2, \phi_3$ it follows that $g(t)$ is given as in formula (3.16) where

$$\begin{aligned}
\cos(\phi_2) &= \frac{\sqrt{c_0 - 2h_0}}{\sqrt{c_0}} & \sin(\phi_2) &= \pm\frac{\sqrt{2h_0}}{\sqrt{c_0}} \\
\cos(\phi_3) &= \mp\cos(\sqrt{c_0 - 2h_0}\, t) & \sin(\phi_3) &= \pm\sin(\sqrt{c_0 - 2h_0}\, t)
\end{aligned}$$

and

$$\dot{\phi}_1(t) = \frac{\sqrt{c_0}}{p_2^2 + p_3^2}\left(p_2\frac{\partial H^{(2,0)}}{\partial p_2} + p_3\frac{\partial H^{(2,0)}}{\partial p_3}\right) = \frac{\sqrt{c_0}}{p_2^2 + p_3^2}\,(p_2p_2 + p_3p_3) = \sqrt{c_0}.$$

Therefore $\phi_1(t) = \int_0^t \dot{\phi}_1(\tau)d\tau = \sqrt{c_0}\, t$. We wish to obtain the trajectories of this system starting from the identity, i.e., $\mathbf{1} \in \mathsf{SO}(3)$. Thus, we translate this trajectory to start at the identity using the left translation

$$g(0)^{-1} = \frac{1}{\sqrt{c_0}}\begin{bmatrix} \sqrt{c_0 - 2h_0} & 0 & -\sqrt{2}\sqrt{h_0} \\ 0 & -\sqrt{c_0} & 0 \\ -\sqrt{2}\sqrt{h_0} & 0 & -\sqrt{c_0 - 2h_0} \end{bmatrix}.$$

Let $x_i(t) = g(0)^{-1}g(t)\,\vec{e}_i,\ i = 1, 2, 3$. The resulting trajectories on the sphere $\mathbb{S}^2$, using the projection $x_1(t) = g(0)^{-1}g(t)\,\vec{e}_1$, are given by

$$x_1(t) = \frac{1}{c_0}\begin{bmatrix} (c_0 - 2h_0) + 2h_0\, \cos(t\sqrt{c_0}) \\ -\sqrt{2c_0h_0}\, \sin(t\sqrt{c_0}) \\ \sqrt{2h_0\,(c_0 - 2h_0)}\,(\cos(t\sqrt{c_0}) - 1) \end{bmatrix}.$$

We also have

$$x_2(t) = \frac{1}{c_0}\begin{bmatrix} \sqrt{2h_0}\left(\sqrt{c_0}\cos(\delta\, t)\sin\left(\sqrt{c_0}\, t\right) + \delta\left(1 - \cos\left(\sqrt{c_0}\, t\right)\right)\sin(\delta\, t)\right) \\ c_0\, \cos\left(\sqrt{c_0}\, t\right)\cos(\delta\, t) + \delta\sqrt{c_0}\, \sin\left(\sqrt{c_0}\, t\right)\sin(\delta\, t) \\ \delta\sqrt{c_0}\, \cos(\delta\, t)\sin\left(\sqrt{c_0}\, t\right) - \sin(\delta\, t)\left(2h_0 + \delta^2\cos\left(\sqrt{c_0}\, t\right)\right) \end{bmatrix}$$

and

$$x_3(t) = \frac{1}{c_0}\begin{bmatrix} \sqrt{2h_0}\left(\sqrt{c_0}\, \sin\left(\sqrt{c_0}\, t\right)\sin(\delta\, t) - \delta\left(1 - \cos\left(\sqrt{c_0}\, t\right)\right)\cos(\delta\, t)\right) \\ c_0\, \cos\left(\sqrt{c_0}\, t\right)\sin(\delta\, t) - \delta\sqrt{c_0}\, \cos(\delta\, t)\sin\left(\sqrt{c_0}\, t\right) \\ \delta\sqrt{c_0}\, \sin\left(\sqrt{c_0}\, t\right)\sin(\delta\, t) + \cos(\delta\, t)\left(2h_0 + \delta^2\cos\left(\sqrt{c_0}\, t\right)\right). \end{bmatrix}$$

where $\delta = \sqrt{c_0 - 2h_0}$. The curve $g(t) \in \mathsf{SO}(3)$ is then given by $g(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) \end{bmatrix}$. We plot the three projections $x_1(t), x_2(t), x_3(t)$ onto $\mathbb{S}^2$ for some specific initial values (see Figure 6.1).

6.1.1 REMARK. In general it is only meaningful to talk about the first projection $x_1(t)$. Only in this subsection do we consider the plots of all three projections as we want to compare the expressions between two cost-equivalent systems.

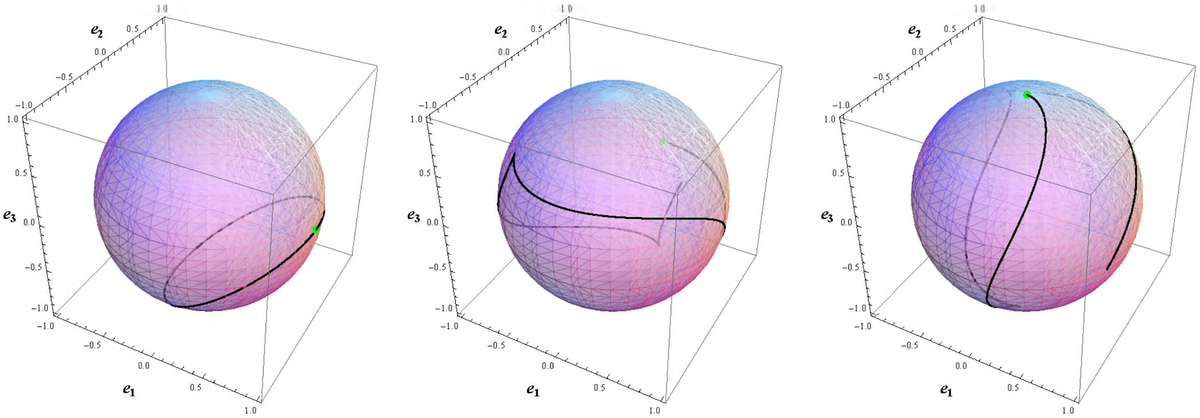


Figure 6.1: $x_1(t), x_2(t), x_3(t)$ over the interval $[0, 2\pi]$ for $c_0 = 6$, $h_0 = 2$

### 6.1.2 An equivalent $(2,0)$ system

We now consider the following optimal control problem given by

$$\begin{aligned} &\dot{g}(t) = g(t)\ (u_1(t)E_1 + u_2(t)E_2)\,, \quad g(\cdot) : [0,T] \to \mathsf{SO}(3),\ u(\cdot) : [0,T] \to \mathbb{R}^2 \\ &g(0) = g_0, \quad g(T) = g_1, \quad g_0, g_1 \in \mathsf{SO}(3), \quad T > 0 \text{ fixed} \\ &\quad \mathcal{J} = \int_0^T (u_1(t)^2 + u_2(t)^2)dt \to \min. \end{aligned} \tag{6.2}$$

Note that this system is not a representative of our classification under cost-extended equivalence. This example is included to illustrate how one can obtain the integral curves of one system from an equivalent system. We calculate the trajectories of this system in two ways. First, we calculate the trajectories by using the equivalence of this system to the system previously investigated. Second, we integrate this system independently. We give an analysis of the results obtained by these two approaches.

#### The transformed system

We begin by noticing that the systems

$$\Sigma^{(2,0)}\ :\ \dot{g} = g(u_1E_2 + u_2E_3) \quad \text{and} \quad \Sigma'\ :\ \dot{g} = g(u_1E_1 + u_2E_2)$$

are detached feedback equivalent. Indeed, there exists $\psi \in \mathsf{Aut}(\mathfrak{so}(3)) \cong \mathsf{SO}(3)$ such that $\psi \cdot \Gamma^{(2,0)} = \Gamma'$. Explicitly this is given by

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Now, every Lie algebra automorphism of $\mathfrak{so}(3)$ is an inner automorphism. That is, any automorphism can be written as $\mathrm{Ad}_g : X \mapsto gXg^{-1}$ for some $g \in \mathsf{SO}(3)$. Using the fact that the $i^{th}$ column of $\psi \in \mathsf{SO}(3)$ is given by $\mathrm{Ad}_h E_i$ (in the basis $E_1, E_2, E_3$) we can calculate the corresponding $h$ explicitly. In this case it follows that

$$h = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

It then follows that the corresponding group automorphism $\phi \in \mathsf{Aut}(\mathsf{SO}(3))$ is given by the conjugation map, i.e., $\phi : g \mapsto hgh^{-1}$. Thus, the projection onto the sphere $\mathbb{S}^2$ of the transformed system is given by $\tilde{x}_1(t) = \phi(g(0)^{-1}g(t))\,\vec{e}_1$. Explicitly, we have

$$\tilde{x}_1(t) = \frac{1}{c_0}\begin{bmatrix} \cos\left(\sqrt{c_0}\,t\right)\cos\left(\delta\,t\right)c_0 + \sin\left(\sqrt{c_0}\,t\right)\sin\left(\delta\,t\right)\sqrt{c_0}\,\delta \\ \cos\left(\delta\,t\right)\sin\left(\sqrt{c_0}\,t\right)\sqrt{c_0}\,\delta - \sin\left(\delta\,t\right)\left(\cos\left(\sqrt{c_0}\,t\right)\delta^2 + 2h_0\right) \\ \sqrt{2}\sqrt{c_0 h_0}\left(\cos\left(\delta\,t\right)\sin\left(\sqrt{c_0}\,t\right) - \left(-1+\cos\left(\sqrt{c_0}\,t\right)\right)\sin\left(\delta\,t\right)\sqrt{1-\frac{2h_0}{c_0}}\right) \end{bmatrix}.$$

Here $\delta = \sqrt{c_0 - 2h_0}$. (The expressions for $\tilde{x}_2(t)$ and $\tilde{x}_3(t)$ are obtained in a similar fashion.) We plot the three projections $\tilde{x}_1(t), \tilde{x}_2(t), \tilde{x}_3(t)$ onto $\mathbb{S}^2$ for some specific initial values (see Figure 6.2).

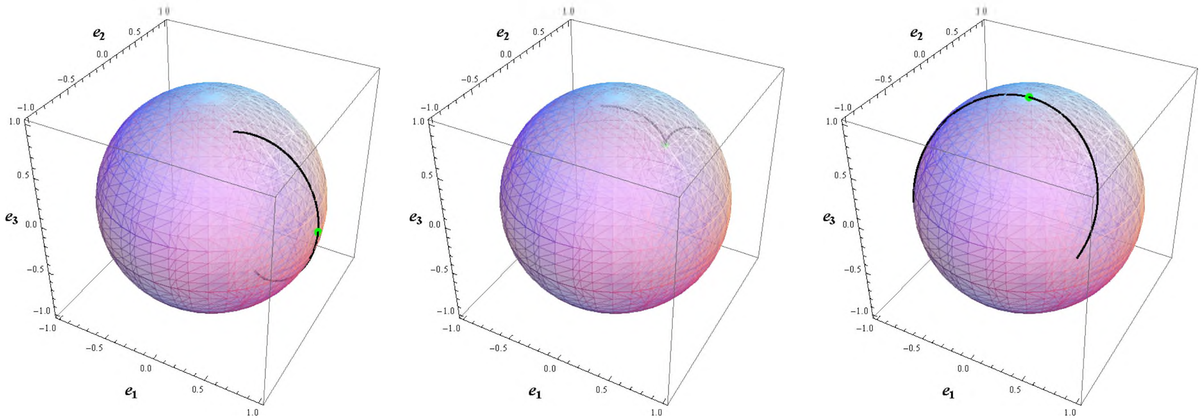


Figure 6.2: $\tilde{x}_1(t), \tilde{x}_2(t), \tilde{x}_3(t)$ over the interval $[-\frac{\pi}{4}, \frac{\pi}{4}]$ for $c_0 = 6$, $h_0 = 2$

### Independent solution

We now solve the optimal control problem (6.2) independently. The associated quadratic Hamilton-Poisson system is given by $H'(p) = \frac{1}{2}(p_1^2 + p_2^2)$ and the optimal controls are given by $(u_1(t), u_2(t)) = (p_1(t), p_2(t))$. It follows that the system $H'$ is equivalent to the system $H_2^{(2,0)}$. Indeed,

$$\psi = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

is a linear Poisson automorphism such that $T\psi \cdot \vec{H}_2^{(2,0)}(p) = (\vec{H}' \circ \psi)(p)$. Let $\tilde{h}_0 = H_2^{(2,0)}(p(0))$ and $\tilde{c}_0 = C(p(0))$. Let $h_0 = H'(\psi(p(0)))$ and $c_0 = C(\psi(p(0)))$. We obtain the relations $\tilde{c}_0 = c_0$ and $\tilde{h}_0 = h_0$. (Recall that $c_0 > 2h_0 > 0$.) Therefore the integral curves of $H'$ are given by

$$\begin{aligned} p_1(t) &= \sqrt{2}\sin\left(\sqrt{c_0 - 2h_0}\, t\right)\sqrt{h_0} \\ p_2(t) &= -\sqrt{2}\cos\left(\sqrt{c_0 - 2h_0}\, t\right)\sqrt{h_0} \\ p_3(t) &= \sqrt{c_0 - 2h_0}. \end{aligned}$$

Using the Euler angles it follows that $g(t)$ is given as in formula (3.16), where

$$\begin{aligned} \cos(\phi_2) &= \sqrt{\frac{2h_0}{c_0}}\, \sin\left(\sqrt{c_0 - 2h_0}\, t\right), & \sin(\phi_2) &= \pm\frac{1}{\sqrt{c_0}}\sqrt{c_0 - 2h_0\, \sin\left(\sqrt{c_0 - 2h_0}\, t\right)^{2}} \\ \cos(\phi_3) &= \pm\frac{\sqrt{c_0 - 2h_0}}{\sqrt{c_0 - 2h_0\, \sin\left(\sqrt{c_0 - 2h_0}\, t\right)^{2}}}, & \sin(\phi_3) &= \pm\frac{\sqrt{2h_0}\cos\left(\sqrt{c_0 - 2h_0}\, t\right)}{\sqrt{c_0 - 2h_0\, \sin\left(\sqrt{c_0 - 2h_0}\, t\right)^{2}}} \end{aligned}$$

and

$$\begin{aligned} \dot{\phi}_1(t) &= \frac{\sqrt{c_0}}{p_2^2 + p_3^2}\left(\frac{\partial H'}{\partial p_2}p_2 + \frac{\partial H'}{\partial p_3}p_3\right) \\ &= \frac{2\cos\left(\sqrt{c_0 - 2h_0}\, t\right)^{2}\sqrt{c_0}h_0}{c_0 - 2h_0 + 2\cos\left(\sqrt{c_0 - 2h_0}\, t\right)^{2}h_0}. \end{aligned}$$

Hence, we have that

$$\phi_1(t) = \int_0^t \dot{\phi}_1(\tau)d\tau = \sqrt{c_0}\, t - \arctan\left(\sqrt{\frac{c_0 - 2h_0}{c_0}}\, \tan\left(\sqrt{c_0 - 2h_0}\, t\right)\right).$$

We wish to obtain the trajectories of this system starting at the identity, i.e., $\mathbf{1} \in \mathsf{SO}(3)$. Therefore we translate this trajectory using the left translation

$$g(0)^{-1} = \frac{1}{\sqrt{c_0}}\begin{bmatrix} 0 & 0 & -\sqrt{c_0} \\ -\sqrt{2h_0} & \sqrt{c_0 - 2h_0} & 0 \\ \sqrt{c_0 - 2h_0} & \sqrt{2h_0} & 0 \end{bmatrix}.$$

Let $x_i(t) = g(0)^{-1}g(t)\,\vec{e}_i$, $i = 1, 2, 3$. The resulting trajectories on the sphere $\mathbb{S}^2$, using the projection $x_1(t) = g(0)^{-1}g(t)\,\vec{e}_1$, are given by

$$x_1(t) = \frac{1}{c_0}\begin{bmatrix} \cos(\phi_1)\sqrt{c_0\left(c_0 - 2h_0\sin\left(\sqrt{c_0 - 2h_0}\, t\right)^{2}\right)} \\ -2h_0\sin\left(\sqrt{c_0 - 2h_0}\, t\right) - \sin(\phi_1)\sqrt{(c_0 - 2h_0)\left(c_0 - 2h_0\sin\left(\sqrt{c_0 - 2h_0}\, t\right)^{2}\right)} \\ \sqrt{2h_0}\left(\sin\left(\sqrt{c_0 - 2h_0}\, t\right)\sqrt{c_0 - 2h_0} - \sin(\phi_1)\sqrt{c_0 - 2h_0\sin\left(\sqrt{c_0 - 2h_0}\, t\right)^{2}}\right) \end{bmatrix}.$$

The curve $g(t) \in \mathsf{SO}(3)$ is given by $g(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) \end{bmatrix}$. We plot the three projections $x_1(t), x_2(t), x_3(t)$ onto $\mathbb{S}^2$ for some specific initial values (see Figure 6.3).

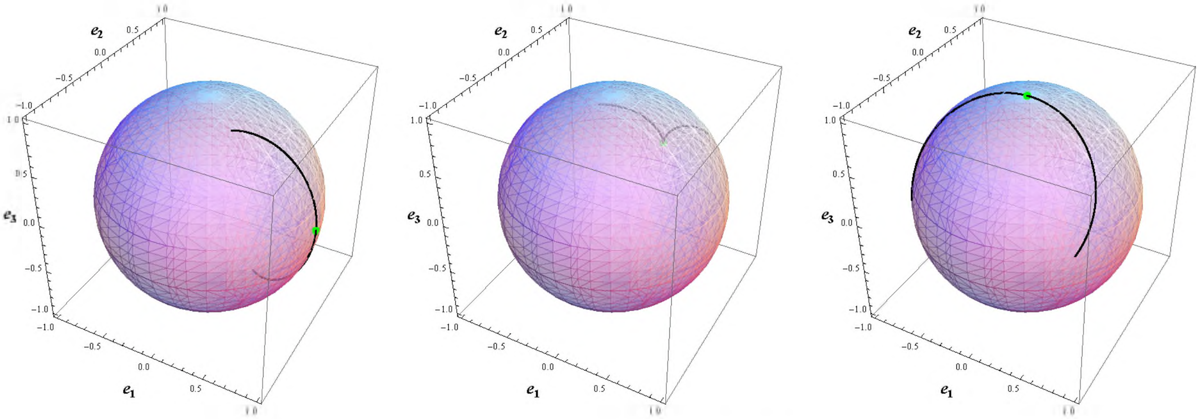


Figure 6.3: $x_1(t), x_2(t), x_3(t)$ over the interval $[-\frac{\pi}{4}, \frac{\pi}{4}]$ for $c_0 = 6,\ h_0 = 2$

Note that the trajectories $x_1(t), x_2(t), x_3(t)$ coincide (at least graphically) with the trajectories $\tilde{x}_1(t), \tilde{x}_2(t), \tilde{x}_3(t)$ corresponding to Figure 6.2. The issue we have with integrating the system (6.2) independently is that the solution is not defined for all values of $t$; this is due to the complexity of the expression for the Euler angle $\phi_1$. However, we can simplify this expression in the following way. Let $\gamma = \arctan\left(\sqrt{1 - \frac{2h_0}{c_0}} \tan\left(\sqrt{c_0 - 2h_0}\, t\right)\right)$. It then follows that

$$\cos(\gamma) = \frac{1}{\sqrt{1 + \left(1 - \frac{2h_0}{c_0}\right) \tan\left(\sqrt{c_0 - 2h_0}\, t\right)^2}} = \frac{\sqrt{c_0} \cos\left(\sqrt{c_0 - 2h_0}\, t\right)}{\sqrt{c_0 - 2h_0 \sin\left(\sqrt{c_0 - 2h_0}\, t\right)^2}}$$

and

$$\sin(\gamma) = \frac{\sqrt{1 - \frac{2h_0}{c_0}} \tan\left(\sqrt{c_0 - 2h_0}\, t\right)}{\sqrt{1 + \left(1 - \frac{2h_0}{c_0}\right) \tan\left(\sqrt{c_0 - 2h_0}\, t\right)^2}} = \frac{\sqrt{c_0 - 2h_0} \sin\left(\sqrt{c_0 - 2h_0}\, t\right)}{\sqrt{c_0 - 2h_0 \sin\left(\sqrt{c_0 - 2h_0}\, t\right)^2}}.$$

Let $x_1^1(t)$ denote the first row of $x_1(t)$ and let $\delta = \sqrt{c_0 - 2h_0 \sin\left(\sqrt{c_0 - 2h_0}\, t\right)^2}$. Thus we have that

$$\begin{aligned}
x_1^1(t) &= \frac{1}{c_0} \cos(\phi_1) \sqrt{c_0 \left(c_0 - 2h_0 \sin\left(\sqrt{c_0 - 2h_0}\, t\right)^2\right)} \\
&= \frac{\sqrt{c_0}}{c_0} \delta \left(\frac{\cos\left(\sqrt{c_0}\, t\right) \cos\left(\sqrt{c_0 - 2h_0}\, t\right) \sqrt{c_0}}{\delta} + \frac{\sin\left(\sqrt{c_0}\, t\right) \sin\left(\sqrt{c_0 - 2h_0}\, t\right) \sqrt{c_0 - 2h_0}}{\delta}\right) \\
&= \cos\left(\sqrt{c_0}\, t\right) \cos\left(\sqrt{c_0 - 2h_0}\, t\right) + \sin\left(\sqrt{c_0}\, t\right) \sin\left(\sqrt{c_0 - 2h_0}\, t\right) \sqrt{\tfrac{c_0 - 2h_0}{c_0}}.
\end{aligned}$$

We can now see that this is exactly the first row of $\tilde{x}_1(t)$. A similar argument shows that $x_2(t) = \tilde{x}_2(t)$ and $x_3(t) = \tilde{x}_3(t)$.

It can be seen in this case that it is thus far simpler to obtain the integral curves for the optimal control problem (6.2) by finding and applying the Lie group automorphism to the integral curves of the optimal control problem (6.1). This approach is extremely useful in the investigation of optimal control problems on $\mathsf{SO}(3)$. Indeed, one can always integrate the simplest representative and then obtain the integral curves for any equivalent systems by finding the corresponding Lie group automorphism taking the one system to the other. (Finding the Lie group automorphism is a far simpler process than integrating each system independently.) Although there is no exact way to know which system in an equivalence class will be the easiest to integrate, experience in working with these systems allows one to make a "well educated guess". In particular, we try to look for the representative which contains the least number of parameters (already done in our classification of cost-extended systems) and most importantly which gives the simplest expression for the differential equation describing the angle $\phi_1$.

### 6.1.3 The system $(\Sigma_\alpha^{(1,1)}, \chi_\eta)$

MATHEMATICA. *The details of the calculations for this system are contained in the Mathematica file:*

*Thesis Mathematica\SO(3)\Applications\Sys11.nb*

Here we consider the cost-extended system $(\Sigma_\alpha^{(1,1)}, \chi_\eta)$. We will only consider the case when $\alpha = 1$ and $\eta = 0$. This system is cost-equivalent to one investigated by Jurdjevic in [33]. This allows for a comparison of our results to previously known ones. However, in [33], Jurdjevic provides sketches of the extremal curves on $\mathbb{S}^2$ but does provide any explicit closed-form solutions. Using our approach we provide closed-form solutions for the extremal trajectories on $\mathbb{S}^2$. (Note that we do not consider the trajectories corresponding to constant controls.)

Thus in this section we are considering the optimal control problem given by

$$\begin{aligned}
&\dot{g}(t) = g(t)\,(E_2 + u_1(t)E_1)\,, \quad g(\cdot) : [0,T] \to \mathsf{SO}(3),\ u(\cdot) : [0,T] \to \mathbb{R} \\
&g(0) = g_0, \quad g(T) = g_1, \quad g_0, g_1 \in \mathsf{SO}(3), \quad T > 0 \text{ fixed} \\
&\mathcal{J} = \int_0^T u_1(t)^2 dt \to \min.
\end{aligned}$$

We know by theorem 2.1.5 that the associated quadratic Hamilton-Poisson system is described by the Hamiltonian function $H_1^1(p) = p_2 + \frac{1}{2}p_1^2$, and that the optimal controls associated to this system are given by $u_1(t) = p_1(t)$. This system is exactly one of the quadratic Hamilton-Poisson systems previously studied and the integral curves are given in theorem 3.3.9.

**Subcase 1:** $c_0 > h_0^2$

We shall first consider the case when $c_0 > h_0^2$. The explicit expressions for the integral curves in this case are given by

$$\begin{aligned}
\bar{p}_1(t) &= \sqrt{2(h_0 + \delta - 1)}\,\mathrm{cn}\,(\Omega\, t,\, k) \\
\bar{p}_2(t) &= h_0 - (h_0 + \delta - 1)\,\mathrm{cn}\,(\Omega\, t,\, k)^2 \\
\bar{p}_3(t) &= \sqrt{2\delta(h_0 + \delta - 1)}\,\mathrm{dn}\,(\Omega\, t,\, k)\,\mathrm{sn}\,(\Omega\, t,\, k)\,.
\end{aligned}$$

Here $\Omega = \sqrt{\delta}$, $k = \sqrt{\frac{h_0+\delta-1}{2\delta}}$, and $\delta = \sqrt{1 + c_0 - 2h_0}$. Using the Euler angles it follows that $g(t)$ is given as in formula (3.16), where

$$\cos(\phi_2) = \sqrt{\tfrac{2(h_0+\delta-1)}{c_0}}\,\mathrm{cn}\,(\Omega\, t,\ k)\ , \qquad \sin(\phi_2) = \pm\tfrac{1}{\sqrt{c_0}}\sqrt{(\delta+1)^2 - 4\delta\,\mathrm{dn}\,(\Omega\, t,\ k)^2}$$
$$\cos(\phi_3) = \pm\tfrac{2k\delta\,\mathrm{dn}(\Omega\, t,k)\,\mathrm{sn}(\Omega\, t,k)}{\sqrt{(\delta+1)^2-4\delta\,\mathrm{dn}(\Omega\, t,k)^2}}\,, \qquad \sin(\phi_3) = \pm\tfrac{\delta+1-2\delta\,\mathrm{dn}(\Omega\, t,k)^2}{\sqrt{(\delta+1)^2-4\delta\,\mathrm{dn}(\Omega\, t,k)^2}}$$

and

$$\dot\phi_1(t) = \frac{\sqrt{c_0}}{p_2^2+p_3^2}\left(p_2\frac{\partial H_1^1}{\partial p_2} + p_3\frac{\partial H_1^1}{\partial p_3}\right) = \sqrt{c_0}\,\frac{\delta+1-2\delta\,\mathrm{dn}\,(\Omega\, t,\ k)^2}{(\delta+1)^2-4\delta\,\mathrm{dn}\,(\Omega\, t,\ k)^2}$$

Therefore we have that

$$\phi_1(t) = \int_0^t \dot\phi_1(\tau)d\tau = \frac{\sqrt{c_0}}{2}\,t - \frac{\sqrt{c_0}(\delta+1)}{2\sqrt{\delta}(\delta-1)}\,\Pi\left(-\frac{4k^2\delta}{(\delta-1)^2},\ \mathrm{am}(\sqrt{\delta}\,t, k^2),\ k\right).$$

(Here $\Pi$ denotes the elliptic integral of the third kind; see appendix D.2.) We now translate this trajectory to start at the identity using the left translation

$$g(0)^{-1} = \frac{1}{\sqrt{c_0}}\begin{bmatrix} 2k\sqrt{\delta} & 0 & -\sigma(\delta-1) \\ 1-\delta & 0 & -\sigma 2k\sqrt{\delta} \\ 0 & \sigma\sqrt{c_0} & 0 \end{bmatrix}, \qquad \sigma = \mathrm{sgn}(\delta-1).$$

The resulting trajectories on the sphere $\mathbb{S}^2$, using the projection $x_1(t) = g(0)^{-1}g(t)\vec{e}_1$ are given by

$$x_1(t) = \frac{1}{\sqrt{c_0}}\begin{bmatrix} 4k^2\delta\,\mathrm{cn}(\sqrt{\delta}\,t,\ k) - \sigma(\delta-1)\cos(\phi_1)\sqrt{(\delta+1)^2-4\delta\,\mathrm{dn}(\sqrt{\delta}\,t,\ k)^2} \\ 2k\sqrt{\delta}\left(\sigma\cos(\phi_1)\sqrt{(\delta+1)^2-4\delta\,\mathrm{dn}(\sqrt{\delta}\,t,\ k)^2} - (\delta-1)\,\mathrm{cn}(\sqrt{\delta}\,t,\ k)\right) \\ \sigma\sqrt{c_0}\ \sin(\phi_1)\sqrt{(\delta+1)^2-4\delta\,\mathrm{dn}(\sqrt{\delta}\,t,\ k)^2} \end{bmatrix}$$

We plot this projection onto the sphere for two sets of specific values (see Figure 6.5, 6.6).

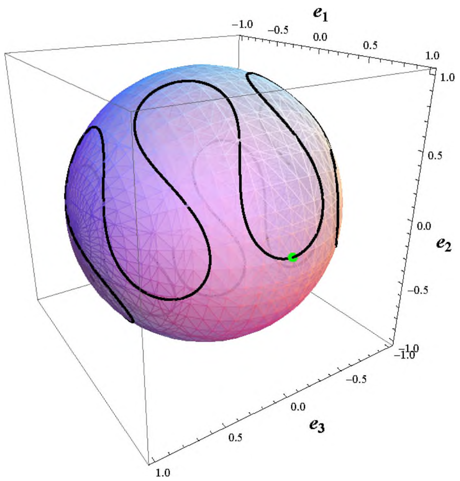


Figure (6.5) $c_0 = 30$, $h_0 = 2$ and $t \in [-3.5\pi, 3.5\pi]$

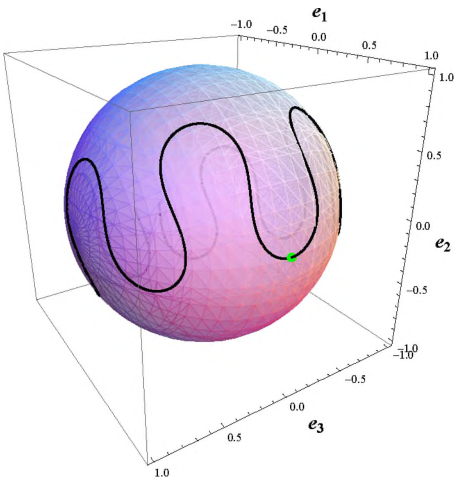


Figure (6.6) $c_0 = 100$, $h_0 = 1.5$ and $t \in [-2.5\pi, 2.5\pi]$

### Subcase 2: $c_0 = h_0^2$

Next, we consider the case when $c_0 = h_0^2$. The explicit expressions for the integral curves in this case are given by

$$\begin{aligned}\bar{p}_1(t) &= 2\sigma\Omega \operatorname{sech}(\Omega t)\\ \bar{p}_2(t) &= h_0 - 2(h_0-1)\operatorname{sech}(\Omega t)^2\\ \bar{p}_3(t) &= 2\sigma(h_0-1)\operatorname{sech}(\Omega t)\tanh(\Omega t).\end{aligned}$$

Here $\Omega = \sqrt{h_0 - 1}$. Using the Euler angles it that follows $g(t)$ is given as in formula (3.16), where

$$\cos(\phi_2) = \frac{2\sigma\Omega}{h_0}\operatorname{sech}(\Omega t) \qquad \sin(\phi_2) = \pm\frac{1}{h_0}\sqrt{h_0^2 - 4(h_0-1)\operatorname{sech}(\Omega t)^2}$$

$$\cos(\phi_3) = \pm\frac{2\sigma(h_0-1)\operatorname{sech}(\Omega t)\tanh(\Omega t)}{\sqrt{h_0^2-4(h_0-1)\operatorname{sech}(\Omega t)^2}} \qquad \sin(\phi_3) = \pm\frac{h_0-2(h_0-1)\operatorname{sech}(\Omega t)^2}{\sqrt{h_0^2-4(h_0-1)\operatorname{sech}(\Omega t)^2}}$$

and

$$\dot{\phi}_1(t) = \frac{h_0}{p_2^2+p_3^2}\left(p_2\frac{\partial H_1^1}{\partial p_2} + p_3\frac{\partial H_1^1}{\partial p_3}\right) = 1 - \frac{2(h_0-2)(h_0-1)}{4+h_0(h_0\cosh(\Omega t)^2-4)}.$$

Therefore we have that $\phi_1(t) = \int_0^t \dot{\phi}_1(\tau)d\tau = t + \arctan\left(\frac{2\Omega\tanh(\Omega t)}{2-h_0}\right)$. We now translate this trajectory to start at the identity using the left translation

$$g(0)^{-1} = \frac{1}{h_0}\begin{bmatrix} 2\sigma\Omega & 0 & \varsigma(2-h_0)\\ 2-h_0 & 0 & -2\varsigma\sigma\Omega\\ 0 & \varsigma h_0 & 0\end{bmatrix}, \quad \text{where} \quad \varsigma = \operatorname{sgn}(h_0-2).$$

The resulting trajectories on the sphere $\mathbb{S}^2$, using the projection $x_1(t) = g(0)^{-1}g(t)\vec{e}_1$ are given by

$$x_1(t) = \frac{1}{h_0^2}\begin{bmatrix}(h_0-2)^2\cos(t) + 4(h_0-1)\operatorname{sech}(\Omega t) + 2\Omega(h_0-2)\sin(t)\tanh(\Omega t)\\ 2\sigma\Omega(h_0-2)(\cos(t) - \operatorname{sech}(\Omega t)) + 4\sigma(h_0-1)\sin(t)\tanh(\Omega t)\\ h_0((h_0-2)\cos(t) - 2\Omega\sin(t)\tanh(\Omega t))\end{bmatrix}.$$

We plot this projection onto the sphere for some specific values (see Figure 6.7, 6.8, 6.9).

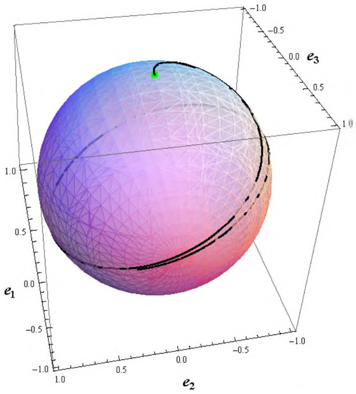


Figure (6.7) $h_0 = \frac{82}{51}$, $\sigma = -1$, and $t \in [0, 3\pi]$

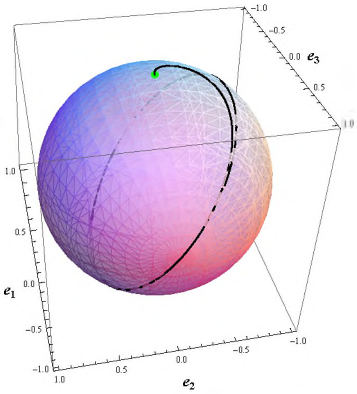


Figure (6.8) $h_0 = \frac{28}{17}$, $\sigma = -1$, and $t \in [0, \frac{5\pi}{2}]$

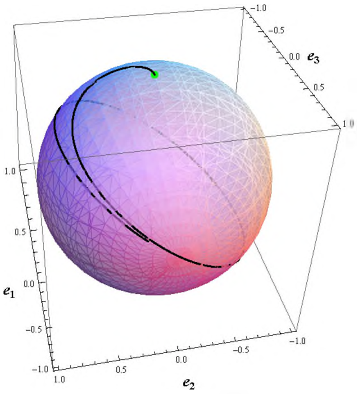


Figure (6.9) $h_0 = \frac{106}{17}$, $\sigma = 1$, and $t \in [0, \frac{27\pi}{10}]$

**Subcase 3:** $c_0 < h_0^2$

Lastly, we consider the case when $0 < c_0 < 2h_0$. The explicit expressions for the integral curves in this case are given by

$$\begin{aligned}
\bar{p}_1(t) &= \sigma\sqrt{2\delta}\,\frac{1 + k\,\mathrm{sn}\,(\Omega\,t,\,k)}{\mathrm{dn}\,(\Omega\,t,\,k)}\\
\bar{p}_2(t) &= h_0 + \delta - \frac{2\delta}{1 - k\,\mathrm{sn}\,(\Omega\,t,\,k)}\\
\bar{p}_3(t) &= -\sigma k\Omega\sqrt{2\delta}\,\frac{\mathrm{cn}\,(\Omega\,t,\,k)}{1 - k\,\mathrm{sn}\,(\Omega\,t,\,k)}.
\end{aligned}$$

Here $\Omega = \sqrt{h_0 - 1 + \delta}$, $k = \sqrt{\frac{h_0-1-\delta}{h_0-1+\delta}}$ and $\delta = \sqrt{{h_0}^2 - c_0}$. Using the Euler angles it that follows $g(t)$ is given as in formula (3.16), where

$$\begin{aligned}
\cos(\phi_2) &= \sigma\sqrt{\frac{2\delta}{c_0}}\,\frac{1 + k\,\mathrm{sn}\,(\Omega\,t,\,k)}{\mathrm{dn}\,(\Omega\,t,\,k)}, \qquad \sin(\phi_2) = \pm\frac{\sqrt{(c_0 - 2\delta) - (c_0 + 2\delta)\,k\,\mathrm{sn}\,(\Omega\,t,\,k)}}{\sqrt{c_0}\sqrt{1 - k\,\mathrm{sn}\,(\Omega\,t,\,k)}}\\
\cos(\phi_3) &= \mp\frac{\sigma\sqrt{2\delta}\,k\Omega\,\mathrm{cn}\,(\Omega\,t,\,k)}{\sqrt{(1 - k\,\mathrm{sn}\,(\Omega\,t,\,k))\,((c_0 - 2\delta) - (c_0 + 2\delta)\,k\,\mathrm{sn}\,(\Omega\,t,\,k))}}\\
\sin(\phi_3) &= \pm\frac{(h_0 - \delta) - (h_0 + \delta)\,k\,\mathrm{sn}\,(\Omega\,t,\,k)}{\sqrt{(1 - k\,\mathrm{sn}\,(\Omega\,t,\,k))\,((c_0 - 2\delta) - (c_0 + 2\delta)\,k\,\mathrm{sn}\,(\Omega\,t,\,k))}}
\end{aligned}$$

and

$$\dot{\phi}_1(t) = \frac{\sqrt{c_0}}{p_2^2 + p_3^2}\left(p_2\frac{\partial H_1^1}{\partial p_2} + p_3\frac{\partial H_1^1}{\partial p_3}\right) = \frac{\sqrt{c_0}\,((h_0 - \delta) - (\delta + h_0)\,k\,\mathrm{sn}\,(\Omega\,t,\,k))}{(c_0 - 2\delta) - (2\delta + c_0)\,k\,\mathrm{sn}\,(\Omega\,t,\,k)}.$$

Therefore we have that

$$\begin{aligned}
\phi_1(t) = \int_0^t \dot{\phi}_1(\tau)d\tau &= \frac{\sqrt{c_0}\,(h_0 + \delta)}{c_0 + 2\delta}\,t - \frac{2\delta\sqrt{c_0}\,(c_0 - 2h_0)}{\Omega\left(c_0^2 - 4\delta^2\right)}\,\Pi\left(\frac{k^2\,(2\delta + c_0)^2}{(c_0 - 2\delta)^2},\,\mathrm{am}\,(\Omega\,t,\,k)\,,\,k\right)\\
&\quad + \frac{1}{2}\arctan\left(\frac{2k\Omega\sqrt{c_0}}{(c_0 - 2h_0)}\,\mathrm{cd}\,(\Omega\,t,\,k)\right) - \frac{1}{2}\arctan\left(\frac{2k\Omega\sqrt{c_0}}{c_0 - 2h_0}\right)
\end{aligned}$$

We now translate this trajectory to start at the identity using the left translation

$$g(0)^{-1} = \frac{1}{\sqrt{c_0(c_0-2\delta)}}\begin{bmatrix} \sigma\sqrt{2\delta\,(c_0 - 2\delta)} & 0 & 2\delta - c_0\\ \sqrt{c_0 - 2\delta}\,(h_0 - \delta) & -\sigma\sqrt{2\delta c_0\,(h_0 - 1 - \delta)} & \sigma\sqrt{2\delta}\,(h_0 - \delta)\\ -\sigma\sqrt{2(c_0 - 2\delta)}\sqrt{\delta\,(h_0 - 1 - \delta)} & \sqrt{c_0}\,(\delta - h_0) & -2\delta\sqrt{h_0 - 1 - \delta} \end{bmatrix}.$$

The resulting trajectories on the sphere $\mathbb{S}^2$, using the projection $x_1(t) = g(0)^{-1}g(t)\vec{e}_1$ are given by

$$x_1(t) = \frac{1}{c_0 R_-(t)}\begin{bmatrix} 2\delta R_+(t) + (c_0 - 2\delta)\,S(t)\cos(\phi_1)\\ \sqrt{2\delta}\sigma\left((h_0 - \delta)\,(R_+(t) - S(t)\cos(\phi_1)) - \sqrt{c_0}\eta\,S(t)\sin(\phi_1)\right)\\ 2\delta\eta S(t)\cos(\phi_1) + \sqrt{c_0}\,(\delta - h_0)\,S(t)\sin(\phi_1) - 2\delta\eta R_+(t) \end{bmatrix}.$$

Here $\eta = \sqrt{h_0 - 1 - \delta}$, $R_\pm(t) = \sqrt{1 \pm k\,\mathrm{sn}\,(\Omega\,t,\,k)}$, and $S(t) = \sqrt{\frac{(c_0-2\delta)-(c_0+2\delta)\,k\ \mathrm{sn}(\Omega\,t,\,k)}{c_0-2\delta}}$.

We plot this projection onto the sphere for some specific values (see Figures 6.10, 6.11, 6.12).

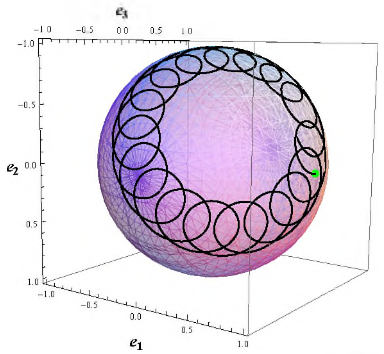


Figure (6.10) $c_0 = 32$, $h_0 = \frac{17}{2}$, $\sigma = -1$ and $t \in [0, 10\pi]$

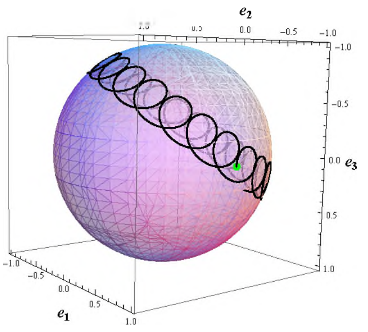


Figure (6.11) $c_0 = 268$, $h_0 = 21$, $\sigma = -1$ and $t \in [0, 8\pi]$

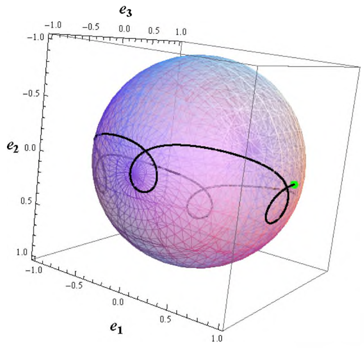


Figure (6.12) $c_0 = 299$, $h_0 = \frac{1750}{101}$, $\sigma = 1$ and $t \in [0, 3\pi]$

6.1.2 Remark. It is worth comparing the plots of this section with the plots given by Jurdjevic in [33]. Although the system investigated in [33] is not the exact system investigated here, it is an equivalent system. The figures are therefore not identical to those in [33], however, the similarities of the corresponding cases can clearly be seen.

### 6.1.4 The system $(\Sigma_\alpha^{(2,1)}, \chi_\eta^3)$

Mathematica. *The details of the calculations for this system are contained in the Mathematica file:*

*Thesis Mathematica\SO(3)\Applications\Sys21.nb*

We now consider the cost-extended system $(\Sigma_\alpha^{(2,1)}, \chi_\eta^3)$. In particular, we consider the case when $\eta = 0$. Thus we are considering the optimal control problem given by

$$\dot{g}(t) = g(t)\ (\alpha E_1 + u_1(t)E_2 + u_2(t)E_3)\,, \quad g(\cdot) : [0, T] \to \mathsf{SO}(3),\ u(\cdot) : [0, T] \to \mathbb{R}^2$$

$$g(0) = g_0, \quad g(T) = g_1, \quad g_0, g_1 \in \mathsf{SO}(3), \quad T > 0 \text{ fixed}$$

$$\mathcal{J} = \int_0^T (u_1(t)^2 + u_2(t)^2)dt \to \min.$$

The associated quadratic Hamilton-Poisson system is given by $H_\alpha^{(2,1)}(p) = \alpha p_1 + \frac{1}{2}(p_2^2 + p_3^2)$, and the optimal controls are given by $(u_1(t), u_2(t)) = (p_2(t), p_3(t))$.

It follows that the system $H^{(2,1)}$ is affinely equivalent to the system $H_0^1(p) = \frac{1}{2}p_1^2$. Indeed,

$$\psi : p \mapsto p \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}$$

is an affine isomorphism such that $T\psi \cdot \vec{H}_0^1(p) = (\vec{H}_\alpha^{(2,1)} \circ \psi)(p)$. The original integral curves for the system $H^1$ can be expressed as follows (see theorem 3.3.5)

$$\begin{aligned}
\bar{p}_1(t) &= \sigma\sqrt{2\tilde{h}_0}\\
\bar{p}_2(t) &= \sigma\sqrt{\tilde{c}_0 - 2\tilde{h}_0}\,\sin(\sqrt{2\tilde{h}_0}\,t)\\
\bar{p}_3(t) &= \sqrt{\tilde{c}_0 - 2\tilde{h}_0}\,\cos(\sqrt{2\tilde{h}_0}\,t).
\end{aligned}$$

Here $\tilde{h}_0 = H^1(p\widetilde{(0)})$ and $\tilde{c}_0 = C(\tilde{p}(0))$. Let $h_0 = H_\alpha^{(2,1)}(\psi(\tilde{p}(0)))$ and $c_0 = C(\psi(\tilde{p}(0)))$. We then obtain the relations $\tilde{c}_0 = c_0 - \alpha\left(\alpha - 2\sigma\sqrt{c_0 - 2h_0 + \alpha^2}\right)$ and $\tilde{h}_0 = \frac{1}{2}\left(c_0 - 2h_0 + \alpha^2\right)$. (Here $\sigma = \operatorname{sgn}(\tilde{p}_1(t))$.) The integral curves of the system $H_\alpha^{(2,1)}$ are obtained by applying $\psi$ to the integral curves of the system $H^1$. Explicitly, we get

$$\begin{aligned}
p_1(t) &= \alpha - \sigma\sqrt{\alpha^2 + c_0 - 2h_0}\\
p_2(t) &= \sigma\sqrt{2}\sqrt{h_0 - \alpha\left(\alpha - \sigma\sqrt{\alpha^2 + c_0 - 2h_0}\right)}\,\sin\left(\sqrt{\alpha^2 + c_0 - 2h_0}\,t\right)\\
p_3(t) &= \sqrt{2}\sqrt{h_0 - \alpha\left(\alpha - \sigma\sqrt{\alpha^2 + c_0 - 2h_0}\right)}\,\cos\left(\sqrt{\alpha^2 + c_0 - 2h_0}\,t\right).
\end{aligned}$$

For convenience, let $\Omega = \sqrt{\alpha^2 + c_0 - 2h_0}$.

Using the Euler angles it follows that $g(t)$ is given as in formula (3.16), where

$$\begin{aligned}
\cos(\phi_2) &= \frac{\alpha - \sigma\Omega}{\sqrt{c_0}} & \cos(\phi_3) &= \cos(\Omega\,t)\\
\sin(\phi_2) &= \frac{\sqrt{2}\sqrt{h_0 - \alpha(\alpha - \sigma\Omega)}}{\sqrt{c_0}} & \sin(\phi_3) &= \sigma\sin(\Omega\,t)
\end{aligned}$$

and

$$\dot{\phi}_1(t) = \frac{\sqrt{c_0}}{p_2^2 + p_3^2}\left(p_2\frac{\partial H_\alpha^{(2,1)}}{\partial p_2} + p_3\frac{\partial H_\alpha^{(2,1)}}{\partial p_3}\right) = \sqrt{c_0}.$$

Therefore, we have that $\phi_1(t) = \int_0^t \dot{\phi}_1(\tau)d\tau = \sqrt{c_0}\,t$. We now translate this trajectory to start at the identity using the left translation

$$g(0)^{-1} = \frac{1}{\sqrt{c_0}}\begin{bmatrix} \alpha - \sigma\Omega & 0 & -\sqrt{2}\sqrt{h_0 - \alpha(\alpha - \sigma\Omega)}\\ 0 & \sqrt{c_0} & 0\\ \sqrt{2}\sqrt{h_0 - \alpha(\alpha - \sigma\Omega)} & 0 & \alpha - \sigma\Omega \end{bmatrix}.$$

The resulting trajectory on the sphere $\mathbb{S}^2$, using the projection $x_1(t) = g(0)^{-1}g(t)\vec{e}_1$, is given by

$$x_1(t) = \frac{1}{c_0}\begin{bmatrix} c_0 - 2\left(h_0 - \alpha(\alpha - \sigma\Omega)\right)\left(1 - \cos\left(\sqrt{c_0}\,t\right)\right)\\ \sqrt{2c_0}\sqrt{h_0 - \alpha(\alpha - \sigma\Omega)}\,\sin\left(\sqrt{c_0}\,t\right)\\ \sqrt{2}\sqrt{h_0 - \alpha(\alpha - \sigma\Omega)}(\alpha - \sigma\Omega)\left(1 - \cos\left(\sqrt{c_0}\,t\right)\right). \end{bmatrix}$$

It then follows that $g(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) \end{bmatrix}$, where

$$x_2(t) = \frac{1}{c_0}\begin{bmatrix} \sqrt{2}\left(2\sqrt{\eta}\sigma(\alpha-\sigma\Omega)\sin(\Omega\,t)\sin^2\left(\frac{\sqrt{c_0}}{2}\,t\right) - \cos(\Omega\,t)\sin(\sqrt{c_0}\,t)\sqrt{\eta c_0}\right) \\ \sigma(-\alpha+\sigma\Omega)\sin(\Omega\,t)\sin(\sqrt{c_0}\,t)\sqrt{c_0} + \cos(\Omega\,t)\cos(\sqrt{c_0}\,t)c_0 \\ \sigma\left(2\eta + (\alpha-\sigma\Omega)^2\cos(\sqrt{c_0}\,t)\right)\sin(\Omega\,t) + (\alpha-\sigma\Omega)\cos(\Omega\,t)\sin(\sqrt{c_0}\,t)\sqrt{c_0} \end{bmatrix}$$

$$x_3(t) = \frac{1}{c_0}\begin{bmatrix} \sqrt{2}\left(2\sqrt{\eta}(\alpha-\sigma\Omega)\cos(\Omega\,t)\sin(\Omega\,t)\sin^2\left(\frac{\sqrt{c_0}}{2}\,t\right) + \sigma\sin(\Omega\,t)\sin(\sqrt{c_0}\,t)\sqrt{\eta c_0}\right) \\ (-\alpha+\sigma\Omega)\cos(\Omega\,t)\sin(\sqrt{c_0}\,t)\sqrt{c_0} - \sigma\cos(\sqrt{c_0}\,t)\sin(\Omega\,t)c_0 \\ \cos(\Omega\,t)\left(2\eta + (\alpha-\sigma\Omega)^2\cos(\sqrt{c_0}\,t)\right) + \sigma(-\alpha+\sigma\Omega)\sin(\Omega\,t)\sin(\sqrt{c_0}\,t)\sqrt{c_0} \end{bmatrix}.$$

Here $\eta = h_0 - \alpha(\alpha - \sigma\Omega)$. We plot the projection of $x_1(t)$ onto the sphere for some specific values in Figures 6.13, 6.14, 6.15.

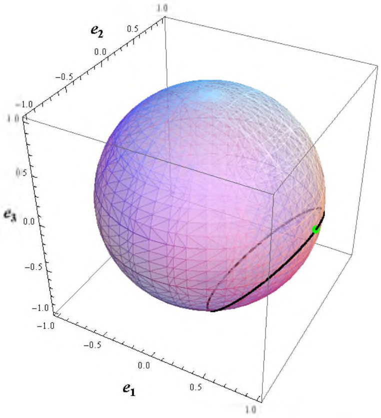


Figure (6.13) $c_0 = 77$, $h_0 = \frac{44}{3}$, $\alpha = \frac{24}{61}$, $\sigma = 1$ and $t \in [0, \pi]$

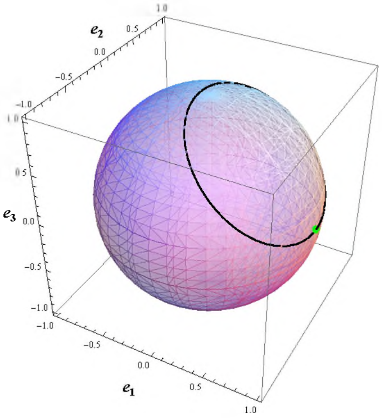


Figure (6.14) $c_0 = \frac{5}{6}$, $h_0 = \frac{38}{13}$, $\alpha = \frac{267}{61}$, $\sigma = 1$ and $t \in [0, \frac{5\pi}{2}]$

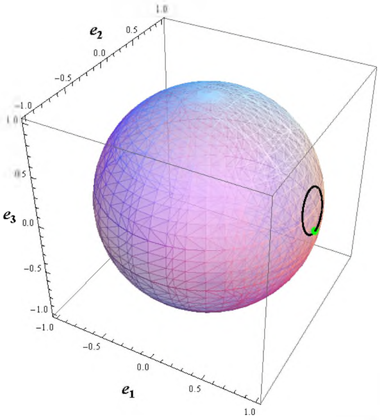


Figure (6.15) $c_0 = 74$, $h_0 = \frac{245}{8}$, $\alpha = \frac{211}{61}$, $\sigma = -1$ and $t \in [0, \pi]$

### 6.1.5 The system $(\Sigma^{(3,0)}, \chi^1_{\alpha\beta\gamma})$

MATHEMATICA. *The details of the calculations for this system are contained in the Mathematica file:*

*Thesis Mathematica\SO(3)\Applications\Sys30.nb*

We begin by considering the cost-extended system $(\Sigma^{(3,0)}, \chi^1_{\alpha\beta\gamma})$. We will consider the case when $\boldsymbol{\alpha} = 0$ and $\gamma = 0$. Thus, we are considering the optimal control problem given by

$$\begin{aligned} &\dot{g}(t) = g(t)\,\left(u_1(t)E_1 + u_2(t)E_2 + u_3(t)E_3(t)\right), \quad g(\cdot) : [0,T] \to \mathsf{SO}(3),\ u(\cdot) : [0,T] \to \mathbb{R}^3 \\ &g(0) = g_0, \quad g(T) = g_1, \quad g_0, g_1 \in \mathsf{SO}(3), \quad T > 0 \text{ fixed} \\ &\mathcal{J} = \int_0^T (u_1(t)^2 + \beta_1 u_2(t)^2 + \beta_2 u_2(t)^2)dt \to \min, \quad 0 < \beta_2 < \beta_1 < 1. \end{aligned}$$

The associated quadratic Hamilton-Poisson system is given by $H^{(3,0)}_{\boldsymbol{\beta}}(p) = \frac{1}{2}(p_1^2 + \frac{1}{\beta_1}p_2^2 + \frac{1}{\beta_2}p_3^2)$, and the optimal controls are given by $(u_1(t), u_2(t), u_3(t)) = (p_1(t), \frac{1}{\beta_1}p_2(t), \frac{1}{\beta_2}p_3(t))$.

The system $H^{(3,0)}$ is linearly equivalent to the system $H^2(\tilde{p}) = \tilde{p}_1^2 + \frac{1}{2}\tilde{p}_2^2$. Indeed,

$$\psi : \tilde{p} \mapsto \tilde{p} \begin{bmatrix} \sqrt{\frac{2\beta_1\beta_2}{(1-\beta_1)(1-\beta_2)}} & 0 & 0 \\ 0 & \frac{\beta_1\sqrt{\beta_2}}{\sqrt{(1-\beta_1)(\beta_1-\beta_2)}} & 0 \\ 0 & 0 & -\frac{\beta_2\sqrt{2\beta_1}}{\sqrt{(1-\beta_2)(\beta_1-\beta_2)}} \end{bmatrix}$$

is a linear isomorphism such that $T\psi \cdot \vec{H}^2(\tilde{p}) = (\vec{H}^{(3,0)}_{\boldsymbol{\beta}} \circ \psi)(\tilde{p})$. The original integral curves for $H^2$ are given in theorem 3.3.7. Let $\tilde{h}_0 = H^2(\tilde{p}(0))$ and $\tilde{c}_0 = C(\tilde{p}(0))$, and let $h_0 = H^{(3,0)}_{\boldsymbol{\beta}}(\psi(\tilde{p}(0)))$ and $c_0 = C(\psi(\tilde{p}(0)))$. We then obtain the relations

$$\tilde{h}_0 = \frac{(1-\beta_1)\,(c_0 - 2\beta_2 h_0)}{2\beta_1\beta_2} \quad \text{and} \quad \tilde{c}_0 = \frac{(1 - 2\beta_1 + \beta_2)c_0 + 2(\beta_1 + (\beta_1 - 2)\beta_2)h_0}{2\beta_1\beta_2}.$$

The integral curves of the system $H^{(3,0)}_{\boldsymbol{\beta}}$ are obtained by applying $\psi$ to the integral curves of the system $H^2$.

**Subcase 1:** $\tilde{c}_0 > 2\tilde{h}_0$
We first consider the case when $\tilde{c}_0 > 2\tilde{h}_0 > 0$. Then the transformed integral curves are given explicitly by

$$\begin{aligned} p_1(t) &= \sqrt{\tfrac{c_0 - 2\beta_2 h_0}{1-\beta_2}}\,\mathrm{cn}(\Omega\, t,\, k) \\ p_2(t) &= \sigma\sqrt{\tfrac{\beta_1(c_0 - 2\beta_2 h_0)}{\beta_1 - \beta_2}}\,\mathrm{sn}(\Omega\, t,\, k) \\ p_3(t) &= -\sigma\sqrt{\tfrac{\beta_2(2h_0 - c_0)}{(1-\beta_2)}}\,\mathrm{dn}(\Omega\, t,\, k). \end{aligned}$$

Here $\Omega = \sqrt{\frac{(\beta_1-\beta_2)(2h_0-c_0)}{\beta_1\beta_2}}$, $k = \sqrt{\frac{(1-\beta_1)(c_0-2\beta_2 h_0)}{(\beta_1-\beta_2)(2h_0-c_0)}}$, and $\sigma \in \{-1, 1\}$. The new conditions on this system become $0 < 2\beta_2 h_0 < c_0 < 2\beta_1 h_0$. Using the Euler angles it follows that $g(t)$ is given as in formula (3.16), where

$$\cos(\phi_2) = \sqrt{\tfrac{c_0 - 2\beta_2 h_0}{c_0(1-\beta_2)}}\,\mathrm{cn}(\Omega\, t,\, k), \qquad \sin(\phi_2) = \sqrt{\frac{\beta_2(2h_0 - c_0) + (c_0 - 2\beta_2 h_0)\,\mathrm{sn}(\Omega\, t,\, k)^2}{c_0(1-\beta_2)}}$$

$$\cos(\phi_3) = \pm\sigma\frac{\sqrt{\beta_2(2h_0 - c_0)}\,\mathrm{dn}(\Omega\, t,\, k)}{\sqrt{\beta_2(2h_0 - c_0) + (c_0 - 2\beta_2 h_0)\,\mathrm{sn}(\Omega\, t,\, k)^2}}$$

$$\sin(\phi_3) = \pm\sigma\frac{\sqrt{\beta_1(1-\beta_2)\,(c_0 - 2\beta_2 h_0)}\,\mathrm{sn}(\Omega\, t,\, k)}{\sqrt{\beta_1 - \beta_2}\sqrt{\beta_2(2h_0 - c_0) + (c_0 - 2\beta_2 h_0)\,\mathrm{sn}(\Omega\, t,\, k)^2}}$$

and

$$\dot{\phi}_1(t) = \tfrac{\sqrt{c_0}}{p_2^2 + p_3^2}\left(p_2\frac{\partial H^{(3,0)}_{\boldsymbol{\beta}}}{\partial p_2} + p_3\frac{\partial H^{(3,0)}_{\boldsymbol{\beta}}}{\partial p_3}\right) = \sqrt{c_0}\,\frac{(2h_0 - c_0) + (c_0 - 2\beta_2 h_0)\;\mathrm{sn}(\Omega\, t,\, k)^2}{\beta_2\,(2h_0 - c_0) + (c_0 - 2\beta_2 h_0)\;\mathrm{sn}(\Omega\, t,\, k)^2}.$$

Therefore we have that

$$\phi_1(t) = \sqrt{c_0}\, t + \frac{c_0\sqrt{\beta_1}(1-\beta_2)\,\Pi\left(\frac{c_0-2\beta_2 h_0}{\beta_2(c_0-2h_0)},\, \mathrm{am}(\Omega\, t, k^2), k^2\right)}{\sqrt{c_0\beta_2(\beta_1-\beta_2)\,(2h_0-c_0)}}.$$

We now translate this trajectory to start at the identity using the left translation

$$g(0)^{-1} = \frac{1}{\sqrt{c_0(1-\beta_2)}}\begin{bmatrix} \sqrt{c_0-2\beta_2 h_0} & 0 & -\sqrt{\beta_2\,(2h_0-c_0)} \\ 0 & -\sigma\sqrt{c_0(1-\beta_2)} & 0 \\ -\sigma\sqrt{\beta_2\,(2h_0-c_0)} & 0 & -\sigma\sqrt{c_0-2\beta_2 h_0} \end{bmatrix}.$$

The resulting trajectory on the sphere $\mathbb{S}^2$, using the projection $x_1(t) = g(0)^{-1}g(t)\vec{e}_1$, is given by

$$x_1(t) = \frac{1}{c_0(1-\beta_2)}\begin{bmatrix} (c_0-2\beta_2 h_0)\ \mathrm{cn}(\Omega\, t,\, k) + \sqrt{\beta_2\,(2h_0-c_0)}\cos(\phi_1)S(t) \\ -\sigma\sqrt{c_0(1-\beta_2)}\ \sin(\phi_1)S(t) \\ \sigma\sqrt{c_0-2\beta_2 h_0}\left(\cos(\phi_1)S(t) - \sqrt{\beta_2\,(2h_0-c_0)}\,\mathrm{cn}(\Omega\, t,\, k)\right) \end{bmatrix}$$

where $S(t) = \sqrt{\beta_2\,(2h_0-c_0) + (c_0-2\beta_2 h_0)\ \mathrm{sn}(\Omega\, t,\, k)^2}$. We plot this projection onto the sphere for some specific values (see Figure 6.16, 6.17, 6.18).

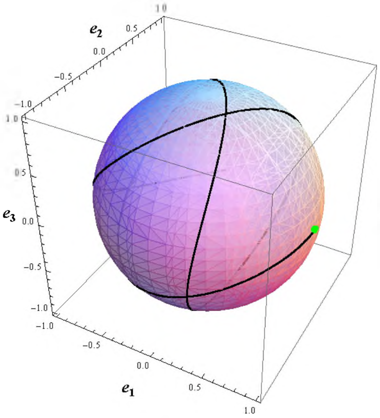


Figure (6.16) $c_0 = 31$, $h_0 = 44$, $\beta_1 = \frac{97}{156}$, $\beta_2 = \frac{52}{287}$ and $t \in [0, \frac{\pi}{4}]$

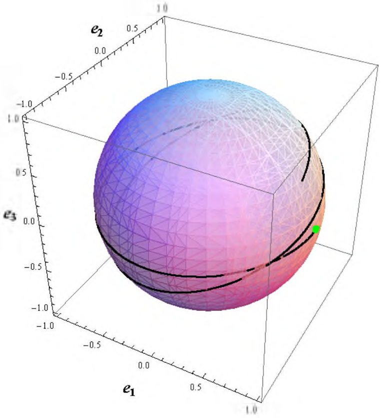


Figure (6.17) $c_0 = 33$, $h_0 = 98$, $\beta_1 = \frac{59}{61}$, $\beta_2 = \frac{3}{20}$ and $t \in [0, \frac{\pi}{9}]$

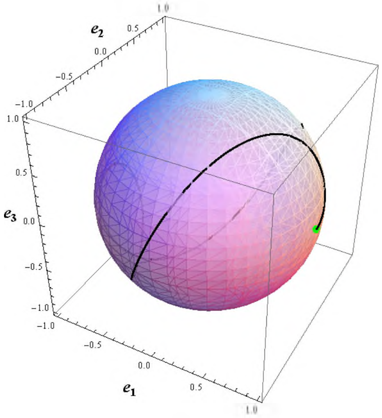


Figure (6.18) $c_0 = 267$, $h_0 = 138$, $\beta_1 = \frac{1547}{1549}$, $\beta_2 = \frac{11}{105}$ and $t \in [0, \frac{\pi}{5}]$

**Subcase 2:** $\tilde{c_0} = 2\tilde{h}_0$

Next, we consider the case when $\tilde{c_0} = 2\tilde{h}_0 > 0$. The transformed integral curves in this case are given explicitly by

$$\begin{aligned} p_1(t) &= \sigma_1\sqrt{\tfrac{2(\beta_1-\beta_2)h_0}{1-\beta_2}}\,\mathrm{sech}\,(\Omega\, t) \\ p_2(t) &= \sigma_1\sigma_2\sqrt{2\beta_1 h_0}\ \tanh(\Omega\, t) \\ p_3(t) &= -\sigma_2\sqrt{\tfrac{2\beta_2 h_0(1-\beta_1)}{1-\beta_2}}\,\mathrm{sech}\,(\Omega\, t). \end{aligned}$$

Here $\Omega = \sqrt{\frac{2h_0(1-\beta_1)(\beta_1-\beta_2)}{\beta_1\beta_2}}$.

The conditions on this system become $c_0 = 2\beta_1 h_0 > 0$. Using the Euler angles it follows that $g(t)$ is given as in formula (3.16), where

$$\cos(\phi_2) = \sigma_1 \sqrt{\tfrac{\beta_1-\beta_2}{\beta_1(1-\beta_2)}}\,\mathrm{sech}\,(\Omega\, t), \qquad \sin(\phi_2) = \frac{\sqrt{\beta_1(1-\beta_2)-(\beta_1-\beta_2)\,\mathrm{sech}\,(\Omega\, t)^2}}{\sqrt{\beta_1(1-\beta_2)}}$$

$$\cos(\phi_3) = \mp\sigma_2 \frac{\sqrt{\beta_2(1-\beta_1)}\,\mathrm{sech}\,(\Omega\, t)}{\sqrt{\beta_1(1-\beta_2)-(\beta_1-\beta_2)\,\mathrm{sech}\,(\Omega\, t)^2}}$$

$$\sin(\phi_3) = \pm\sigma_1\sigma_2 \frac{\sqrt{\beta_1(1-\beta_2)}\ \tanh(\Omega\, t)}{\sqrt{\beta_1(1-\beta_2)-(\beta_1-\beta_2)\,\mathrm{sech}\,(\Omega\, t)^2}}$$

and

$$\dot{\phi}_1(t) = \tfrac{\sqrt{2\beta_1 h_0}}{p_2^2+p_3^2}\left(p_2 \frac{\partial H_{\boldsymbol{\beta}}^{(3,0)}}{\partial p_2} + p_3 \frac{\partial H_{\boldsymbol{\beta}}^{(3,0)}}{\partial p_3}\right) = \sqrt{2\beta_1 h_0}\, \frac{1-\beta_2-(\beta_1-\beta_2)\,\mathrm{sech}\,(\Omega\, t)^2}{\beta_1(1-\beta_2)-(\beta_1-\beta_2)\,\mathrm{sech}\,(\Omega\, t)^2}.$$

Therefore we get $\phi_1(t) = \arctan\left(\tfrac{\sqrt{\beta_1-\beta_2}}{\sqrt{\beta_2(1-\beta_1)}}\ \tanh(\Omega\, t)\right) + \sqrt{\tfrac{2h_0}{\beta_1}}\, t$. We are specifically interested in finding $\cos(\phi_1(t))$ and $\sin(\phi_1(t))$. These expressions can be simplified using the following results:

$$\begin{aligned}\cos\left(\arctan(\sqrt{\tfrac{\beta_1-\beta_2}{\beta_2(1-\beta_1)}}\,\tan(\Omega\, t))\right) &= \frac{1}{\sqrt{1+\tfrac{\beta_1-\beta_2}{\beta_2(1-\beta_1)}\ \tanh(\Omega\, t)^2}}\\ &= \frac{\sqrt{2}\cosh(\Omega\, t)\sqrt{(1-\beta_1)\,\beta_2}}{\sqrt{2\beta_2-\beta_1\,(1+\beta_2)+\beta_1\,(1-\beta_2)\cosh(2\Omega\, t)}}\end{aligned}$$

and

$$\begin{aligned}\sin\left(\arctan(\sqrt{\tfrac{\beta_1-\beta_2}{\beta_2(1-\beta_1)}}\,\tan(\Omega\, t))\right) &= \frac{\tanh(\Omega\, t)}{\sqrt{\tfrac{\beta_2-\beta_1\beta_2}{\beta_1-\beta_2}+\tanh(\Omega\, t)^2}}\\ &= \frac{\sqrt{2}\sinh(\Omega\, t)\sqrt{\beta_1-\beta_2}}{\sqrt{2\beta_2-\beta_1\,(1+\beta_2)+\beta_1\,(1-\beta_2)\cosh(2\Omega\, t)}}.\end{aligned}$$

We can then break up $\cos(\phi_1(t))$ and $\sin(\phi_1(t))$ using the addition formulas for trigonometric functions. We now translate this trajectory to start at the identity using the left translation

$$g(0)^{-1} = \tfrac{1}{\sqrt{\beta_1(1-\beta_2)}}\begin{bmatrix} \sigma_1\sqrt{\beta_1-\beta_2} & 0 & -\sqrt{\beta_2(1-\beta_1)} \\ 0 & -\sigma_2\sqrt{\beta_1(1-\beta_2)} & 0 \\ -\sigma_2\sqrt{\beta_2(1-\beta_1)} & 0 & -\sigma_1\sigma_2\sqrt{\beta_1-\beta_2} \end{bmatrix}.$$

The resulting trajectories on the sphere $\mathbb{S}^2$, using the projection $x_1(t) = g(0)^{-1}g(t)\vec{e}_1$ are given by

$$x_1(t) = \tfrac{1}{\alpha_2(1-\alpha_1)}\begin{bmatrix} \alpha_1(1-\alpha_2)\cos(\omega\, t)+\alpha_3\,\mathrm{sech}\,(\Omega\, t)-\sqrt{\alpha_1\alpha_3(1-\alpha_2)}\sin(\omega\, t)\ \tanh(\Omega\, t) \\ -\sigma_2\sqrt{\alpha_2(1-\alpha_1)}\left(\sqrt{\alpha_1(1-\alpha_2)}\sin(\omega\, t)+\sqrt{\alpha_3}\cos(\omega\, t)\ \tanh(\Omega\, t)\right) \\ \sigma_1\sigma_2\sqrt{\alpha_1\alpha_3(1-\alpha_2)}\left(\cos(\omega\, t)-\mathrm{sech}\,(\Omega\, t)-\sqrt{\tfrac{\alpha_3}{\alpha_1(1-\alpha_2)}}\sin(\omega\, t)\ \tanh(\Omega\, t)\right) \end{bmatrix}$$

where $\omega = \sqrt{\frac{2h_0}{(1-\alpha_1)}}$, $\Omega = \sqrt{\frac{2\alpha_1\alpha_3 h_0}{(1-\alpha_1)(1-\alpha_2)}}$, $\alpha_1 = 1-\beta_1$, $\alpha_2 = 1-\beta_2$ and $\alpha_3 = \beta_1 - \beta_2$. We plot this projection onto the sphere for some specific values (see Figure 6.19, 6.20, 6.21).

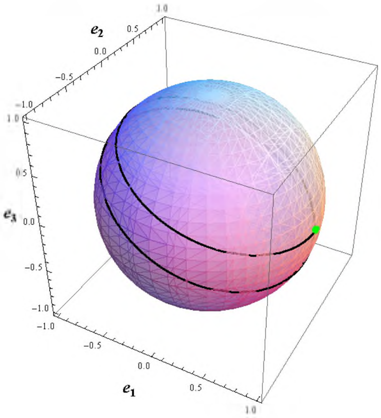


Figure (6.19) $h_0 = \frac{139}{8}$, $\beta_1 = \frac{33}{302}$, $\beta_2 = \frac{82}{925}$, $\sigma_2 = -\sigma_1 = 1$, $t \in [0, \frac{\pi}{5}]$

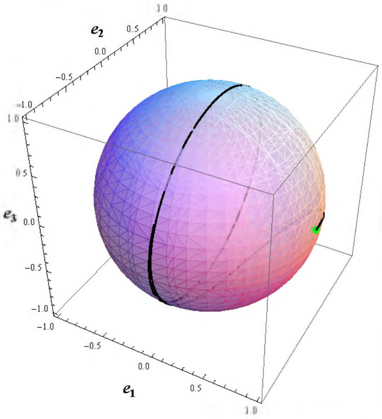


Figure (6.20) $h_0 = \frac{139}{8}$, $\beta_1 = \frac{267}{302}$, $\beta_2 = \frac{1}{23}$, $\sigma_2 = \sigma_1 = -1$, $t \in [0, \frac{\pi}{2}]$

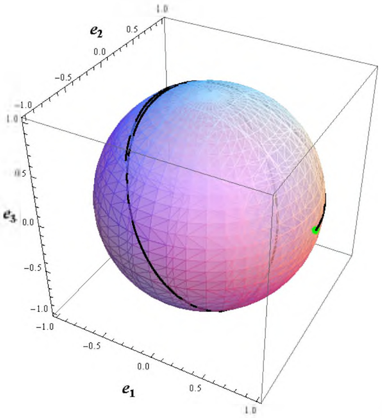


Figure (6.21) $h_0 = \frac{173}{16}$, $\beta_1 = \frac{31}{302}$, $\beta_2 = \frac{29}{984}$, $\sigma_1 = -\sigma_2 = 1$, $t \in [0, \frac{\pi}{5}]$

**Subcase 3:** $0 < \tilde{c}_0 < 2\tilde{h}_0$

Lastly, we consider the case when $0 < \tilde{c}_0 < 2\tilde{h}_0$. Then the transformed integral curves are given explicitly by

$$\begin{aligned} p_1(t) &= \sigma\sqrt{\tfrac{c_0 - 2\beta_2 h_0}{1-\beta_2}}\,\mathrm{dn}(\Omega\, t,\, k) \\ p_2(t) &= \sqrt{\tfrac{\beta_1(2h_0 - c_0)}{1-\beta_1}}\,\mathrm{sn}(\Omega\, t,\, k) \\ p_3(t) &= -\sigma\sqrt{\tfrac{\beta_2(2h_0-c_0)}{1-\beta_2}}\,\mathrm{cn}(\Omega\, t,\, k). \end{aligned}$$

Here $\Omega = \sqrt{\frac{(1-\beta_1)(c_0-2\beta_2 h_0)}{\beta_1\beta_2}}$, $k = \sqrt{\frac{(\beta_1-\beta_2)(2h_0-c_0)}{(1-\beta_1)(c_0-2\beta_2 h_0)}}$, and $\sigma \in \{-1, 1\}$. The new conditions on this system become $0 < 2\beta_2 h_0 < 2\beta_1 h_0 < c_0 < 2h_0$. Using the Euler angles it follows that $g(t)$ is given as in formula (3.16), where

$$\cos(\phi_2) = \sigma\sqrt{\tfrac{c_0-2\beta_2 h_0}{c_0(1-\beta_2)}}\,\mathrm{dn}(\Omega\, t,\, k), \qquad \sin(\phi_2) = \pm\sqrt{\frac{c_0(1-\beta_2) - (c_0 - 2\beta_2 h_0)\,\mathrm{dn}(\Omega\, t,\, k)^2}{c_0(1-\beta_2)}}$$

$$\cos(\phi_3) = \pm - \frac{\sigma\sqrt{\beta_2\,(2h_0 - c_0)}\,\mathrm{cn}(\Omega\, t,\, k)}{\sqrt{c_0(1-\beta_2) - (c_0 - 2\beta_2 h_0)\,\mathrm{dn}(\Omega\, t,\, k)^2}}$$

$$\sin(\phi_3) = \pm\frac{\sqrt{\beta_1(1-\beta_2)\,(2h_0 - c_0)}\,\mathrm{sn}(\Omega\, t,\, k)}{\sqrt{(1-\beta_1)\,(c_0(1-\beta_2) - (c_0 - 2\beta_2 h_0)\,\mathrm{dn}(\Omega\, t,\, k)^2)}}$$

and

$$\dot{\phi}_1(t) = \tfrac{\sqrt{c_0}}{p_2^2+p_3^2}\left(p_2\frac{\partial H_{\beta}^{(3,0)}}{\partial p_2} + p_3\frac{\partial H_{\beta}^{(3,0)}}{\partial p_3}\right) = \sqrt{c_0}\,\frac{2h_0(1-\beta_2) - (c_0 - 2\beta_2 h_0)\,\mathrm{dn}(\Omega\, t,\, k)^2}{c_0(1-\beta_2) - (c_0 - 2\beta_2 h_0)\,\mathrm{dn}(\Omega\, t,\, k)^2}.$$

Therefore we have that

$$\phi_1(t) = \sqrt{c_0}\, t + \frac{\sqrt{\beta_1 c_0}(1-\beta_2)\,\Pi\left(-\frac{\beta_1-\beta_2}{\beta_2(1-\beta_1)},\, \mathrm{am}(\Omega\, t, k^2), k^2\right)}{\beta_2\sqrt{(1-\beta_1)\,(c_0-2\beta_2 h_0)}}.$$

We now translate this trajectory to start at the identity using the left translation

$$g(0)^{-1} = \frac{1}{\sqrt{c_0(1-\beta_2)}}\begin{bmatrix} \sigma\sqrt{c_0-2\beta_2 h_0} & 0 & -\sqrt{\beta_2\,(2h_0-c_0)} \\ 0 & -\sigma\sqrt{c_0(1-\beta_2)} & 0 \\ -\sigma\sqrt{\beta_2\,(2h_0-c_0)} & 0 & -\sqrt{c_0-2\beta_2 h_0} \end{bmatrix}.$$

The resulting trajectories on the sphere $\mathbb{S}^2$, using the projection $x_1(t) = g(0)^{-1}g(t)\vec{e_1}$ are given by

$$x_1(t) = \frac{1}{c_0(1-\beta_2)}\begin{bmatrix} (c_0-2\beta_2 h_0)\,\mathrm{dn}(\Omega\, t,\, k) + \sqrt{\beta_2\,(2h_0-c_0)}\,\cos(\phi_1)S(t) \\ -\sigma\sqrt{c_0(1-\beta_2)}\,\sin(\phi_1)S(t) \\ \sqrt{c_0-2\beta_2 h_0}\left(\cos(\phi_1)S(t) - \sqrt{\beta_2\,(2h_0-c_0)}\,\mathrm{dn}(\Omega\, t,\, k)\right) \end{bmatrix}$$

where $S(t) = \sqrt{c_0(1-\beta_2) - (c_0-2\beta_2 h_0)\,\mathrm{dn}(\Omega\, t,\, k)^2}$. We plot this projection onto the sphere for some specific values (see Figure 6.22, 6.23, 6.24).

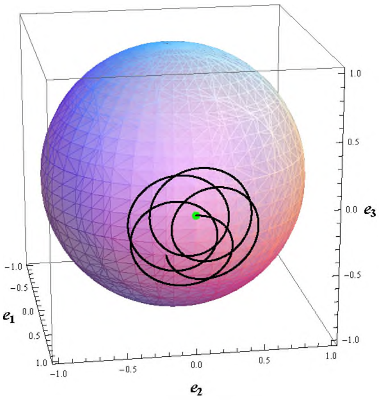


Figure (6.22) $c_0 = \frac{298}{13}$, $h_0 = \frac{35}{3}$, $\beta_1 = 0.8$, $\beta_2 = 0.3$ and $t \in [0, 2\pi]$

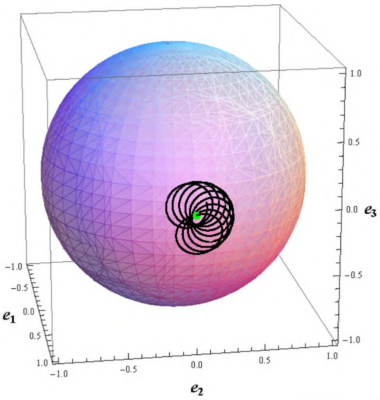


Figure (6.23) $c_0 = \frac{149}{2}$, $h_0 = \frac{939}{25}$, $\beta_1 = 0.9$, $\beta_2 = 0.12$ and $t \in [0, \frac{6\pi}{6}]$

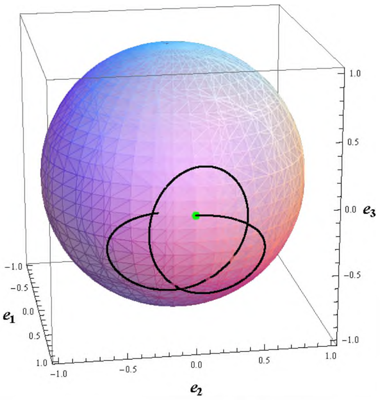


Figure (6.24) $c_0 = \frac{248}{7}$, $h_0 = \frac{95}{3}$, $\beta_1 = 0.32$, $\beta_2 = 0.1$ and $t \in [0, \frac{\pi}{8}]$

## 6.2 A four-input control system on $\mathsf{SO}(4)$

Mathematica. *The details of the calculations for this system are contained in the Mathematica file:*

*Thesis Mathematica\SO(4)\Applications\Sys40.nb*

Let us consider the optimal control problem on $\mathsf{SO}(4)$ , $(\Sigma_1^{4,0}, \chi^4)$, specified by

$$\begin{aligned}
&\dot{g}(t) = g(t)\ \left(u_1(t)E_2 + u_2(t)E_3 + u_3(t)E_5 + u_4(t)E_6\right)\\
&g(\cdot) : [0,T] \to \mathsf{SO}(4), \quad u(\cdot) : [0,T] \to \mathbb{R}^4\\
&g(0) = g_0, \quad g(T) = g_1, \quad g_0, g_1 \in \mathsf{SO}(4), \quad T > 0 \text{ fixed}\\
\mathcal{J}(u(\cdot)) = &\int_0^T \left(\sum_{i=1}^4 c_i u_i^2(t)\right) dt \to \min, \quad c_1, \ldots, c_4 > 0.
\end{aligned}$$

We shall consider the case when $c_1 = c_2 = c_3 = c_4 = 1$. In this case the associated quadratic Hamilton-Poisson system is given by $H_4(p) = \frac{1}{2}(p_2^2 + p_3^2 + p_5^2 + p_6^2)$. This system in turn is linearly equivalent to the system $\tilde{H}(p) = \frac{1}{2}(p_1^2 + p_4^2)$. Indeed,

$$-(H_4 - \frac{1}{2}C_1 - \frac{1}{2}C_2)(p) = H'(p).$$

That is we have used the properties $(\mathfrak{C}2)$ and $(\mathfrak{C}3)$ of lemma 2.2.24. Thus the linear isomorphism $\psi = \operatorname{diag}(-1,-1,-1,-1,-1,-1)$ yields $T\psi \cdot \vec{\tilde{H}}(p) = (H_4 \circ \psi)(p)$.

The functions $\tilde{H}^1(p) = \frac{1}{2}p_1^2$ and $\tilde{H}^2(p) = \frac{1}{2}p_4^2$ are both integrals of motion for the system $\tilde{H}$. Similarly, for the system $H_4$ the functions $H^1(p) = \frac{1}{2}(p_2^2 + p_3^2)$ and $H^2(p) = \frac{1}{2}(p_5^2 + p_6^2)$ are both integrals of motion. Let $\tilde{h}_{10} = \tilde{H}^1(\tilde{p}(0))$, $\tilde{h}_{20} = \tilde{H}^2(\tilde{p}(0))$, $\tilde{c}_{10} = C_1(\tilde{p}(0))$, and $\tilde{c}_{20} = C_2(\tilde{p}(0))$. Let $h_{10} = H^1(\psi(\tilde{p}(0)))$, $h_{20} = H^2(\psi(\tilde{p}(0)))$, $c_{10} = C_1(\psi(\tilde{p}(0)))$, and $c_{20} = C_2(\psi(\tilde{p}(0)))$.

We then obtain the relations $c_{10} = \tilde{c}_{10}$, $c_{20} = \tilde{c}_{20}$, $\tilde{h}_{10} = \frac{1}{2}c_{10} - h_{10}$ , and $\tilde{h}_{20} = \frac{1}{2}c_{20} - h_{20}$. The transformed integral curves for the system $H_4$ are then given by

$$\begin{aligned}
p_1(t) &= -\sigma_1\sqrt{c_{10} - 2h_{10}} & p_4(t) &= -\sigma_2\sqrt{c_{20} - 2h_{20}}\\
p_2(t) &= -\sigma_1\sqrt{2h_{10}}\ \sin(t\sqrt{c_{10} - 2h_{10}}) & p_5(t) &= -\sigma_2\sqrt{2h_{20}}\ \sin(t\sqrt{c_{20} - 2h_{20}})\\
p_3(t) &= -\sqrt{2h_{10}}\ \cos(t\sqrt{c_{10} - 2h_{10}}) & p_6(t) &= -\sqrt{2h_{20}}\ \cos(t\sqrt{c_{20} - 2h_{20}}).
\end{aligned}$$

For convenience we shall assume that $\sigma_1 = \sigma_2 = -1$.

Let $(\varphi_1, \varphi_2, \varphi_3)$ and $(\varphi_4, \varphi_5, \varphi_6)$ denote the coordinates of two points in $\mathsf{SU}(2)$. It then follows that $g_1(t) = \exp(\varphi_1 F_1)\exp(\varphi_2 F_2)\exp(\varphi_3 F_1)$ and $g_2(t) = \exp(\varphi_4 F_1)\exp(\varphi_5 F_2)\exp(\varphi_6 F_1)$, where

$$\begin{aligned}
\cos(\frac{\varphi_2}{2}) &= \frac{\sqrt{1+\delta_1}}{\sqrt{2}} & \cos(\frac{\varphi_5}{2}) &= \frac{\sqrt{1+\delta_2}}{\sqrt{2}}\\
\sin(\frac{\varphi_2}{2}) &= \frac{\sqrt{1-\delta_1}}{\sqrt{2}} & \sin(\frac{\varphi_5}{2}) &= \frac{\sqrt{1-\delta_2}}{\sqrt{2}}
\end{aligned}$$

$$e^{\frac{1}{2}i\varphi_3} = \left|\sin(\frac{\delta_1\sqrt{c_{10}}}{2}\,t)\right| + i\left|\cos(\frac{\delta_1\sqrt{c_{10}}}{2}\,t)\right| \qquad e^{\frac{1}{2}i\varphi_6} = \left|\sin(\frac{\delta_2\sqrt{c_{20}}}{2}\,t)\right| + i\left|\cos(\frac{\delta_2\sqrt{c_{20}}}{2}\,t)\right|.$$

Here $\delta_1 = \sqrt{1 - \frac{2h_{10}}{c_{10}}}$ and $\delta_2 = \sqrt{1 - \frac{2h_{20}}{c_{20}}}$. We also have

$$\dot{\phi}_1(t) = \frac{\sqrt{c_{10}}}{p_2^2 + p_3^2}\left(p_2\frac{\partial H}{\partial p_2} + p_3\frac{\partial H}{\partial p_3}\right) = \sqrt{c_{10}}.$$

Therefore $\phi_1(t) = \int_0^t \dot{\phi}_1 dt = \sqrt{c_{10}}\, t$. Similarly, we have $\phi_4(t) = \sqrt{c_{20}}\, t$. For convenience, we shall assume that $t \in \left[0, \min\left(\frac{\pi}{\delta_1\sqrt{c_{10}}}, \frac{\pi}{\delta_2\sqrt{c_{20}}}\right)\right]$. We now use theorem 4.1.8 in order to find the corresponding trajectory on the group $\mathsf{SO}(4)$. We have

$$g_1(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} ie^{\frac{1}{2}it\sqrt{c_{10}}(1-\delta_1)}\sqrt{1+\delta_1} & -ie^{\frac{1}{2}it\sqrt{c_{10}}(1+\delta_1)}\sqrt{1-\delta_1} \\ -ie^{-\frac{1}{2}it\sqrt{c_{10}}(1+\delta_1)}\sqrt{1-\delta_1} & -ie^{-\frac{1}{2}it\sqrt{c_{10}}(1-\delta_1)}\sqrt{1+\delta_1} \end{bmatrix}$$

$$g_2(t)^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} -ie^{-\frac{1}{2}it\sqrt{c_{20}}(1-\delta_2)}\sqrt{1+\delta_2} & ie^{\frac{1}{2}it\sqrt{c_{20}}(1+\delta_2)}\sqrt{1-\delta_2} \\ ie^{-\frac{1}{2}it\sqrt{c_{20}}(1+\delta_2)}\sqrt{1-\delta_2} & ie^{\frac{1}{2}it\sqrt{c_{20}}(1-\delta_2)}\sqrt{1+\delta_2} \end{bmatrix}$$

First consider $W_1 = \frac{1}{2}g_1(t) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} g_2(t)^{-1}$. This matrix can be written in the form

$$W_1 = \frac{1}{2} \begin{bmatrix} w_0 + iw_1 & w_2 + iw_3 \\ -w_2 + iw_3 & w_0 - iw_1 \end{bmatrix}$$

where

$$\begin{aligned} w_0 &= \frac{1}{2(c_{10}c_{20})^{\frac{1}{4}}} \left( \sqrt{\rho_{1-}\rho_{2-}} \cos\left(\tfrac{\rho_{1+}-\rho_{2+}}{2}\, t\right) + \sqrt{\rho_{1+}\rho_{2+}} \cos\left( (\tfrac{\rho_{1-}-\rho_{2-}}{2}\, t\right)\right) \\ w_1 &= \frac{1}{2(c_{10}c_{20})^{\frac{1}{4}}} \left( \sqrt{\rho_{1-}\rho_{2-}} \sin\left(\tfrac{\rho_{1+}-\rho_{2+}}{2}\, t\right) + \sqrt{\rho_{1+}\rho_{2+}} \sin\left(\tfrac{\rho_{1-}-\rho_{2-}}{2}\, t\right)\right) \\ w_2 &= \frac{1}{2(c_{10}c_{20})^{\frac{1}{4}}} \left( \sqrt{\rho_{1-}\rho_{2+}} \cos\left(\tfrac{\rho_{1+}+\rho_{2-}}{2}\, t\right) - \sqrt{\rho_{1+}\rho_{2-}} \cos\left(\tfrac{\rho_{1-}+\rho_{2+}}{2}\, t\right)\right) \\ w_3 &= \frac{1}{2(c_{10}c_{20})^{\frac{1}{4}}} \left( \sqrt{\rho_{1-}\rho_{2+}} \sin\left(\tfrac{\rho_{1+}+\rho_{2-}}{2}\, t\right) - \sqrt{\rho_{1+}\rho_{2-}} \sin\left(\tfrac{\rho_{1-}+\rho_{2+}}{2}\, t\right)\right). \end{aligned}$$

Here, $\rho_{1+} = \sqrt{c_{10}}\,(1+\delta_1)$, $\rho_{1-} = \sqrt{c_{10}}\,(1-\delta_1)$, $\rho_{2+} = \sqrt{c_{20}}\,(1+\delta_2)$, and $\rho_{2-} = \sqrt{c_{20}}\,(1-\delta_2)$. We then get that the first column of $\mathsf{SO}(4)$ is given by $\hat{w}_1 = \begin{bmatrix} w_0 & w_1 & w_2 & w_3 \end{bmatrix}^\top$. The columns $\hat{w}_2$, $\hat{w}_3$, and $\hat{w}_4$, can be found in a similar fashion, where each is defined through the equations given below (refer to theorem 4.1.8)

$$\begin{aligned} \frac{1}{2}g_1(t) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} g_2(t)^{-1} &= W_1 \mapsto \hat{w}_1 & \qquad \frac{1}{2}g_1(t) \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} g_2(t)^{-1} &= W_2 \mapsto \hat{w}_2 \\ \frac{1}{2}g_1(t) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} g_2(t)^{-1} &= W_3 \mapsto \hat{w}_3 & \qquad \frac{1}{2}g_1(t) \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} g_2(t)^{-1} &= W_4 \mapsto \hat{w}_4. \end{aligned}$$

It follows that

$$\hat{w}_2 = \frac{1}{2(c_{10}c_{20})^{\frac{1}{4}}} \begin{bmatrix} \sqrt{\rho_{1-}\rho_{2-}} \sin\left(\tfrac{\rho_{1+}-\rho_{2+}}{2}\, t\right) - \sqrt{\rho_{1+}\rho_{2+}} \sin\left(\tfrac{\rho_{1-}-\rho_{2-}}{2}\, t\right) \\ \sqrt{\rho_{1+}\rho_{2+}} \cos\left(\tfrac{\rho_{1-}-\rho_{2-}}{2}\, t\right) - \sqrt{\rho_{1-}\rho_{2-}} \cos\left(\tfrac{\rho_{1+}-\rho_{2+}}{2}\, t\right) \\ \sqrt{\rho_{2-}\rho_{1+}} \sin\left(\tfrac{\rho_{1-}+\rho_{2+}}{2}\, t\right) + \sqrt{\rho_{1-}\rho_{2+}} \sin\left(\tfrac{\rho_{2-}+\rho_{1+}}{2}\, t\right) \\ -\sqrt{\rho_{2-}\rho_{1+}} \cos\left(\tfrac{\rho_{1-}+\rho_{2+}}{2}\, t\right) - \sqrt{\rho_{1-}\rho_{2+}} \cos\left(\tfrac{\rho_{2-}+\rho_{1+}}{2}\, t\right) \end{bmatrix}$$

$$\hat{w}_3 = \frac{-1}{2(c_{10}c_{20})^{\frac{1}{4}}} \begin{bmatrix} \sqrt{\rho_{2-}\rho_{1+}}\,\cos\left(\frac{\rho_{2+}-\rho_{1-}}{2}\,t\right) - \sqrt{\rho_{1-}\rho_{2+}}\,\cos\left(\frac{\rho_{1+}-\rho_{2-}}{2}\,t\right) \\ -\sqrt{\rho_{2-}\rho_{1+}}\,\sin\left(\frac{\rho_{2+}-\rho_{1-}}{2}\,t\right) - \sqrt{\rho_{1-}\rho_{2+}}\,\sin\left(\frac{\rho_{1+}-\rho_{2-}}{2}\,t\right) \\ \sqrt{\rho_{1-}\rho_{2-}}\,\cos\left(\frac{\rho_{1+}+\rho_{2+}}{2}\,t\right) + \sqrt{\rho_{1+}\rho_{2+}}\,\cos\left(\frac{\rho_{1-}+\rho_{2-}}{2}\,t\right) \\ \sqrt{\rho_{1-}\rho_{2-}}\,\sin\left(\frac{\rho_{1+}+\rho_{2+}}{2}\,t\right) + \sqrt{\rho_{1+}\rho_{2+}}\,\sin\left(\frac{\rho_{1-}+\rho_{2-}}{2}\,t\right) \end{bmatrix}$$

$$\hat{w}_4 = \frac{1}{2(c_{10}c_{20})^{\frac{1}{4}}} \begin{bmatrix} \sqrt{\rho_{1-}\rho_{2+}}\,\sin\left(\frac{\rho_{1+}-\rho_{2-}}{2}\,t\right) - \sqrt{\rho_{2-}\rho_{1+}}\,\sin\left(\frac{\rho_{2+}-\rho_{1-}}{2}\,t\right) \\ -\sqrt{\rho_{2-}\rho_{1+}}\,\cos\left(\frac{\rho_{2+}-\rho_{1-}}{2}\,t\right) - \sqrt{\rho_{1-}\rho_{2+}}\,\cos\left(\frac{\rho_{1+}-\rho_{2-}}{2}\,t\right) \\ \sqrt{\rho_{1+}\rho_{2+}}\,\sin\left(\frac{\rho_{1-}+\rho_{2-}}{2}\,t\right) - \sqrt{\rho_{1-}\rho_{2-}}\,\sin\left(\frac{\rho_{1+}+\rho_{2+}}{2}\,t\right) \\ \sqrt{\rho_{1-}\rho_{2-}}\,\cos\left(\frac{\rho_{1+}+\rho_{2+}}{2}\,t\right) - \sqrt{\rho_{1+}\rho_{2+}}\,\cos\left(\frac{\rho_{1-}+\rho_{2-}}{2}\,t\right) \end{bmatrix}.$$

We then have that $g(t) = \begin{bmatrix} \hat{w}_1 & \hat{w}_2 & \hat{w}_3 & \hat{w}_4 \end{bmatrix}$.

# Conclusion

In this thesis we have considered a class of invariant optimal control problems on the orthogonal groups $\mathsf{SO}(3)$ and $\mathsf{SO}(4)$. The objective was to find ways to study large subclasses of these problems effectively. To this end, we introduced several appropriate equivalence relations. We then classified classes of objects (e.g., control systems, optimal control problems, Hamilton-Poisson systems) under these equivalence relations in order to obtain finite lists of representatives (or normal forms). We needed then only consider each of these normal forms, rather than arbitrary objects.

In general, a left-invariant control affine system on $\mathsf{SO}(3)$ will depend on up to twelve parameters. Under detached feedback equivalence (see proposition 3.2.1) this class of control systems has been reduced to two representatives containing no parameters and two single-parameter families of representatives. This reduces the complexity of the control systems we need consider dramatically. An arbitrary cost function, associated to an $\ell$-input control system, will depend on up to $\ell^2 + \ell$ parameters; thus a maximum of twelve for a cost-extended system on $\mathsf{SO}(3)$. For our classification of cost-extended systems, under cost equivalence, we obtained the following results. For single-input inhomogeneous systems we reduced the class of cost functions to a single-parameter family of representatives. For two-input homogeneous systems, we were able to reduce the class of cost functions to a single-parameter family and a three-parameter family of representatives. On the other hand, for two-input inhomogeneous systems we reduced the class of cost functions to three families of representatives containing one, two, and three parameters. Lastly, for three-input homogeneous systems we were able to reduce the class of cost functions to six families of representatives; containing between one and five parameters.

An arbitrary quadratic Hamilton-Poisson system on $\mathfrak{so}(3)^*_-$ will also depend on up to twelve parameters. Using affine equivalence, we reduced this class of systems to three homogeneous representatives (containing no parameters) and nine families of inhomogeneous representatives; containing at most three parameters. We further divided these families into two types according to the structure of their equilibria. For type I systems we provided a complete description of the (Lyapunov) stability nature of their equilibria. We also found explicit expressions for the integral curves of the associated Hamiltonian vector fields in terms of (possibly rational functions of) basic Jacobi elliptic functions. For type II systems we provided an almost complete analysis of the stability nature of their equilibria. (There were only a few isolated equilibrium states for which we were unable to determine their stability nature.) We now elaborate on how these results have contributed to the study of optimal control problems on $\mathsf{SO}(3)$ and Hamilton-Poisson systems on $\mathfrak{so}(3)^*_-$.

Over the last few decades, there have been a number of papers dealing with optimal control problems on $\mathsf{SO}(3)$ and Hamilton-Poisson systems on $\mathfrak{so}(3)^*_-$ (see, e.g., [29, 6, 11, 50]). Typically these papers have only dealt with very specific problems and have primarily focused on the free rigid body problem. Dăniasă et. al. ([29]) have performed a more general analysis of quadratic Hamilton-Poisson systems on $\mathfrak{so}(3)^*_-$, using the notion of *orthogonal equivalence*. Their paper, however, is only concerned with those Hamilton-Poisson systems on $\mathfrak{so}(3)^*_-$ which are homogeneous; despite this, it makes for an interesting comparison with the results obtained in this thesis. In particular, they also obtain three classes of homogeneous systems. However, unlike in [29], our approach of using affine equivalence produces homogeneous representatives which contain no parameters but still cover all possible cases effectively. This simplifies the necessary work required as well as the complexity of the expressions for the integral curves of the normal forms.

To date there has been little research devoted to the study of inhomogeneous quadratic Hamilton-Poisson systems on $\mathfrak{so}(3)^*_-$, and in particular, the calculation of explicit expressions for the integral curves of the associated Hamiltonian vector fields. The approach we used to determine the critical energy states provides a useful way of partitioning the cases of qualitatively distinct integral curves. That is, each region (which results as the partitioning of the energy states) usually corresponds to integral curves with different explicit expressions. The typical configurations are then the graphs of the intersections of the Casimir and Hamiltonian functions for typical values of each region. These graphs provide a practical means of visualizing the behaviour of the integral curves, as well as how they change as they pass through critical values. (This knowledge generally proves useful in obtaining explicit expressions for the integral curves.) We believe the approach we have taken could be applied to the integration of quadratic Hamilton-Poisson systems on other Lie-Poisson spaces (see, e.g., [5]).

To the best of our knowledge our results pertaining to the stability nature of the equilibria, as well as the explicit expressions for the integral curves, of inhomogeneous quadratic Hamilton-Poisson systems on $\mathfrak{so}(3)^*_-$ are original. However, that is not to say that none of these systems have ever been considered. For example, in [33], Jurdjevic considers an inhomogeneous quadratic Hamilton-Poisson system on $\mathfrak{so}(3)^*_-$, which is affinely equivalent to our system $H_1^1(p) = p_2 + \frac{1}{2}p_1^2$. However, his interest lies mainly with an optimal control problem associated to this Hamilton-Poisson system. Thus, although he provides the intersections of the level sets of the Casimir and Hamiltonian functions, he does not investigate the stability nature of the equilibrium states nor does he provide any explicit expressions for the integral curves of the associated Hamiltonian vector field. We were very recently made aware of a paper by Montaldi, [46], dealing with bifurcations of relative equilibria of Hamiltonian systems. The plots he provides for the equilibria seem to make for an interesting comparison with the results obtained in this thesis for inhomogeneous systems on $\mathfrak{so}(3)^*_-$. However, as far as we can tell, Montaldi was not able to determine the Lyapunov stability nature for a number of equilibrium states that we were.

In section 6.1 we considered several examples of optimal control problems on $\mathsf{SO}(3)$. To the best of our knowledge, the explicit expressions for extremal curves on the group $\mathsf{SO}(3)$ and $\mathbb{S}^2$ are original. However, the following contributions merit acknowledgement. As mentioned in subsection 6.1.3 similar results, particularly the plots of the extremal curves, were obtained by Jurdjevic in [33]. However, he did not provide the final explicit expressions for the extremal curves on the group $\mathsf{SO}(3)$ and the homogeneous space $\mathbb{S}^2$. Recently, an article by Beschastnyi and Sachkov [14], dealing with the sub-Riemannian geodesics on $\mathsf{SO}(3)$ and $\mathbb{S}^2$, has also been made available. In this paper, they find explicit expressions of the Euler angles (as do we) for a class of sub-Riemannian problems on $\mathsf{SO}(3)$. (The sub-Riemannian problem can be treated by realising it as on optimal control problem on the corresponding Lie group.) The class of systems they investigate are, under affine equivalence, equivalent to the system $H^2(p) = p_1^2 + \frac{1}{2}p_2^2$, and the optimal control problems they investigate are related to the system $(\Sigma^{(3,0)}, \chi^1_{\alpha\beta\gamma})$ we consider in subsection 6.1.1. Their paper makes for a good comparison with the work done in this thesis (in particular, comparing the expressions they obtain for the Euler angles and the figures of the extremal curves). It is worth mentioning that their paper goes further in that it discusses under what conditions the extremal trajectories are periodic, as well as considering the conditions and constraints under which these extremal trajectories are in fact optimal.

An arbitrary left-invariant control affine system on $\mathsf{SO}(4)$ will depend on up to forty-two parameters. In this thesis we provided a classification of all such systems, under $\mathfrak{L}$-equivalence. We also determined exactly which of these systems had full rank, thus also obtaining a classification under detached feedback equivalence. The largest family of representatives obtained in our classification contains six parameters. Thus, although this approach is not as effective as it for control systems on $\mathsf{SO}(3)$, this is a very useful reduction from an arbitrary system. Our classification of control systems on $\mathsf{SO}(4)$ is an original contribution and has been published in [4].

We were also able to obtain a partial classification of homogeneous quadratic Hamilton-Poisson space on $\mathfrak{so}(4)^*_-$. We found arriving at a complete classification in this case was not possible due to computational difficulties. However, using the fact that $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, we were able to arrive at a number of conclusive results concerning the stability nature of the equilibria for a large subclass of Hamilton-Poisson systems (specifically, those systems which are shown to decompose as the product of two such systems on $\mathfrak{so}(3)^*_-$, see section 5.1). In this case, we showed how already known results for Hamilton-Poisson systems on $\mathfrak{so}(3)^*_-$ can be used to determine results for such systems on $\mathfrak{so}(4)^*_-$. An investigation of the stability nature of the equilibria for a simple class of indecomposable systems was also provided (see section 5.2).

There have been a number of papers published concerned with optimal control problems on $\mathsf{SO}(4)$ and quadratic Hamilton-Poisson systems on $\mathfrak{so}(4)^*_-$ (see, e.g., [12], [17], [27], [28], [55]). Again, as for $\mathsf{SO}(3)$, these papers typically deal only with specific problems. In [27], the authors provide a detailed investigation of the stability nature of the equilibria for a class of Hamilton-Poisson systems on $\mathfrak{so}(4)^*_-$ relating to the free rigid body problem. In their work, energy-Casimir methods fail to always determine the stability nature of the equilibria. We also encountered this problem in our investigation of indecomposable systems on $\mathfrak{so}(4)^*_-$. Hence there remain many open problems for such systems on $\mathfrak{so}(4)^*_-$. Although there is still much research to be done on $\mathsf{SO}(4)$ and $\mathfrak{so}(4)^*_-$, we believe that our approach of classifying control systems, as well as quadratic Hamilton-Poisson systems, is a useful starting point for the study of these objects. Classifying systems also helps to gain a better understanding of the different possible types of qualitative structures of these systems (for example, being able to determine when a system can be decomposed into two simpler systems, as in section 5.1).

Finally, in section 6.2, we provide an example of an optimal control problem corresponding to a four-input control system on $\mathsf{SO}(4)$. This example shows how the approach presented in section 4.1 can be used to obtain the extremal trajectories of an optimal control on $\mathsf{SO}(4)$ via the double cover $\mathsf{SU}(2) \times \mathsf{SU}(2)$. This approach has also been considered in [42] in the investigation of a similar optimal control problem on $\mathsf{SO}(4)$. Our example helps to strengthen the practical usefulness of this approach. In conjunction with our classification of systems, we believe that with further research, there is potential for the application of this approach to the calculation of the extremal trajectories for a number of optimal control problems on $\mathsf{SO}(4)$.

## Outlook

This thesis makes some important contributions to the study of optimal control problems on the orthogonal groups $\mathsf{SO}(3)$ and $\mathsf{SO}(4)$. However, there are a number of ideas that can be investigated further. The following are topics of possible future research.

The ideas used in [14] to consider when the extremal trajectories obtained are periodic, and under what conditions they are in fact optimal, would be an worthwhile extension of the work done in this thesis on $\mathsf{SO}(3)$.

An investigation into alternative theoretical approaches or techniques in order to refine our classification of quadratic Hamilton-Poisson systems on $\mathfrak{so}(4)^*_-$ could be of particular interest. Ideally, this could be done, in order to obtain a complete classification (with distinct non-equivalent representatives) and hopefully to provide tools for investigating such systems on other higher-dimensional Lie-Poisson spaces, e.g, $\mathfrak{se}(3)^*_-$.

The integration of Hamilton-Poisson systems on $\mathfrak{so}(4)^*_-$ is also a topic which requires further investigation. In this thesis, we have essentially restricted our consideration to those systems whose solutions are expressible in terms of basic Jacobi elliptic functions. It is known, at least in theory, that solutions for integrable systems on $\mathfrak{so}(4)^*_-$ can be obtained in terms of hyper-elliptic functions (consider the remarkable results of Kowalevski). Jurdjevic, in [35], discusses the results of Kowalevski from a more modern perspective. Thus, a better understanding of the nature and role of hyper-elliptic functions and how they can be used to solve more challenging optimal control problems on $\mathsf{SO}(4)$ would be an interesting field of research.

An extension of our investigation of Hamilton-Poisson systems on $\mathfrak{so}(4)^*_-$ to include inhomogeneous systems would also be an exciting area to investigate. For example, it should not be too difficult to find expressions for the integral curves and determine the stability nature of the equilibria for those inhomogeneous system which decompose as the product of inhomogeneous systems on $\mathfrak{so}(3)^*_-$. As mentioned in this thesis, one of the concerns when dealing with systems on higher-dimensional Lie-Poisson spaces is whether the system is actually integrable (generally in the sense of Liouville) or not. In this regard, especially for inhomogeneous systems on $\mathfrak{so}(4)^*_-$, Sokolov and Wolf [57] have done some interesting work using the ideas of Kowalevski exponents and using the tools of modern computer algebra. Further investigation into the integrability and explicit integration of such systems would be a viable topic for future research.

# Appendix A

# Lie groups

## A.1 Preliminaries

A **Lie group** $\mathsf{G}$ is a group equipped with the structure of a (finite-dimensional, real) smooth manifold such that the product map $\mu : \mathsf{G} \times \mathsf{G} \to \mathsf{G}$, $(g, h) \mapsto gh$ is smooth. By the implicit function theorem, it then follows that the inverse map $g \mapsto g^{-1}$ is also smooth. In this thesis we are interested in restricting our attention to real finite-dimensional matrix Lie groups. Let $\mathsf{H}$ be an abstract subgroup of $\mathsf{G}$. $\mathsf{H}$ is called a *Lie subgroup* of $\mathsf{G}$ if it is an immersed submanifold of $\mathsf{G}$. If $\mathsf{H}$ is also an embedded submanifold of $\mathsf{G}$, then it is called a *closed Lie subgroup* of $\mathsf{G}$.

A.1.1 Definition. A (real, finite-dimensional) **matrix Lie group** is any closed Lie subgroup of some general linear group $\mathsf{GL}(n, \mathbb{R})$.

A map $\phi : \mathsf{G} \to \mathsf{H}$ between Lie groups $\mathsf{G}$ and $\mathsf{H}$ is a *Lie group homomorphism* if it simultaneously a homomorphism of abstract groups and a smooth map. A Lie group homomorphism $\phi : \mathsf{G} \to \mathsf{H}$ is called a *Lie group isomorphism* if it is simultaneously an isomorphism of abstract groups and a diffeomorphism of manifolds.

A.1.2 Definition. A **(real) Lie algebra** consists of a vector space $V$, together with a bilinear map $[\cdot, \cdot] : V \times V \to V$ called the **Lie bracket**, such that for all $x, y, z \in V$,

$$[x, y] = -[y, x], \qquad \text{(Skew symmetry)}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \qquad \text{(Jacobi identity).}$$

To each matrix Lie group $\mathsf{G}$ we can naturally associate a Lie algebra $\mathfrak{g}$, which is the tangent space at identity $T_{\mathbf{1}}\mathsf{G} = \{\dot{g}(0) \mid g(\cdot) \text{ is a smooth curve in } \mathsf{G}, g(0) = \mathbf{1}\}$ equipped with a Lie bracket given by the *matrix commutator* $[A, B] = AB - BA$. A *Lie subalgebra* of $\mathfrak{g}$ is a subset $\mathfrak{h} \subset \mathfrak{g}$ that is a Lie algebra in its own right. An *ideal* of $\mathfrak{g}$ is a Lie subalgebra $\mathfrak{h}$ such that for every $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$ we have $[X, Y] \in \mathfrak{h}$. Let $\Gamma$ be a subset of a Lie algebra $\mathfrak{g}$. The *Lie algebra generated by* $\Gamma$ is denoted $\mathrm{Lie}(\Gamma)$. That is, $\mathrm{Lie}(\Gamma)$ is the smallest Lie subalgebra of $\mathfrak{g}$ containing $\Gamma$.

A linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ is called a *Lie algebra homomorphism* if it preserves the Lie bracket, i.e., $\phi([X, Y]_{\mathfrak{g}}) = [\phi(X), \phi(Y)]_{\mathfrak{h}}$ for all $X, Y \in \mathfrak{g}$. If $\phi$ is also bijective, then it is called a *Lie algebra isomorphism*.

Using left (or right) translations one can construct natural isomorphisms between the tangent spaces of a Lie group $\mathsf{G}$ at different points. Let $L_g : \mathsf{G} \to \mathsf{G}$, $h \mapsto gh$ denote the *left translation.* Then for $\xi \in T_h\mathsf{G}$ we have

$$g\xi = T_h L_g \cdot \xi \in T_{gh}\mathsf{G}.$$

In particular for matrix Lie groups the "product" $g\xi$ coincides with matrix multiplication. The Lie algebra of a Lie group may be characterised in terms of left-invariant vector fields. A vector field $X \in \mathrm{Vec}(\mathsf{G})$ is termed left-invariant if $T_h L_g \cdot X(h) = gX(h) = X(gh)$ for all $g, h \in \mathsf{G}$. (Consequently, every left-invariant vector field is of the form $X(g) = gA$ for some $A \in \mathfrak{g}$.) It is well known that the set $\mathrm{Vec}_L(\mathsf{G})$ of all left-invariant vector fields on $\mathsf{G}$, together with the Lie bracket of vector fields (defined by $[X, Y][f] = X[Y[f]] - Y[X[f]]$, $f \in C^\infty(\mathsf{G})$), is isomorphic (as Lie algebras) to the Lie algebra $\mathfrak{g}$. Indeed, we have the correspondence

$$X \in \mathcal{X}(\mathsf{G}),\ X(g) = gA \leftrightarrow X(\mathbf{1}) = A \in \mathfrak{g}.$$

We will denote the *centre of a Lie group* $\mathsf{G}$ as $\mathsf{Z}(\mathsf{G})$ and the *centre of a Lie algebra* as $\mathsf{Z}(\mathfrak{g})$. These centres are explicitly given by

$$\mathsf{Z}(\mathsf{G}) = \{g \in \mathsf{G} \mid \forall h \in \mathsf{G},\, gh = hg\} \quad \mathsf{Z}(\mathfrak{g}) = \{A \in \mathfrak{g} \mid \forall B \in \mathfrak{g},\, [A, B] = 0\}\,.$$

Note that $\mathsf{Z}(\mathsf{G})$ is a normal closed Lie subgroup with Lie algebra $\mathsf{Z}(\mathfrak{g})$, an ideal of $\mathfrak{g}$.

### Topology of Lie groups

Recall that a Lie group $\mathsf{G}$ is called **simply connected** if for every two smooth curves $g(\cdot) : [0, 1] \to \mathsf{G}$ and $h(\cdot) : [0, 1] \to \mathsf{G}$ with the same endpoints, there exists a continuous function $H : \mathsf{G} \times [0, 1] \to \mathsf{G}$ such that $H(\cdot, 0) = g(\cdot)$ and $H(\cdot, 1) = h(\cdot)$. (That is, the curve $g(\cdot)$ can be continuously deformed into $h(\cdot)$.) Alternatively, $\mathsf{G}$ is called simply connected if it is connected and the **fundamental group** of $\mathsf{G}$, denoted $\pi_1(\mathsf{G})$, is the trivial group (i.e., $\pi_1(\mathsf{G}) = \{\mathbf{1}\}$).

A.1.3 PROPOSITION. ([30]) *A simply connected Lie group is determined up to isomorphisms by its Lie algebra.*

Let $\mathsf{N}$ be a normal closed subgroup of $\mathsf{G}$. The quotient $\mathsf{G}/\mathsf{N}$ may be given a smooth structure such that it is a Lie group. (However, $\mathsf{G}/\mathsf{N}$ may not be a matrix Lie group.) Indeed, the following results hold.

A.1.4 PROPOSITION. ([30]) *Let* $\mathsf{H}$ *be a closed Lie subgroup of a Lie group* $\mathsf{G}$*. The set* $\mathsf{G}/\mathsf{H}$ *of left cosets of* $\mathsf{H}$ *in* $\mathsf{G}$ *possess a unique differentiable structure for which the canonical map* $p : \mathsf{G} \to \mathsf{G}/\mathsf{H}$, $g \mapsto g\mathsf{H}$ *is a quotient map.*

A.1.5 PROPOSITION. ([30]) *Let* $\mathsf{N}$ *be a normal closed Lie subgroup of a Lie group* $\mathsf{G}$*. Then the quotient group* $\mathsf{G}/\mathsf{N}$ *with the differentiable structure of proposition A.1.4 is a Lie group.*

Let $\mathsf{G}$ and $\mathsf{H}$ be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. A **covering homomorphism** from $\mathsf{G}$ onto $\mathsf{H}$ is a Lie group homomorphism $\phi : \mathsf{G} \to \mathsf{H}$ such that $T_{\mathbf{1}}\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra isomorphism. Equivalently, $\phi$ is a covering homomorphism if $\ker \phi = \{g \in \mathsf{G} \mid \phi(g) = \mathbf{1}\}$ is discrete. A covering homomorphism is called a **double cover** if it is also a 2-to-1 map. We note that for every Lie group $\mathsf{G}$ there exists a simply connected Lie group $\tilde{\mathsf{G}}$ with isomorphic Lie algebra. Moreover, we have the following result.

A.1.6 Theorem. ([30]) *Every connected Lie group* $\mathsf{G}$ *is isomorphic to a quotient* $\tilde{\mathsf{G}}/\mathsf{N}$ *where* $\tilde{\mathsf{G}}$ *is a simply connected Lie group and* $\mathsf{N}$ *is a discrete normal subgroup. The pair* $(\tilde{\mathsf{G}}, \mathsf{N})$ *is determined by these conditions up to an isomorphism.*

A Lie group $\tilde{\mathsf{G}}$ satisfying the conditions of theorem A.1.6 is called the **universal covering Lie group** of the Lie group $\mathsf{G}$. A covering homomorphism $q : \tilde{\mathsf{G}} \to \mathsf{G}$ is then called a **universal covering** (homomorphism) of $\mathsf{G}$.

### Representations

Let $\mathsf{G}$ be a Lie group with Lie algebra $\mathfrak{g}$ and Let $V$ be a vector space over $\mathbb{R}$. A Lie group homomorphism $\rho : \mathsf{G} \to \mathsf{GL}(V)$ is called a *linear representation of* $\mathsf{G}$ in $V$. By the *dual representation* of a linear representation $\rho$, we mean the linear representation $\rho^*$ of the group $\mathsf{G}$ in the dual space $V^*$ defined by $(\rho^*(g) \cdot \mu)(v) = \mu(\rho(g)^{-1}v)$ for $\mu \in V^*$ and $v \in V$. A *linear representation of* $\mathfrak{g}$ is a Lie algebra homomorphism $\psi : \mathfrak{g} \to \mathfrak{gl}(V)$.

For a Lie group $\mathsf{G}$ there exists a natural linear representation (and associated dual representation) of the group (and its Lie algebra) in its Lie algebra $\mathfrak{g}$. The adjoint representation (respectively coadjoint representation) of a Lie group $\mathsf{G}$ in its Lie algebra $\mathfrak{g}$, denoted by Ad (respectively Ad$^*$) are linear representations defined by

$$\begin{aligned} \mathrm{Ad} : \ & \mathsf{G} \to \mathsf{GL}(\mathfrak{g}) & \qquad \mathrm{Ad}^* : \ & \mathsf{G} \to \mathsf{GL}(\mathfrak{g}^*) \\ & g \mapsto \mathrm{Ad}_g & & g \mapsto \mathrm{Ad}^*_{g^{-1}} \\ \mathrm{Ad}_g : \ & \mathfrak{g} \to \mathfrak{g} & \mathrm{Ad}^*_g : \ & \mathfrak{g}^* \to \mathfrak{g}^* \\ & X \mapsto gXg^{-1} & & \mu \mapsto \mu \circ \mathrm{Ad}_g \end{aligned}$$

Their tangent maps at identity are linear representations of $\mathfrak{g}$, denoted by ad and ad$^*$ respectively, given by

$$\begin{aligned} \mathrm{ad} : \ & \mathfrak{g} \to \mathrm{Der}(\mathfrak{g}) & \qquad \mathrm{ad}^* : \ & \mathfrak{g} \to \mathrm{Der}(\mathfrak{g}^*) \\ & X \mapsto \mathrm{ad}_X & & X \mapsto -\mathrm{ad}^*_X \\ \mathrm{ad}_X : \ & \mathfrak{g} \to \mathfrak{g} & \mathrm{ad}^*_X : \ & \mathfrak{g}^* \to \mathfrak{g}^* \\ & Y \mapsto [X, Y] & & \mu \mapsto \mu \circ \mathrm{ad}_X \end{aligned}$$

The *adjoint orbit* through a point $X \in \mathfrak{g}$ is given by $\{\mathrm{Ad}_g(X) \mid g \in \mathsf{G}\}$. Similarly, the *coadjoint orbit* through a point $\mu \in \mathfrak{g}^*$ is given by $\left\{\mathrm{Ad}^*_{g^{-1}}(\mu) \mid g \in \mathsf{G}\right\}$. Note that these orbits are independent of the connected Lie group chosen (i.e., they are solely properties of the Lie algebra).

### Lie group and Lie algebra homomorphisms

A.1.7 Theorem. (Uniqueness) *A Lie group homomorphism* $\phi$ *from a connected Lie group* $\mathsf{G}$ *to a Lie group* $\mathsf{H}$ *is uniquely determined by its tangent map* $T_{\mathbf{1}}\phi$.

A.1.8 THEOREM. (EXISTENCE) *Let* $\mathsf{G}, \mathsf{H}$ *be Lie groups with* $\mathsf{G}$ *simply connected. Then for every Lie algebra homomorphism* $\psi : \mathfrak{g} \to \mathfrak{h}$ *there exists a Lie group homomorphism* $\phi : \mathsf{G} \to \mathsf{H}$ *such that* $T_{\mathbf{1}}\phi = \psi$.

Let $\mathsf{G}$ be a Lie group, let $\mathsf{Aut}(\mathsf{G})$ be its group of automorphisms and let $\mathsf{Aut}(\mathfrak{g})$ be the group of automorphisms of its Lie algebra $\mathfrak{g}$. If $\mathsf{G}$ is connected, then the map

$$\mathbf{d} : \mathsf{Aut}(\mathsf{G}) \to \mathsf{Aut}(\mathfrak{g}), \quad \phi \mapsto T_{\mathbf{1}}\phi$$

is an injection. Moreover, if $\mathsf{G}$ is simply connected, then it is a Lie group isomorphism. The group $\mathsf{Aut}(\mathfrak{g})$ is a matrix Lie group. In general we have the following.

A.1.9 PROPOSITION. *For any connected Lie group* $\mathsf{G}$, *the group* $\mathbf{d}\mathsf{Aut}(\mathsf{G})$ *is a closed Lie subgroup of the Lie group* $\mathsf{Aut}(\mathfrak{g})$.

If $\mathsf{G}$ is connected, then $\mathsf{Int}(\mathfrak{g}) = \mathrm{Ad}(\mathsf{G})$ depends only on the Lie algebra $\mathfrak{g}$ and is a normal subgroup of $\mathsf{Aut}(\mathfrak{g})$. It is called the **group of inner automorphisms of the algebra** $\mathfrak{g}$.
The following proposition is useful for determining the group $\mathbf{d}\mathsf{Aut}(\mathsf{G})$.

A.1.10 PROPOSITION. ([31]) *Suppose that* $\mathsf{G}$ *is a connected Lie group with Lie algebra* $\mathfrak{g}$ *and suppose that* $q : \tilde{\mathsf{G}} \to \mathsf{G}$ *is a universal covering. Then* $\mathbf{d}\mathsf{Aut}(\mathsf{G}) \subset \mathsf{Aut}(\mathfrak{g})$. *Moreover, if* $\psi \in \mathsf{Aut}(\mathfrak{g})$, *then* $\psi \in d\mathsf{Aut}(\mathsf{G})$ *if and only if* $\phi(\ker q) = \ker q$, *where* $\phi \in \mathsf{Aut}(\tilde{G})$ *is the unique automorphism such that* $T_{\mathbf{1}}\phi = (T_{\mathbf{1}}q^{-1}) \cdot \psi \cdot T_{\mathbf{1}}q$.

### Simple and semi-simple Lie algebras

A.1.11 DEFINITION. A (non-abelian) Lie algebra is called:

- **simple** if it contains no ideal other than the trivial ideal $\{0\}$ and itself $\mathfrak{g}$.
- **semisimple** it has no nontrivial abelian ideals.

Clearly any simple Lie algebra is also semisimple.

A bilinear form $\mathcal{B}$ on a Lie algebra $\mathfrak{g}$, of a Lie group $\mathsf{G}$, is said to be **invariant** if

$$\mathcal{B}(\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y)) = \mathcal{B}(X, Y)$$

for all $g \in \mathsf{G}$, $X, Y \in \mathfrak{g}$. For a connected Lie group, this definition is equivalent to the bilinear form $\mathcal{B}$ satisfying the property

$$\mathcal{B}([X, Y], Z) + \mathcal{B}(X, [Y, Z]) = 0$$

for all $X, Y, Z \in \mathfrak{g}$. Consider now the (real) bilinear form

$$\mathcal{K}_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, \quad \mathcal{K}_{\mathfrak{g}}(X, Y) = \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$$

on a finite-dimensional Lie algebra $\mathfrak{g}$. This form is called the **(Cartan) Killing form** of $\mathfrak{g}$.

A.1.12 Proposition. ([30]) *A Lie algebra* $\mathfrak{g}$ *is semisimple if and only if its Cartan Killing form is nondegenerate.*

A.1.13 Proposition. ([30]) *A Lie algebra* $\mathfrak{g}$ *is semisimple if and only if* $\mathfrak{g}$ *can be decomposed into a direct sum*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$$

*where* $\mathfrak{g}_i$, $i = \overline{1, s}$, *are simple nonabelian ideals. Any ideal of the Lie algebra* $\mathfrak{g}$ *is a sum of the ideals* $\mathfrak{g}_i$. *In particular, the above decomposition is unique.*

A Lie group $\mathsf{G}$ is called *compact* if it is compact as a topological space. A (real) finite-dimensional Lie algebra $\mathfrak{g}$ is said to be **compact** if there exists a definite invariant scalar product on $\mathfrak{g}$. It is well known that the Lie algebra $\mathfrak{g}$ of any compact Lie group $\mathsf{G}$ is compact.

A.1.14 Proposition. ([30]) *The Killing form of a compact Lie algebra is negative definite. A (real) Lie algebra is semisimple and compact if and only if its Killing form is negative definite.*

## A.2 The Lie group $\mathsf{SU}(2)$

The special unitary group of dimension three is given by

$$\mathsf{SU}(2) = \left\{ \begin{bmatrix} \alpha & \bar{\beta} \\ -\beta & \bar{\alpha} \end{bmatrix} \mid \alpha, \beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

The (real) Lie algebra of $\mathsf{SU}(2)$ is given by

$$\mathfrak{su}(2) = \left\{ \frac{1}{2} \begin{bmatrix} ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & -ix_1 \end{bmatrix} \mid x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

Let

$$F_1 = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad F_2 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad F_3 = \frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

be the standard (ordered) basis for the Lie algebra $\mathfrak{su}(2)$. The commutator relations are given by

$$[F_1, F_2] = F_3, \quad [F_2, F_3] = F_1, \quad [F_3, F_1] = F_2.$$

The group $\mathsf{SU}(2)$ is a simply-connected, compact Lie group.

A.2.1 Proposition. *The Lie group* $\mathsf{SU}(2)$ *is a double-cover of the Lie group* $\mathsf{SO}(3)$. *In fact, it is the universal covering group of* $\mathsf{SO}(3)$.

There is an isomorphism between the vector spaces $\mathbb{R}^3 \to \mathfrak{so}(3) \to \mathfrak{su}(2)$ given explicitly by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \mapsto \frac{1}{2}\begin{bmatrix} ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & -ix_1 \end{bmatrix}.$$

A.2.2 Proposition. *The Lie group* $\mathsf{SU}(2) \times \mathsf{SU}(2)$ *is a double-cover of the Lie group* $\mathsf{SO}(4)$. *In fact, it is the universal cover of* $\mathsf{SO}(4)$.

# Appendix B

# Hamilton-Poisson formalism

In this thesis we wish to investigate the optimal controls associated to cost-extended control systems. It is shown that these controls are affinely related to the integral curve of a vector field $\vec{H}$ associated to some function $H$ on the dual space of the Lie algebra. In this section we introduce the terminology and results required for the investigation of such systems, called Hamilton-Poisson systems.

Let $M$ be a smooth manifold. A *Poisson structure (or Poisson bracket)* on $M$ is a bilinear, skew-symmetric map $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ satisfying:

- $\{F,\{G,H\}\} + \{G,\{H,F\}\} + \{H,\{F,G\}\} = 0$ (Jacobi identity)
- $\{FG,H\} = F\{G,H\} + \{F,H\}G$ (derivation property)

for all $F,G,H \in C^\infty(M)$. The pair $(M,\{\cdot,\cdot\})$ is called a *Poisson manifold*. If the underlying manifold is a vector space, then we will refer to the pair as a *Poisson space*.

B.0.3 Definition. Let $(M,\{\cdot,\cdot\})$ be a Poisson manifold and $H \in C^\infty(M)$. The vector field $\vec{H}$ defined by

$$\vec{H}[F] = \{F,H\}, \quad F \in C^\infty(M)$$

is called the *Hamiltonian vector field* associated to the *Hamiltonian function* $H$. (Here $\vec{H}[F]$ is the Lie-derivative of the function $F$ along the vector field $\vec{H}$.) The triple $(M,\{\cdot,\cdot\},H)$ is called a **Hamilton-Poisson system**.

If the Poisson manifold $(M,\{\cdot,\cdot\})$ is fixed, we identify a Hamilton-Poisson system with its Hamiltonian function. Given a Hamiltonian vector field $\vec{H}$ on $M$, an *integral curve* of $\vec{H}$ is an absolutely continuous curve $p(\cdot)$ that satisfies the *equations of motion*, i.e., $\dot{p}(t) = \vec{H}(p(t))$. By the Caratheodory existence and uniqueness theorem for ordinary differential equations, there exists a unique solution to the Cauchy problem

$$\dot{p}(t) = \vec{H}(p(t)), \quad p(0) = p_0 \in M.$$

As such, integral curves always exist locally. The vector field $\vec{H}$ is said to be *complete* if the domain of every integral curve can be extended to $\mathbb{R}$. A function $C \in C^\infty(M)$ is called a *Casimir function* if $\{C,F\} = 0$ for every $F \in C^\infty(M)$. That is, $C$ is constant along the flow of all Hamiltonian vector fields, or equivalently, $\vec{C} = 0$. Note, however, that nontrivial Casimir functions are not guaranteed to exist; furthermore, they may not be defined globally.

Suppose $(M, \{\cdot,\cdot\})$ admits a (global) Casimir function $C$ and let $\vec{H}$ be a Hamiltonian vector field. Since $C$ is a constant of motion, every integral curve $p(\cdot)$ of $\vec{H}$ (through some $p(0)$) evolves on the level set $C^{-1}(c_0)$, where $c_0 = C(p(0))$. Similarly, as $H$ is a constant of motion, it follows that $p(\cdot)$ evolves on the level set $H^{-1}(h_0)$, where $h_0 = H(p(0))$. Thus $p(\cdot)$ evolves along the intersection $C^{-1}(c_0) \cap H^{-1}(h_0)$.

### Lie-Poisson structure

Let $\mathfrak{g}$ be a (real, $n$-dimensional) Lie algebra, with dual space $\mathfrak{g}^*$, and let $[\cdot,\cdot]$ denote the Lie bracket on $\mathfrak{g}$. The *(minus) Lie-Poisson structure* (or *(minus) Lie-Poisson bracket*) on $\mathfrak{g}^*$ is given by

$$\{F, G\}(p) = -p\left([\mathbf{d}F(p), \mathbf{d}G(p)]\right).$$

Here $p \in \mathfrak{g}^*$, $F, G \in C^\infty(\mathfrak{g}^*)$, and $\mathbf{d}F(p), \mathbf{d}G(p) \in \mathfrak{g}^{**} \cong \mathfrak{g}$. A **Lie-Poisson space** is a pair $(\mathfrak{g}^*, \{\cdot,\cdot\})$ where $\{\cdot,\cdot\}$ is the (minus) Lie-Poisson bracket on $\mathfrak{g}^*$; we denote $\mathfrak{g}^*_- = (\mathfrak{g}^*, \{\cdot,\cdot\})$.

A *linear Poisson automorphism* is a linear isomorphism $\psi : \mathfrak{g}^* \to \mathfrak{g}^*$ that preserves the Poisson bracket, i.e., $\{F, G\} \circ \psi = \{F \circ \psi, G \circ \psi\}$ for every $F, G \in C^\infty(\mathfrak{g}^*)$. Linear Poisson automorphisms are exactly the dual maps of Lie algebra automorphisms.

Let $\vec{H}$ be the Hamiltonian vector field associated to the Hamiltonian function $H \in C^\infty(\mathfrak{g}^*)$. Specifically we have $\vec{H}(p) = \mathrm{ad}^*_{\mathbf{d}H(p)}(p)$. The equations of motion can be expressed componentwise as

$$\dot{p}_i = -p\left([E_i, \mathbf{d}H(p)]\right), \quad i = 1, \ldots, n$$

where $(E_i)_{i=1}^n$ is a basis for $\mathfrak{g}$.

# Appendix C

# Stability

## C.1 Nonlinear stability

In this thesis we are interested in the Lyapunov (nonlinear) stability of nonlinear dynamical systems evolving on the dual spaces of Lie algebras. The reduced extremal equations on $\mathfrak{g}^*$, corresponding to optimal control problems on Lie groups, form such a class of systems.

C.1.1 Definition. Let $X$ be a smooth vector field on $\mathfrak{g}^*$. We say that a point $p_e$ in $\mathfrak{g}^*$ is an **equilibrium state** of $X$ if $X(p_e) = 0$.

C.1.2 Definition. Let $p_e$ be an equilibrium state of a smooth vector field $X$, and let $DX(p_e)$ denote the linearization of $X$ at $p_e$. We say that $p_e$ is

1. *(Lyapunov) stable* if for every neighbourhood $U$ of $p_e$ in $\mathfrak{g}^*$ there exists a neighbourhood $V \subset U$ of $p_e$ such that for every integral curve $p(\cdot)$ of $X$ with $p(0) \in V$ we have $p(t) \in U$ for all $t > 0$.

2. *(Lyapunov) unstable* if it is not stable. That is, there exists a neighbourhood $U$ of $p_e$ in $\mathfrak{g}^*$ such that for every neighbourhood $V \subset U$ of $p_e$ there exists an integral curve $p(t)$ of $X$ with $p(0) \in V$ such that $p(t_1) \notin U$ for some $t_1 > 0$.

3. *spectrally stable* if the eigenvalues of $DX(p_e)$ all have non-positive real part.

4. *spectrally unstable* if it is not spectrally stable.

The most common method for proving the instability of an equilibrium state of a nonlinear system is to prove that the state is spectrally unstable; as spectral instability implies Lyapunov instability. This method, however, is sometimes insufficient to prove instability and one may have to refer to the definition of Lyapunov instability, i.e., one must find a specific integral curve that satisfies the definition. For low-dimensional Lie groups such as $\mathsf{SO}(3)$ this is generally not too difficult. Looking at the intersection of the level sets of the Casimir and Hamiltonian functions, on which the solution curves evolve, can give one a good idea as to whether or not certain equilibrium states are (nonlinearly) stable or unstable.

## C.2 Energy-Casimir methods

The *energy-Casimir method* [32] gives sufficient conditions for Lyapunov stability of equilibrium states of certain types of Hamilton-Poisson dynamical systems (cf. [43, 49]). The method is restricted to certain types of systems, since its implementation relies on an abundant supply of Casimir functions.

The standard energy-Casimir method states that if $p_e$ is an equilibrium point of a Hamiltonian vector field $\vec{H}$ (associated with an energy function $H$) and if there exists a Casimir function $C$ such that $p_e$ is a critical point of $H + C$ and $\mathbf{d}^2(H + C)(p_e)$ is (positive or negative) definite, then $p_e$ is Lyapunov stable. Ortega and Ratiu have obtained a generalization of the standard energy-Casimir method (cf. [48, 47]).

Generalized energy-Casimir method *Let* $(M, \{\cdot,\cdot\}, H)$ *be a Poisson dynamical system. Let* $p_e$ *be an equilibrium state of* $\vec{H}$ *and* $C_1, \dots, C_k : M \to \mathbb{R}$ *conserved quantities of* $\vec{H}$*, that is* $\{C_i, H\} = 0,\ i = 1, \dots, k$*. Assume that there exist constants* $\lambda_0, \lambda_1, \dots, \lambda_k$ *such that*

$$\mathbf{d}(\lambda_0 H + \lambda_1 C_1 + \cdots + \lambda_k C_k)(p_e) = 0$$

*and the quadratic form*

$$\mathbf{d}^2(\lambda_0 H + \lambda_1 C_1 + \cdots + \lambda_k C_k)(p_e)\ |_{W \times W} = 0$$

*is positive definite, where* $W = \ker \mathbf{d}H(p_e) \cap \ker \mathbf{d}C_1(p_e) \cap \cdots \cap \ker \mathbf{d}C_k(p_e)$*. Then* $p_e$ *is (Lyapunov) stable.*

Continuous energy-Casimir method *If* $C_1, C_2, \dots, C_k$ *are (locally) conserved quantities such that* $\cap_{i=1}^k C_i^{-1}(C_i(p_e)) = \{p_e\}$ *then the equilibrium state* $p_e$ *is (Lyapunov) stable.*

# Appendix D

# Integration

## D.1 Integrability

In this section we include the theory relevant to the study of integrability of dynamical systems, and specifically, for the integrability of Hamilton-Poisson systems.

### Liouville integrability

A system of differential equations is said to be *integrable by quadratures* if its solutions can be found after a finite number of steps involving algebraic operations (including the inverting of functions) and integration of given functions.

Let $T^*\mathsf{G}$ be the cotangent bundle of a Lie group $\mathsf{G}$ with $n = \dim \mathsf{G}$. A family $\Phi$ of functions on $T^*\mathsf{G}$ (or in fact any symplectic manifold) is said to be *involutive* if $\{f, g\} = 0$ for any $f, g \in \Phi$. The functions $f_1, \ldots, f_m$ are said to be *functionally independent* (or just, *independent*) at a point $\xi \in T^*\mathsf{G}$ if their differentials are linearly independent. It is well known that the maximal number of independent functions at $\xi$ that are also involutive cannot exceed the dimension of $\mathsf{G}$. In general, we are seeking enough integrals of motion such that the system can be explicitly integrated; in our case the Hamiltonian plus an additional $n-1$ first integrals $\{H, f_1, \ldots, f_{n-1}\}$, which must be involutive and independent. For the system $\dot{x}(t) = \vec{H}(x(t))$, these first integrals (which are involutive with respect to the Poisson bracket) automatically provide $n-1$ symmetries of the form

$$\frac{dx_i}{dt_j} = \{x_i, f_j\}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, n-1.$$

Taken together with the Hamiltonian $H$, we then have a set of integrals and symmetries which are sufficient to integrate the system by quadratures. Thus we call such a Hamiltonian system, admitting $n-1$ additional first integrals, *completely integrable*. We provide a more specialised statement of this idea in the following definition.

D.1.1 Definition. ([8]) Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold of rank $2r$ and let $\mathbf{F} = \{f_1, \ldots, f_s\}$ be involutive and independent, with $s = \dim(M) - r$. Then we say that $\mathbf{F}$ is *completely integrable* and that $(M, \{\cdot,\cdot\}, \mathbf{F})$ is a **completely integrable system**. The vector fields $\vec{f_i}$ are then called integrable vector fields and the map $\mathbf{F}$ is called the *momentum map*.

We say that $r$ is the number of *degrees of freedom of the integrable systems* and we call $2r$ its *rank*.

D.1.2 REMARK. Notice that $2r$ is the dimension of the symplectic leaves of maximal dimension (typically the generic leaf) and that $r$ is the number of independent commuting Hamiltonian vector fields on such a leaf.

## D.2 Jacobi elliptic functions

Given the modulus $k \in [0,1]$, the basic *Jacobi elliptic functions* $\mathrm{sn}(\cdot,k)$, $\mathrm{cn}(\cdot,k)$, and $\mathrm{dn}\,(\cdot,k)$ can be defined as

$$\begin{aligned} \mathrm{sn}(x,k) &= \sin\,\mathrm{am}(x,k) \\ \mathrm{cn}(x,k) &= \cos\,\mathrm{am}(x,k) \\ \mathrm{dn}(x,k) &= \sqrt{1-k^2\sin^2\mathrm{am}(x,k)} \end{aligned}$$

where $\mathrm{am}(\cdot,k) = F(\cdot,k)^{-1}$ is the amplitude and $F(\varphi,k) = \int_0^\varphi \frac{dt}{\sqrt{1-k^2\sin^2 t}}$. (For the degenerate cases $k=0$ and $k=1$ we recover the circular functions and the hyperbolic functions, respectively.) The number $K$ is defined as $K = F(\frac{\pi}{2},k)$ and the complementary modulus is defined as $k' = \sqrt{1-k^2}$. (The functions $\mathrm{sn}(\cdot,k)$ and $\mathrm{cn}(\cdot,k)$ are $4K$ periodic whereas $\mathrm{dn}(\cdot,k)$ is $2K$ periodic.) Nine other elliptic functions are defined by taking reciprocals and quotients; in particular, we get $\mathrm{ns}(\cdot,k) = \frac{1}{\mathrm{sn}(\cdot,k)}$, $\mathrm{nd}(\cdot,k) = \frac{1}{\mathrm{dn}(\cdot,k)}$ and $\mathrm{cd}(\cdot,k) = \frac{\mathrm{cn}(\cdot,k)}{\mathrm{dn}(\cdot,k)}$.

The elliptic integrals of the second and third kind are given respectively by

$$\begin{aligned} E(\varphi,k) &= \int_0^\varphi \sqrt{1-k^2\sin^2\varphi}\,d\varphi \\ \Pi(n,\varphi,k) &= \int_0^\varphi \frac{d\varphi}{(1+n\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}}. \end{aligned}$$

The complete elliptic integrals of the second and third kind are defined as $E(\frac{\pi}{2},k)$ and $\Pi(n,\frac{\pi}{2},k)$, respectively.

### Reduction to standard form

For the purposes of this thesis, we are interested in solving an integral of the form $\int \frac{dx}{\omega}$, where $\omega^2$ is a quartic (or cubic) in $x$. We briefly outline how an integral of this form may be reduced to a standard form, as discussed in [10].

First, any quartic (or cubic) $\omega^2$ in $x$ (with no repeated factors) can be expressed in the form

$$\left(A_1(x-r_1)^2 + B_1(x-r_2)^2\right)\left(A_2(x-r_1)^2 + B_2(x-r_2)^2\right)$$

where $A_1, B_1, A_2, B_2, r_1, r_2 \in \mathbb{R}$. We can express $\omega^2$ as the product of two quadratics (which, if the roots are real, are arranged to have non-interlacing roots)

$$\omega^2 = X_1X_2 = (a_1x^2 + 2b_1x + c_1)(a_2x^2 + 2b_2x + c_2)$$

with $a_i, b_i, c_i \in \mathbb{R}$, $i = \overline{1,2}$. Consider the polynomial $X_1 - \lambda X_2$. We solve for the values of $\lambda$ which make a perfect square in $x$. Let the roots of this quadratic (in $\lambda$) be $\lambda_1$ and $\lambda_2$. Then there exist $r_1$ and $r_2$ such that

$$X_1 - \lambda_1 X_2 = (a_1 - \lambda_1 a_2)(x - r_1)^2, \qquad X_1 - \lambda_2 X_2 = (a_1 - \lambda_2 a_2)(x - r_2)^2.$$

(We note that $\lambda_1$ and $\lambda_2$ can be shown to be real and distinct under our assumptions, unless $a_1 b_2 = a_2 b_1$, in which case $X_1 = a_1(x - r_1)^2 + B_1, X_2 = a_2(x - r_1)^2 + B_2$.) In other words our quartic takes the given form

$$X_1 X_2 = \left(A_1(x - r_1)^2 + B_1(x - r_2)^2\right)\left(A_2(x - r_1)^2 + B_2(x - r_2)^2\right),$$

where

$$A_1 = \frac{\lambda_2\,(a_1 - a_2\lambda_1)}{\lambda_2 - \lambda_1}, \qquad A_2 = \frac{a_1 - a_2\lambda_1}{\lambda_2 - \lambda_1}, \qquad B_1 = \frac{\lambda_1\,(a_1 - a_2\lambda_2)}{\lambda_1 - \lambda_2}, \qquad B_2 = \frac{a_1 - a_2\lambda_2}{\lambda_1 - \lambda_2}.$$

Thus we have

$$\int \frac{dx}{\omega} = \int \frac{dx}{\sqrt{\left(A_1(x - r_1)^2 + B_1(x - r_2)^2\right)\left(A_2(x - r_1)^2 + B_2(x - r_2)^2\right)}}.$$

Consider the change in variable $s = \frac{x - r_1}{x - r_2}$. Then (provided $A_1 \neq 0$ and $A_2 \neq 0$) we have that

$$\begin{aligned}\frac{dx}{\omega} &= \frac{(r_1 - r_2)ds}{(s-1)^2\sqrt{\left(A_1(\frac{r_2 s - r_1}{s-1} - r_1)^2 + B_1(\frac{r_2 s - r_1}{s-1} - r_2)^2\right)\left(A_2(\frac{r_2 s - r_1}{s-1} - r_1)^2 + B_2(\frac{r_2 s - r_1}{s-1} - r_2)^2\right)}}\\ &= \frac{ds}{(r_1 - r_2)\sqrt{\sigma A_1 A_2}\sqrt{\sigma\left(s^2 + \frac{B_1}{A_1}\right)\left(s^2 + \frac{B_2}{A_2}\right)}}\end{aligned}$$

for some $\sigma \in \{-1, 1\}$. Thus we can express the integral $\int \frac{dx}{\omega}$ as

$$\int \frac{dx}{\omega} = \frac{1}{(r_1 - r_2)\sqrt{\sigma A_1 A_2}} \int \frac{ds}{\sqrt{\sigma\left(s^2 + \frac{B_1}{A_1}\right)\left(s^2 + \frac{B_2}{A_2}\right)}}.$$

Once the integral has been reduced to this form we can use one of the well-known elliptic integral formulas. Indeed, simple elliptic integrals can be expressed in terms of appropriate inverse (elliptic) functions. The following formulas hold true (see [10, 41]):

$$\int_x^b \frac{dt}{\sqrt{(a^2 + t^2)(b^2 - t^2)}} = \frac{1}{\sqrt{a^2+b^2}}\,\mathrm{cn}^{-1}\left(\tfrac{1}{b}\,x,\ \tfrac{b}{\sqrt{a^2+b^2}}\right), \quad 0 \le x \le b \tag{D.1}$$

$$\int_x^a \frac{dt}{\sqrt{(a^2 - t^2)(t^2 - b^2)}} = \frac{1}{a}\,\mathrm{dn}^{-1}\left(\tfrac{1}{a}\,x,\ \tfrac{\sqrt{a^2-b^2}}{a}\right), \quad b \le x \le a \tag{D.2}$$

$$\int_x^\infty \frac{dt}{\sqrt{(t^2 - a^2)(t^2 - b^2)}} = \frac{1}{a}\,\mathrm{ns}^{-1}\left(\tfrac{1}{a}\,x,\ \tfrac{b}{a}\right), \quad b < a \le x. \tag{D.3}$$

# Appendix E

# Controllability of a system on $\mathsf{SO}(4)$

In this appendix we show, using an example, how we determine which control systems on $\mathsf{SO}(4)$ are controllable (or, equivalently, have full rank).

Mathematica. *The full-rank calculations for the remaining systems can be found in the Mathematica file:*

*Thesis Mathematica\SO(4)\Lequivalence\FullRankSys.nb*

We shall consider the two-input homogeneous systems as an example. Consider the family of systems $\Sigma^{(2,0)}_{2,\alpha} : u_1(E_1 + \alpha_1 E_4) + u_2(E_2 + \alpha_2 E_5)$. We construct a certain matrix $Z \in \mathbb{R}^{m\times 6}$. The first three rows are given by

$$\begin{aligned} z_1 &= \begin{bmatrix} 1 & 0 & 0 & \alpha_1 & 0 & 0 \end{bmatrix} \\ z_2 &= \begin{bmatrix} 0 & 1 & 0 & 0 & \alpha_2 & 0 \end{bmatrix} \\ z_3 &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & \alpha_1\alpha_2 \end{bmatrix}. \end{aligned}$$

These are just elements that span the trace $\Gamma^{(2,0)}_{2,\alpha}$ along with the Lie bracket of these two elements, identified with vectors in $\mathbb{R}^6$. We then add the rows corresponding to the Lie brackets of the elements $z_1, z_2, z_3$. That is, we add the rows corresponding to $z_4 = [z_1, z_3]$ and $z_5 = [z_2, z_3]$. These are given by

$$\begin{aligned} z_4 &= \begin{bmatrix} 0 & -1 & 0 & 0 & -\alpha_1^2\alpha_2 & 0 \end{bmatrix} \\ z_5 &= \begin{bmatrix} 1 & 0 & 0 & \alpha_1\alpha_2^2 & 0 & 0 \end{bmatrix}. \end{aligned}$$

We then keep adding, as rows, the Lie brackets of all the elements we have obtained thus far, i.e, $[z_i, z_j]$, $i, j = 1, \ldots, 5$. (Excluding all bracketed terms which result in the zero vector.)

The following lemma provides us with the conditions of how many iterated Lie brackets of elements we need to consider. For convenience we introduce the following notation,

$$\begin{aligned}\Xi_{u_1} &= \Xi(\mathbf{1}, u_1)\\ \Xi_{u_1 u_2 \ldots u_r} &= [\Xi_{u_1 u_2 \ldots u_{r-1}}, \Xi_{u_r}].\end{aligned}$$

E.0.1 LEMMA. *Let* $\Sigma = (\mathsf{G}, \Xi)$ *be a left-invariant control affine system. Then the Lie algebra generated by* $\Gamma$*, where* $\Gamma$ *is the trace of* $\Sigma$*, is given by*

$$\mathrm{Lie}\,(\Gamma) = \mathrm{span}\left\{\Xi_{u_1}, \Xi_{u_1,u_2}, \ldots, \Xi_{u_1,u_2,\ldots,u_{n-m+1}} \mid u_1, u_2, \ldots, u_{n-m+1} \in \mathbb{R}^{\ell}\right\}$$

*where* $n = \dim(\mathsf{G})$ *and* $m = \dim(\mathrm{span}\,(\Gamma))$*.*

In this case $\dim(\mathsf{SO}(4)) = 6$ and $\dim(\mathrm{span}\,(\Gamma)) = 2$. Thus we need only consider up to the 5-th order of iterations of Lie brackets. Collecting all the vectors as rows, it follows that

$$Z = \begin{bmatrix} -1 & 0 & 0 & -\alpha_1^3\alpha_2^2 & 0 & 0\\ -1 & 0 & 0 & -\alpha_1\alpha_2^4 & 0 & 0\\ 0 & -1 & 0 & 0 & -\alpha_1^2\alpha_2 & 0\\ 0 & 0 & -1 & 0 & 0 & -\alpha_1^3\alpha_2\\ 0 & 0 & -1 & 0 & 0 & -\alpha_1\alpha_2^3\\ 0 & 0 & 1 & 0 & 0 & \alpha_1\alpha_2\\ 0 & 1 & 0 & 0 & \alpha_2 & 0\\ 0 & 1 & 0 & 0 & \alpha_1^4\alpha_2 & 0\\ 0 & 1 & 0 & 0 & \alpha_1^2\alpha_2^3 & 0\\ 1 & 0 & 0 & \alpha_1 & 0 & 0\\ 1 & 0 & 0 & \alpha_1\alpha_2^2 & 0 & 0 \end{bmatrix}.$$

If $\det(ZZ^{\top}) = 0$, it follows that the system does not have full rank. On the other hand, if $\det(ZZ^{\top}) \neq 0$, then the system does have full rank. In this case we have that

$$\begin{aligned}\det(ZZ^{\top}) = {} & 2\alpha_1^4\alpha_2^4\left(3 + 3\alpha_1^8 - 2\alpha_1^2\left(1+\alpha_2^2\right) - 2\alpha_1^6\left(1+\alpha_2^2\right) + \alpha_1^4\left(1 - 2\alpha_2^2 + 3\alpha_2^4\right)\right)\\ & \left(3 - 2\left(1+\alpha_1^2\right)\alpha_2^2 + \left(1 - 2\alpha_1^2 + 3\alpha_1^4\right)\alpha_2^4 +\right.\\ & \left.-2\left(1+\alpha_1^2\right)\alpha_2^6 + 3\alpha_2^8\right)\left(1 + \alpha_1^4 - \alpha_2^2 + \alpha_2^4 - \alpha_1^2\left(1+\alpha_2^2\right)\right).\end{aligned}$$

It follows that $\det(ZZ^{\top}) = 0$ if and only if $\alpha_2 = 0$ or $\alpha_1 = \alpha_2 = 1$. Thus $\Sigma_{2,\boldsymbol{\alpha}}^{(2,0)}$ is not controllable exactly when $\alpha_2 = 0$ or $\alpha_1 = \alpha_2 = 1$.

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