RHODES UNIVERSITY

DEPARTMENT OF MATHEMATICS

EXTENSION THEOREMS ON L-TOPOLOGICAL SPACES
AND
AND L-FUZZY VECTOR SPACES

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Abstract

A non-trivial example of an L-topological space, the L-fuzzy real line is examined. Various L-topological properties and their relationships are developed. Extension theorems on the L-fuzzy real line as well as extension theorems on more general L-topological spaces follow. Finally, a theory of L-fuzzy vector spaces leads up to a fuzzy version of the Hahn-Banach theorem.

KEYWORDS: L-topological space, L-fuzzy real line, L-fuzzy unit interval, L-fuzzy vector space, fuzzy normed space, Extension theorems.

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Contents

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PREFACE

Fuzzy set theory and its applications have seen a tremendous growth over the last thirty years. Particularly pleasing has been the development of non-trivial aspects of topology into the L-fuzzy setting. Throughout this text an effort was made to concentrate on lattices L, more general than the original unit interval. Many of the L-fuzzy statements and proofs needed to be adapted from their (I-valued) fuzzy counterparts.

Zadeh introduced the concept of a *fuzzy set* in [64]. Subsequently, many attempts have been made to extend many mathematical notions to the fuzzy (and more recently the L-fuzzy) setting. The concept of a *fuzzy topology* was first put forward by Chang in [7].

We provide an introduction to lattice theory which contains some basic facts necessary for dealing with L-fuzzy sets. We also include the notion of *residual implication* as well as some associated properties.

Chapters 2 and 3 are introductory chapters to *L-fuzzy sets* and *L-fuzzy topology* respectively. We define the notion of *neighbourhood* which is useful in the theory of L-fuzzy vector spaces.

In chapter 4 we consider the *L-fuzzy real line* and show how the real number line can be embedded into it. We then define *left-hand*, *right-hand* and *ordinary L-topologies* on the L-fuzzy real line and explain how they respectively correspond to the left-hand, right-hand and ordinary topologies on the real line.

In chapter 5 we look at some of the various L-topological analogues of the classical notions of *density, regularity, normality* and the *separation axioms* as well as their relationships to each other. This enables us to state and prove theorems of continuous extension on dense sets of an L-topological space in chapter 6 where we consider both the questions of existence as well as uniqueness of a continuous extension from a dense subset to the whole space. We also provide Hutton's L-fuzzy version of Urysohn's Lemma [17] and Kubiak's L-fuzzy version of the Tietze Extension Principle [33]. For the latter it is necessary to first prove a L-fuzzy insertion theorem. For completeness, we also present an alternative derivation of this insertion theorem.

In chapter 7 we define and examine the important notions regarding *L-fuzzy vector spaces.* The theory of *fuzzy vector spaces* was established and developed by Katsaras and Liu [25] and Katsaras [26, 27]. The theory provided in this text is adapted to the L-fuzzy situation. We define *convex*, *balanced*, and *absorbing* L-fuzzy sets and this leads us to the notion of an *L-normed space.* This provides the setting for us to state and prove a fuzzy version of the Hahn-Banach theorem from classical topology.

Chapter 1

Lattices

1.1 Introduction

Lattice theory will constantly be appealed to throughout the main text. We use standard notation and terminology accepted in lattice theory. We present the main definitions as well as some necessary theory that will be needed in what is to follow. For access to the deeper ideas of lattice theory the reader is referred to [4] and [12]. Also, [46] and [59] were helpful in the writing of this introductory chapter.

All lattices considered are assumed to be bounded with a smallest element 0 and a largest element 1.

1.1.1 Definition

A lattice L ($= (L, \leq)$) is called *complete*, if each subset $D \subseteq L$ has a *join* (the *supremum*): $\vee D \in L$. (By the duality principle this is equivalent to the requirement that each $D \subseteq L$ has a *meet* (the *infimum*): $\wedge D \in L$.)

1.1.2 Definition

A lattice is called *distributive* if $(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c)$, or equivalently, $(a \vee b) \wedge (a \vee c) = a \vee (b \wedge c)$ for any $a, b, c \in L$. A lattice is said to be *infinitely distributive*, or a *frame* (*complete Heyting algebra*) if $a \wedge (\bigvee_j b_j) = \bigvee_j (a \wedge b_j)$ for any $a \in L, \{b_j : j \in J\} \subseteq L$

A complete lattice is called *completely distributive*, if

$$\wedge\{\vee\{a_{j_\kappa} : \kappa \in \mathrm{K}_j\} : j \in J\} = \vee\{\wedge\{a_{j_{f(j)}} : j \in J\} : f \in \prod_{j \in J} \mathrm{K}_j\}$$

1.1.3 Definition

A mapping $' : L \longrightarrow L$ is called an *involution*, if $\forall a \in L, (a')' = a$ and any lattice with an involution is said to satisfy *the law of double negation.* An involution is said to be *order reversing* if $a \leq b$ implies $b' \leq a'$. If L is equipped with an order reversing involution then we say that L is *de Morgan.*

Assumption: All lattices considered in this thesis are assumed to be complete and infinitely distributive. otherwise stated.

Unless otherwise indicated, lattices will be assumed to be equipped with an order reversing involution $' : L \longrightarrow L$.

1.1.4 Definition

An equivalence relation ρ on a lattice L is a *congruence relation* iff for $a_i, b_i \in L$ $(i = 1, 2)$

$$a_i \rho b_i (i=1,2) \Rightarrow (a_1 \vee a_2)\rho(b_1 \vee b_2) \text{ and } (a_1 \wedge a_2)\rho(b_1 \wedge b_2).$$

Then $L/_\rho$ becomes a lattice with

$$[a] \vee [b] = [a \vee b] \text{ and } [a] \wedge [b] = [a \wedge b].$$

1.1.5 Definition

Let L and M be lattices. A function $\varphi : L \longrightarrow M$ is a *lattice homomorphism* if it preserves supremum and infimum, that is, $\forall a, b \in L$

$$\varphi(a \vee b) = \varphi(a) \vee \varphi(b) \text{ and } \varphi(a \wedge b) = \varphi(a) \wedge \varphi(b).$$

If L and M are frames then we say $\varphi : L \longrightarrow M$ is a *frame homomorphism* if it preserves arbitrary sup and finite inf.
If L and M are complete lattices then we say $\varphi : L \longrightarrow M$ is a *complete lattice homomorphism* if it preserves arbitrary sup and arbitrary inf.
$\varphi : L \longrightarrow M$ is a *lattice* (resp., *frame, complete lattice*) *embedding* if it is an injective lattice (resp., frame, complete lattice) homomorphism.

1.2 Residual Implication

1.2.1 Definition

If L is a frame and $a, b \in L$ we define *residual implication* by:

$$a \to b = \bigvee \{l \in L : a \wedge l \leq b\}.$$

We list the following useful properties of residual implication.

1.2.2 Lemma

(1) $a \leq (b \to c) \Leftrightarrow a \wedge b \leq c$;
(2) $a \wedge (a \to b) \leq b$;
(3) $(a \to b) \to b \geq a$;
(4) $a \leq b \Rightarrow$ (a) $a \to c \geq b \to c$;
(b) $c \to a \leq c \to b$;
(5) $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$;
(6) $(a \vee b) \to c = (a \to c) \wedge (b \to c)$;
(7) $a \to 1 = 1$ and $1 \to a = a$.

Proof.

(1)

$\Rightarrow$:

Let $a \leq (b \to c) \Leftrightarrow a \leq \bigvee\{l \in L : b \wedge l \leq c\}$
$\Rightarrow (b \wedge a) \leq b \wedge \bigvee\{l \in L : b \wedge l \leq c\}$
$= \bigvee\{b \wedge l : b \wedge l \leq c\} \leq c.$ (since L is a frame)

$\Leftarrow$:

Let $a \wedge b \leq c \Rightarrow a \in \{l \in L : b \wedge l \leq c\}$
$\Rightarrow a \leq \bigvee\{l \in L : b \wedge l \leq c\} = (b \to c).$

(2)

$$(a \to b) \wedge a = a \wedge (a \to b) \leq b \Leftrightarrow (a \to b) \leq (a \to b).$$
(from (1))

(3)

$$a \leq (a \to b) \to b \Leftrightarrow a \wedge (a \to b) \leq b. \text{ (from (1))}$$
and from (2) we have the result.

(4)

(a)

$$a \leq b \Rightarrow \forall l \in L, a \wedge l \leq b \wedge l.$$

Thus

$$\bigvee\{l \in L : a \wedge l \leq c\} \geq \bigvee\{l \in L : b \wedge l \leq c\}$$

i.e.

$$a \to c \geq b \to c.$$

(b)

Assume that we have $a \leq b$. This implies that $\forall l \in L$,

$$c \wedge l \leq a \Rightarrow c \wedge l \leq b.$$

Thus

$$\bigvee\{l \in L : c \wedge l \leq a\} \leq \bigvee\{l \in L : c \wedge l \leq b\}.$$

That is

$$c \to a \leq c \to b.$$

(5)

We have

$$a \to (b \wedge c) \leq a \to b, a \to c$$

and thus

$$a \to (b \wedge c) \leq (a \to b) \wedge (a \to c).$$

On the other hand for $d \leq (a \to b) \wedge (a \to c)$ we have
$d \leq (a \to b)$ and $d \leq (a \to c)$
$\Rightarrow d \wedge a \leq b$ and $d \wedge a \leq c$ (from (1))
$\Rightarrow d \wedge a \leq b \wedge c$.
$\Rightarrow d \leq a \to (b \wedge c)$. (from (1))
Now by letting $d := (a \to b) \wedge (a \to c)$ we have the desired result.

(6)

Choose $a, b \in L$. Then from (4) we have

$$(a \vee b) \to c \leq a \to c, b \to c$$

and hence

$$(a \vee b) \to c \leq (a \to c) \wedge (b \to c).$$

Consider $d \in L$ such that $d \leq (a \to c) \wedge (b \to c)$
$\Rightarrow d \leq (a \to c)$ and $d \leq (b \to c)$
$\Rightarrow d \wedge a \leq c$ and $d \wedge b \leq c$ (from (1))
$\Rightarrow (d \wedge a) \vee (d \wedge b) \leq c$
$\Rightarrow d \wedge (a \vee b) \leq c$ (distributivity)
Now (1) yields

$$d \leq (a \vee b) \to c$$

and by setting $d = (a \to c) \wedge (b \to c)$ the proof is complete.

(7)

$\forall l, a \in L$ we have $l \wedge a \leq 1$. Especially $a \wedge 1 = a \leq 1$ for all $a \in L$ and thus

$$1 \in \{l \in L : a \wedge l \leq 1\}.$$

So we have

$$a \to 1 = \bigvee\{l \in L : a \wedge l \leq 1\} = 1.$$

and since $\forall l \in L, 1 \wedge l = l$ we have

$$1 \to a = \bigvee\{l \in L : 1 \wedge l \leq a\} = a.$$

Note: In general, we do **not** have $(a \wedge b) \to c = (a \to c) \vee (b \to c)$ (This is important since if $c = 0$ and $a \to 0 = a'$ then this would mean that the second de Morgan law holds!)
Consider the following diagramatic representation of a lattice.

Then $(a \wedge b) \to 0 = 1$ but $(a \to 0) \vee (b \to 0) = c \neq 1$.

Remark 1: $a \to 0$ defines a kind of complement. It is in general **order-reversing**, but **not** involutary, i.e. we have $a \leq (a \to 0) \to 0$ but not equality. With $a' := a \to 0$ we do have the following from Lemma 1.2.2 (1)

(i) $a \leq b' \Leftrightarrow a \wedge b = 0.$

If we demand that also $a = (a \to 0) \to 0$, i.e the law of double negation should hold, then L is a Boolean algebra (see [58]). But we then have that $a' := a \to 0$ is an order reversing involution. Moreover we find in this case

(ii) $a \leq b' \Leftrightarrow a' \vee b' = 1.$

Remark 2: If we have an order-reversing involution on L, then a' and $a \to 0$ need not have any relation in general.

1.2.3 Example

The lattice $(I, \leq)$ (where I is the unit interval $[0,1]$) has an order-reversing involution $' : I \longrightarrow I$ defined in the following way:

$$\forall a \in I, a' = 1 - a.$$

Consider $a \in I$.
We have that

$$a \to 0 = \bigvee\{l \in L : a \wedge l \leq 0\}$$

$$= \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{if } a \neq 0 \end{cases}$$

which is quite clearly not related to a'.

Chapter 2

L-Fuzzy Sets

uniform

2.1 Introduction

Set theory forms the foundation of mathematics and every mathematical object is a set or a class. Fuzzy set theory is a departure from bivalent logic and has become very much accepted by the mathematical community over the last few decades. Fuzzy set theory was effectively started when Zadeh published his now famous paper [64] in 1965.

Bivalent logic does not enable us to offer accurate answers to certain types of question. For example, who is a member of the set of tall people? Using bivalent logic we would have to agree on some minimal height required by a person for them to be included in that set. This way of answering such a question is unsatisfactory because it is not in accordance with our intuitive notion of tallness. The fuzzy way of answering that question would be to say that to each height a person may have, there corresponds a degree of membership to the set of tall people.

The idea of a fuzzy set originates from the fact that "ordinary", crisp sets are in one to one correpondence with functions from a universal set, X, into the set $\{0, 1\}$ and hence we can view a set as a function. A fuzzy set is also a function from a universal set, X, but into a more general set. Originally, a fuzzy set was defined as a function from X into the unit interval, $[0, 1]$. More recently L-fuzzy sets have become more prevalent in the literature, where an L-fuzzy set is a function from X into L for any complete lattice L. The value yielded by an L-fuzzy set at a particular element can be viewed as the degree of membership of the element with respect to the L-fuzzy set.

We have a set calculus of $\mathcal{P}(X)$ (the power set of X) and so it is natural to ask whether this calculus can be extended to L-fuzzy sets. We thus need to define notions which are analogues of subset, union, intersection and complement in such a way that when the fuzzy sets are in fact crisp sets then the respective notions reduce to the usual crisp ones.

Throughout, unless otherwise stated, L will denote a complete lattice with supremum 1 and infimum 0.

2.1.1 Definition

Let X be a set and L a complete lattice. An *L-fuzzy set* on X (*L-subset of X*) is a map from X into L. That is, if μ is a L-fuzzy subset of X then $\mu \in L^X$, where L^X denotes the collection of all maps from X into L.

In the case when L is the unit interval I then we refer to L-fuzzy sets simply as *fuzzy sets*.

Order-structure of L-subsets

In classical set theory we define for a subset A of a universal set X the *characteristic function* of A, denoted by 1_A by

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Note that $1_A \in \{0,1\}^X = 2^X$ and there is a natural bijection between $\mathcal{P}(X)$ and 2^X. If A^c denotes the complement of A, we see that:
$\forall x \in X, 1_\phi(x) = 0;$
$\forall x \in X, 1_X(x) = 1;$
$\forall x \in X, 1_{A^c}(x) = 1 - 1_A(x);$
$\forall x \in X, 1_{A \cup B}(x) = 1_A(x) \vee 1_B(x);$
$\forall x \in X, 1_{A \cap B}(x) = 1_A(x) \wedge 1_B(x);$
$A \subseteq B \Leftrightarrow \forall x \in X, 1_A(x) \leq 1_B(x).$

Now on L^X we define correspondingly:

The *empty fuzzy set* 1_ϕ is: $\forall x \in X, 1_\phi(x) = 0;$
The *whole fuzzy set* 1_X is: $\forall x \in X, 1_X(x) = 1;$
$\mu = \nu \Leftrightarrow \forall x \in X, \mu(x) = \nu(x);$
$\mu \leq \nu \Leftrightarrow \forall x \in X, \mu(x) \leq \nu(x);$
$(\mu \vee \nu)(x) \equiv \mu(x) \vee \nu(x), x \in X;$
$(\mu \wedge \nu)(x) \equiv \mu(x) \wedge \nu(x), x \in X;$
$(\bigvee_{j \in J} \mu_j)(x) \equiv \bigvee_{j \in J} \mu_j(x), x \in X;$
$(\bigwedge_{j \in J} \mu_j)(x) \equiv \bigwedge_{j \in J} \mu_j(x), x \in X;$
$\mu'(x) \equiv \mu(x)', x \in X.$

If $A \in \mathcal{P}(X)$ and $d \in L$ we define

$$d1_A(x) = \begin{cases} d & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

So

$$d1_{\{x\}} = \begin{cases} d & \text{on } x \\ 0 & \text{elsewhere} \end{cases}$$

We call $d1_{\{x\}}$ a *fuzzy point* with *support* at x and *value* d.

Thus we equip L^X with an order structure induced by L, and since L is a complete lattice, so is L^X. Furthermore if L is de Morgan, so is L^X and if L is a frame, so is L^X. If $L = I$ we could, for example, define an order reversing involution in the following way: $\mu'(x) = 1 - \mu(x), x \in X$.

2.1.2 Theorem
(1) If $f \in L^{X \times Y}$ then,

$$\sup_{(x,y) \in X \times Y} f(x,y) = \sup_{x \in X} \sup_{y \in Y} f(x,y);$$

$$\inf_{(x,y) \in X \times Y} f(x,y) = \inf_{x \in X} \inf_{y \in Y} f(x,y);$$

$$\sup_{x \in X} \inf_{y \in Y} f(x,y) \leq \inf_{y \in Y} \sup_{x \in X} f(x,y).$$

(2) If $X, Y \subseteq L$ then,

$$\sup X \wedge \sup Y = \sup_{x \in X} \sup_{y \in Y} x \wedge y;$$

$$\inf X \wedge \inf Y = \inf_{x \in X} \inf_{y \in Y} x \wedge y.$$

If $\mu, \nu \in L^X$ then,

$$\sup_{x \in X} (\mu \wedge \nu)(x) \leq \sup_{x \in X} \mu(x) \wedge \sup_{x \in X} \nu(x).$$

(4) If $\nu \in L^X$ and $A, B \subseteq X$ then,

$$\sup_{x \in A} \nu(x) \wedge \sup_{y \in B} \nu(y) = \sup_{x \in A} \sup_{y \in B} (\nu(x) \wedge \nu(y)).$$

For a given L-fuzzy set we associate collections of crisp subsets of X with it.

If $\mu \in L^X$ and $d \in L$ we define,

$$\mu^d = \{x \in X : \mu(x) > d\};$$

$$\mu_d = \{x \in X : \mu(x) \geq d\}.$$

These crisp sets are referred to as d-levels (or cuts), strong and weak respectively. When a crisp theory is to be fuzzified, very often we expect that if μ is a fuzzy set that has a certain fuzzy property then μ^d has the crisp property which is analogous to that particular fuzzy property. For an L-fuzzy set μ, we call μ^0 the *support* of μ.

2.1.3 Lemma

If $\mu_j \in L^X, j \in J$ then $\quad (\bigwedge_{j \in J} \mu_j)_d = \bigcap_{j \in J} (\mu_j)_d$.

Proof.

$$\begin{aligned} x \in (\textstyle\bigwedge_{j \in J} \mu_j)_d &\Leftrightarrow \textstyle\bigwedge_{j \in J} \mu_j(x) \geq d \\ &\Leftrightarrow \forall j \in J, \mu_j(x) \geq d \\ &\Leftrightarrow \forall j \in J, x \in (\mu_j)_d \\ &\Leftrightarrow x \in \textstyle\bigcap_{j \in J} (\mu_j)_d. \end{aligned}$$

2.2 L-Fuzzy Sets Induced by Maps

For a function

$$f : X \longrightarrow Y$$

there corresponds a function

$$f^{\rightarrow} : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$$

where $f^{\rightarrow}(A) = \{f(x) : x \in A\}$ is called the *direct image* of $A \subseteq X$; and a function

$$f^{\leftarrow} : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$$

where $f^{\leftarrow}(B) = \{x \in X : f(x) \in B\}$ is called the *pre-image* of $B \subseteq Y$.

In Zadeh's historical paper [64] he defined fuzzy analogues to these functions and this idea has subsequently been extended to the L-fuzzy situation. For X and Y sets, $f : L^X \longrightarrow L^Y$, $\mu \in L^X$ and $\nu \in L^Y$ we define the *direct image* of μ, denoted by $f[\mu]$ and the *pre-image* of ν, denoted by $f^{\leftarrow}[\nu]$ as follows:

For $y \in Y$,

$$f[\mu](y) = \sup_{f(x)=y} \mu(x)$$

with the convention that $\sup \phi = 0$ and

$$f^{\leftarrow}[\nu] = \nu \circ f.$$

It is a simple matter to confirm that these definitions reduce to the usual crisp ones in the case where μ and ν are crisp. The next theorem gives a useful list of properties of maps on fuzzy sets. A proof of this theorem is given in [46] for the special case $L := I$ but the same proof is also valid for the more general case. Also confer [16].

2.2.1 Theorem

Let X, Y, Z be sets and let $f \in Y^X$, $g \in Z^Y$, $\mu \in L^X$, $\nu \in L^Y$ and $\lambda \in L^Z$. Let $(\mu_j : j \in J) \in (L^X)^J$ and $(\nu_j : j \in J) \in (L^Y)^J$. Then

(1) $(g \circ f)[\mu] = g[f[\mu]]$;
(2) $(g \circ f)^{\leftarrow}[\lambda] = f^{\leftarrow}[g^{\leftarrow}[\lambda]]$;
(3) $f^{\leftarrow}[\bigvee_{j\in J} \nu_j] = \bigvee_{j\in J} f^{\leftarrow}[\nu_j]$;
(4) $f^{\leftarrow}[\bigwedge_{j\in J} \nu_j] = \bigwedge_{j\in J} f^{\leftarrow}[\nu_j]$.
(5) $f^{\leftarrow}[\nu'] = (f^{\leftarrow}[\nu])'$;
(6) $\nu_1 \leq \nu_2 \Rightarrow f^{\leftarrow}[\nu_1] \leq f^{\leftarrow}[\nu_2]$;
(7) $f[\bigvee_{j\in J} \mu_j] = \bigvee_{j\in J} f[\mu_j]$;
(8) $f[\bigwedge_{j\in J} \mu_j] \leq \bigwedge_{j\in J} f[\mu_j]$;
(9) $f[\mu]' \leq f[\mu']$;
(10) $\mu_1 \leq \mu_2 \Rightarrow f[\mu_1] \leq f[\mu_2]$;
(11) $f[f^{\leftarrow}[\nu]] \leq \nu$, with equality if f is surjective;
(12) $\mu \leq f^{\leftarrow}[f[\mu]]$, with equality if f is injective;
(13) $f[f^{\leftarrow}[\nu] \wedge \mu] \leq f[\mu]$, with equality if f is injective.

Finally, we present a natural definition of the cartesian product of L-fuzzy sets. when we reach L-fuzzy vector spaces (chapter 7).

2.2.2 Definition (Katsaras and Liu, [25])

For all $i \in I$ let μ_i be an L-fuzzy sets in X_i. We define $\prod_{i\in I} \mu_i$ to be the L-fuzzy set μ in $\prod_{i\in I} X_i$ given by

$$\mu((x_i)) = \bigwedge_{i\in I} \mu_i(x_i).$$

In the case that I is finite, $I = \{1, \ldots n\}$, we write $\mu_1 \times \cdots \times \mu_n$.

For further information with regard to L-fuzzy sets, the reader is referred to [10, 13, 15, 39, 40, 42, 64].

Chapter 3

L-Topological Spaces

3.1 Introduction

Shortly after fuzzy set theory was developed, mathematicians began to fuzzify various areas of classical mathematics. The concept of a fuzzy topological space (also an L-topological space) follows naturally from the corresponding classical notion.

In [7] Chang introduced the notion of a fuzzy topology in the following way:

3.1.1 Definition
A *fuzzy topology* on X is a subset τ of I^X satisfying

(1) $1_\phi, 1_X \in \tau$;
(2) $\mu, \nu \in \tau \Rightarrow \mu \wedge \nu \in \tau$;
(3) $\forall j \in J, \mu_j \in \tau \Rightarrow \bigvee_{j \in J} \mu_j \in \tau$.

The pair (X, τ) is called a *fuzzy topological space* (*fts*) and the members of τ the *fuzzy open sets* of X.

In [37] Lowen defines a subset $\tau \subseteq L^X$ to be a fuzzy topology on X if (1), (2), (3) hold as well as:
(4) $\forall a \in I, a1_X \in \tau$.

J.A. Goguen proposed, in [14], the natural generalization of Chang's definition by substituting L-fuzzy sets for fuzzy sets. That is, Goguen defined an *L-fuzzy topology* (or simply an *L-topology*) on X to be a family τ of L-fuzzy sets (i.e. $\tau \subseteq L^X$) which satisfies conditions (1)-(3) above. Naturally, the pair (X, τ) is called an *L-fuzzy topological space* (*L-topological space*) and L-fuzzy sets $\mu \in \tau$ are called *open* in this space. In this context, we require that L be any bounded complete lattice

3.1.2 Example
(1) The discrete L-topology on X: $\tau = L^X$.

(2) The indiscrete L-topology on X: $\tau = \{1_\phi, 1_X\}$.

(3) Any ordinary (crisp) topology T on X generates an L-topology on X - simply identify with the open sets, their characteristic functions.

If τ_1 and τ_2 are L-topologies on a set X then we say that τ_1 is *smaller* (*coarser*) than τ_2 (or equivalently τ_2 is *bigger* (*finer*) than τ_1) iff $\tau_1 \subset \tau_2$.

As in General Topology we define the concepts of a base and subbase.

3.1.3 Definition

Let (X, τ) be an L-topological space. A set $B \subseteq L^X$ is called a *base* for τ iff each element of τ is the supremum of members of B.

Also $S \subseteq L^X$ is called a *subbase* for τ iff the family of all finite infima of members of S is a base for τ.

3.1.4 Lemma

Let $S \subseteq L^X$. If L is a frame then

$$\langle S \rangle = \{ \bigvee_{J \in K} \bigwedge_{j \in J} \mu_j : \forall J \in K, J finite; \quad \forall J \in K, \forall j \in J, \mu_j \in S\} \bigcup \{1_X, 1_\phi\}$$

is an L-topology on X. We call it the L-topology *generated* by S.

Proof.

(i)

Let $\mu, \nu \in \langle S \rangle$. Then $\mu = \bigvee_{J \in K_1} \bigwedge_{j \in J} \mu_j$ and $\nu = \bigvee_{I \in K_2} \bigwedge_{i \in I} \mu_i$ with each $J \in K_1$, $I \in K_2$ finite and each $\mu_i, \mu_j \in S$.

So $\mu \wedge \nu = \bigvee_{I \in K_2} (\bigwedge_{i \in I} \mu_i \wedge (\bigvee_{J \in K_1} \bigwedge_{j \in J} \mu_j))$ (since L is a frame)
$= \bigvee_{I \in K_2} \bigvee_{J \in K_1} ((\bigwedge_{i \in I} \mu_i) \wedge (\bigwedge_{j \in J} \mu_j))$ (since L is a frame)
$= \bigvee_{I \in K_1 \cup K_2} \bigwedge_{i \in I} \mu_i \in \langle S \rangle$.

(ii)

Let $\{\nu_i : i \in I\} \subseteq \langle S \rangle$. Then by the definition of $\langle S \rangle$ we have that for each $i \in I$,

$$\nu_i = \bigvee_{J \in K} \bigwedge_{j \in J} \mu_{ij}$$

with each $J \in K$ finite and $\forall i \in I, \forall j \in J$, $\mu_{ij} \in S$.

Then

$$\bigvee_{i \in I} \nu_i = \bigvee_{i \in I} [\bigvee_{J \in K} \bigwedge_{j \in J} \mu_{ij}] = \bigvee_{i \in I, J \in K} \bigwedge_{j \in J} \mu_{ij} \in \langle S \rangle.$$

3.1.5 Lemma

For a set X and for each $i \in I$ let τ_i be an L-topology on X. Then $\tau = \bigcap_{i \in I} \tau_i$ is an L-topology.

Proof.

(i)

$1_\phi, 1_X \in \tau$ trivially.

(ii)

Let $\mu, \nu \in \tau$. Then $\forall i \in I$, $\mu, \nu \in \tau_i$.
$\Leftrightarrow \forall i \in I, \mu \wedge \nu \in \tau_i$
$\Leftrightarrow \mu \wedge \nu \in \tau$.

(iii) For each $j \in J$, let $\mu_j \in \tau$. That is $\forall j \in J, \forall i \in I$, $\mu_j \in \tau_i$
$\Leftrightarrow \forall i \in I, \bigvee_{j \in J} \mu_j \in \tau_i$
$\Leftrightarrow \bigvee_{j \in J} \mu_j \in \tau$.

3.1.6 Lemma
If we define for τ_1, τ_2 L-topologies on a set X and with

$$\tau_1 \vee \tau_2 := \langle \tau_1 \cup \tau_2 \rangle$$

and

$$\tau_1 \wedge \tau_2 := \tau_1 \cap \tau_2$$

then the collection of L-topologies on X is a lattice.

3.1.7 Lemma
Let $S \subseteq L^X$. Then

$$\langle S \rangle = \bigcap_{\delta \supseteq S, \delta \in T} \delta$$

where T is the collection of all L-topologies on X.

Proof.

(i)
Let $\mu \in \langle S \rangle$. Then $\mu \in \bigvee_{J \in K} \bigwedge_{j \in J} \mu_j$ where each $J \in K$ is finite and each $\mu_j \in S$.
Let δ be an L-topology on X such that $\delta \supseteq S$ then $\forall J \in K, \forall j \in J$, $\mu_j \in \delta$
and thus for each $J \in K$, $\bigwedge_{j \in J} \mu_j \in \delta$
$\Rightarrow \bigvee_{J \in K} \bigwedge_{j \in J} \mu_j \in \delta$
$\Rightarrow \mu \in \bigcap_{\delta \in T, \delta \supseteq S} \delta$.

(ii)
$\mu \in \bigcap_{\delta \in T, \delta \supseteq S} \delta$ implies that for all δ such that $\delta \supseteq S$ and $\delta \in T$ we have that $\mu \in \delta$. Hence $\mu \in \langle S \rangle$, as $\langle S \rangle$ is an L-topology which contains S.

3.2 Basic L-Topological Notions

3.2.1 Definition
The *L-fuzzy interior* μ° of an L-fuzzy set μ is the join of all members of τ contained in μ. i.e.,

$$\mu^\circ = \bigvee \{\nu \in L^X : \nu \in \tau, \nu \leq \mu\}.$$

This is the largest open L-fuzzy set contained in μ and, trivially,

$$\mu \text{ is open iff } \mu = \mu^\circ.$$

As mentioned above, the concept of an L-topological space is reasonable for any complete, bounded lattice L to develop a substantial theory. Often, however, one needs the lattice L to satisfy some additional requirements. The most frequently required conditions are (1) L has infinite distributivity (L is a frame) and (2) L be equipped with an order reversing involution $'$, i.e. L is de Morgan.

The assumption for an order reversing involution enables us to give reasonable definitions of closedness and related notions.

3.2.2 Definition
An L-fuzzy set μ in an L-topological space (X, τ) is *τ-closed* (*L-fuzzy closed*) iff $\mu' \in \tau$.

It then follows trivially from the definition of a closed L-fuzzy set that the collection of closed L-fuzzy sets $\mathcal{C}$ satisfies the following properties:

(1) $1_\phi, 1_X \in \mathcal{C}$;

(2) If $\mu, \nu \in \mathcal{C}$ then $\mu \vee \nu \in \mathcal{C}$;

(3) If $\{\mu_j : j \in J\} \subseteq \mathcal{C}$, then $\bigwedge_{j \in J} \mu_j \in \mathcal{C}$.

The concept of closure leads, naturally, to the notion of a closure operator.

3.2.3 Definition
The *L-fuzzy closure* $\overline{\mu}$ of an L-fuzzy set is the meet of all τ-closed sets which contain μ. That is,

$$\overline{\mu} = \bigwedge \{\nu \in L^X : \nu' \in \tau, \mu \leq \nu\}.$$

Therefore $\overline{\mu}$ is the smallest τ-closed set which contains μ and

$$\mu \text{ is closed iff } \mu = \overline{\mu}.$$

It is clear that the closure operator in L-fuzzy topology can be treated the same as its classical analogue as is illustrated by the following two propositions.

3.2.4 Proposition
Let (X, τ) be an L-topological space and $\mu, \nu \in L^X$. Then the closure operator $\bar{\ }: L^X \longrightarrow L^X$ has the following properties:

(1) $\overline{1_\phi} = 1_\phi$;

(2) $\overline{\mu \vee \nu} = \overline{\mu} \vee \overline{\nu}$;

(3) $\overline{\overline{\mu}} = \overline{\mu}$;

(4) $\mu \leq \overline{\mu}$;

3.2.5 Proposition
Let (X, τ) be an L-topological space and $\mu \in L^X$. Then

(1) $\overline{\mu}' = (\mu')^\circ$;

(2) $(\mu^\circ)' = \overline{(\mu')}$.

Proof.

(1)

$$\begin{aligned} \overline{\mu}' &= [\bigwedge \{\nu \in L^X : \nu' \in \tau, \mu \leq \nu\}]' \\ &= \bigvee \{\nu' \in L^X : \nu' \in \tau, \mu \leq \nu\} \\ &= \bigvee \{\nu' \in L^X : \nu' \in \tau, \mu' \geq \nu'\} \\ &= (\mu')^\circ. \end{aligned}$$

(2)

Simply replace μ with μ' in (1).

3.3 Continuous Functions

The notion of (fuzzy) continuity was first introduced in [7] by Chang in 1968.

3.3.1 Definition
Let (X, τ_1) and (Y, τ_2) be two L-topological spaces. A function $f : (X, \tau_1) \longrightarrow (Y, \tau_2)$ is *continuous* iff $\forall \nu \in \tau_2, f^{\leftarrow}[\nu] \in \tau_1$.

3.3.2 Proposition

Let (X, τ_1), (Y, τ_2) and (Z, τ_3) be L-topological spaces. If $f : (X, \tau_1) \longrightarrow (Y, \tau_2)$ and $g : (Y, \tau_2) \longrightarrow (Z, \tau_3)$ are continuous functions then $g \circ f : (X, \tau_1) \longrightarrow (Z, \tau_3)$ is continuous.

3.3.3 Theorem

Let (X, τ_1) and (Y, τ_2) be L-topological spaces and $f : (X, \tau_1) \longrightarrow (Y, \tau_2)$ a function. Then the following are equivalent:

(1) f is continuous,

(2) For each τ_2-closed ν, $f^{\leftarrow}[\nu]$ is τ_1-closed,

(3) For each $\nu \in L^Y, \overline{f^{\leftarrow}[\nu]} \leq f^{\leftarrow}[\overline{\nu}]$,

(4) For each $\mu \in L^X, f[\overline{\mu}] \leq \overline{f[\mu]}$,

(5) For each $\nu \in L^Y, f^{\leftarrow}[\nu^{\circ}] \leq (f^{\leftarrow}[\nu])^{\circ}$.

Proof.

(1) $\Rightarrow$ (2):

Let $\nu' \in \tau_2$. Then $f^{\leftarrow}[\nu'] \in \tau_1$ and $(f^{\leftarrow}[\nu])' = f^{\leftarrow}[\nu'] \in \tau_1$. Thus $f^{\leftarrow}[\nu]$ is τ_1-closed.

(2) $\Rightarrow$ (1):

Let $\nu \in \tau_2$. Then ν' is τ_2-closed and so $f^{\leftarrow}[\nu'] = (f^{\leftarrow}[\nu])'$ is τ_1-closed. Therefore we have $f^{\leftarrow}[\nu] \in \tau_1$ and hence f is continuous.

(2) $\Rightarrow$ (4):

For $\mu \in L^X$,

$$\overline{f[\mu]} = \bigwedge\{\nu \in L^Y : \nu' \in \tau_2, \nu \geq f[\mu]\}.$$

Therefore

$$f^{\leftarrow}[\overline{f[\mu]}] = \bigwedge\{f^{\leftarrow}[\nu] : \nu \in L^Y, \nu' \in \tau_2, \nu \geq f[\mu]\}$$

and

$$\overline{\mu} = \bigwedge\{\sigma \in L^X : \sigma' \in \tau_1, \sigma \geq \mu\}.$$

But

$$\nu' \in \tau_2 \Rightarrow (f^{\leftarrow}[\nu])' \in \tau_1$$

by (2) and

$$\nu \geq f[\mu] \Rightarrow \mu \leq f^{\leftarrow}[f[\mu]] \leq f^{\leftarrow}[\nu].$$

Thus

$$f^{\leftarrow}[\overline{f[\mu]}] \geq \overline{\mu}$$

and hence

$$\overline{f[\mu]} \geq f[f^{\leftarrow}[\overline{f[\mu]}]] \geq f[\overline{\mu}].$$

Thus we have,

$$\overline{f[\mu]} \geq f[\overline{\mu}].$$

(4) $\Rightarrow$ (3):

For $\nu \in L^Y, f^{\leftarrow}[\nu] \in L^X$.
Therefore

$$f[\overline{f^{\leftarrow}[\nu]}] \leq \overline{f[f^{\leftarrow}[\nu]]} \leq \overline{\nu}$$

and so

$$f^{\leftarrow}[\overline{\nu}] \geq f^{\leftarrow}[f[\overline{f^{\leftarrow}[\nu]}]] \geq \overline{f^{\leftarrow}[\nu]}.$$

Thus we have

$$f^{\leftarrow}[\overline{\nu}] \geq \overline{f^{\leftarrow}[\nu]}.$$

(3) ⇒ (2):

Let $\nu' \in \tau_2$. Then $\nu \in L^Y$ and

$$\overline{f^{\leftarrow}[\nu]} \leq f^{\leftarrow}[\overline{\nu}] = f^{\leftarrow}[\nu].$$

Therefore

$$\overline{f^{\leftarrow}[\nu]} = f^{\leftarrow}[\nu]$$

and hence

$$f^{\leftarrow}[\nu] \text{ is } \tau_1\text{-closed.}$$

(1) ⇒ (5):

Choose $\nu \in L^Y$.

Then by definition of the L-fuzzy interior we have $\nu^\circ \in \tau_2$. Thus $f^{\leftarrow}[\nu^\circ] \in \tau_1$ from (1).

We always have that $\nu^\circ \leq \nu$ so

$$f^{\leftarrow}[\nu^\circ] \leq f^{\leftarrow}[\nu]$$

and since $f^{\leftarrow}[\nu^\circ]$ is open and we have

$$f^{\leftarrow}[\nu^\circ] \leq (f^{\leftarrow}[\nu])^\circ.$$

(5) ⇒ (3):

For $\nu \in L^Y, \exists \mu \in L^Y$ such that $\mu = \nu'$.

Now by (5) we have

$$f^{\leftarrow}[\mu^\circ] \leq (f^{\leftarrow}[\mu])^\circ.$$

Thus

$$(f^{\leftarrow}[\mu^\circ])' \geq ((f^{\leftarrow}[\mu])^\circ)'$$

so

$$f^{\leftarrow}[(\mu^\circ)'] \geq ((f^{\leftarrow}[\mu])^\circ)'$$

By proposition 3.2.5 we arrive at

$$f^{\leftarrow}[\overline{\mu'}] \geq \overline{f^{\leftarrow}[\mu]'} = \overline{f^{\leftarrow}[\mu']}$$

so

$$f^{\leftarrow}[\overline{\nu''}] \geq \overline{f^{\leftarrow}[\nu'']}$$

i.e.

$$\overline{f^{\leftarrow}[\nu]} \leq f^{\leftarrow}[\overline{\nu}].$$

3.3.4 Theorem

Let $(X, \tau_1), (Y, \tau_2)$ be L-topological spaces. A function $f : (X, \tau_1) \longrightarrow (Y, \tau_2)$ is continuous iff $\forall \mu \in S$, $f^{\leftarrow}[\mu] \in \tau_1$, where S is any subbase of τ_2.

Proof.

Let S be a subbase of τ_2.

⇒:

Let $\mu \in S$. Then $\mu \in \tau_2$ and trivially $f^{\leftarrow}[\mu] \in \tau_1$.

⇐:

Let $\mu \in \tau_2$. We have that μ is a supremum of finite infima of elements of S. That is, for each $j \in J$, there exists a finite K_j such that

(i) $\forall j \in J, \forall k \in K_j$, $\mu_k \in S$.

(ii) $\mu = \bigvee_{j \in J} \bigwedge_{k \in K_j} \mu_k$.

Now

$$f^{\leftarrow}[\mu] = f^{\leftarrow}[\bigvee_{j \in J} \bigwedge_{k \in K_j} \mu_k]$$

$$= \bigvee_{j \in J} f^{\leftarrow}[\bigwedge_{k \in K_j} \mu_k]$$

by Theorem 2.2.1 (2)

$$= \bigvee_{j \in J} \bigwedge_{k \in K_j} f^{\leftarrow}[\mu_k]$$

by Theorem 2.2.1 (1) and since $\forall j \in J, \forall k \in K_j$, $f^{\leftarrow}[\mu_k] \in \tau_1$ we have that $f^{\leftarrow}[\mu] \in \tau_1$.

3.3.5 Theorem

Consider a family of L-topological spaces $\{(X_j, \delta_j) : j \in J\}$ and a set X without an L-topology. For each $j \in J$ let

$$f_j : X \longrightarrow (X_j, \delta_j)$$

be a mapping.

Now consider the subbase $S = \{f_j^{\leftarrow}[\mu_j] : \mu_j \in \delta_j, j \in J\}$. Let τ_1 be the L-topology generated by S. We call τ_1 the *initial L-topology* and it is the smallest L-topology on X such that all mappings f_j will be continuous.

$f_j : X \longrightarrow X_j$ be mappings for
Let (Y, τ_2) be an L-topological space and let $f : (Y, \tau_2) \longrightarrow (X, \tau_1)$ a mapping. Then f is continuous iff $\forall j \in J$, $f_j \circ f$ is continuous.

Proof.

$\Rightarrow$:

Trivial.

$\Leftarrow$:

By the preceeding theorem it is enough to show that each element of any subbase of τ_1 has a preimage in τ_2. Let S be a subbase of τ_1 and $\mu \in S$. Then $\mu = f_j^{\leftarrow}[\mu_j]$ with $\mu_j \in \delta_j$ for some $j \in J$ and we have

$$f^{\leftarrow}[\mu] = f^{\leftarrow}[f_j^{\leftarrow}[\mu_j]] = (f_j \circ f)^{\leftarrow}[\mu_j] \in \tau_2.$$

We list the following special cases of initial L-topologies as examples.

3.3.6 Examples

(1) **Product Spaces**

For $X = \prod_{j \in J}$ where (X_j, τ_j) are L-topological spaces and $f_j = p_j$ for $j \in J$ (the projection maps), i.e. $\forall j \in J$, $p_j((x_i : i \in J)) = x_j$.

Now for each $j \in J$, let $\mu_j \in \tau_j$. For $j_1 \in J$ we have

$$f_{j_1}^{\leftarrow}[\mu_{j_1}] = \prod_{j \in J} \mu_j$$

where $\mu_j = 1_X$ for $j \neq j_1$.

Hence the initial L-topology is the one generated by the subbase $\{f_j^{\leftarrow}[\mu_j] : j \in J;\ \ \forall j \in J, \mu_j \in \tau_j\}$. This L-topology is referred to as the *product L-topology* on X.

(2) **Subspaces**
For (X, τ) an L-topological space with $A \subseteq X$ let

$$f = i_A = \begin{cases} A \longrightarrow X \\ x \longmapsto x \end{cases}$$

Then $i_A^{\leftarrow}[\mu] = \mu|_A$ for $\mu \in L^X$. The initial L-topology $\tau_A = \{\mu|_A : \mu \in \tau\}$ is the *subspace L-topology*, that is, the collection of elements of τ restricted to A.

3.3.7 Lemma

Let (X, τ) be an L-topological space, let $A \subseteq X$ and let τ_A be the subspace L-topology on A. Then for $\mu \in L^X$,

(1) $\mu^\circ|_A \leq (\mu|_A)^{\circ A}$,

(2) $\overline{\mu|_A}^A \leq \overline{\mu}|_A$

where μ° is the interior of $\mu \in L^X$ with respect to τ,
$\mu^{\circ A}$ is the interior of $\mu \in L^A$ with respect to τ_A,
$\overline{\mu}$ is the closure of $\mu \in L^X$ with respect to τ
and $\overline{\mu}^A$ is the closure of $\mu \in L^A$ with respect to τ_A.

Proof.

(1)

$$\mu^\circ|_A = \bigvee \{\nu \in L^X : \nu \in \tau, \nu \leq \mu\}|_A$$

and

$$(\mu|_A)^{\circ A} = \bigvee \{\nu \in L^A : \nu \in \tau_A, \nu \leq \mu|_A\}.$$

Let $A = \{\nu \in L^X : \nu \in \tau, \nu \leq \mu\}|_A$ and $B = \{\nu \in L^A : \nu \in \tau_A, \nu \leq \mu|_A\}$. Then $A = \{\nu|_A : \nu \in L^X, \nu \in \tau, \nu \leq \mu\}$. We know that

(i) $\nu \in L^X \Rightarrow \nu|_A \in L^A$,
(ii) $\nu \in \tau \Rightarrow \nu|_A \in \tau_A$,
(iii) $\nu \leq \mu \Rightarrow \nu|_A \leq \mu|_A$.

So $A \subseteq B$ and therefore $\mu^\circ|_A \leq (\mu|_A)^{\circ A}$.

(2)

$$\overline{\mu|_A}^A = \bigwedge \{\nu \in L^A : \nu' \in \tau_A, \mu|_A \leq \nu\}$$

and

$$\overline{\mu}|_A = \bigwedge \{\nu \in L^X : \nu' \in \tau, \mu \leq \nu\}|_A.$$

Let $C = \{\nu \in L^A : \nu' \in \tau_A, \mu|_A \leq \nu\}$ and $D = \{\nu \in L^X : \nu' \in \tau, \mu \leq \nu\}|_A$. Now $D = \{\nu|_A : \nu \in L^X, \nu' \in \tau, \mu \leq \nu\}$. In addition to (i) and (iii) above, we know that

(iv) $\nu' \in \tau \Rightarrow (\nu|_A)' = \nu'|_A \in \tau|_A$.

So $D \subseteq C$ and thus $\overline{\mu|_A}^A \leq \overline{\mu}|_A$.

The inequalities of the preceeding lemma cannot be replaced by equalities as is illustrated in the following example.

3.3.8 Example

Let X be a set and $A \subseteq X$. Consider the following I-topology τ on X defined as follows:

$$\tau := \{\mu \in L^X : \mu \geq a\} \cup \{1_\phi\}$$

for some $a \in (0, 1]$.

Then $\tau_A = \{\nu \in L^A : \mu \geq a\} \bigcup \{1_A\}$. Let $b > a$ and choose c such that $0 < c < a$. Now define

$$\omega(x) = \begin{cases} b & \text{if } x \in A \\ c & \text{if } x \notin A \end{cases}$$

Now $\omega^\circ = 1_\phi$ but $(\omega|_A)^\circ \neq 1_\phi$.

We could similarly construct an example that shows that the inequality of Lemma 3.3.7 (2) cannot be replaced by equality either.

For further information regarding fuzzy topology and L-topology, the reader is referred to [20, 43, 44, 45, 59].

Chapter 4

The L-Fuzzy Real Line

4.1 Introduction

This chapter leads on from the previous one in the sense that the L-fuzzy real line $\mathbb{R}(L)$ is an interesting example of an L-topological space. We will later see that this particular L-topological space provides the setting for two of the most important L-fuzzy extension theorems, namely, Urysohn's Lemma and the Tietze Extension Principle. The L-fuzzy unit interval was introduced by Hutton in [17] in 1975 and this idea was subsequently generalised to the L-fuzzy real line by Gantner et al. in [11]. In this chapter we summarize the properties of the L-fuzzy real line.

4.2 Preliminaries

For L a lattice, $L^{\mathbb{R}}$ is ordered pointwisely, ie. $f \leq g \Leftrightarrow \forall t \in \mathbb{R}, f(t) \leq g(t)$

The lattice theoretic properties of L are inherited by $L^{\mathbb{R}}$
e.g. if L is complete, so is $L^{\mathbb{R}}$
if L is a frame, so is $L^{\mathbb{R}}$, etc.

4.2.1 Definition
$\mathbb{R}_L = \{f \in L^{\mathbb{R}} : \bigwedge_{t\in\mathbb{R}} f(t) = 0, \bigvee_{t\in\mathbb{R}} f(t) = 1, f$ non-increasing $\}$

and $H_L = \{\varphi \in L^I : \varphi$ is non-increasing $\}$.

4.2.2 Proposition
$\mathbb{R}_L$ is a sublattice of $L^{\mathbb{R}}$ if L is a frame with order-reversing involution $'$ (so arbitrary infima also distribute over finite suprema).

Proof.
Assume L is a frame and that $f, g \in \mathbb{R}_L$. Clearly $f \vee g$ and $f \wedge g$ are non-increasing. It is also obvious that

$$\bigvee_{t\in\mathbb{R}} f \vee g(t) = 1 \text{ and } \bigwedge_{t\in\mathbb{R}} f \wedge g(t) = 0.$$

We only have to show that $\bigwedge f \vee g(t) = 0$ and $\bigvee f \wedge g(t) = 1$

$$\begin{aligned} \text{e.g. } \bigwedge f \vee g(t) &= \bigwedge\{f \vee g(t) : t \in \mathbb{R}\} \\ &= \bigwedge\{f(t) \vee g(s) : s, t \in \mathbb{R}\} && \text{(since } f \text{ and } g \text{ are non-increasing)} \\ &= \bigwedge_{t\in\mathbb{R}} f(t) \vee \bigwedge_{s\in\mathbb{R}} g(s) && (L \text{ is a frame with } ') \\ &= 0 \vee 0 = 0. \end{aligned}$$

Note: $\mathbb{R}_L$ is not necessarily complete.

4.2.3 Definition

If $f \in \mathbb{R}_L, f^+, f^- \in L^{\mathbb{R}}$ are defined in the following way:

$\forall t \in \mathbb{R}, f^+(t) = f(t+) = \bigvee_{s>t} f(s)$ (the "right hand limit")

$\forall t \in \mathbb{R}, f^-(t) = f(t-) = \bigwedge_{s<t} f(s)$ (the "left hand limit").

Clearly (i) $f \leq g \Rightarrow f^+ \leq g^+$ and $f^- \leq g^-$.
(ii) f^+ and f^- are non-increasing.
(iii) $\forall t \in \mathbb{R}, f^+(t) \leq f(t) \leq f^-(t)$.

4.2.4 Proposition

Let L be complete. Then for all $f, g \in \mathbb{R}_L$

(1) $^+, ^-$ map $\mathbb{R}_L$ into $\mathbb{R}_L$ monotonically.
(2) $f^{-+} = f^+ = f^{++}$.
(3) $f^{+-} = f^- = f^{--}$.
(4) $f^+ \leq g^+ \Leftrightarrow f^- \leq g^- \Leftrightarrow f^+ \leq g^-$.
(5) $f^+ = g^+ \Leftrightarrow f^- = g^-$.

Proof.

(1)

From (i), (ii) and (iii) we have

$$\bigvee_{t\in\mathbb{R}} f^-(t) = 1 \text{ and } \bigwedge_{t\in\mathbb{R}} f^+(t) = 0.$$

It remains to be shown that $\bigwedge_{t\in\mathbb{R}} f^-(t) = 0$ and $\bigvee_{t\in\mathbb{R}} f^+(t) = 1$.
e.g. for $s < t$ $(s, t \in \mathbb{R})$ we have $f(s) \geq f^-(t)$.
So $f(s) \geq \bigwedge_{s<t} f^-(t)$
$\geq \bigwedge_{t\in\mathbb{R}} f^-(t)$ (since f^- is non-increasing).
Thus

$$0 = \bigwedge_{s\in\mathbb{R}} f(s) \geq \bigwedge_{t\in\mathbb{R}} f^-(t).$$

Also for $s \in \mathbb{R}$ we have $f(s) \leq \bigvee_{t<s} f^+(t)$
$= \bigvee_{t\in\mathbb{R}} f^+(t)$ (since f^+ is non-increasing).
So

$$1 = \bigvee_{s\in\mathbb{R}} f(s) \leq \bigvee_{t\in\mathbb{R}} f^+(t).$$

(2)

If $t \in \mathbb{R}$ then for $s > t$ we have $f^+(t) \geq f^-(s)$. So

$$f^+(t) \geq \bigvee_{s>t} f^-(s) = f^{-+}(t).$$

Since $f \leq f^-$ we have $f^+ \leq f^{-+}$
Therefore $f^+ = f^{-+}$

Also

$$f^+(t) = \bigvee_{s>t} f(s) = \bigvee_{u>t} \bigvee_{s>u} f(s) = \bigvee_{u>t} f^+(u) = f^{++}(t).$$

(3)

If $t \in \mathbb{R}$ then for $s < t$ we have $f^+(s) \geq f^-(t)$. So

$$f^{+-}(t) = \bigwedge_{s<t} f^+(s) \geq f^-(t).$$

That is, $f^{+-} \geq f^-$. Since $f^+ \leq f$ we have $f^{+-} \leq f^-$ and hence $f^{+-} = f^-$.

Also
$$f^-(t) = \bigwedge_{s<t} f(s) = \bigwedge_{u<t} \bigwedge_{s<u} f(s) = \bigwedge_{u<t} f^-(u) = f^{--}(t).$$

(4)

$$\begin{aligned} f^+ \leq g^+ &\Rightarrow f^- = f^{+-} \leq g^{+-} = g^- \\ &\Rightarrow f^+ \leq g^- \\ &\Rightarrow f^+ = f^{++} \leq g^{-+} = g^+. \qquad \text{by (2)} \end{aligned}$$

(5)

Follows from (4).

4.2.5 Corollary

$f(t+) = \bigvee_{s>t}\{f(s-)\}$;
$f(t-) = \bigwedge_{s<t}\{f(s+)\}$.

4.2.6 Proposition

With L a frame ${}^+, {}^- : \mathbb{R}_L \longrightarrow \mathbb{R}_L$ are lattice homeomorphisms, that is

$$(f \vee g)^- = f^- \vee g^-, \qquad (f \wedge g)^- = f^- \wedge g^-,$$
$$(f \vee g)^+ = f^+ \vee g^+, \qquad (f \wedge g)^+ = f^+ \wedge g^+.$$

Proof.

(i)

$$\begin{aligned} f^-(t) \vee g^-(t) &= \textstyle\bigwedge_{s<t} f(s) \vee \bigwedge_{s<t} g(s) \\ &= \textstyle\bigwedge_{s_1,s_2<t} f(s_1) \vee g(s_2) \qquad (L \text{ is a frame}) \\ &= \textstyle\bigwedge_{s<t} f(s) \vee g(s) \qquad (\text{since } f \text{ and } g \text{ are non-increasing}) \\ &= (f \vee g)^-(t). \end{aligned}$$

(ii)

$$\begin{aligned} f^-(t) \wedge g^-(t) &= \textstyle\bigwedge_{s<t} f(s) \wedge \bigwedge_{s<t} g(s) \\ &= \textstyle\bigwedge_{s_1,s_2<t} f(s_1) \wedge g(s_2) \\ &= \textstyle\bigwedge_{s<t} f(s) \wedge g(s) \qquad (\text{since } f \text{ and } g \text{ are non-increasing}) \\ &= (f \wedge g)^-(t). \end{aligned}$$

(iii)

$$\begin{aligned} f^+(t) \wedge g^+(t) &= \textstyle\bigvee_{s>t} f(s) \vee \bigvee_{s>t} g(s) \\ &= \textstyle\bigvee_{s_1,s_2>t} f(s_1) \vee g(s_2) \qquad (L \text{ is a frame}) \\ &= \textstyle\bigvee_{s>t} f(s) \vee g(s) \qquad (\text{since } f \text{ and } g \text{ are non-increasing}) \\ &= (f \vee g)^+(t). \end{aligned}$$

(iv)

$$\begin{aligned} f^+(t) \wedge g^+(t) &= \textstyle\bigvee_{s>t} f(s) \wedge \bigvee_{s>t} g(s) \\ &= \textstyle\bigvee_{s_1,s_2>t} f(s_1) \wedge g(s_2) \\ &= \textstyle\bigvee_{s>t} f(s) \wedge g(s) \qquad (\text{since } f \text{ and } g \text{ are non-increasing}) \\ &= (f \wedge g)^+(t). \end{aligned}$$

4.2.7 Definition

$f \sim g \Leftrightarrow f^+ = g^+ (\Leftrightarrow f^- = g^-)$ on $\mathbb{R}_L$.

4.2.8 Proposition

This is an equivalence relation on $\mathbb{R}_L$, and a congruence relation on $\mathbb{R}_L$ if L is a frame. (An equivalence relation on a lattice is a congruence relation if it preserves $\vee$ and $\wedge$.)

Proof.

Consider $f_1, f_2, g_1, g_2 \in \mathbb{R}_L$ such that
$$f_1 \sim f_2 \text{ and } g_1 \sim g_2.$$

From the definition of $\sim$ we have $f_1^+ = f_2^+$ and $g_1^+ = g_2^+$ and thus

$$(f_1 \vee g_1)^+ = f_1^+ \vee g_1^+ = f_2^+ \vee g_2^+ = (f_2 \vee g_2)^+.$$

4.2.9 Definition

The *L-fuzzy real line*, $\mathbb{R}(L)$, is defined as follows:

$$\mathbb{R}(L) = \mathbb{R}_L/_\sim = \{[f]_\sim : f \in \mathbb{R}_L\}.$$

Remark: $\mathbb{R}(L)$ is partially ordered with $[f] \leq [g] \Leftrightarrow f^+ \leq g^+$.

4.2.10 Theorem

For L a frame we have that $\mathbb{R}(L)$ is a lattice since $[f] \vee [g] = [f \vee g]$ and $[f] \wedge [g] = [f \wedge g]$.

Proof.

$[f] \vee [g] = [f \vee g]$:

Let L be a frame and $f, g \in \mathbb{R}(L)$. Put $[h] = [f] \vee [g]$. We now have to show that $h^+ = (f \vee g)^+$
Now

$$[f] \leq [h] \text{ and } [g] \leq [h]$$

therefore $f^+ \leq h^+$ and $g^+ \leq h^+$ and hence

$$(f \vee g)^+ = f^+ \vee g^+ \leq h^+.$$

That is

$$[f \vee g] \leq [f] \vee [g].$$

Conversely, $f^+ \leq f^+ \vee g^+ = (f \vee g)^+$ since L is a frame.
Thus $[f] \leq [f \vee g]$ and likewise $[g] \leq [f \vee g]$.
So

$$[f] \vee [g] \leq [f \vee g].$$

$[f] \wedge [g] = [f \wedge g]$:

For L a frame and $f, g \in \mathbb{R}(L)$. Put $[h] = [f] \wedge [g]$. Now

$$[f] \geq [h] \text{ and } [g] \geq [h]$$

so $f^+ \geq h^+$ and $g^+ \geq h^+$ and hence

$$(f \wedge g)^+ = f^+ \wedge g^+ \geq h^+.$$

Thus

$$[f \wedge g] \geq [f] \wedge [g].$$

Also, since L is a frame $f^+ \geq f^+ \wedge g^+ = (f \wedge g)^+$.
Thus $[f] \geq [f \wedge g]$ and likewise $[g] \geq [f \wedge g]$.
So

$$[f] \wedge [g] \leq [f \wedge g].$$

4.3 Embedding of $\mathbb{R}$ in $\mathbb{R}(L)$

By virtue of the embedding below, the name of $\mathbb{R}(L)$ is justified. Indeed, $\mathbb{R}(L)$ is analogous to the "ordinary" real line.

$$e : \mathbb{R} \longrightarrow \mathbb{R}(L)$$

$$e(t) = < t >$$

where $f \in < t >$ iff $f^+(t) = 0$ and $f^-(t) = 1$.

4.3.1 Proposition

$f \in < t >$ iff $f^+(x) = 1_{(-\infty,t)}(x)$ and $f^-(x) = 1_{(-\infty,t]}(x)$, i.e. $f^-(x)' = 1_{(t,\infty)}(x)$.

4.3.2 Proposition

If L is a frame, e is a lattice embedding. (If $L = \{0, 1\}$, e is a lattice isomorphism, i.e. $\mathbb{R}(\{0, 1\}) \approx \mathbb{R}$.)

Proof.

Let $f \in < t >, g \in < s >, h \in < t \vee s >$.

Then $(f \vee g)^+(x) = f^+(x) \vee g^+(x)$ $\qquad$ (L is a frame)

$$= 1_{(-\infty,t)}(x) \vee 1_{(-\infty,s)}(x)$$

$$= 1_{(-\infty,t\vee s)}(x) = h^+(x).$$

Therefore

$$(f \vee g) \sim h \Rightarrow [h] = [f \vee g] = [f] \vee [g] \Rightarrow < t \vee s > = < t > \vee < s > .$$

4.4 The L-Fuzzy Real Line as an L-Topological Space

We are going to define L-topologies on $\mathbb{R}(L)$.

4.4.1 Definition

For each $t \in \mathbb{R}$ let $R_t, L_t : \mathbb{R}(L) \longrightarrow L$ be defined by:

$$R_t[f] = f^+(t), \qquad\qquad L_t[f] = f^-(t)'.$$

We extend these definitions to ∞ and $-\infty$ in the natural way:

$$R_{-\infty} := 1 \qquad \text{and} \qquad R_\infty := 0$$

$$L_\infty := 1 \qquad \text{and} \qquad L_{-\infty} := 0.$$

4.4.2 Theorem

$\mathcal{R} = \{R_t : t \in \mathbb{R}\} \bigcup \{1_\phi, 1_X\}$ and
$\mathcal{L} = \{L_t : t \in \mathbb{R}\} \bigcup \{1_\phi, 1_X\}$ are L-fuzzy topologies on $\mathbb{R}(L)$.

Proof.

Choose $s, t \in \mathbb{R}$. For $f \in \mathbb{R}(L)$, $R_{t\vee s}[f] = f^+(t \vee s)$
$= \bigvee_{r > t\vee s} f(r)$.

Now <u>if $t \vee s = t$:</u>
$R_{t\vee s}[f] = \bigvee_{r>t} f(r)$
$= f^+(t)$
$= R_t[f]$ and also $R_t[f] \leq R_s[f]$ since f^+ is non-increasing.

<u>if $t \vee s = s$:</u>
$R_{t\vee s}[f] = R_s[f]$ and $R_s[f] \leq R_t[f]$ since f^+ is non-increasing.

Therefore $R_{t\vee s} = R_s \wedge R_t$.

Let $\{R_{t_j} : j \in J\} \subseteq \mathcal{R}$. Then we'll show that $\bigvee_{j\in J} R_{t_j} = R_{\bigwedge_{j\in J} t_j}$:
We have

$$\begin{aligned}(\bigvee_{j\in J} R_{t_j})[f] &= \bigvee_{j\in J} \bigvee_{s>t_j} f(s) \\ &= \bigvee_{s > \bigwedge_{j\in J} t_j} f(s) \\ &= f^+(\bigwedge_{j\in J} t_j) \in \mathcal{R}.\end{aligned}$$

Likewise $L_t \wedge L_s = L_{t\vee s}$.

Also

$$\begin{aligned}(\bigwedge_{j\in J} L_{t_j})[f] &= \bigvee_{j\in J} f^-(t_j)' \\ &= \bigvee_{j\in J} [\bigwedge_{s<t_j} f(s)]' \\ &= \bigvee_{j\in J} \bigvee_{s<t_j} (f'(s))\end{aligned}$$

(de Morgan)

$$
\begin{aligned}
&= \bigvee_{s<\bigvee_{j\in J} t_j} (f'(s))\\
&= [\bigwedge_{s<\bigvee_{j\in J} t_j} f(s)]'\\
&= f^-(\bigvee_{j\in J} t_j)'\\
&= L_{\bigvee_{j\in J} t_j}[f].
\end{aligned}
$$

Therefore

$$\bigwedge_{j\in J} L_{t_j} = L_{\bigvee_{j\in J} t_j} \in \mathcal{L}.$$

The proof is complete.

$\mathcal{R}$ and $\mathcal{L}$ are called the *left-* and *right-hand topology*, respectively.

4.4.3 Definition
Let $\mathcal{U}$ be the smallest L-topology containing $\mathcal{R}$ and $\mathcal{L}$. This will be called the *natural L-topology* on $\mathbb{R}(L)$.

Note: $\mathcal{U} = \bigcap\{L\text{-topologies containing } \mathcal{R}\cup\mathcal{L}\}$
$= \text{the join of } \mathcal{R} \text{ and } \mathcal{L} \text{ in the lattice of } L\text{-topologies on } \mathbb{R}(L).$

4.4.4 Theorem
With L a frame, $(\mathbb{R}(L),\mathcal{U},\leq)$ is an L-topological lattice, i.e. the operations $\vee$ and $\wedge$ are continuous.

Proof.
Consider

$$s : \mathbb{R}(L)\times\mathbb{R}(L) \longrightarrow \mathbb{R}(L)$$

given by $s([f],[g]) = [f]\vee[g] = [f\vee g]$ with the product L-topology on $\mathbb{R}(L)\times\mathbb{R}(L))$, ie. the smallest L-topology under which the projection maps p_1 and p_2 onto $\mathbb{R}(L)$ are L-continuous.

$$
\begin{aligned}
\text{Now, } s^{\leftarrow}[L_t]([f],[g]) &= L_t[f\vee g]\\
&= (f\vee g)^-(t)'\\
&= (f^-\vee g^-)(t)' \qquad \text{(by Proposition 4.2.6)}\\
&= f^-(t)'\wedge g^-(t)' \qquad \text{(de Morgan)}\\
&= L_t[f]\wedge L_t[g]\\
&= L_t\circ p_1([f],[g])\wedge L_t\circ p_2([f],[g])\\
&= p_1^{\leftarrow}(L_t)\wedge p_2^{\leftarrow}(L_t)([f],[g]),
\end{aligned}
$$

therefore

$$s^{\leftarrow}(L_t) = p_1^{\leftarrow}(L_t)\wedge p_2^{\leftarrow}(L_t) \in \mathcal{U}\times\mathcal{U}.$$

Likewise

$$s^{\leftarrow}(R_t) \in \mathcal{U}\times\mathcal{U}.$$

Then apply Theorem 3.3.4 and we have that s is a continuous mapping.

Similarly $i : \mathbb{R}(L)\times\mathbb{R}(L) \longrightarrow \mathbb{R}(L)$ where $i([f],[g]) = [f]\wedge[g] = [f\wedge g]$ is continuous.

We remind the reader of the most well known classical topologies on the real line, that is, the *right-hand*, *left-hand* and *ordinary topologies* on $\mathbb{R}$: $\tau_r = \langle\{[t,\infty) : t\in\mathbb{R}\}\rangle$, $\tau_l = \langle\{(-\infty,t] : t\in\mathbb{R}\}\rangle$ and $\tau_{ord} = \langle\{(a,b) : a,b\in\mathbb{R}, a<b\}\rangle$ repectively (where $\langle S\rangle$ denotes the topology generated by the $S\subseteq\mathcal{P}(\mathbb{R})$ in "classical" sense).

The L-topologies $\mathcal{R}, \mathcal{L}$ and $\mathcal{U}$ on $\mathbb{R}(L)$ correspond to the right-hand, left-hand and ordinary topologies on $\mathbb{R}$ (τ_r, τ_l and τ_{ord}) in the following sense:

4.4.5 Theorem

The embedding

$$e : \mathbb{R} \longrightarrow \mathbb{R}(L)$$
$$e(t) =< t >$$

is a continuous embedding of

(1) $(\mathbb{R}, \mathcal{X}(\tau_r))$ into $(\mathbb{R}(L), \mathcal{R})$;
(2) $(\mathbb{R}, \mathcal{X}(\tau_l))$ into $(\mathbb{R}(L), \mathcal{L})$;
(3) $(\mathbb{R}, \mathcal{X}(\tau_{ord}))$ into $(\mathbb{R}(L), \mathcal{U})$.

Where $\mathcal{X}(\tau_j), j = r, l, ord$ consist of the L-topologies of the characteristic functions of the τ_j. In fact, in case (3), e is also open. So e is a topological embedding in that case.

Proof.

We shall just prove case (1) since the other cases are similar.

$$\begin{aligned} e^{\leftarrow}[R_t](r) &= R_t < r > \\ &= \begin{cases} 0 & \text{if } t \geq r \\ 1 & \text{if } t < r \end{cases} \\ &= 1_{(t,\infty)}(r) \text{ which is open in } (R, \mathcal{X}(\tau_r)). \end{aligned}$$

So e is continuous.

Recall that if $f \in < t >$ then $f^+(x) = 1_{(-\infty,t)}(x)$ and$f^-(x)' = 1_{(t,\infty)}(x)$. (Proposition 4.3.1)
So for $a, b \in \mathbb{R}$ $(a < b)$ we have

$$\begin{aligned} e^{\rightarrow}(1_{(a,b)})([f]) &= e^{\rightarrow}(1_{(a,\infty)} \wedge 1_{(-\infty,b)})([f]) \\ &= \bigvee_{<r>=[f]} (1_{(a,\infty)}(r) \wedge 1_{(-\infty,b)}(r)) \\ &= \bigvee_{<r>=[f]} (f^-(r)' \wedge f^+(r)) \\ &= \bigvee_{<r>=[f]} (L_r[f] \wedge R_r[f]) \\ &= \bigvee_{<r>=[f]} (L_r \wedge R_r)[f] \end{aligned}$$

which is open in $\mathcal{U}$. Therefore e is open in case (3).

4.4.6 Corollary

If $L = \{0, 1\}$, e is a homeomorphism, ie. $\mathbb{R}(\{0, 1\}) \approx \mathbb{R}$.

4.4.7 Definition

The *L-unit interval* $I(L)$ $(= [0, 1](L))$ is the sublattice of $\mathbb{R}(L)$ consisting of all $[f]$ such that $f(t) = 1$ for $t < 0$ and $f(t) = 0$ for $t > 1$. The L-topologies used on $I(L)$ are the subspace L-topologies induced by $\mathcal{R}, \mathcal{L}$ and $\mathcal{U}$. That is $\mathcal{R}_{I(L)} = \{R_t : t \in I\} \bigcup \{1_\phi, 1_{[0,1]}\}$, $\mathcal{L}_{I(L)} = \{L_t : t \in I\} \bigcup \{1_\phi, 1_{[0,1]}\}$ and $\mathcal{U}_{I(L)} = \langle \mathcal{R}_{I(L)} \cup \mathcal{L}_{I(L)} \rangle$ respectively.

Also $(0, 1)(L)$ is the set of all $[f] \in \mathbb{R}(L)$ such that $f(t) = 1$ for $t \leq 0$ and $f(t) = 0$ for $t \geq 1$.

For further reading with regard to the L-fuzzy real line, the reader is referred to [2, 18, 19, 21, 22, 34, 41, 49, 50, 51, 52, 53, 54, 55, 56, 57].

Chapter 5

L-Topological Properties

5.1 Introduction

In order to establish the required extension theorems we will need to define and examine certain properties of L-topological spaces. The book [16], Chapter 3 provides a comprehensive overview of L-Fuzzy Topology with contributions from many experts in the various areas. As is often the case when fuzzifying general topological notions - many crisp topological properties have a number of L-fuzzy analogues.

5.2 Denseness

We first consider a frame L (without order reversing involution). Let (X, τ) be an L-topological space. Define for $\mu \in \tau$

$$\mu^* = \bigvee\{\nu \in \tau : \nu \wedge \mu = 1_\phi\}.$$

We find that for $\mu \in \tau$
$\mu \wedge \mu^* = \mu \wedge \bigvee\{\nu \in \tau : \mu \wedge \nu = 1_\phi\}$
$\quad = \bigvee\{\mu \wedge \nu : \nu \in \tau, \mu \wedge \nu = 1_\phi\} = 1_\phi$.

(Since L is a frame finite meets distribute over arbitrary joins.) This μ^* is 'something like a complement in τ' (bearing in mind that we do not have a complement in L). Note that $\mu^* \in \tau$ by Definition 3.1.1.

There are a number of L-fuzzy analogues to the crisp concept of denseness.

5.2.1 Definition ([16], Chapter 3)
For an L-topological space (X, τ) then $D \subseteq X$ is *strictly dense* iff $\forall \mu \in \tau, \forall x \in X$,

$$\mu(x) \leq \bigvee\{\mu(y) : y \in D\}.$$

In what follows we again assume that the frame L has an order reversing involution $'$.

We reproduce another definition of the concept of denseness given by Höhle and Sostak in [16], Chapter 3, p. 187 which we shall refer to as dense.

5.2.2 Definition (Höhle & Sostak, [16], Chapter 3)
For an L-topological space (X, τ) and $D \subseteq X$ then D is *dense* iff $\forall \mu \in \tau$,

$$(\forall x \in D, \mu(x) = 0) \Rightarrow (\forall x \in X, \mu(x) = 0).$$

Obviously: strictly dense $\Rightarrow$ dense.

Next, we reproduce the definition of "ordinary" topological denseness and for convenience we shall refer to this as weak denseness throughout the rest of this thesis.

5.2.3 Definition
For an L-topological space (X, τ) then $D \subseteq X$ is *weakly dense* iff $\overline{1_D} = 1_X$.

5.2.4 Proposition
For an L-topological space (X, τ) and $D \subseteq X$ then D is dense $\Rightarrow$ D is weakly dense.

Proof.
Let D be dense and consider $\overline{1_D}(x)$ and $\mu \in \tau$ such that $\mu \geq 1_D$. We have that $\mu' \leq (1_D)' = 1_{X \backslash D}$ and μ' is open. Now for $x \in D, \mu'(x) \leq 1_{X \backslash D}(x) = 0$. Hence $\forall x \in X, \mu'(x) = 0$ and thus for $x \in X, \mu(x) = 1$. Hence $\forall x \in X, \overline{1_D}(x) = 1$.

So we have the following situation:

$$\boxed{\textbf{strictly dense} \Rightarrow \textbf{dense} \Rightarrow \textbf{weakly dense}}$$

5.3 Separation axioms

5.3.1 Definition (Kolmogoroff - separation, [16], Chapter 3)
An L-fuzzy set (X, τ) is T_0 iff $\forall x, y \in X; x \neq y, \exists \mu \in \tau$ such that $\mu(x) \neq \mu(y)$.

5.3.2 Definition (Haussdorf - separation, [16], Chapter 3)
An L-topological space (X, τ) is T_2 iff $\forall x, y \in X; x \neq y; \exists \mu, \nu \in \tau$ such that

$$\mu(x) \wedge \nu(y) \neq 0$$

and

$$\mu \wedge \nu = 1_\phi.$$

5.3.3 Definition
An L-topological space (X, τ) is *strong* T_2 iff $\forall x, y \in X; x \neq y; \exists \mu, \nu \in \tau$

$$\mu(x) = \nu(y) = 1 \text{ and } \mu \wedge \nu = 1_\phi.$$

5.3.4 Definition (Kubiak's T_2-axiom, [36])
An L-topological space (X, τ) is K-T_2 iff $\forall x, y \in X; x \neq y; \exists \mu, \nu \in \tau$ such that

$$\mu(x) \not\leq \mu(y), \nu(y) \not\leq \nu(x)$$

and

$$\mu \leq \nu'.$$

5.4 Regularity

5.4.1 Definition (Höhle & Sostak, [16], Chapter 3)
An L-topological space (X, τ) is *regular* iff

$$\forall \mu \in \tau, \mu \geq \bigvee \{\nu \in \tau : \nu^* \wedge \mu = 1_X\}.$$

5.4.2 Lemma (Höhle & Sostak, [16], Chapter 3)
If an L-topological space (X, τ) is regular then

$$(X,\tau) \text{ is } T_0 \Leftrightarrow (X,\tau) \text{ is } T_2.$$

Proof.

$\Rightarrow$:

Assume (X,τ) is T_0 and not T_2, i.e. there is a pair $x, y \in X; x \neq y$ such that for all $\mu, \nu \in \tau$ we do **not** have

$$\mu(x) \wedge \nu(y) \neq 0 \text{ and } \mu \wedge \nu = 1_\phi.$$

By T_0 $\exists \mu \in \tau$ such that $\mu(x) \neq \mu(y)$. Without loss if generality, we assume that $\mu(x) \not\leq \mu(y)$. Let now $\nu \in \tau$ such that $\nu^* \vee \mu = 1_X$.
Thus $\nu(x) = 1_X(y) \wedge \nu(x) = [\nu^*(y) \wedge \nu(x)] \vee [\mu(y) \wedge \nu(x)]$.
As $\nu^* \wedge \nu = 1_\phi$, we must have $\nu^*(y) \wedge \nu(x) = 0$ which implies (by the fact that (X,τ) is not T_2)

$$\nu(x) = 0 \vee [\mu(y) \wedge \mu(x)] \leq \mu(y).$$

By regularity we have $\mu(x) = \bigvee\{\nu(x) : \nu \in \tau, \nu^* \vee \mu = 1_X\} \leq \mu(y)$, a contradiction.

$\Leftarrow$:

Let $x, y \in X; x \neq y$. By T_2, $\exists \mu, \nu \in \tau$ such that $\mu(x) \wedge \nu(y) \neq 0$ and $\mu \wedge \nu = 1_\phi$.
From $\mu \wedge \nu = 1_\phi$ we infer $\nu \leq \mu^*$ and therefore

$$\mu(x) \wedge \mu^*(y) > 0.$$

Hence $\mu(x) \neq \mu(y)$ (else $\mu(x) \wedge \mu^*(y) = \mu(y) \wedge \mu^*(y) = 0$).

5.4.3 Definition (Höhle & Sostak, [16], Chapter 3)
An L-topological space (X,τ) is *star-regular* iff $\forall \mu \in \tau$,

$$\mu = \bigvee\{\nu \in \tau : \nu^* \to 0 \leq \mu\}$$

with $\to$ denoting residual implication as given in Definition 1.2.1.

5.4.4 Lemma
If an L-topological space (X,τ) is star-regular then

$$(X,\tau) \text{ is } T_0 \Leftrightarrow (X,\tau) \text{ is } T_2.$$

Proof.

$\Rightarrow$:

Assume (X,τ) is not T_2, i.e. $\exists x, y \in X; x \neq y$ such that for all $\mu, \nu \in \tau$ we do **not** have $\mu(x) \wedge \nu(y) \neq 0$ and $\nu \wedge \nu = 1_\phi$.
By the T_0 property, $\exists \mu \in \tau$ such that $\mu(x) \neq \mu(y)$. We assume $\mu(x) \not\leq \mu(y)$. Let now $\nu \in \tau$ such that $\nu^* \to 0 \leq \mu$.
Then for all $z \in X$,

$$\bigvee\{\alpha \in L : \nu^*(z) \wedge \alpha = 0\} \leq \mu(z).$$

Especially,

$$\bigvee\{\alpha \in L : \nu^*(y) \wedge \alpha = 0\} \leq \mu(y).$$

We know $\nu \wedge \nu^* = 1_\phi$ and hence we conclude

$$\nu(x) \wedge \nu^*(y) = 0.$$

therefore $\nu(x) \leq \mu(y)$.
By star-regularity,

$$\mu(x) = \bigvee\{\nu(x) : \nu \in \tau, \nu^* \to 0 \leq \mu\} \leq \mu(y),$$

a contradiction.

$\Leftarrow$:

As before in Lemma 5.4.2.

We now offer an alternative definition of regularity first proposed by Kubiak. For convenience we shall refer to this property as K-regularity.

5.4.5 Definition (Kubiak, [16], Chapter 6)

An L-topological space (X, τ) is *K-regular* iff

$$\forall \mu \in \tau, \mu = \bigvee \{\nu \in \tau : \overline{\nu} \leq \mu\}.$$

5.4.6 Lemma

Let (X, τ) be an L-topological space and let L be a Boolean algebra (see [58]). Then

$$(X, \tau) \text{ is star-regular} \Leftrightarrow (X, \tau) \text{ is } K\text{-regular.}$$

Proof.

Since L is Boolean we have with $a' := a \to 0$,

$$\begin{aligned} \nu^* \to 0 &= (\nu^*)' && \text{(since } L \text{ is Boolean)} \\ &= (\bigvee \{\sigma \in \tau : \nu \wedge \sigma = 1_\phi\})' \\ &= (\bigvee \{\sigma \in \tau : \nu' \vee \sigma' = 1_X\})' && \text{(de Morgan)} \\ &= \bigwedge \{\sigma' : \sigma \in \tau : \nu' \vee \sigma' = 1_X\} && \text{(de Morgan)} \\ &= \bigwedge \{\sigma' : \sigma \in \tau : \nu \leq \sigma'\} && \text{(since } L \text{ is Boolean)} \\ &= \overline{\nu}. \end{aligned}$$

So $\bigvee \{\nu \in \tau : \nu^* \to 0 \leq \mu\} = \bigvee \{\nu \in \tau : \overline{\nu} \leq \mu\}$.
Thus star-regular $\Leftrightarrow$ K-regular.

5.4.7 Lemma (Kubiak, [16], Chapter 6)

If an L-topological space (X, τ) is K-regular then (X, τ) is $T_0 \Leftrightarrow (X, \tau)$ is K-T_2.

Proof.

$\Leftarrow$:

Assume (X, τ) is not T_0. Then there exist $x, y \in X$ $(x \neq y)$ such that for all $\mu \in \tau$, $\mu(x) = \mu(y)$. That is, (X, τ) is not K-T_2.

$\Rightarrow$:

For the converse, we assume that (X, τ) is T_0. Choose $x, y \in L$ such that $x \neq y$ then we have $\exists \mu \in \tau$ such that $\mu(x) \neq \mu(y)$. We can assume without loss of generality that $\mu(x) \not\leq \mu(y)$. Hence by regularity we have

$$\bigvee_{\nu \in \tau, \overline{\nu} \leq \mu} \nu(x) \not\leq \bigvee_{\nu \in \tau, \overline{\nu} \leq \mu} \nu(y).$$

Now we have that $\bigvee \{\nu \in \tau : \overline{\nu} \leq \mu\} = \bigvee_{\nu \in \tau, \overline{\nu} \leq \mu} \overline{\nu}$ (since $\bigvee_{\nu \in \tau, \overline{\nu} \leq \mu} \overline{\nu} \leq \mu = \bigvee_{\nu \in \tau, \overline{\nu} \leq \mu} \nu$ and in general $\bigvee \overline{\nu} \geq \bigvee \nu$).
Hence

$$\bigvee_{\nu \in \tau, \overline{\nu} \leq \mu} \nu(x) \not\leq \bigvee_{\nu \in \tau, \overline{\nu} \leq \mu} \overline{\nu}(y).$$

Thus there exists $\nu \in \tau, \overline{\nu} \leq \mu$ such that $\nu(x) \not\leq \overline{\nu}(y)$. Therefore also $\nu(x) \not\leq \nu(y)$ (or else we have $\nu \leq \overline{\nu}$, a contradiction), also $\overline{\nu}(x) \not\leq \overline{\nu}(y)$ by the same argument and therefore $\overline{\nu}'(y) \not\leq \overline{\nu}'(x)$.
So we choose $\gamma = \overline{\nu}' = (\nu')^\circ \in \tau$ (from Proposition 3.2.5) and find

$$\nu(x) \not\leq \nu(y) \text{ and } \gamma(y) \not\leq \gamma(x).$$

Finally $\gamma = (\nu')^\circ \leq \nu'$ and therefore the pair ν, γ is the desired 'K-T_2-pair'.

5.5 Normality

We now mention an L-topological analogue to the concept of normality.

5.5.1 Definition ([20])
A fuzzy topological space is *normal* if for every closed set ν and open set μ such that $\nu \leq \mu$, there exists a set σ such that

$$\nu \leq \sigma^\circ \leq \overline{\sigma} \leq \mu.$$

The following is an L-fuzzy analogue of Katětov [23, 24] and Tongs' [60] useful characterization of the classical notion of normality. The proof is omitted since it follows trivially.

5.5.2 Theorem
Let (X, τ) be an L-topological space. Then the following statements are equivalent:

(1) (X, τ) is normal.
(2) For every $\gamma', \mu \in \tau$ such that $\gamma \leq \mu$, there exists $\nu \in \tau$ such that $\gamma \leq \nu \leq \overline{\nu} \leq \mu$.
(3) For every $\gamma', \mu \in \tau$ such that $\gamma \leq \mu$, there exists $\sigma', \nu \in \tau$ such that $\gamma \leq \nu \leq \sigma \leq \mu$.

We will need the natural result that a closed crisp set inherits normality with respect to the subspace L-topology.

5.5.3 Lemma
Let (X, τ) be a normal L-topological, and $1'_A \in \tau$ be crisp. Then (A, τ_A) is normal.

Proof.
Choose $\mu_A, (\nu_A)' \in \tau_A$ such that $\nu_A \leq \mu_A$. Now we have $\mu_A = \mu|_A$ for some $\mu \in \tau$. Also, there exists $\omega \in \tau$ such that $\omega|_A = (\nu_A)'$. Then $(\omega|_A)' = (\nu|_A)'' = \nu|_A$. Hence $(\omega')|_A = \nu|_A$. So let $\tilde{\nu} := \omega'$. Then let $\nu := \tilde{\nu} \wedge 1_A$ is τ-closed and $\nu \leq \mu$.
Since (X, τ) is normal we have that $\exists \sigma$ such that

$$\nu \leq \sigma^\circ \leq \overline{\sigma} \leq \mu.$$

We consider $\sigma_A = \sigma|_A$ and by Lemma 3.3.7

$$\nu_A \leq (\sigma^\circ)|_A \leq (\sigma_A)^{\circ A} \leq \overline{\sigma|_A}^A \leq \overline{\sigma}|_A \leq \mu_A.$$

That is, (A, τ_A) is normal.

The crisp concept of perfect normality also has an L-topological analogue.

5.5.4 Definition ([17])
An L-topological space is *perfectly normal* if for every closed set ν and open set μ such that $\nu \leq \mu$, there exists a continuous function $f : X \longrightarrow I(L)$ such that for every $x \in X$

$$\nu(x) = f(x)(1-) \leq f(x)(0+) = \mu(x).$$

Chapter 6

Extension Theorems

6.1 Introduction

Given a continuous function defined on a crisp subset A of X where (X, T) is an "ordinary" topological space - it is natural to question what conditions are required to guarantee a continuous extension of the function to X. This kind of problem has long been contemplated in classical topology and yielded many extension theorems. Foremost amongst these are Urysohn's Lemma and the Tietze Extension Principle.

In this chapter we will look at ways of dealing with the same type of problem in the L-topological setting. Indeed, all classical extension theorems can be applied to a particular class of L-topological space by virtue of the next lemma. We first need to recall a well-known classical definition.

6.1.1 Definition
Let (X, T) be a topological space. A function $\mu : (X, T) \longrightarrow (I, |\cdots|)$ is *lower semicontinuous* iff $\forall a \in I$,

$$\{x \in X : \mu(x) > a\} = \mu^{\leftarrow}[(a, 1]] \in T.$$

6.1.2 Definition (Lowen, [38])
Let (X, T) be a topological space. $\omega(T)$ is defined as the set of functions $\mu \in I^X$ that are lower semicontinuous.

Lowen's proof of the following lemma is different to the given one and can be found in [38].

6.1.3 Lemma (Kubiak)
Let (X, T) be a topological space and Y a set and $f : Y \longrightarrow (X, T)$. If $f^{\leftarrow}(T)$ is the initial topology on Y with respect to f and $f^{\leftarrow}(\omega(T))$ the corresponding initial L-topology on X with respect to f then

$$\omega(f^{\leftarrow}(T)) = f^{\leftarrow}(\omega(T)).$$

Proof.
Let $h \in \omega(f^{\leftarrow}(T))$. This implies that $\forall a \in I$,

$$h^{\leftarrow}[(a, 1]] \in f^{\leftarrow}(T).$$

Thus $\exists G_a \in T$ such that

$$h^{\leftarrow}[(a, 1]] = f^{\leftarrow}[G_a].$$

We want to show that $h \in f^{\leftarrow}(\omega(T))$
that is $h = f^{\leftarrow}[g] = g \circ f$ for some $g \in \omega(T)$.

Now $h = \bigvee_{a \in I}(a \wedge 1_{h^{\leftarrow}[(a,1]]})$

$$\begin{aligned}&= \bigvee_{a\in I}(a \wedge 1_{f^{\leftarrow}[G_a]})\\&= \bigvee_{a\in I}(a \wedge f^{\leftarrow}[1_{[G_a]}])\\&= \bigvee_{a\in I}(a \wedge 1_{[G_a]}) \circ f\\&= g \circ f\end{aligned}$$

where $g = \bigvee_{a\in I}(a \wedge 1_{[G_a]}) : X \longrightarrow I$ and

$$g^{\leftarrow}[(b, 1]] = \bigcup_{a>b} G_a \in T.$$

On the other hand, let $h \in f^{\leftarrow}(\omega(T))$.
Then $\exists g \in \omega(T)$ such that $h = f^{\leftarrow}[g] = g \circ f$. Now for $a \in I$,

$$\begin{aligned}h^{\leftarrow}[(a, 1]] &= (g \circ f)^{\leftarrow}[(a, 1]]\\&= f^{\leftarrow}[g^{\leftarrow}[(a, 1]]].\end{aligned}$$

Since $g^{\leftarrow}[(a, 1]] \in T$ we have that

$$f^{\leftarrow}[g^{\leftarrow}[(a, 1]]] \in f^{\leftarrow}(T).$$

Therefore $h \in \omega(f^{\leftarrow}(T))$.

The previous result provides us with the following useful corollary.

6.1.4 Corollary

Let (X, T) be a topological space and $A \subseteq X$. If T_A the subspace topology on A and $\omega(T)|_A$ the subspace L-topology on A . Then $\omega(T_A) = \omega(T)|_A$.

Proof.
Let $i : A \longrightarrow (X, T)$ be the inclusion mapping. Then $T_A = i^{\leftarrow}(T)$. Now by the previous lemma

$$\omega(T_A) = \omega(i^{\leftarrow}(T)) = i^{\leftarrow}(\omega(T)) = \omega(T)|_A.$$

So $\omega(T_A) = \omega(T)|_A$.

6.1.5 Lemma

Let $(X, T_1), (Y, T_2)$ be topological spaces. If $f : (X, T_1) \longrightarrow (Y, T_2)$ is continuous then
$f : (X, \omega(T_1)) \longrightarrow (Y, \omega(T_2))$ is also continuous.

Proof.
Let $\nu \in \omega(T_2)$. We need to show that $\forall a \in I$,

$$\mu^{\leftarrow}(a, 1] \in T_1$$

where $\mu = f^{\leftarrow}[\nu]$.
Choose $a \in I$. We now have

$$\begin{aligned}\mu^{\leftarrow}(a, 1] &= (f^{\leftarrow}[\nu])^{\leftarrow}(a, 1]\\&= (\nu \circ f)^{\leftarrow}(a, 1]\\&= (f^{\leftarrow} \circ \nu^{\leftarrow})(a, 1] \in T_1\\&= f^{\leftarrow}(\nu^{\leftarrow}((a, 1])) \in T_1 \qquad \text{(since } \nu^{\leftarrow}((a, 1]) \in T_2\text{)}.\end{aligned}$$

Thus μ is open in $(X, \omega(T_1))$.

The situation that we have is the following:

Given two topological spaces (X, T_1) and (Y, T_2), $A \subseteq X$ and a continuous function

$$f : (A, \omega(T_1)|_A) \longrightarrow (Y, \omega(T_2)).$$

If a "classsical" extension theorem holds, i.e.

$$f : (A, T_1|_A) \longrightarrow (Y, T_2)$$

has extension

$$g : (A, T_1) \longrightarrow (Y, T_2)$$

such that $g|_A = f$ then by Corollary 6.1.4 and Lemma 6.1.5

$$g : (X, \omega(T_1)) \longrightarrow (Y, \omega(T_2))$$

is an extension of

$$f : (A, \omega(T_1|_A)) \longrightarrow (Y, \omega(T_2)).$$

This result is, however, not very helpful because it can only be applied to a very specific class of L-topological spaces (namely, the class of L-topological spaces where $L := I$ and the L-topologies are topologically generated spaces). L-topological counterparts do exist for many classical extension theorems including Urysohn's Lemma and the Tietze Extension Principle which are presented in this chapter.

6.2 Continuous Extension: dense subspaces

Given a continuous function defined on a dense set $D \subseteq X$, where (X, τ_1) is an L-topological space, we want to examine the conditions required to extend that function continuously to X. Firstly we will deal with the situation in which we already have the required continuous extension but want to know what is required for that extension to be unique.

6.2.1 Theorem (Uniqueness, [16], Chapter 6)
Let (X, τ_1) and (Y, τ_2) be L-topological spaces, $\phi \neq D \subseteq X$ dense, (Y, τ_2) T_2 and $\varphi, \psi : X \longrightarrow Y$ continuous such that $\varphi|_D = \psi|_D$. Then

$$\varphi \equiv \psi.$$

Proof.
Let $x \in X$ and assume $\varphi(x) \neq \psi(x)$. Now by T_2, $\exists \mu, \nu \in \tau_2$ such that

$$\varphi^{\leftarrow}[\mu](x) \wedge \psi^{\leftarrow}[\nu](x) = \mu(\varphi(x)) \wedge \nu(\psi(x)) \neq 0 \text{ and } \mu \wedge \nu = 1_{\phi}.$$

φ, ψ continuous implies

$$\varphi^{\leftarrow}[\mu], \psi^{\leftarrow}[\nu] \in \tau_1 \text{ and hence } \varphi^{\leftarrow}[\mu] \wedge \psi^{\leftarrow}[\nu] \in \tau_1.$$

From the fact that $\mu \wedge \nu = 1_{\phi}$ we conclude

$$\forall z \in D, \mu \wedge \nu(\varphi(z)) = 0$$

and since $z \in D$ we have
$$\begin{aligned} \mu \wedge \nu(\varphi(z)) &= \mu(\varphi(z)) \wedge \nu(\varphi(z)) \\ &= \mu(\varphi(z)) \wedge \nu(\psi(z)) \\ &= \varphi^{\leftarrow}[\mu](z) \wedge \psi^{\leftarrow}[\nu](z). \end{aligned}$$

We conclude by denseness that

$$\forall x \in X, \varphi^{\leftarrow}[\mu](x) \wedge \psi^{\leftarrow}[\nu](x) = 0$$

a contradiction to $\mu(\varphi(x)) \wedge \nu(\psi(x)) > 0$ (from T_2)
Hence for all $x \in X$, $\varphi(x) = \psi(x)$.

The next statement regarding uniqueness of a continuous extension is given as a corollary - but we need to establish some preliminary facts first.

6.2.2 Lemma
If (Y, τ_2) is a strong T_2 L-topological space then $\triangle_Y := \{(y, y) : y \in Y\}$ is closed in $(Y \times Y, \tau_2 \times \tau_2)$

Proof.
We show that $(1_{\triangle_Y})' = 1_{\triangle_Y^c}$ is open in $Y \times Y$ where $\triangle_Y^c$ denotes the complement of $\triangle_Y$.
Let $(x, y) \in \triangle_Y^c$. By strong T_2 we have $\exists \mu_x, \nu_y \in \tau_2$ such that

$$\mu_x(x) \wedge \nu_y(y) = 1 \text{ and } \mu_x \wedge \nu_y = 1_{\phi}.$$

Thus $\mu_x \times \nu_y(x, y) = 1$ and $\forall z \in Y$, $\mu_x \times \nu_y(z, z) = 0$.
Hence

$$1_{\triangle_Y^c} = \bigvee_{x, y \in Y, x \neq y} (\mu_x \times \nu_y) \in \tau_2 \times \tau_2$$

(for $\mu, \nu \in \tau_2$, the sets $\mu \times \nu$ form a subbase of $\tau_2 \times \tau_2$).

6.2.3 Lemma

If

$$\varphi, \psi : (X, \tau_1) \longrightarrow (Y, \tau_2)$$

are continuous then the mapping

$$\varphi \times \psi = \begin{cases} (X, \tau_1) \longrightarrow (Y \times Y, \tau_2 \times \tau_2) \\ x \longmapsto (\varphi(x), \psi(x)) \end{cases}$$

is continuous.

Proof.

$$p_1 \circ (\varphi \times \psi)(x) = p_1(\varphi(x), \psi(x)) = \varphi(x)$$

and

$$p_2 \circ (\varphi \times \psi)(x) = p_2(\varphi(x), \psi(x)) = \psi(x)$$

where p_1 and p_2 are the respective projection mappings.
Hence $p_1 \circ (\varphi \times \psi), p_2 \circ (\varphi \times \psi)$ are continuous. Therefore by Theorem 3.3.5 $\varphi \times \psi$ is continuous.

By the previous two lemmas we have that

$$(\varphi \times \psi)^{\leftarrow}[\triangle_Y] = \{x \in X : (\varphi(x), \psi(x)) \in \triangle_Y\}$$

$$= \{x \in X : \varphi(x) = \psi(x)\} = H$$

is closed in X.

Now consider a weakly dense subset D of X and $\varphi, \psi : X \longrightarrow Y$ continuous such that $\varphi|_D = \psi|_D$.
We have that $D \subseteq H$ and this implies:

$$1_X = \overline{1_D} \leq \overline{1_H} \leq 1_H.$$

We thus have the following corollary:

6.2.4 Corollary (Uniqueness)

Let (X, τ_1) and (Y, τ_2) be L-topological spaces, $\phi \neq D \subseteq X$ weakly dense, (Y, τ_2) strong T_2 and $\varphi, \psi : X \longrightarrow Y$ continuous such that $\varphi|_D = \psi|_D$. Then

$$\varphi \equiv \psi.$$

The next two theorems show that the conditions required for uniqueness of a continuous extension are weaker than those required to ensure existence of an extension (from a dense set to the whole space). This, of course, means that the extensions guaranteed by the following two theorems are unique. Both of these theorems are stated and proved in [16], Chapter 6.

6.2.5 Theorem (Principle of Continuous Extension 1)
Let $(X,\tau_1),(Y,\tau_2)$ be L-topological spaces and let (Y,τ_2) be T_2 and regular, $\phi \neq D \subseteq X$ strictly dense, $\varphi : (D,(\tau_1)|_D) \longrightarrow (Y,\tau_2)$ continuous. Then the following two conditions are equivalent:

(1) $\exists \psi : X \longrightarrow Y$ continuous, $\psi|_D = \varphi$;

(2) $\forall x \in X, \exists y \in Y$ satisfying the following condition: $\forall \nu \in \tau_2, \exists \mu \in \tau_1$ such that

$$\text{(a) } \nu(y) \leq \mu(x),$$

$$\text{(b) } \mu|_D = \varphi^{\leftarrow}[\nu].$$

Proof.
(1) $\Rightarrow$ (2): ψ continuous and $\psi|_D = \varphi$. Let $x \in X$. Choose $y = \psi(x)$. Thus for $\nu \in \tau_2$ we find by continuity that $\mu = \psi^{\leftarrow}[\nu] \in \tau_1$ and $\psi^{\leftarrow}[\nu](x) = \nu(\psi(x)) = \nu(y)$, i.e (a).
Moreover, as $\psi|_D = \psi \circ i_D = \varphi$ we find for $x \in D$,

$$\varphi^{\leftarrow}[\nu](x) = \nu(\varphi(x)) = \nu(\psi|_D(x)) = \nu(\psi(x)) = \psi^{\leftarrow}[\nu](x).$$

Hence $\varphi^{\leftarrow}[\nu] = \psi^{\leftarrow}[\nu]|_D$. So (b) is established.

(2) $\Rightarrow$ (1): We will prove this in three steps.

step 1: Let $x \in X$. Firstly we will show that $y \in Y$ of assertion (b) is uniquely determined: To this end let $\nu \in \tau_2$ and assume y_1, y_2 fulfill requirement (2) and that $y_1 \neq y_2$. Then by the T_2 property, we have that $\exists \nu_1, \nu_2 \in \tau_2$ such that

$$\nu_1(y_1) \wedge \nu_2(y_2) > 0 \text{ and } \nu_1 \wedge \nu_2 = 1_{\phi}.$$

By condition (2) there are $\mu_1, \mu_2 \in \tau_1$ such that

$$\nu_1(y_1) \leq \mu_1(x) \text{ and } \nu_2(y_2) \leq \mu_2(x)$$

with $\mu_1|_D = \varphi^{\leftarrow}[\nu_1]$, $\mu_2|_D = \varphi^{\leftarrow}[\nu_2]$.

$$\begin{aligned} \text{Now we have } \nu_1(y_1) \wedge \nu_2(y_2) &\leq \mu_1 \wedge \mu_2(x) \\ &\leq \bigvee\{\mu_1 \wedge \mu_2(z) : z \in D\} \qquad \text{(since } D \text{ is strictly dense)} \\ &= \bigvee\{(\varphi^{\leftarrow}[\nu_1] \wedge \varphi^{\leftarrow}[\nu_2])(z) : z \in D\} \\ &= \bigvee\{(\nu_1 \wedge \nu_2)(\varphi(z)) : z \in D\} = 0 \qquad \text{(by } T_2\text{).} \end{aligned}$$

A contradiction. Hence $y_1 = y_2$.

step 2: Next, we can now define a mapping:

$$\psi = \begin{cases} X \longrightarrow Y & \\ x \longmapsto y & \text{uniquely determined by (2)} \end{cases}$$

We now show that $\psi|_D = \varphi$: Let $x \in D$. Then we have that $y = \psi(x)$ fulfills

$$\nu(\psi(x)) \leq \mu(x) = \mu|_D(x) = \varphi^{\leftarrow}[\nu](x) = \nu(\varphi(x))$$

with a pair $(\nu, \mu = \mu_\nu)$ according to condition (2). With a similar argument as in step 1, we assume $\psi(x) \neq \varphi(x)$. Then, by T_2, there are $\nu_1, \nu_2 \in \tau_2$ with

$$\nu_1(\psi(x)) \wedge \nu_2(\varphi(x)) > 0 \text{ and } \nu_1 \wedge \nu_2 = 1_\phi.$$

Also we have by (2) that there exists $\mu_1, \mu_2 \in \tau_1$ such that for $i \in \{0, 1\}$,

(a) $\nu_i(\psi(x)) \leq \mu_i(x)$

(b) $\mu_i|_D = \varphi^{\leftarrow}[\nu_i]$.

But then
$$\begin{aligned} \nu_1(\psi(x)) \wedge \nu_2(\varphi(x)) &\leq \nu_1(\varphi(x)) \wedge \nu_2(\varphi(x)) && \text{(from (a) and (b))} \\ &\leq \mu_1(x) \wedge \mu_2(x) \\ &= (\mu_1 \wedge \mu_2)(x) \\ &\leq \bigvee\{(\mu_1 \wedge \mu_2)(z) : z \in D\} && \text{(strictly dense)} \\ &= \bigvee\{(\nu_1 \wedge \nu_2)(\varphi(z)) : z \in D\} = 0 && \text{(by } T_2\text{)}, \end{aligned}$$
a contradiction. hence $\varphi(x) = \psi(x)$ on D and ψ is an extension, i.e $\psi|_D = \varphi$.

Step 3: ψ is continuous (here we make use of regularity).
Let $\nu \in \tau_2$. Consider $\gamma \in \tau_2$ such that $\gamma^* \vee \nu = 1_Y$. For $x_0 \in X, \exists \mu_{x_0} \in \tau_1$ such that

$$\gamma(\psi(x_0)) \leq \mu_{x_0}(x_0) \text{ and } \mu_{x_0}|_D = \varphi^{\leftarrow}[\gamma].$$

We will show

$$(*) \qquad \forall x \in X, \gamma^*(\psi(x)) \wedge \mu_{x_0}(x) = 0.$$

For $x \in X$ we find $\tilde{\mu_x} \in \tau_1$ such that $\gamma^*(\psi(x)) \leq \tilde{\mu_x}(x)$ and $\tilde{\mu_x}|_D = \varphi^{\leftarrow}[\gamma^*]$.
This implies
$$\begin{aligned} \gamma^*(\psi(x)) \wedge \mu_{x_0}(x) &\leq \tilde{\mu_x}(x) \wedge \mu_{x_0}(x) \\ &= \tilde{\mu_x} \wedge \mu_{x_0}(x) \\ &= \bigvee\{\tilde{\mu_x} \wedge \mu_{x_0}(z) : z \in D\} \text{ (strict denseness)} \\ &= \bigvee\{\gamma^*(\varphi(z)) \wedge \gamma(\varphi(z)) : z \in D\} = 0 \text{ (since } \gamma^* \wedge \gamma = 1_\phi). \end{aligned}$$
Now $\gamma^* \vee \nu = 1_Y$ and thus

$$\begin{aligned} \mu_{x_0}(x) &= (\gamma^* \vee \nu)(\psi(x)) \wedge \mu_{x_0}(x) \\ &= [\gamma^*(\psi(x)) \wedge \mu_{x_0}(x)] \vee [\nu(\psi(x)) \wedge \mu_{x_0}(x)] \\ &= \nu(\psi(x)) \wedge \mu_{x_0}(x) \leq \nu(\psi(x)). \end{aligned}$$

From this we get

$$\psi^{\leftarrow}[\gamma](x) = \gamma(\varphi(x)) \leq \bigvee_{x_0 \in X} \mu_{x_0}(x) \leq \nu(\psi(x)) = \psi^{\leftarrow}[\nu](x)$$

and, by regularity, we have

$$\psi^{\leftarrow}[\nu](x) = \nu(\psi(x)) = \bigvee\{\gamma(\psi(x)) : \gamma \in \tau_2, \gamma^* \vee \nu = 1_Y\}$$

$$\leq \bigvee_{x_0 \in X} \nu_{x_0}(x) \leq \psi^{\leftarrow}[\nu](x),$$

i.e.
$$\psi^{\leftarrow}[\nu] = \bigvee_{x_0 \in X} \mu_{x_0} \in \tau_1.$$

Hence ψ is continuous.

6.2.6 Theorem (Principle of Continuous Extension 2 (star-regular case))

Let $(X, \tau_1), (Y, \tau_2)$ be L-topological spaces and let (Y, τ_2) be T_2 and star-regular, $\phi \neq D \subseteq X$ strictly dense, $\varphi : D \longrightarrow Y$ continuous. Then the following two conditions are equivalent:

(1) $\exists \psi : X \longrightarrow Y$ continuous, $\psi|_D = \psi \circ i_D = \varphi$;

(2) $\forall x \in X, \exists y \in Y$ satisfying the following condition: $\forall \nu \in \tau_1, \exists \mu \in \tau_1$ such that

$$\text{(a) } \nu(y) \leq \mu(x),$$

$$\text{(b) } \mu|_D = \varphi^{\leftarrow}[\nu].$$

Proof.
(1) $\Rightarrow$ (2): as in the case of the preceeding Theorem.

(2) $\Rightarrow$ (1):
step 1: as before (no regularity used).
step 2: as before (no regularity used).
step 3: Let $\nu \in \tau_2$. Consider $\gamma \in \tau_2$ such that $\gamma^* \to 0 \leq \nu$.
For $x_0 \in X, \exists \mu_{x_0} \in \tau_1$ such that

$$\nu(\psi(x_0)) \leq \mu_{x_0}(x_0) \text{ and } \mu_{x_0}|_D = \varphi^{\leftarrow}[\gamma].$$

We will again show

$$(**) \qquad \forall x \in X, \gamma^*(\psi(x)) \wedge \mu_{x_0}(x) = 0.$$

For $x \in X$ we find $\tilde{\mu_x} \in \tau_1$ such that

$$\gamma^*(\psi(x)) \leq \tilde{\mu_x}(x) \text{ and } \tilde{\mu_x}|_D = \varphi^{\leftarrow}[\gamma^*].$$

then again, we find

$$\begin{aligned} \gamma^*(\psi(x)) \wedge \mu_{x_0}(x) &\leq \tilde{\mu_x} \wedge \mu_{x_0}(x) \\ &= \bigvee\{\tilde{\mu_x} \wedge \mu_{x_0}(z) : z \in D\} \\ &= \bigvee\{\gamma^*(\varphi(z)) \wedge \gamma(\varphi(z)) : z \in D\} = 0. \end{aligned}$$

Hence $\mu_{x_0}(x) \leq \gamma^*(\psi(x)) \to 0 \leq \nu(\psi(x))$ and therefore, again as before

$$\psi^{\leftarrow}[\gamma](x) = \gamma(\psi(x)) \leq \bigvee_{x_0 \in X} \mu_{x_0}(x) \leq \nu(\psi(x)) = \psi^{\leftarrow}[\nu](x).$$

Now by star-regularity

$$\psi^{\leftarrow}[\nu](x) = \bigvee\{\gamma(\psi(x)) : \gamma \in \tau_2, \gamma^* \to 0 \leq \nu\}$$

$$\leq \bigvee_{x_0 \in X} \mu_{x_0}(x) \leq \psi^{\leftarrow}[\nu](x).$$

i.e. $\psi^{\leftarrow}[\nu] \in \tau_1$.

6.3 Urysohn's Lemma

This section is motivated by Urysohn's Lemma in general topology. A proof of this lemma can be found in most standard texts on general topology (e.g. [28]).

We once again assume L to be equipped with an order reversing involution.

6.3.1 Lemma (Urysohn's Lemma)
A topological space (X,T) is normal iff for any pair of closed, disjoint sets A and B, there exists a continuous function $f:(X,T)\longrightarrow[0,1]$, such that f is zero on A and one on B.

This can be viewed as an extension theorem since what this theorem actually says, is that the continuous function

$$g:(A\cup B,T|_{A\cup B})\longrightarrow[0,1],$$

$$g(x)=\begin{cases}0 & \text{if } x\in A\\ 1 & \text{if } x\in B\end{cases}$$

can be continuously extended to the whole space X.

For the continuity of $g(x)$ refer to [5], p. 38, Ex I, Section 3.3, 4 b) or [28], Chapter 3, Problem B. When Hutton introduced the concept of the L-fuzzy unit interval in [17] he produced a version of Urysohn's lemma for the L-fuzzy unit interval. Even though the following result is not an extension theorem, we feel its inclusion in this chapter is justified by the classical situation mentioned above.

We now state Hutton's version of Urysohn's lemma.

6.3.2 Theorem (L-Fuzzy Urysohn's Lemma)
An L-topological space (X,τ) is normal iff for every closed set ν and open set μ such that $\nu\le\mu$, there exists a continuous function $f:X\longrightarrow I(L)$ such that for every $x\in X$

$$\nu(x)\le f(x)(1-)\le f(x)(0+)\le\mu(x).$$

i.e.

$$\nu\le f^{\leftarrow}[(L_1)']\le f^{\leftarrow}[R_0]\le\mu.$$

Proof.

$\Leftarrow$:
Choose $x\in X$. Since

$$\nu(x)\le f(x)(1-)\le f(x)(0+)\le\mu(x),$$

and for any $t\in(0,1)$

$$f(x)(1-)\le f(x)(t+)\le f(x)(t-)\le f(x)(0+),$$

we have

$$\nu(x)\le f(x)(t+)\le f(x)(t-)\le\mu(x).$$

Now $f^{\leftarrow}[(L_t)'](x)=f(x)(t-)$ and $f^{\leftarrow}[R_t](x)=f(x)(t+)$. Since f is continuous we have $f^{\leftarrow}[(L_t)']$ is closed and $f^{\leftarrow}[R_t]$ is open. Hence

$$\nu\le f^{\leftarrow}[R_t]\le f^{\leftarrow}[(L_t)']\le\mu,$$

so we have that (X,τ) is normal.

$\Rightarrow$:
Conversely, construct $\{\sigma_r:r\in(0,1)\}$ so that for each $r\in(0,1)$, $\nu\le\sigma_r\le\mu$ and $r<s$ implies $\overline{\sigma_s}\le\sigma_r^{\circ}$. Define $f(x)(t)=\sigma_t(x)$. Clearly

$$\nu(x)\le f(x)(1-)\le f(x)(0+)\le\mu(x).$$

Now

$$f^{\leftarrow}[R_t] = \bigvee_{r>t} \sigma_r.$$

For any $s \in (0,1)$ choose $r \in (0,s)$. Then $\sigma_s \leq \overline{\sigma_s} \leq \sigma_r^{\circ}$ and so

$$\bigvee_{r>t} \sigma_r = \bigvee_{r>t} \sigma_r^{\circ}$$

is open and

$$f^{\leftarrow}[(L_t)'] = \bigwedge_{r<t} \sigma_r = \bigwedge_{r<t} \overline{\sigma_r}$$

is closed. Hence f is continuous.

In [17], Hutton claims as a trivial consequence of the definition of perfect normality (Definition 5.5.4):

An L-topological space is perfectly normal iff it is normal and every closed set is a countable intersection of open sets.

This is not clear to us. However, for (X, τ) an L-topological space we certainly have that (X, τ) is perfectly normal (Definition 5.5.4) $\Rightarrow$ (X, τ) is normal and every closed set is a countable intersection of open sets:

Let $\mu, \nu \in L^X$, μ open, ν closed such that $\nu \leq \mu$. Then by perfect normality we have that there exists a continuous $f : X \longrightarrow I(L)$ such that $\forall x \in X$,

$$\nu(x) = f(x)(1-) \leq f(x)(0+) = \mu(x).$$

Now

$$f(x)(1-) = L_1'[f(x)]$$

(from Corollary 4.2.5).

$R_a[f]$ is non-increasing in s. For $x \in X$ and for any $s \in [0,1)$ choose $r \in (s,1) \cap \mathbb{Q}$. Then

$$R_r[f(x)] \leq R_s[f(x)]$$

and thus

$$\begin{aligned} \bigwedge_{s \in [0,1)} \{R_s[f(x)]\} &= \bigwedge_{s \in [0,1) \cap \mathbb{Q}} \{R_s[f(x)]\} \\ &= \bigwedge_{s \in [0,1) \cap \mathbb{Q}} \{R_s[f(x)]\} \\ &= \bigwedge_{s \in [0,1) \cap \mathbb{Q}} \{f^{\leftarrow}[R_s](x)\}. \end{aligned}$$

Now since x was arbitrarily chosen we have that

$$\nu = \bigwedge_{s \in [0,1) \cap \mathbb{Q}} \{f^{\leftarrow}[R_s]\}.$$

So ν is a countable intersection of open sets and since ν was arbitrarily chosen we have the result.

6.4 The Tietze Extension Principle

Kubiak's L-fuzzy version of The Tietze Extension Principle, unlike the L-fuzzy Urysohn's lemma, cannot be proven by simply adapting the classical proof using Urysohn's Lemma. The method we use to prove this Tietze Extension Principle is via the Insertion Theorem which characterizes normality.

The Insertion Theorem

The Katětov-Tong Insertion Theorem is a charaterization of normality in general topology given independently by Katětov [23], [24] and Tong [60]. Kubiak extended the Insertion Theorem to the L-fuzzy real line for L-topological spaces, by adapting the proof used by Katětov. The following work, until otherwise indicated, is due to Kubiak [33].

6.4.1 Definition

Let (X, τ) be an L-topological space. A function $f : X \longrightarrow \mathbb{R}(L)$ is called *lower (resp. upper) semicontinuous* if $f^{\leftarrow}[R_t]$ (resp. $f^{\leftarrow}[L_t]$) is open for each $t \in \mathbb{R}$.

Equivalently, f is lower (resp. upper) semicontinuous iff it is continuous with respect to the right-hand (resp. left-hand) L-topology on $\mathbb{R}(L)$, where the right-hand (resp. left-hand) topology is generated from the base $\mathcal{R} = \{R_t : t \in \mathbb{R}\}$ (resp. $\mathcal{L} = \{L_t : t \in \mathbb{R}\}$).

6.4.2 Lemma

For each $j \in J$ let $f_j : (X, \tau) \longrightarrow (\mathbb{R}(L), \mathcal{U})$ be a lower (resp. upper) semicontinuous mapping. Then $f = \bigvee_{j\in J} f_j$ (resp. $h = \bigwedge_{j\in J} f_j$) and $g = \bigwedge_{j=1}^{n} f_j$ (resp. $m = \bigvee_{j=1}^{n} f_j$) are lower (resp. upper) semicontinuous.

Proof

(i)

Let $t \in \mathbb{R}$ and $x \in X$. Then
$$\begin{aligned} f^{\leftarrow}[R_t](x) &= R_t[f(x)] \\ &= f(x)^{+}(t) \\ &= \textstyle\bigvee_{s>t} f(x)(s) \\ &= \textstyle\bigvee_{s>t}(\bigvee_{j\in J} f_j(x))(s) \\ &= \textstyle\bigvee_{s>t}\bigvee_{j\in J}(f_j(x))(s) \\ &= \textstyle\bigvee_{j\in J}\bigvee_{s>t}(f_j(x))(s) \\ &= \textstyle\bigvee_{j\in J}(\bigvee_{s>t}(f_j(x))(s) \\ &= \textstyle\bigvee_{j\in J} f_j(x)^{+}(t) \\ &= \textstyle\bigvee_{j\in J} f_j^{\leftarrow}[R_t](x) \in \tau. \end{aligned}$$

(ii)

It suffices to show that $g = f_1 \wedge f_2$ is lower semicontinuous. Choose $x \in X$. Now
$$\begin{aligned} f^{\leftarrow}[R_t](x) &= R_t(f(x)) \\ &= (f(x))^{+}(t) \\ &= (f_1(x) \wedge f_2(x))^{+}(t) \\ &= (f_1(x)^{+} \wedge f_2(x)^{+})(t) \qquad \text{(Proposition 4.2.6)} \\ &= f_1(x)^{+}(t) \wedge f_2(x)^{+}(t) \\ &= R_t(f_1)(x)(t) \wedge R_t(f_2)(t). \end{aligned}$$

Thus
$$f^{\leftarrow}[R_t] = f_1^{\leftarrow}[R_t] \wedge f_2^{\leftarrow}[R_t] \in \tau.$$

The corresponding statements in paretheses can be established in a similar way using L_t instead of R_t and making use of the de Morgan laws.

Remarks:

(1) For the rest of this section, continuity of f means continuity with respect to the natural

L-topology on $\mathbb{R}(L)$.
(2) Lower and upper semicontinuous functions with values in $I(L)$ are defined in the obvious way.
(3) f is continuous iff it is both lower and upper semicontinuous.
(4) In the case that $L = \{0, 1\}$ we get the usual semicontinuities of real-valued functions.

6.4.3 Lemma

Let (X, τ) be an L-topological space, let $\mu \in L^X$, and let $f : X \longrightarrow \mathbb{R}(L)$ be such that $\forall x \in X$,

$$f(x)(t) = \begin{cases} 1 & \text{if } t < 0, \\ \mu(x) & \text{if } 0 \leq t \leq 1, \\ 0 & \text{if } t > 1, \end{cases}$$

Then f is lower (resp. upper) semicontinuous iff μ is open (resp. closed).

Proof.
It is suffucient to observe that

$$\begin{aligned} f^{\leftarrow}[R_t] &= 1 && \text{if } t < 0, \\ &= \mu && \text{if } 0 \leq t < 1, \\ &= 0 && \text{if } t \geq 1, \end{aligned}$$

and

$$\begin{aligned} f^{\leftarrow}[L_t'] &= 1 && \text{if } t \leq 0, \\ &= \mu && \text{if } 0 < t \leq 1, \\ &= 0 && \text{if } t > 1. \end{aligned}$$

6.4.4 Lemma

Let (X, τ) be a normal L-topological space, and let $\{\mu_i\}_{i=1}^{\infty}$ and $\{\nu_j\}_{j=1}^{\infty}$ be countable families of elements of L^X. If there exists $\mu, \nu \in L^X$ such that for all $i, j = 1, 2, \ldots$, we have $\overline{\mu_i} \leq \overline{\mu} \leq \nu_j^{\circ}$ and $\overline{\mu_i} \leq \nu^{\circ} \leq \nu_j^{\circ}$, then there exists $\sigma \in L^X$ such that for all $i, j = 1, 2, \ldots$, $\overline{\mu_i} \leq \sigma^{\circ}$ and $\overline{\sigma} \leq \nu_j^{\circ}$.

Proof.
We begin by showing by induction that for all $n \geq 2$ there exists a collection
$\{\sigma_i, \lambda_i : 1 \leq i < n\} \subseteq L^X$ such that the following conditions hold for all $i, j = 1, 2, \ldots, n-1$:

$$\begin{aligned} \overline{\mu_i} &\leq \sigma_i^{\circ}, \\ \overline{\lambda_j} &\leq \nu_j^{\circ}, \\ \overline{\mu} &\leq \lambda_j^{\circ}, \\ \overline{\sigma_i} &\leq \nu^{\circ}, \\ \overline{\mu_i} &\leq \nu_j^{\circ}. \end{aligned} \tag{P_n}$$

Clearly (P_2) follows immediately from the normality of X. Now suppose that for some $n \geq 2$ we have defined $\sigma_i, \lambda_i \in L^X$ $(i < n)$ such that (P_n) holds. Since $\overline{\mu_n} \leq \overline{\mu} \leq \lambda_j^{\circ}$ $(j < n)$ and $\overline{\mu_n} \leq \nu^{\circ}$, by normality of X there exists $\sigma_n \in L^X$ such that $\overline{\mu_n} \leq \sigma_n^{\circ} \leq \overline{\sigma_n} \leq \bigwedge_{j<n}(\lambda_j \wedge \nu)$. Similarly, since $\overline{\mu} \leq \nu_n^{\circ}$ and $\overline{\sigma_i} \leq \nu_n^{\circ}$ $(i \leq n)$ there exists $\lambda_n \in L^X$ such that $\bigvee_{i \leq n}(\overline{\sigma_i} \vee \overline{\mu}) \leq \lambda_n^{\circ} \leq \overline{\lambda_n} \leq \nu_n^{\circ}$. Thus (P_{n+1}) holds.
Now set $\sigma = \bigvee_{i=1}^{\infty} \sigma_i$. Then for all $i = 1, 2, \ldots$, $\overline{\mu_i} \leq \sigma_i^{\circ} \leq \sigma^{\circ}$. Since $\overline{\sigma_i} \leq \lambda_j^{\circ}$ $(i, j = 1, 2, \ldots)$, we have $\sigma_i \leq \lambda_j$, so that for all $j = 1, 2, \ldots$, we have $\overline{\sigma} \leq \overline{\lambda_j} \leq \nu_j^{\circ}$. This completes the proof.

6.4.5 Lemma

Let (X, τ) be a normal L-topological space. Let $\{\theta_r\}_{r\in\mathbb{Q}}$ and $\{\eta_r\}_{r\in\mathbb{Q}}$ be monotone increasing collections of, respectively, closed and open L-fuzzy sets on X ($\mathbb{Q}$ is the set of all rational numbers) such that whenever $r < s$ we have $\theta_r \leq \eta_s$. Then there exists a collection $\{\omega_r\}_{r\in\mathbb{Q}} \subseteq L^X$ such that whenever $r < s$ we have $\theta_r \leq \omega_s^\circ$, $\overline{\omega_r} \leq \omega_s^\circ$, and $\overline{\omega_r} \leq \eta_s$.

Proof.

Firstly, we arrange all the rational numbers into a sequence $\{r_n\}$ (without any repititions). For every $n \geq 2$ we define inductively a collection $\{\omega_{r_i} : 1 \leq i < n\} \subseteq L^X$ such that for all $1 \leq i, j < n$,

$$\begin{aligned} &\theta_r \leq \omega_{r_i}^\circ && \text{if } r < r_i, \\ &\overline{\omega_{r_i}} \leq \eta_r && \text{if } r_i < r, \qquad\qquad (S_n) \\ &\overline{\omega_{r_i}} \leq \omega_{r_j}^\circ && \text{if } r_i < r_j, \end{aligned}$$

Now, let us observe that the countable collections $\{\theta_r\}_{r\in\mathbb{Q}}$ and $\{\eta_r\}_{r\in\mathbb{Q}}$ together with θ_{r_1} and η_{r_1} satisfy all the hypotheses of Lemma 6.4.4, so there exists $\sigma_1 \in L^X$ such that for all $r < r_1$, $\theta_r \leq \sigma_1^\circ$ and for all $r > r_1$, $\overline{\sigma_1} \leq \eta_r$. By setting $\omega_{r_1} = \sigma_1$, we get (S_2).
Assume that the L-fuzzy sets ω_{r_2} are already defined for $i < n$ and satisfy (S_n). Define

$$\mu = \bigvee\{\omega_{r_i} : i < n, r_i < r_n\} \vee \theta_{r_n}$$

and

$$\nu = \bigwedge\{\omega_{r_j} : j < n, r_j > r_n\} \wedge \eta_{r_n}.$$

Then we have that whenever $r_i < r_n < r_j \quad (i, j < n)$,

$$\overline{\omega_{r_i}} \leq \overline{\mu} \leq \omega_{r_j}^\circ \qquad \text{and} \qquad \overline{\omega_{r_i}} \leq \nu^\circ \leq \omega_{r_j}^\circ$$

and also, whenever $r < r_n < s$,

$$\theta_r \leq \overline{\mu} \leq \eta_s \qquad \text{and} \qquad \theta_r \leq \nu^\circ \leq \eta_s.$$

This means that the countable collections

$$\{\omega_{r_i} : i < n, r_i < r_n\} \cup \{\theta_r : r < r_n\}$$

and

$$\{\omega_{r_j} : j < n, r_j > r_n\} \cup \{\eta_r : r > r_n\}$$

together with μ and ν fulfill all hypotheses of Lemma 6.4.4. Hence there exists $\sigma_n \in L^X$ such that

$$\begin{aligned} &\theta_r \leq \sigma_n^\circ && \text{if } r < r_n, \\ &\overline{\omega_{r_i}} \leq \sigma_n^\circ && \text{if } r_i < r_n, \\ &\overline{\sigma_n} \leq \eta_r && \text{if } r_n < r, \\ &\overline{\sigma_n} \leq \omega_{r_j}^\circ && \text{if } r_n < r_j, \end{aligned}$$

where $1 \leq i, j \leq n - 1$. By setting $\omega_{r_n} = \sigma_n$ we obtain L-fuzzy sets $\omega_{r_1}, \omega_{r_2}, \ldots, \omega_{r_n}$ that satisfy (S_{n+1}).
Thus, the collection $\{\omega_{r_i} : i = 1, 2, \ldots\}$ has the required properties. This completes the proof.

The final tool that we will need to prove the L-fuzzy Tietze extension principle is the following theorem.

6.4.6 Theorem (L-Fuzzy Insertion Theorem)
Let (X,τ) be an L-topological space, Then the following statements are equivalent:

(1) (X,τ) is normal.
(2) If $g,h : X \longrightarrow \mathbb{R}(L)$, g is upper semicontinuous, h is lower semicontinuous, and $g \leq h$, then there exists a continuous function $f : X \longrightarrow \mathbb{R}(L)$ such that $g \leq f \leq h$.

Proof.

(2) $\Rightarrow$ (1): Let $\gamma', \sigma \in \tau$ such that $\gamma \leq \sigma$.
Define $g,h : X \longrightarrow \mathbb{R}(L)$ by

$$\begin{aligned} g(x)(t) &= 1 && \text{if } t < 0, \\ &= \gamma(x) && \text{if } 0 \leq t \leq 1, \\ &= 0 && \text{if } t > 1, \end{aligned}$$

and

$$\begin{aligned} h(x)(t) &= 1 && \text{if } t < 0, \\ &= \sigma(x) && \text{if } 0 \leq t \leq 1, \\ &= 0 && \text{if } t > 1, \end{aligned}$$

for each $x \in X$. By Lemma 6.4.3, g is upper semicontinuous and h is lower semicontinuous. Certainly $g \leq h$ holds, so that there exists a continuous function $f : X \longrightarrow \mathbb{R}(L)$ such that $g \leq f \leq h$. Now, suppose $t \in (0,1)$. Then we have

$$\gamma = g^{\leftarrow}[R_t] \leq f^{\leftarrow}[R_t] \leq f^{\leftarrow}[L'_t] \leq h^{\leftarrow}[L'_t] = \sigma,$$

which means that (X,τ) is normal.

(1) $\Rightarrow$ (2): We begin by defining two mappings $H, G : \mathbb{Q} \longrightarrow L^X$ by

$$H(r) = H_r = h^{\leftarrow}[R'_r]$$

and

$$G(r) = G_r = g^{\leftarrow}[L_r]$$

for all $r \in \mathbb{Q}$. Clearly, H and G are monotone increasing.
$\{H'_r, G_r : r \in \mathbb{Q}\} \subseteq \tau$, and whenever $r < s$ we have $H_r \leq G_s$. By Lemma 6.4.5 there exists a mapping $F : \mathbb{Q} \longrightarrow L^X$ such that

$$\begin{aligned} H_r &\leq F_s^{\circ}, \\ \overline{F_r} &\leq F_s^{\circ}, \\ \overline{F_r} &\leq G_s, \end{aligned}$$

whenever $r < s$ $(r,s \in \mathbb{Q})$. We now let

$$V_t = \textstyle\bigwedge_{r<t} F'_r \qquad \text{for all } t \in \mathbb{R},$$

and we define a monotone decreasing family $\{V_t : t \in \mathbb{R}\} \subseteq L^X$. Further we have

$$\overline{V_t} \leq V_s^{\circ} \qquad \text{whenever } s < t.$$

Now, for $s < r < r' < t$ $(s,t \in \mathbb{R}$ and $r,r' \in \mathbb{Q})$ we have $V'_s \leq \overline{F_r} \leq F^{\circ}_{r'} \leq V'_t$, hence $\overline{V_t} \leq V_s^{\circ}$. We also have

$$\begin{aligned}
\textstyle\bigvee_{t\in\mathbb{R}} V_t &= \textstyle\bigvee_{t\in\mathbb{R}} \bigwedge_{r<t} F'_r \\
&\geq \textstyle\bigvee_{t\in\mathbb{R}} \bigwedge_{r<t} G'_r \\
&= \textstyle\bigvee_{t\in\mathbb{R}} \bigwedge_{r<t} g^{\leftarrow}[L'_t] \\
&= \textstyle\bigvee_{t\in\mathbb{R}} g^{\leftarrow}[L'_t] \\
&= g^{\leftarrow}(\textstyle\bigvee_{t\in\mathbb{R}} L'_t) = 1;
\end{aligned}$$

similarly,

$$\bigwedge_{t\in\mathbb{R}} V_t = 0.$$

We now define a function $f : X \longrightarrow \mathbb{R}(L)$ satisfying the required properties.
Let

$$f(x)(t) = V_t(x)$$

for all $x \in X$ and $t \in \mathbb{R}$ (Theorem 6.3.2). We have thus shown that f is well defined, i.e., $\forall x \in X, f(x) \in \mathbb{R}(L)$. We now show that f is continuous. Observe that

$$\bigvee_{s>t} V_s = \bigvee_{s>t} V_s^{\circ}$$

and

$$\bigwedge_{s<t} V_s = \bigwedge_{s<t} \overline{V_s}.$$

Then

$$f^{\leftarrow}[R_t] = \bigvee_{s>t} V_s = \bigvee_{s>t} V_s^{\circ}$$

is open. Now

$$f^{\leftarrow}[L'_t] = \bigwedge_{s<t} V_s = \bigwedge_{s<t} \overline{V_s},$$

is closed so that f is continuous. We now need only show that $g \leq f \leq h$. To this end we first show $\forall t \in \mathbb{R}$,

$$g^{\leftarrow}[L'_t] \leq f^{\leftarrow}[L'_t] \leq h^{\leftarrow}[L'_t]$$

and

$$g^{\leftarrow}[R_t] \leq f^{\leftarrow}[R_t] \leq h^{\leftarrow}[R_t].$$

We now have

$$\begin{aligned}
g^{\leftarrow}[L'_t] &= \textstyle\bigwedge_{s<t} g^{\leftarrow}[L'_s] \\
&= \textstyle\bigwedge_{s<t} \bigwedge_{r<s} g^{\leftarrow}[L'_r] \\
&= \textstyle\bigwedge_{s<t} \bigwedge_{r<s} G'_r \\
&\leq \textstyle\bigwedge_{s<t} \bigwedge_{r<s} F'_r \\
&= \textstyle\bigwedge_{s<t} V_s = f^{\leftarrow}[L'_t],
\end{aligned}$$

and

$$f^{\leftarrow}[L'_t] = \textstyle\bigwedge_{s<t} V_s$$

$$
\begin{aligned}
&= \textstyle\bigwedge_{s<t} \bigwedge_{r<s} F'_r \\
&\leq \textstyle\bigwedge_{s<t} \bigwedge_{r<s} H'_r \\
&= \textstyle\bigwedge_{s<t} \bigwedge_{r<s} h^{\leftarrow}[R_r] \\
&= \textstyle\bigwedge_{s<t} h^{\leftarrow}[L'_s] \\
&= h^{\leftarrow}[L'_t].
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
g^{\leftarrow}[R_t] &= \textstyle\bigvee_{s>t} g^{\leftarrow}[R_s] \\
&= \textstyle\bigvee_{s>t} \bigvee_{r>s} g^{\leftarrow}[L'_r] \\
&= \textstyle\bigvee_{s>t} \bigvee_{r>s} G'_r \\
&\leq \textstyle\bigvee_{s>t} \bigwedge_{r<s} F'_r \\
&= \textstyle\bigvee_{s>t} V_s = f^{\leftarrow}[R_t],
\end{aligned}
$$

and

$$
\begin{aligned}
f^{\leftarrow}[R_t] &= \textstyle\bigvee_{s>t} V_s \\
&= \textstyle\bigvee_{s>t} \bigwedge_{r<s} F'_r \\
&= \textstyle\bigvee_{s>t} \bigvee_{r>s} H'_r \\
&= \textstyle\bigvee_{s>t} \bigvee_{r>s} h^{\leftarrow}[R_r] \\
&= \textstyle\bigvee_{s>t} h^{\leftarrow}[R_s] = h^{\leftarrow}[R_t].
\end{aligned}
$$

$\forall x \in X$, $g^{\leftarrow}[R_t](x) \leq f^{\leftarrow}[R_t](x)$ implies $g(x)^{+}(t) \leq f(x)^{+}(t)$. But then $g(x) \leq f(x)$ by virtue of the remark after Definition 4.2.9. This completes the proof.

Note: In the the case that $L = \{0, 1\}$, the above theorem reduces to the charaterizaion of normality due to Katětov [23].

An alternative approach to the Insertion Theorem

For completeness we now digress by presenting a different approach to proving the Insertion Theorem that we will need to yield the L-fuzzy Tietze Extension Principle. This method was used by Kotzé and Kubiak in [29]. The following work, until otherwise indicated, is from [29].

We remind the reader that for $t \in I$, $< t >= [1_{[0,t)}] \in I(L)$. We now adopt some notation that was used in [29]. Given $t \in I$ then $\mathbf{t}$ stands for the constant map taking all of X to $< t >$ (see Section 4.3). Let us write (a) for the member of $I(L)$ generated by a member of L^I whose constant value is $a \in L$.

Next, we define a concept of a characteristic function of an L-set.

6.4.7 Definition ([32])
For $\mu \in L^X$, $1_\mu : X \longrightarrow I(L)$ is defined by:

$$1_\mu(x) = (\mu(x))$$

for every $x \in X$.

6.4.8 Lemma ([32], Remark 7.5))
If X is an L-topological space, then μ is open (resp., closed) iff 1_μ is lower semicontinuous (resp., upper semicontinuous).

Proof

$\Leftarrow$:
Let τ be an L-topology on X. Assume 1_μ is lower semicontinuous. That is $\forall t \in I$,

$$1_\mu^{\leftarrow}[R_t] \in \tau.$$

For $x \in X$, $\begin{aligned}1_\mu^{\leftarrow}[R_t](x) &= R_t(1_\mu(x)) \\ &= R_t((\mu(x))). \\ &= (\mu(x)). \qquad \text{(since } \mu(x) \text{ is constant)}\end{aligned}$

So

$$1_\mu^{\leftarrow}[R_t](x) = (\mu(x)) = \begin{cases} 1 & \text{if } t < 0, \\ \mu(x) & \text{if } 0 \leq t < 1, \\ 0 & \text{if } t \geq 1, \end{cases}$$

By Lemma 6.4.3 we have that μ is open.

$\Rightarrow$:
Simply follow the steps of $\Leftarrow$ backwards.

6.4.9 Lemma
For every $t \in I$, $\mathbf{t}$ is continuous.

Proof.
Let $t \in I$. Choose $s \in I$ and we have that $R_s \in \mathcal{U}_{I(L)}$. For $x \in X$ we have

$$\begin{aligned}\mathbf{t}^{\leftarrow}[R_s](x) &= R_s(\mathbf{t}(x)) \\ &= R_s(< t >) \\ &= R_s[1_{[0,t)}] \\ &= \textstyle\bigvee_{s>r} 1_{[0,t)}(s)\end{aligned}$$

$$= \begin{cases} 1 & \text{if } s < t \\ 0 & \text{if } s \geq t \end{cases}$$

Thus for all $s \in I$,

$$\mathbf{t}^{\leftarrow}[R_s] \in \{1_X, 1_\phi\} \subseteq \tau.$$

Similarly

$$\mathbf{t}^{\leftarrow}[L_s] \in \tau.$$

Hence $\mathbf{t}$ is continuous.

We now state and prove the following decompositions for a member of $I(L)^X$ which are analagous to those well-known decompositions of members of I^X.

6.4.10 Lemma
For a set X and for $\mu \in I(L)^X$ the following holds:
(1) $\mu = \bigvee\{r \wedge 1_{R_r[\mu]} : r \in \mathbb{Q} \cap I\} = \bigvee\{r \wedge 1_{L'_r[\mu]} : r \in \mathbb{Q} \cap I\}$;
(2) $\mu = \bigwedge\{r \vee 1_{R_r[\mu]} : r \in \mathbb{Q} \cap I\} = \bigwedge\{r \vee 1_{L'_r[\mu]} : r \in \mathbb{Q} \cap I\}$.

Proof.
We shall just prove (1) since (2) follows easily from (1) by the fact that L has an order reversing involution; (see [32], Remark 7.5). It is enough to show that for every $[f] \in I(L)$ we have

$$[f] = \bigvee\{< r > \wedge(f^+(r)) : r \in \mathbb{Q} \cap I\} = \bigvee\{< r > \wedge(f^-(r)) : r \in \mathbb{Q} \cap I\}.$$

Let $t \in I \backslash \{1\}$ and $\lambda_r \in (f^+(r))$. Then

$$(\bigvee_r (1_{[0,1)} \wedge \lambda_r))^+(t) = \bigvee_r (1_{[0,r)}(t) \wedge f^+(r)) = \bigvee_{r>t} f^+(r) = f^{++}(t) = f^+(t).$$

One can prove the second equality in (1) in a similar way using the fact the $f^{-+} = f^+$ (see [32]).

We will use the following result later. Alexandrov and Pasynkov [1] present if for the case of sets and it is presented in [35] for the case of real-valued functions (in topological spaces).

6.4.11 Lemma
Let X be an L-topological space and $a \leq b$ in $I(L)^X$. For all $n \in \mathbb{N}$ let g_n, h_n be lower semicontinuous and f_n, k_n upper semicontinuous, $g_n \leq f_n$, and $h_n \leq k_n$. If $a \leq \bigvee\{g_n : n \in \mathbb{N}\} \leq \bigvee\{f_n : n \in \mathbb{N}\} \leq b$ and $a \leq \bigwedge\{h_n : n \in \mathbb{N}\} \leq \bigwedge\{k_n : n \in \mathbb{N}\} \leq b$, then there is a lower semicontinuous l and an upper semicontinuous u such that $a \leq l \leq u \leq b$.

Proof.
Let $l_1 = g_1$ and for $n \geq 2$, $l_n = g_n \wedge \bigwedge\{h_i : i < n\}$ We have

$$a \leq \bigvee_{n\in\mathbb{N}} g_n \wedge \bigwedge_{n\in\mathbb{N}} h_n = \bigvee_{n\in\mathbb{N}} (g_n \wedge \bigwedge_{n\in\mathbb{N}} h_n) \leq g_1 \wedge \bigvee_{n\geq 2} (g_n \wedge \bigwedge_{i<n} h_i) = \bigvee_{n\in\mathbb{N}} l_n = l.$$

l is lower semicontinuous by Lemma 6.4.2. And thus $l_m \leq g_m \leq \bigvee\{f_i : i \leq n\}$ if $m \leq n$, and $l_m \leq h_m \leq k_m$ if $m > n$.
That is for every $m \in \mathbb{N}$,

$$l_m \leq \bigwedge_{n\in\mathbb{N}} (k_n \vee \bigvee_{i\leq n} f_i) = u$$

which is upper semicontinuous (by Lemma 6.4.2). Therefore,

$$l \leq u \leq \bigwedge_{n\in\mathbb{N}} (k_n \vee \bigvee_{n\in\mathbb{N}} f_n) \leq b.$$

6.4.12 Corollary ([33], Lemma 3.5)
Let X be a normal L-topological space. Let μ, θ_n ($n \in \mathbb{N}$) be closed, let ν, η_n ($n \in \mathbb{N}$) be open, and for all $m, n \in \mathbb{N}$ let $\theta_n \leq \mu \leq \eta_m$ and $\theta_n \leq \nu \leq \eta_m$. Then there exist an open σ and a closed ω such that for all $m, n \in \mathbb{N}$,

$$\theta_n \leq \sigma \leq \omega \leq \eta_m.$$

Proof.
From the definition of normality we have that for every $n \in \mathbb{N}$ there exists α_n, β_n (open), and μ_n, ω_n (closed) such that $\theta_n \leq \alpha_n \leq \mu_n \leq \nu$ and $\mu \leq \beta_n \leq \omega_n \leq \eta_n$. Then $a = \bigvee_{n\in\mathbb{N}} 1_{\theta_n} = 1_{\bigvee_{n\in\mathbb{N}} \theta_n}$, $b = \bigwedge_{n\in\mathbb{N}} 1_{\eta_n} = 1_{\bigwedge_{n\in\mathbb{N}}} \eta_n$, $g_n = 1_{\alpha_n}$, $h_n = 1_{\beta_n}$, $f_n = 1_{\mu_n}$, and $k_n = 1_{\omega_n}$ satisfy all the hypotheses of Theorem 6.4.11. We thus have

$$a \leq l \leq u \leq b$$

with l that is lower semicontinuous and u that is upper semicontinuous. Now $\bigvee_{n\in\mathbb{N}} \theta_n = R_{\frac{1}{2}}[a] \leq R_{\frac{1}{2}}[l] \leq L'_{\frac{1}{2}}[u] \leq L'_{\frac{1}{2}}[b] = \bigwedge_{n\in\mathbb{N}} \eta_n$. Put $\sigma = R_{\frac{1}{2}}[l]$ and $\omega = L'_{\frac{1}{2}}[u]$.

We can now generate a function from X to $I(L)$ from a non-increasing family of L-subsets of X. The idea was first presented by Hutton [17].

Consider $\{F_r : r \in \mathbb{Q} \cap I\}$ a family of non-decreasing L-subsets of a set X. For every $x \in X$, let $f(x) \in I(L)$ be the equivalence class (under relation) generated by $\varphi_x \in H_L$ defined by

$$\varphi_x(t) = \bigwedge\{F_r'(x) : r < t, r \in \mathbb{Q} \cap I\}.$$

Then we say that the function $f : X \longrightarrow I(L)$ is generated by the family $\{F_r\}$.

We are now in a position to provide the alternative proof of the Insertion Pinciple given in [29].

6.4.13 Theorem (L-Fuzzy Insertion Theorem)
Let (X, τ) be an L-topological space. The following two conditions are equivalent:

(1) X is normal
(2) For $a, b : X \longrightarrow I(L)$, a upper semicontinuous and b lower semicontinuous such that $a \leq b$, there exists a continuous $f : X \longrightarrow I(L)$ such that

$$a \leq f \leq b.$$

Proof. (**Kotzé & Kubiak, [29]**)
For every $r \in \mathbb{Q} \cap I$, $\mu_r = R_r[a] = \bigvee\{L'_{r+\frac{1}{n}}[a] : n \in \mathbb{N}\}$ and $\nu_r = L'_r[b] = \bigwedge\{R_{r-\frac{1}{n}}[b] : n \in \mathbb{N}\}$. By Corrolary 6.4.12, since $\mu_r \leq L'_r[a] \leq \nu_r$ and $\mu_r \leq R_r[b] \leq \nu_r$, hence there exist σ_r (open) and ω_r (closed) such that $\mu_r \leq \sigma_r \leq \omega_r \leq \nu_r$. Let $\{r_n\}$ be an enumeration of $\mathbb{Q} \cap I$. Now let $g_n = r_n \wedge 1_{\sigma_{r_n}}, f_n = r_n \wedge 1_{\omega_{r_n}}, h_n = r_n \vee 1_{\sigma_{r_n}}$, and $k_n = r_n \vee 1_{\omega_{r_n}}$. By Lemma 6.4.10, we have

$$a = \bigvee_{r \in \mathbb{Q} \cap I} (r \wedge 1_{\mu_r}) \leq \bigvee_{n \in \mathbb{N}} g_n \leq \bigvee_{n \in \mathbb{N}} f_n \leq \bigvee_{r \in \mathbb{Q} \cap I} (r \wedge 1_{\nu_r}) = b$$

and

$$a = \bigwedge_{r \in \mathbb{Q} \cap I} (r \vee 1_{\mu_r}) \leq \bigwedge_{n \in \mathbb{N}} h_n \leq \bigwedge_{n \in \mathbb{N}} k_n \leq \bigwedge_{r \in \mathbb{Q} \cap I} (r \vee 1_{\nu_r}) = b.$$

Therefore, by Lemma 6.4.11, there is a lower semicontinuous l_0 and an upper semicontinuous u_0 such that $a \leq l_0 \leq u_0 \leq b = l_1$. By the normality of X, the insertion process may be continued by inductively defining two families (as in the case of the classical Urysohn's Lemma, see [9], 1.5.10), $\{l_r : r \in \mathbb{Q} \cap I\}$ of lower semicontinuous functions and $\{u_r : r \in \mathbb{Q} \cap I\}$ of upper semicontinuous functions such that $l_r \leq u_r \leq l_s$ whenever $0 \leq r < s \leq 1$.
Now, for every $r \in \mathbb{Q} \cap I$ let $\omega_r = R'_r[l_{1-r}]$ and $\sigma_r = L_r[u_{1-r}]$. Then if $r \leq s$ we have that

$$\omega_r \leq \sigma_s \leq \omega_s.$$

Let f be generated by $\{\omega_r\}$. Then for every $t \in I$, $L_t[f] = \bigvee\{F_r : r < t\} = \bigvee\{\sigma_r : r < t\}$ (open) and $R'_t[f] = \bigwedge\{F_r : r > t\}$ (closed). Hence f is continuous. Since $a \leq l_{1-r} \leq b$ and thus $R'_r[b] \leq F_r \leq R'_r[a]$. Therefore

$$L_t[b] = \bigvee_{r<t} R'_r[b] \leq \bigvee_{r<t} F_r = L_t[f] \leq \bigvee_{r<t} R'_r[a] = L_t[a],$$

That is $a \leq f \leq b$. The converse is trivial and has already been proved.

Remark: Lemma 6.4.11 and its proof are valid for functions with values in the L-fuzzy real line $\mathbb{R}(L)$ [11].

It is now simple to extend Theorem 6.4.13 to $\mathbb{R}(L)$-valued functions.

6.4.14 Corollary ([33])
Let (X, τ) be an L-topological space. The following two conditions are equivalent:

(1) X is normal
(2) For $a, b : X \longrightarrow \mathbb{R}(L)$, a upper semicontinuous and b lower semicontinuous such that $a \leq b$, there exists a continuous $f : X \longrightarrow \mathbb{R}(L)$ such that

$$a \leq f \leq b.$$

Proof.
By adding one detail to the proof of Theorem 6.4.13, this becomes evident. We assume a and b to be $\mathbb{R}(L)$-valued. Let $h : \mathbb{R}(L) \longrightarrow (0,1)(L)$ be an increasing homeomorhism (see [53]). We have that the compositions ha and hb take values in $(0,1)(L) \subseteq I(L)$.
We have thus proved that $\exists f$ continuous such that $ha \leq f \leq hb$. So f is $(0,1)(L)$-valued and $h^{-1} \circ f$ is the required function.

For more information on the Insertion Theorem (in the crisp case), the reader is referred to [30, 35].

Remark:

We will briefly mention the concept of the σ-ring. This more general structure can be used to prove the statements of the previous section. For a more comprehensive look at σ-rings we refer the reader to [29].
For this section we will assume L (with an order reversing involution $'$) to be complete (not necessarily infinitely distributive).
Definition
A *ring* $\mathcal{A}$ in L^X is a subset of L^X closed under finite sup and finite inf.

A *σ-ring* $\mathcal{A}$ in L^X is a ring in L^X which is closed under countable sup.

Definition
A σ-ring is normal if, given $\mu, \nu \in \mathcal{A}$ with $\mu' \leq \nu$, there exist $\omega, \gamma \in \mathcal{A}$ such that

$$\mu' \leq \omega \leq \gamma' \leq \nu.$$

Note: If $\mathcal{A}$ is an L-topology then this definition of normality reduces to the usual L-topological definition.
Definition
A function f from X to $\mathbb{R}(L)$ is called *lower* (resp., *upper*) *$\mathcal{A}$-measurable* if for every $t \in \mathbb{R}$, $R_t[f] \in \mathcal{A}$ (resp., $L_t[f] \in \mathcal{A}$).
We say that f is *$\mathcal{A}$-measurable* if it is both lower and upper $\mathcal{A}$-measurable.

We say that $\mu, \nu \in L^X$ are completely *$\mathcal{A}$-separated* in X if there is an $\mathcal{A}$-measurable function f on X and some $s < t$ in $\mathbb{R}$ such that $\mu \leq R'_s[f]$ and $\nu \leq L'_t[f]$.
Note: The previous proof of the Insertion theorem did not involve uncountable operations in L^X. Therefore, the results are still valid if we replace a normal L-topology with a normal σ-ring in L^X and continuity with measurability.

The L-Fuzzy Tietze Theorem is, in fact a simple consequence of the L-Fuzzy Insertion Theorem.

6.4.15 Theorem (L-fuzzy Tietze Theorem, [33])
Let (X, τ) be a normal L-topological space, let $A' \in \tau$ be crisp, and let $f : (A, \tau_A) \longrightarrow I(L)$ be continuous. Then there exists a continuous function $F : (X, \tau) \longrightarrow I(L)$ such that $F|_A = f$.

Proof.
Define two functions $g, h : X \longrightarrow I(L)$ by

$$\begin{aligned} g(x) &= f(x) \qquad \text{if } x \in A, \\ &= [\alpha_0] \qquad \text{if } x \notin A, \end{aligned}$$

and

$$\begin{aligned} h(x) &= f(x) \qquad \text{if } x \in A, \\ &= [\alpha_1] \qquad \text{if } x \notin A, \end{aligned}$$

where $[\alpha_0]$ and $[\alpha_1]$ are equivalence classes determined by $\alpha_0, \alpha_1 : \mathbb{R} \longrightarrow L$ such that

$$\alpha_0(t) = 1, \qquad t < 0;$$

$$= 0, \qquad t > 0;$$

and

$$\alpha_1(t) = 1, \qquad t < 1;$$

$$= 0, \qquad t > 1.$$

We show g and h are, respectively, upper and lower semicontinuous. Let $t > 0 (t \in \mathbb{R})$. Then

$$g^{\leftarrow}[L_t](x) = f^{\leftarrow}[L_t](x), \qquad x \in A,$$

$$= 1, \qquad x \notin A,$$

where $f^{\leftarrow}[L_t]$, being open in (A, τ_A) is of the form $\mu_t|_A$, where $\mu_t \in \tau$, so that

$$g^{\leftarrow}[L_t] = \mu_t \vee A'$$

is open in (X, τ). Clearly, $g^{\leftarrow}[L_t] = 0$ for each $t \leq 0$. Thus by Lemma 6.4.3 g is upper semicontinuous. Similarly, we obtain

$$h^{\leftarrow}[R_t] = \nu_t \cup A', \qquad t < 1,$$

$$= 0, \qquad t \geq 1,$$

where $\nu_t \in \tau$ is such that $f^{\leftarrow}[R_t] = \nu_t|_A$. Thus by Lemma 6.4.3 h is lower semicontinuous. We then have $g \leq h$. Now by Theorem 6.4.6 there exists a continuous function $F : (X, \tau) \longrightarrow I(L)$ such that $\forall x \in X$,

$$g(x) \leq F(x) \leq h(x).$$

Hence, for each $x \in A$ we get

$$f(x) \leq F(x) \leq f(x),$$

so that F is the required extension of f on (X, τ).

Chapter 7

L-Fuzzy Vector Spaces

7.1 Introduction

In this chapter, we provide an introduction to L-fuzzy vector spaces and their fundamental properties. Much of the foundations of this area of mathematics was formulated by Katsaras and Liu in [25]. Katsaras then went on to extend these ideas in the papers [26] and [27]. We define and characterize the concept of an L-fuzzy subspace of a real vector space, noting some salient features of L-fuzzy sets defined on a real vector space. We examine the notions of convex, balanced and absorbing L-fuzzy sets and lead up to the notions of translation, the L-fuzzy seminorm, the L-norm and the L-normed space.

In this text we will be more general where possible and restate this theory for L-fuzzy vector spaces. For many results this can be easily done since the translation from fuzzy vector theory to L-fuzzy vector theory is both simple and natural. Certain results, however, are true for only the I-fuzzy case. These results are marked with a $(*)$.

Throughout, E will denote a real vector space over $\mathbb{R}$ and I will denote the unit interval $[0,1]$.

7.2 Preliminaries

7.2.1 Definition

Let $f : E^n \longrightarrow E, f(x_1, \ldots, x_n) = x_1 + \cdots + x_n$. We define $\mu_1 + \cdots + \mu_n = f(\mu)$.

For a a scalar and ν an L-fuzzy set in E, we define $a\nu = g(\nu)$ where $g : E \longrightarrow E, g(x) = ax$.

7.2.2 Definition ([25])

For $\mu \in L^E, t \in \mathbb{R}$ and $x \in E$ we define

$t\mu(x) = \mu(\frac{x}{t})$ for $t \neq 0$

If $t = 0$:

$$t\mu(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \sup \mu & \text{if } x = 0 \end{cases}$$

This is indeed the natural way in which to define $t\mu$. Let $E := \mathbb{R}$. Let $\mu = 1_{[a,b]}$ for $a, b \in \mathbb{R}, a \leq b$ then for $x \in \mathbb{R}$ and $t \in \mathbb{R}$ $(t > 0)$:

$$t\mu = 1_{[a,b]}(\frac{x}{t}) = \begin{cases} 1 & \text{if } \frac{x}{t} \in [a,b] \\ 0 & \text{if } \frac{x}{t} \notin [a,b] \end{cases}$$

$\frac{x}{t} \in [a,b] \Leftrightarrow a \leq \frac{x}{t} \leq b \Leftrightarrow ta \leq x \leq tb$ and hence

$$t\mu = (\frac{x}{t}) = \begin{cases} 1 & \text{if } x \in [ta, tb] \\ 0 & \text{if } x \notin [ta, tb] \end{cases}$$

That is, the set $[a, b]$ is stretched by a factor of t. For $t < 0$, via a similar argument, we have

$$t\mu = 1_{[a,b]}(\frac{x}{t}) = \begin{cases} 1 & \text{if } x \in [tb, ta] \\ 0 & \text{if } x \notin [tb, ta] \end{cases}$$

If, on the other hand, we have that $t = 0$ then $0 \cdot 1_{[a,b]} = 1_{\{0\}}$ i.e. we have the fuzzy point with support 0 and value 1.

7.2.3 Lemma

Let $s, t \in \mathbb{R}$ and let μ, μ_1 and μ_2 be L-fuzzy sets in E. Then

(1) $s(t\mu) = t(s\mu) = (st)\mu$.
(2) $\mu_1 \leq \mu_2 \Rightarrow t\mu_1 \leq t\mu_2$.

Proof.

(1)

if $s, t \neq 0$:

$$s(t\mu)(x) = (t\mu)(\frac{x}{s}) = (\mu(\frac{x}{st})) = (s\mu)(\frac{x}{t}) = t(s\mu)(x).$$

Also

$$(st)\mu(x) = \mu(\frac{x}{st}).$$

if $s = 0$ and $t \neq 0$:

$$0(t\mu)(x) = \begin{cases} \sup(t\mu) & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

$$= \begin{cases} \sup \mu & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

As $\sup_{x \in E} \mu(x) = \sup_{x \in E} \mu(\frac{x}{t})$ (replace x by tx).

$$t(0 \cdot \mu)(x) = (0 \cdot \mu)(\frac{x}{t}) = \begin{cases} \sup \mu & \text{if } \frac{x}{t} = 0 \\ 0 & \text{if } \frac{x}{t} \neq 0 \end{cases}$$

$$= \begin{cases} \sup \mu & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

$$(0t)\mu(x) = 0 \cdot \mu(x) = \begin{cases} \sup \mu & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

Obviously, the case where $t = 0$ and $s \neq 0$ is the same as the preceeding case.

When $t = s = 0$ then we have

$$0 \cdot (0 \cdot \mu)(x) = \begin{cases} \sup(0 \cdot \mu) & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

.

$$= \begin{cases} \sup \mu & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

.

$$= 0 \cdot \mu(x).$$

(2)

Choose $x \in X$. We have that $\mu_1(x) \leq \mu_2(x)$. If $t \neq 0$ then

$$t\mu_1(x) = \mu_1(\frac{x}{t}) \leq \mu_2(\frac{x}{t}) = t\mu_2(x).$$

If $t = 0$ and $x = 0$ then $0 \cdot \mu_1(0) = \sup \mu_1$ and $0 \cdot \mu_2(0) = \sup \mu_2$. Since we have that $\sup \mu_1 \leq \sup \mu_2$ we have that $0 \cdot \mu_1(0) \leq 0 \cdot \mu_2(0)$. If $t = 0$ and $x \neq 0$ then $0 \cdot \mu_1(x) = 0 = 0 \cdot \mu_2(x)$.

7.2.4 Lemma

Let E, F be real vector spaces and let $f : E \longrightarrow F$ be a linear mapping. Let $\mu, \nu \in L^E$ and let $k \in \mathbb{R}$. Then

(1) $f[k\mu] = kf[\mu]$
(2) $f[\mu + \nu] = f[\mu] + f[\nu]$

Proof.

Let $y \in F$.

(1)

<u>if $k \neq 0$:</u>

$$f[k\mu](y) = \sup_{z:f(z)=y} k\mu(z)$$

$$= \sup_{z:f(z)=y} \mu(\frac{z}{k})$$

and

$$kf[\mu](y) = k \sup_{z:f(z)=y} \mu(z)$$

$$= \sup_{z:f(z)=y} \mu(\frac{z}{k}).$$

<u>if $k = 0$:</u>

$$f[0\mu](y) = \sup_{z:f(z)=y} 0\mu(z)$$

$$= \begin{cases} \sup \mu & \text{if } f(0) = y \\ 0 & \text{if } f(0) \neq y \end{cases}$$

and

$$0f[\mu](y) = 0 \cdot \sup_{z:f(z)=y} \mu(z)$$

$$= \begin{cases} \sup\{\sup_{z:f(z)=y} \mu(z)\} & \text{if } f(0) = y \\ 0 & \text{if } f(0) \neq y \end{cases}$$

$$= \begin{cases} \sup \mu & \text{if } f(0) = y \\ 0 & \text{if } f(0) \neq y. \end{cases}$$

(2)

$$f[\mu + \nu](y) = \sup_{z:f(z)=y} (\mu + \nu)(z)$$

$$= \sup_{z:f(z)=y} \sup_{z_1+z_2=z} \{\mu(z_1) \wedge \nu(z_2)\}$$

$$= \sup_{f(z_1+z_2)=y} \{\mu(z_1) \wedge \nu(z_2)\}$$

$$= \sup_{f(z_1)+f(z_2)=y} \{\mu(z_1) \wedge \nu(z_2)\}$$

(since f is linear)

$$= \sup_{x_1+x_2=y} \{ \sup_{f(z_1)=x_1, f(z_2)=x_2} \{\mu(z_1) \wedge \nu(z_2)\}\}$$

$$= \sup_{x_1+x_2=y} \{ \sup_{f(z_1)=x_1} \sup_{f(z_2)=x_2} \{\mu(z_1) \wedge \nu(z_2)\}\}$$

(from Theorem 2.1.2 (1))

$$= \sup_{x_1+x_2=y} \{ \sup_{f(z_1)=x_1} \mu(z_1) \wedge \sup_{f(z_2)=x_2} \nu(z_2)\}$$

(L is a frame)

$$= \sup_{x_1+x_2=y} \{f[\mu](z_1) \wedge f[\nu](z_2)\}$$

$$= (f[\mu] + f[\nu])(y).$$

7.2.5 Definition

$\mu \in L^E$ is called a *L-fuzzy subspace* of E if $\forall a, b \in \mathbb{R}$ and $\forall x, y \in E$

$$\mu(ax + by) \geq \mu(x) \wedge \mu(y).$$

7.2.6 Proposition

Let μ be an L-fuzzy subspace of E then:

(1) $\mu(0) = \sup_{x \in E} \mu(x)$.
(2) For each $d \in L, \mu_d$ is a linear subspace of E.
(3) $x \in E, a \neq 0 \Rightarrow \mu(ax) = \mu(x)$.

Proof.

(1)

$x \in E \Rightarrow \mu(0) = \mu(0 \cdot x + 0 \cdot x) \geq \mu(x) \wedge \mu(x) = \mu(x)$.

(2)

Choose $d \in L$. If $\mu_d = \phi$ then it is a linear subspace of E. If not then choose $x, y \in \mu_d$. Then

$$\mu(x) \geq d \text{ and } \mu(x) \geq d.$$

Since μ is an L-fuzzy subspace we have $\forall a, b \in \mathbb{R}$,

$$\mu(ax + by) \geq \mu(x) \wedge \mu(y) \geq d \wedge d = d.$$

Hence $ax + by \in \mu_d$ and so μ_d is a linear subspace.

(3)

$x \in E, a \neq 0 \Rightarrow \mu(ax) = \mu(ax + 0x) \geq \mu(x) \wedge \mu(x) = \mu(x)$.
Now replace x by ax and a by $\frac{1}{a}$ to get $\mu(x) \geq \mu(ax)$. Equality follows.

7.2.7 Proposition ([25])

Let $\mu, \mu_1, ..., \mu_n$ be L-fuzzy sets in E and $r_1, ..., r_n \in \mathbb{R}$, then the following assertions are equivalent.

(1) $r_1\mu_1 + \cdots + r_n\mu_n \leq \mu$.
(2) $\forall x_1, ..., x_n \in E$ we have

$$\mu(r_1x_1 + \cdots + r_nx_n) \geq \min\{\mu_1(x_1), \ldots, \mu_n(x_n)\}.$$

Proof.

(1) $\Rightarrow$ (2) :

$$\mu(r_1x_1+\cdots+r_nx_n)\geq(r_1\mu_1+\cdots+r_n\mu_n)(r_1x_1+\cdots+r_nx_n)$$
$$\geq\min\{r_1\mu_1(r_1x_1),\ldots,r_n\mu_n(r_nx_n)\}\qquad\text{(from Definition 2.2.2)}$$
$$\geq\min\{\mu_1(x_1),\ldots,\mu_n(x_n)\}.\qquad\text{(from Definition 7.2.2)}$$

$(2)\Rightarrow(1)$:

By rearranging the order if necessary, we may assume that $r_i\neq 0$ for $i=1,\ldots,k$, and $r_i=0$ for $k<i\leq n$. If $\forall i=1,\ldots,n$, $r_i\neq 0$ then this method of proof is still valid. Let $x_1,\ldots,x_k$ be elements of E. For all $y_1,\ldots,y_{n-k}$ in E. We have

$$\mu(r_1x_1+\cdots+r_kx_k)\geq\min\{\mu_1(x_1),\ldots,\mu_{k+1}(y_1),\ldots,\mu_n(y_{n-k})\}\qquad\text{(from (2))}$$

Since $0\mu_j(0)=\sup_{y\in E}\mu_j(y)$, we get

$$\mu(r_1x_1+\cdots+r_kx_k)\geq\min\{\mu_1(x_1),\ldots,\mu_k(x_k),0\mu_{k+1}(0),\ldots,0\mu_n(0)\}.$$

Now, $(r_1\mu_1+\cdots+r_n\mu_n)(z)=\sup_{x_1+\cdots+x_k=z}[\min\{r_1\mu_1(x_1),\ldots,r_n\mu_n(x_n)\}]$

(from Definition 2.2.2)

$$=\sup_{x_1+\cdots+x_k=z}[\min\{r_1\mu_1(x_1),\ldots,r_k\mu_k(x_k),0\mu_{k+1}(0),\ldots,0\mu_n(0)]$$
$$=\sup_{x_1+\cdots+x_k=z}[\min\{\mu_1((\tfrac{1}{r_1})x_1),\ldots,\mu_k((\tfrac{1}{r_k})x_k),0\mu_{k+1}(0),\ldots,0\mu_n(0)\}]$$
$$\leq\sup_{x_1+\cdots+x_k=z}\mu(r_1(\tfrac{1}{r_1})x_1+\cdots+r_k(\tfrac{1}{r_k})x_k)=\mu(z).$$

We are now in a position to give a characterization of an L-fuzzy subspace.

7.2.8 Lemma ([25])

Let μ be an L-fuzzy set in E then the following are equivalent:

(1) μ is an L-fuzzy subspace.
(2) $\forall k,m\in\mathbb{R}$, we have $k\mu+m\mu\leq\mu$.
(3) The following two conditions hold:

(i) $\mu+\mu\leq\mu$.
(ii) $\forall t\in\mathbb{R},t\mu\leq\mu$.

Proof.
$(3)\Rightarrow(2)$ trivially, also (1) and (2) are equivalent by Proposition 7.2.7.

$(2)\Rightarrow(3)$:
$\mu+\mu=1\cdot\mu+1\cdot\mu\leq\mu$
and $k\mu=k\mu+0\cdot\mu\leq\mu$.

7.2.9 Proposition $(*)$

Let $u,v\in E$ and μ an I-fuzzy subspace such that $\mu(u)>\mu(v)$. Then $\mu(u+v)=\mu(v)$.

Proof.
Since $\mu(u)>\mu(v)$ we have $\mu(u+v)\geq\mu(v)$. Also $\mu[(u+v)-u]=\mu(v)\geq\mu(u+v)\wedge\mu(u)$. Since $\mu(u)>\mu(v)$ we have $\mu(u+v)\leq\mu(v)$. Consequently $\mu(u+v)=\mu(v)$.

7.2.10 Proposition ($*$)

If μ is an I-fuzzy subspace over E and $v, w \in E$ with $\mu(v) \neq \mu(w)$ then $\mu(v+w) = \mu(v) \wedge \mu(w)$.

Proof.

Apply Proposition 7.2.9.

The next two propositions are adapted from [25].

7.2.11 Proposition

If μ and ν are L-fuzzy subspaces of E and $k \in \mathbb{R}$ then $k\mu$ and $\mu + \nu$ are L-fuzzy subspaces.

Proof.

(i)

We have that for $x, y \in E$ and $a, b \in \mathbb{R}$

$$\mu(ax+by) \geq \mu(x) \wedge \mu(y).$$

Let $k \in \mathbb{R}$ and assume that $k \neq 0$. Then

$$\begin{aligned} k\mu(ax+by) &= \mu(\tfrac{1}{k}(ax+by)) \\ &= \mu(\tfrac{a}{k}x + \tfrac{b}{k}y) \\ &\geq \mu(\tfrac{x}{k}) \wedge \mu(\tfrac{y}{k}) \\ &\geq k\mu(x) \wedge k\mu(y). \end{aligned}$$

If on the other hand we have that $k = 0$ then

$$0 \cdot \mu(ax+by) = \begin{cases} 0 & \text{if } ax+by \neq 0 \\ \sup \mu & \text{if } ax+by = 0 \end{cases}$$

if $ax+by=0$:

$0 \cdot \mu(ax+by) = \sup \mu \geq \mu(x) \wedge \mu(y)$.

if $ax+by \neq 0$:

We have that $0 \cdot \mu(ax+by) = 0$. We must show that $(0 \cdot \mu(x)) \wedge (0 \cdot \mu(y)) = 0$. Assume that $(0 \cdot \mu(x)) \wedge (0 \cdot \mu(y)) \neq 0$. Then

$$0 \cdot \mu(x) > 0 \text{ and } 0 \cdot \mu(y) > 0.$$

So $y = x = 0$. A contradiction.

(ii)

$$(\mu+\nu)(ax+by) = \bigvee_{z_1+z_2=ax+by} \mu(z_1) \wedge \nu(z_2).$$

Now if $x_1 + x_2 = x$ and $y_1 + y_2 = y$ for $x_1, x_2, y_1, y_2 \in E$ then

$$(ax_1+by_1) + (ax_2+by_2) = ax+by.$$

So

$$(\mu+\nu)(ax+by) \geq \bigvee_{x_1+x_2=x, y_1+y_2=y} \{(\mu(ax_1+by_1) \wedge \nu(ax_2+by_2))\}$$

$$\geq \bigvee_{x_1+x_2=x, y_1+y_2=y} \{(\mu(x_1) \wedge \mu(y_1)) \wedge (\nu(x_2) \wedge \nu(y_2))\}$$

(since μ and ν are both L-fuzzy subspaces)

$$= \bigvee_{x_1+x_2=x, y_1+y_2=y} \{(\mu(x_1) \wedge \nu(x_2)) \wedge (\mu(y_1) \wedge \nu(y_2))\}$$

$$= \bigvee_{x_1+x_2=x} \bigvee_{y_1+y_2=y} \{(\mu(x_1) \wedge \nu(x_2)) \wedge (\mu(y_1) \wedge \nu(y_2))\}$$

(by Theorem 2.1.2(1))

$$= \bigvee_{x_1+x_2=x} \{\mu(x_1) \wedge \nu(x_2)\} \wedge \bigvee_{y_1+y_2=y} \{\mu(y_1) \wedge \nu(y_2)\}$$

(since L is a frame)

$$= (\mu + \nu)(x) \wedge (\mu + \nu)(y).$$

7.2.12 Proposition

If $(\mu_j)_{j\in J}$ is a collection of L-fuzzy subspaces of E then $\bigwedge_{j\in J} \mu_j$ is also an L-fuzzy subspace.

Proof.

For $m, k \in \mathbb{R}$ and $x, y \in E$

$$\begin{aligned} \text{then } (\textstyle\bigwedge_{j\in J} \mu_j)(mx + ky) &= \textstyle\bigwedge_{j\in J} \mu_j(mx + ky) \\ &\geq \textstyle\bigwedge_{j\in J}(\mu_j(x) \wedge \mu_j(y)) \qquad \text{(the } \mu_j\text{'s are } L\text{-fuzzy subspaces)} \\ &= (\textstyle\bigwedge_{j\in J} \mu_j(x)) \wedge (\textstyle\bigwedge_{j\in J} \mu_j(y)) \\ &= (\textstyle\bigwedge_{j\in J} \mu_j)(x) \wedge (\textstyle\bigwedge_{j\in J} \mu_j)(y). \end{aligned}$$

7.3 Properties of L-Fuzzy Vector Spaces

All the definitions and propositions of this section are adapted from the work of Katsaras and Liu in [25]. To clarify these ideas we provide justifications of these definitions by considering what they mean when the L-fuzzy sets are crisp.

7.3.1 Definition

An L-fuzzy set μ on E is *convex if* $\mu(kx + (1-k)y) \geq \mu(x) \wedge \mu(y)$ whenever $x, y \in E$ and $0 \leq k \leq 1$.

Remark: Let μ be convex and crisp. That is $\mu = 1_A$ for some $A \subseteq E$. Let $x, y \in A$ and $k \in [0, 1]$ then

$$1_A(kx + (1-k)y) \geq 1_A(x) \wedge 1_A(y).$$

But $1_A(x) = 1_A(y) = 1$ so we have

$$kx + (1-k)y \in A.$$

So A is convex in the classical sense. We thus have that our definition of convexity reduces to the classical notion of convexity in the crisp case.

7.3.2 Proposition

Let μ be an L-fuzzy set in E then the following three assertions are equivalent:

(1) μ is convex.
(2) $\forall k \in [0, 1], k\mu + (1-k)\mu \leq \mu$.
(3) $\forall d \in L, \mu_d$ is convex.

Proof.

The equivalence of (1) and (2) follows from Proposition 7.2.7 with

$$\begin{aligned} r_1 &:= k, & & r_2 := 1-k, \\ x_1 &:= x & \text{and} \quad & x_2 := y. \end{aligned}$$

(1) ⇒ (3):

Choose $k \in [0, 1]$ and $d \in L$. Let $x, y \in \mu_d$ then $\mu(x) \geq d$ and $\mu(y) \geq d$ thus

$$\mu(x) \wedge \mu(y) \geq d.$$

So from the convexity of μ we have

$$\mu(kx + (1-k)y) \geq \mu(x) \wedge \mu(y) \geq d.$$

Thus $kx + (1-k)y \in \mu_d$. i.e. μ_d is convex.

(3) $\Rightarrow$ (1):
Choose $k \in [0,1]$. Let $x, y \in E$ and let $d := \mu(x) \wedge \mu(y) \in L$. Then $x, y \in \mu_d$. By the convexity of μ_d we have

$$kx + (1-k)y \in \mu_d.$$

Hence

$$\mu(kx + (1-k)y) \geq d = \mu(x) \wedge \mu(y).$$

7.3.3 Proposition
Let E, F be vector spaces in $\mathbb{R}$ and let $f : E \longrightarrow F$ be a linear map. If μ is a convex L-fuzzy set in E, then $f[\mu]$ is a convex L-fuzzy set in F. Similarly, $f^{\leftarrow}[\nu]$ is a convex L-fuzzy set in E whenever ν is a convex L-fuzzy set in F.

Proof.
Let $k \in [0,1]$ and μ a convex L-fuzzy set in E. Then by Lemma 7.2.4 we have
$f[k\mu + (1-k)\mu] = f[k\mu] + f[(1-k)\mu]$
$= kf[\mu] + (1-k)f[\mu]$.
By Proposition 7.3.2 we have that $k\mu + (1-k)\mu \leq \mu$.
Now by Theorem 2.2.1 (10) we have $f[k\mu + (1-k)\mu] \leq f[\mu]$ which implies
$f[k\mu] + f[(1-k)\mu] \leq f[\mu]$ and thus

$$kf[\mu] + (1-k)f[\mu] \leq f[\mu].$$

So $f[\mu]$ is convex by Proposition 7.3.2.

Now assume that ν is a convex L-fuzzy set in F and let $k \in [0,1]$. Set $M = kf^{\leftarrow}[\nu] + (1-k)f^{\leftarrow}[\nu]$

Then $f(M) = f[kf^{\leftarrow}[\nu] + (1-k)f^{\leftarrow}[\nu]]$
$= f[kf^{\leftarrow}[\nu]] + f[(1-k)f^{\leftarrow}[\nu]]$ (by Lemma 7.2.4 (2))
$= kf[f^{\leftarrow}[\nu]] + (1-k)f[f^{\leftarrow}[\nu]]$ (by Lemma 7.2.4 (1))
$\leq k\nu + (1-k)\nu$ (by Theorem 2.2.1 (11))
$\leq \nu$ (by Proposition 7.3.2)

Now by Theorem 2.2.1 (6) we have $f^{\leftarrow}[f[M]] \leq f^{\leftarrow}[\nu]$ and hence by Theorem 2.2.1 (12) we have $M \leq f^{\leftarrow}[\nu]$.

7.3.4 Proposition
If μ, ν are convex L-fuzzy sets in E, then $\mu + \nu$ is a convex L-fuzzy set in E.

Proof.
Let μ, ν be convex L-fuzzy sets. Let $x, y \in E$ and choose $k \in [0,1]$. Then

$$(\mu + \nu)(kx + (1-k)y) = \bigvee_{z_1 + z_2 = kx + (1-k)y} \{\mu(z_1) \wedge \nu(z_2)\}$$

If $x_1 + x_2 = x$ and $y_1 + y_2 = y$ for $x_1, x_2, y_1, y_2 \in E$ then
$(kx_1 + (1-k)y_1) + (kx_2 + (1-k)y_2) = kx + (1-k)y$ and

$$(\mu + \nu)(kx + (1-k)y) \geq \bigvee_{x_1 + x_2 = x, y_1 + y_2 = y} \{\mu(kx_1 + (1-k)y_1) \wedge \nu(kx_2 + (1-k)y_2)\}$$

$$\geq \bigvee_{x_1+x_2=x, y_1+y_2=y} \{(\mu(x_1) \wedge \mu(y_1)) \wedge ((\nu(x_2) \wedge \nu(y_2))\}$$

(since μ and ν are convex)

$$\geq \bigvee_{x_1+x_2=x, y_1+y_2=y} \{(\mu(x_1) \wedge \nu(x_2)) \wedge ((\mu(y_1) \wedge \nu(y_2))\}$$

$$\geq \bigvee_{x_1+x_2=x} \bigvee_{y_1+y_2=y} \{(\mu(x_1) \wedge \nu(x_2)) \wedge ((\mu(y_1) \wedge \nu(y_2))\}$$

(from Theorem 2.1.2 (1))

$$\geq \bigvee_{x_1+x_2=x} \{\mu(x_1) \wedge \nu(x_2)\} \wedge \bigvee_{y_1+y_2=y} \{\mu(y_1) \wedge \nu(y_2)\}$$

(since L is a frame)

$$= (\mu + \nu)(x) \wedge (\mu + \nu)(y).$$

7.3.5 Definition

An L-fuzzy set μ on E is *balanced* if $\mu(kx) \geq \mu(x)$ whenever $x \in E$, $|k| \leq 1$.

Remark: Let μ be balanced and crisp. So $\mu = 1_A$ for some $A \subseteq E$. Let $x \in E$ and $k \in \mathbb{R}$ such that $|k| \leq 1$. Consider the case where $k \neq 0$. We then have

$1_A(x) \geq k1_A(x)$

$\quad \geq 1_A(\frac{x}{k})$.

So

$$1_A(\frac{x}{k}) = 1 \Rightarrow 1_A(x) = 1.$$

Thus $\frac{x}{k} \in A \Rightarrow x \in A$ and hence $x \in kA \Rightarrow x \in A$. Therefore $kA \subseteq A$. We thus have A is balanced in the classical sense.

7.3.6 Proposition

An L-fuzzy set μ is balanced $\Rightarrow \mu(0) = \bigvee_{x \in E} \mu(x)$.

Proof.

Choose $x \in E$ then

$$\mu(0) = \mu(0 \cdot x) \geq \mu(x).$$

Since x is arbitrarily chosen, the proof is complete.

7.3.7 Proposition

Let μ be an L-fuzzy set in E then the following three assertions are equivalent:

(1) μ is balanced.
(2) $\forall k \in \mathbb{R}$ such that $|k| \leq 1$ we have $k\mu \leq \mu$.
(3) $\forall d \in L, \mu_d$ is balanced.

Proof.

(1) $\Rightarrow$ (2):

(i) Let $x \in E$ and let $k \in [-1, 1]$ be such that $k \neq 0$ then we have

$$\frac{1}{k}\mu(x) = \mu(kx) \geq \mu(x)$$

and by Lemma 7.2.3 we have $k\frac{1}{k}\mu(x) \geq k\mu(x)$ and therefore

$$k\mu(x) \leq \mu(x).$$

(ii) If, on the other hand, we have that $k = 0$ and $x \neq 0$ then we have that $0 \cdot \mu(x) = 0$ and hence

$$0 \cdot \mu(x) \leq \mu(x).$$

(iii) Lastly, if $k = 0$ and $x = 0$ then $0 \cdot \mu(0) = \sup \mu = \mu(0)$ by Proposition 7.3.6. Therefore we have (2).

(2) $\Rightarrow$ (3):
Choose $d \in L$. Choose $k \in \mathbb{R}$ such that $|k| \leq 1$ and choose $x \in \mu_d$. Then $k\mu \leq \mu$ by (2).

(i) If $k \neq 0$ then

$$k\mu(x) \leq \mu(x) \Rightarrow \mu(\frac{x}{k}) \leq \mu(x).$$

So $\mu(\frac{x}{k}) \geq d \Rightarrow \mu(x) \geq d$ and thus $\frac{x}{k} \in \mu_d \Rightarrow x \in \mu_d$. i.e $x \in k\mu_d \Rightarrow x \in \mu_d$

(ii) If $k = 0$ and $x = 0$ then (2) trivially

$$0 \in \mu_d \Rightarrow 0 \in 0\mu_d = \{0\}.$$

(iii) If $k = 0$ and $x \neq 0$ then

$$x \in 0\mu_d = \{0\} \Rightarrow x = 0$$

a contradiction.

From (i), (ii) and (iii) we have that μ_d is balanced.

(3) $\Rightarrow$ (1):
Choose $k \in \mathbb{R}$ such that $|k| \leq 1$ and let $d \in L$. We have that

$$x \in \mu_d \Rightarrow kx \in \mu_d.$$

Thus $\mu(x) \geq d \Rightarrow \mu(kx) \geq d$ and hence $\mu(x) \leq \mu(kx)$.

7.3.8 Proposition
Let E, F be vector spaces in $\mathbb{R}$ and let $f : E \longrightarrow F$ be a linear map. If μ is a balanced L-fuzzy set in E, then $f[\mu]$ is a balanced L-fuzzy set in F. Similarly, $f^{\leftarrow}[\nu]$ is a balanced L-fuzzy set in E whenever ν is a balanced L-fuzzy set in F.

Proof.
Choose $k \in \mathbb{R}$ such that $|k| \leq 1$ and let $\mu \in L^E$ be balanced. We have by Proposition 7.3.7 that $k\mu \leq \mu$. Now by Theorem 2.2.1 (10) we have $f[k\mu] \leq f[\mu] \Leftrightarrow kf[\mu] \leq f[\mu]$ (from Lemma 7.2.4 (1)).
Now by Proposition 7.3.7 we have that $f[\mu]$ is balanced in F.

Let $\nu \in L^F$ be balanced and choose $k \in \mathbb{R}$ such that $|k| \leq 1$. By Proposition 7.3.7 we have $k\nu \leq \nu$ and by Theorem 2.2.1 (6) we have $f^{\leftarrow}[k\nu] \leq f^{\leftarrow}[\nu]$. Now if $k \neq 0$ then for $x \in X$,

$$\begin{aligned} f^{\leftarrow}[k\nu](x) \leq f^{\leftarrow}[\nu](x) &\Leftrightarrow k\nu(f(x)) \leq \nu(f(x)) \\ &\Leftrightarrow \nu(\tfrac{1}{k}f(x)) \leq \nu(f(x)) \\ &\Leftrightarrow \nu(f(\tfrac{x}{k})) \leq \nu(f(x)) \\ &\Leftrightarrow f^{\leftarrow}[\nu](\tfrac{x}{k}) \leq f^{\leftarrow}[\nu](x) \\ &\Leftrightarrow kf^{\leftarrow}[\nu](x) \leq f^{\leftarrow}[\nu](x). \end{aligned}$$

If, on the other hand, $k = 0$ then

$$0 \cdot f^{\leftarrow}[\nu](x) = 0 \cdot \nu(f(x)) = \begin{cases} \sup \nu & \text{if } f(x) = 0 \\ 0 & \text{if } f(x) \neq 0 \end{cases}$$

We have from Proposition 7.3.6 that $\nu(0) = \sup \nu$ and as

$$f^{\leftarrow}[\nu](x) = \nu(f(x)) = \begin{cases} \nu(0) & \text{if } f(x) = 0 \\ \nu(f(x)) & \text{if } f(x) \neq 0 \end{cases}$$

we have $0 \cdot f^{\leftarrow}[\nu] \leq f^{\leftarrow}[\nu]$.

7.3.9 Proposition

If μ, ν are balanced L-fuzzy sets in E, then $\mu + \nu$ is a balanced L-fuzzy set in E.

Proof.
Let μ, ν be balanced and choose $k \in \mathbb{R}$ such that $|k| \leq 1$. Choose $x \in E$. Now

$$(\mu + \nu)(x) = \bigvee_{x_1 + x_2 = x} \{\mu(x_1) \wedge \nu(x_2)\}$$

$$\leq \bigvee_{x_1 + x_2 = x} \{\mu(kx_1) \wedge \nu(kx_2)\}$$

(since μ is balanced)

$$\leq \bigvee_{kx_1 + kx_2 = kx} \{\mu(kx_1) \wedge \nu(kx_2)\}$$

(since $x_1 + x_2 = x \Rightarrow kx_1 + kx_2 = kx$ for $x_1, x_2 \in E$)

$$\leq \bigvee_{z_1 + z_2 = kx} \{\mu(z_1) \wedge \nu(z_2)\} = (\mu + \nu)(kx).$$

7.3.10 Proposition

If $(\mu_j)_{j \in J}$ is a family of convex (balanced) L-fuzzy sets in E, then $\mu = \bigwedge_{j \in J} \mu_j$ is a convex (balanced) L-fuzzy set in E.

Proof.
Let $d \in L$ then

$$\mu_d = \{x \in E : \mu(x) \geq d\} = \bigcap_{j \in J} \{x \in E : \mu_j(x) \geq d\}.$$

Since the intersection of ordinary convex (balanced) L-subsets of E is convex (balanced), the result follows from Propositions 7.3.2 and 7.3.7.

7.3.11 Definition

An L-fuzzy set μ on E is *absorbing* if $\bigvee_{t>0} t\mu = 1_E$.

Remark:
Let μ be crisp and absorbing. That is $\mu = 1_A$ for some $A \subseteq E$. Unlike the notions of convexity and balancedness, the notion absorbing does not reduce to the classical notion. This is illustrated by he following example.

7.3.12 Example

Consider the set $\mathbb{R} \times \mathbb{R}$ with the usual product L-topology. Let

$$A := \{(0,0)\} \bigcup \{(x, y) \in \mathbb{R} \times \mathbb{R} : 1 \leq x^2 + y^2 \leq 2\}.$$

Now

$$\bigvee_{t>0} t1_A = 1$$

but for $x = (1,1)$ $\nexists q \in \mathbb{R}$, $q > 0$ such that $\forall s \in \mathbb{R}$, $|s| < q$, $sx \in A$ and hence A is not absorbing in the classical sense.

Note: μ absorbing $\Rightarrow \sup_{t>0} t\mu(0) = \mu(0) = 1$.

7.3.13 Proposition

Let $f : E \longrightarrow F$ be a linear map for E, F real vector spaces and μ an absorbing L-fuzzy set in F. Then $f^{\leftarrow}[\mu]$ is an absorbing L-fuzzy set in E.

Proof.

Let $x \in E$.

$$\begin{aligned} tf^{\leftarrow}[\mu](x) &= f^{\leftarrow}[\mu](\tfrac{x}{t}) \\ &= \mu(f(\tfrac{x}{t})) \\ &= \mu(\tfrac{1}{t}f(x)) \\ &= t\mu(f(x)). \end{aligned}$$

So

$$\bigvee_{t>0} tf^{\leftarrow}[\mu](x) = \bigvee_{t>0} t\mu(f(x)) = 1.$$

as μ is absorbing.

7.3.14 Proposition (∗)

Let $\mu \in I^E$ be convex, balanced and absorbing. Then μ^0 is convex, balanced and absorbing.

Proof.

(i)

Choose $k \in [0, 1]$ and choose $x, y \in \mu^0$. Then

$$\mu(x) > 0 \text{ and } \mu(y) > 0$$

We have from the convexity of μ that

$$\mu(kx + (1-k)y) \geq \mu(x) \wedge \mu(y) > 0.$$

So $kx + (1-k)y \in \mu^0$.

(ii)

Choose $m \in \mathbb{R}$ such that $|m| \leq 1$. Let $x \in \mu^0$. As μ is balanced so we have $\mu(mx) \geq \mu(x) > 0$ thus $\mu(x) > 0 \Rightarrow \mu(mx) > 0$. So

$$x \in \mu^0 \Rightarrow mx \in \mu^0$$

and hence

$$\frac{1}{m}x \in \mu^0 \Rightarrow m\frac{1}{m}x \in \mu^0$$

That is

$$\frac{1}{m}x \in \mu^0 \Rightarrow x \in \mu^0$$

or equivalently

$$x \in m\mu^0 \Rightarrow x \in \mu^0.$$

So μ^0 is balanced.

(iii)

Choose $x \in \mu^0$. Since μ is absorbing we have that $(\bigvee_{t>0} \mu)(tx) = 1$. There exists $q \in \mathbb{R}$, $q > 0$ such that $\mu(qx) > 0$. If this were not the case then we would have $\bigvee_{t>0} t\mu(x) = 0$, a contradiction. Choose $s \in \mathbb{R}$ such that $|s| \leq q$. Since $|s| \leq q$ we have that $|\frac{s}{q}| \leq 1$. Because μ is balanced we have that $\mu(\frac{s}{q}x) \geq \mu(x) \Leftrightarrow \mu(\frac{s}{q}x) \geq \mu(\frac{q}{q}x)$

$$\Leftrightarrow q\mu(sx) \geq q\mu(qx)$$
$$\Leftrightarrow \tfrac{1}{q}q\mu(sx) \geq \tfrac{1}{q}q\mu(qx)$$
$$\Leftrightarrow \mu(sx) \geq \mu(qx) > 0$$

So $sx \in \mu^0$.
Therefore μ^0 is absorbing.

7.4 L-Fuzzy Topological Vector Spaces and L-Normed Spaces

We now extend Katsaras's definition of a fuzzy topological vector space (fuzzy linear space) to the L-topological situation. The following three definitions are adapted from their fuzzy analogues given in [25].

The fuzzy norm and fuzzy seminorm were first formulated by Katsaras in [27] and we now present his motivation and definitions. If p is a seminorm on a vector space E, then the set $V = \{x : p(x) < 1\}$ is convex, balanced, absorbing and the family $\{tV : t > 0\}$ is a base at zero for a linear topology. Further, p is a norm iff $\bigcap_{t>0} tV = \{0\}$. Conversely, if W is a balanced, convex, absorbing subset of E, then the Minkowski functional p of W,

$$p(x) = \inf\{t > 0 : x \in tW\}$$

is a seminorm on E. We also have

$$\{x : p(x) < 1\} \subseteq W \subseteq \{x : p(x) \leq 1\}$$

so we have that the linear topology generated by p coincides with the linear topology which has as a base at zero the family $\{tW : t > 0\}$. This leads us to the following definition:

7.4.1 Definition ([27])
A convex, balanced and absorbing $\rho \in L^E$ is called an *L-fuzzy seminorm* on E. If in addition $\forall x \neq 0, \inf_{t>0} t\rho(x) = 0$, ρ is called an *L-fuzzy norm* (*L-norm*).

An *L-seminormed space* is a pair (E, ρ), E a vector space, ρ an L-seminorm on E. An *L-normed space* is a pair (E, ρ), E a vector space, ρ an L-norm on E.

7.4.2 Definition
Given $x \in E$ and $\mu \in L^E$ then $x + \mu \in L^E$ is defined as $\forall y \in E, (x + \mu)(y) = \mu(y - x)$.

7.4.3 Definition ($*$)
A *linear I-topology* on a vector space E over $\mathbb{R}$ is an I-topology (containing all the constant L-sets) such that the two mappings

$$+ : E \times E \longrightarrow E, \qquad (x, y) \longmapsto x + y,$$

$$\cdot : \mathbb{R} \times E \longrightarrow E, \qquad (t, y) \longmapsto ty,$$

are continuous when $\mathbb{R}$ is equipped with the $\omega(\tau_{ord})$, the I-topology generated (in the sense of Lowen [37]) by the usual topology of $\mathbb{R}$ and $\mathbb{R} \times E$, $E \times E$ have the corresponding product I-topologies.

A vector space E with a linear I-topology is called an *I-fuzzy topological vector space* (*I-fuzzy topological linear space*).

7.4.4 Definition ($*$)
A collection B of fuzzy sets in E is a *base at zero* for a fuzzy linear topology if the collection

$$N_0 = \{\mu \in I^E : \exists \nu \in B, \mu \geq \nu, \mu(0) = \nu(0)\}$$

is a collection of neighbourhoods of zero for a fuzzy linear topology. This means that for each $\mu \in N_0$, there exists a $\sigma \in \tau$ such that $\sigma \leq \mu$ and $\sigma(0) = \mu(0)$ (Warren [62]).
For each $x \in E$, we define N_x the collection of all neighbourhoods of x in the following way:

$$N_x = \{x + \mu : \mu \in N_0\}.$$

These collections generate an I-topology in the following way:

$\sigma \in I^E$ is open iff $\forall x \in E$, $\sigma(x) > 0$ implies $\exists \mu \in N_x$ such that $\mu \leq \sigma$ and $\mu(x) = \sigma(x)$ (Warren [62]).

7.4.5 Theorem ([26], ∗)

Let B be a family of balanced fuzzy sets in E. Then B is a base at zero for a fuzzy linear topology iff B satisfies the following conditions:

(1) For each $\mu \in B$, $\mu(0) > 0$.
(2) For each non-zero constant fuzzy set c in E and and $l \in (0, c)$ there exists $\mu \in B$ with $\mu \leq c$ and $\mu(0) > l$.
(3) If $\mu_1, \mu_2 \in B$ and $l \in (0, \mu_1(0) \wedge \mu_2(0))$ then there exists $\mu \in B$ with $\mu \leq \mu_1 \wedge \mu_2$ and $\mu(0) > l$.
(4) If $\mu \in B$ and $t \in \mathbb{R}$, $t \neq 0$ then for each $l \in (0, \mu(0))$ there exists $\mu_1 \in B$ with $\mu_1 \leq t\mu$ and $\mu_1(0) > l$.
(5) Let $\mu \in B$ and let $l \in (0, \mu(0))$. Then there exists $\mu_1 \in B$ such that $\mu_1(0) > l$ and $\mu_1 + \mu_1 \leq \mu$.
(6) Let $\mu \in B$ and $x_0 \in E$. If $l \in (0, \mu(0))$ then there exists a positive number s such that for all $t \in \mathbb{R}$ such that $|t| \leq s$ we have $\mu(tx_0) > l$.
(7) For each $\mu \in B$ there exists a fuzzy set μ_1 in E with $\mu_1 \leq \mu$ and $\mu_1(0) = \mu(0)$ and such that for each $x_0 \in E$ for which $\mu_1(x_0) > 0$ and each n such that $0 < n < \mu_1(x_0)$ there exists $\sigma \in B$ with $\sigma \leq -x_0 + \mu_1$ and $\sigma(0) > n$.

7.4.6 Theorem ([27], ∗)

If ρ is an I-fuzzy seminorm on E, then the family

$$B_\rho = \{l \wedge (t\rho) : t > 0, l \in (0, 1]\}$$

is a base at zero for a fuzzy linear topology τ_ρ.

Proof.
It is trivial to show that the elements of B_ρ are balanced. We need only show that B_ρ satisfies conditions (1) - (7) of Theorem 7.4.5.

(1)
Let $\mu \in B_\rho$. Then $\mu = l \wedge t\rho$ for some $l \in (0, 1]$ and $0 < t \in \mathbb{R}$. Now $t\rho(0) = 1$ since ρ is absorbing.
Hence we have $\mu(0) = (l \wedge t\rho)(0)$
$= l \wedge t\rho(0)$
$= l > 0.$

(2)
Let c be a non-zero constant fuzzy set in E and let $l \in (0, c)$. Let $m := \frac{l+c}{2} < c$ then $\mu := m \wedge \rho$ is the function we need, since $\mu = m \wedge \rho \leq m \leq c$ and $\mu(0) = c \wedge \rho(0) = c \wedge 1$ (since μ is absorbing) and thus $\mu(0) = c > l$.

(3)
Let $\mu_1, \mu_2 \in B_\rho$ and let l be such that $0 < l < \mu_1(0) \wedge \mu_2(0)$. Then $\mu_1 = m \wedge t\rho$ and $\mu_2 = n \wedge s\rho$ where $m, n \in (0, 1]$ and $0 < s, t \in \mathbb{R}$. Choose q such that $l < q < \mu_1(0) \wedge \mu_2(0)$. Choose r such that $r \leq s, t$. Now let $\mu := q \wedge (r\rho)$. We have now that $|\frac{r}{s}| < 1$ and because ρ is balanced we have $(\frac{r}{s})\rho \leq \rho$ and so $s(\frac{r}{s})\rho \leq s\rho$, i.e. $r\rho \leq s\rho$. Similarly $r\rho \leq t\rho$. So we have

$$\mu = q\rho \leq (m \wedge t\rho) \wedge (n \wedge s\rho) = \mu_1 \wedge \mu_1.$$

Also $\mu(0) = q \wedge r\rho(0) = q \wedge \rho(0) = q \wedge 1 = q$.

(4)
Let $\mu \in B_\rho$ and $t \in \mathbb{R}$, $t \neq 0$ and choose $l \in (0, \mu(0))$. We have that $\mu = m \wedge s\rho$ for some $m \in (0,1]$ and $0 < s \in \mathbb{R}$. Now $t\mu = t(m \wedge s\rho) = tm \wedge (st)\rho = m \wedge (st)\rho \in B_\rho$. Let $\mu_1 := t\mu$. Then $\mu_1 \leq t\mu$. Also $\mu_1(0) = t\mu(0) = \mu(0) = m > l$.

(5)
Let $\mu \in B_\rho$. We have that $\mu = m \wedge s\rho$ for some $m \in (0,1]$ and $0 < s \in \mathbb{R}$. Let $l \in (0, \mu(0))$. Now let $s_1 = \frac{s}{2}$ and let $\mu_1 = m \wedge s_1\rho$. Choose any $x \in E$. Then we have:

$$(\mu_1 + \mu_1)(x) = \bigvee_{x_1+x_2=x} (m \wedge s_1\rho(x_1)) \wedge (m \wedge s_1\rho(x_2))$$

$$= \bigvee_{x_1+x_2=x} m \wedge (s_1\rho(x_1) \wedge s_1\rho(x_2))$$

$$= \bigvee_{y \in E} m \wedge (s_1\rho(y) \wedge s_1\rho(x-y))$$

$$= \bigvee_{y \in E} m \wedge (\frac{s}{2}\rho(y) \wedge \frac{s}{2}\rho(x-y))$$

$$= \bigvee_{y \in E} m \wedge (\rho(\frac{2y}{s}) \wedge \rho(\frac{2x-2y}{s}))$$

$$\leq \bigvee_{y \in E} m \wedge \rho((\frac{1}{2})(\frac{2y}{s}) + (\frac{1}{2})(\frac{2x-2y}{s}))$$

(from the convexity of ρ)

$$= \bigvee_{y \in E} m \wedge \rho(\frac{y+x-y}{s})$$

$$= \bigvee_{y \in E} m \wedge \rho(\frac{x}{s})$$

$$= m \wedge s\rho(x) = \mu(x).$$

Also $\mu_1(0) = m \wedge \frac{s}{2}\rho(0) = m \wedge \rho(0) = m \wedge 1 = m = \mu(0) > l$.

(6)
Let $\mu \in B_\rho$. We have that $\mu = m \wedge r\rho$ for some $m \in (0,1]$ and $0 < r \in \mathbb{R}$. Let $x_0 \in E$ and let $l \in (0, \mu(0))$.

$$\bigvee_{t>0} t\rho(x_0) = \bigvee_{t>0} \rho(tx_0) = 1$$

(ρ is absorbing). We thus have that there exists $s \in R$, $s > 0$ such that $\rho(sx_0) > l$. Choose t such that $|t| \leq s$. Then $|\frac{t}{s}| \leq 1$ and since ρ is balanced we have

$$\begin{aligned}
\rho(\tfrac{t}{s}x_0) \geq \rho(x_0) &\Leftrightarrow s\rho(tx_0) \geq \rho(x_0) \\
&\Leftrightarrow (\tfrac{1}{s})s\rho(tx_0) \geq \rho(x_0) \\
&\Leftrightarrow (\tfrac{1}{s})s\rho(tx_0) \geq \tfrac{1}{s}\rho(x_0) \\
&\Leftrightarrow \rho(tx_0) \geq \rho(sx_0) > l.
\end{aligned}$$

(7)
Let $\mu = l \wedge (t\rho) \in B_\rho$ with $l \in (0,1]$ and $0 < t \in \mathbb{R}$. For $x \in E$ define $\tilde{\rho} : E \longrightarrow \mathbb{R}$ by

$$\tilde{\rho}(x) = \bigvee_{s>1} \rho(sx)$$

and take $\mu_1 = l \wedge (t\tilde{\rho})$. For each $m \in (0,1)$ we have $m\rho \leq \tilde{\rho} \leq \rho$. Further, $\mu_1 \leq \mu$ and $\mu_1(0) = l = \mu(0)$. Choose $x_0 \in E$ with $\mu_1(x_0) > 0$ and choose $n \in \mathbb{R}$ such that $0 < n < \mu_1(x_0)$. Choose n_1 such that $n < n_1 < \mu_1(x_0)$. Since $(t\tilde{\rho})(x_0) = \tilde{\rho}(\frac{x_0}{t}) > n_1$, there exists $s_0 > 1$ such that $\rho(s_0 x_0) > n_1$. Choose $s \in \mathbb{R}$ such that $1 < s < s_0$. Then

$$\tilde{\rho}(\frac{sx_0}{t}) \geq \rho(\frac{s_0 x_0}{t}) > n_1$$

and so $\mu_1(\frac{sx_0}{t}) > n_1$. Since μ_1 is convex, with $q = \frac{1}{s}$, we have

$$\begin{aligned}
\mu_1(x + x_0) &= \mu_1(q(sx_0) + (1-q)(\frac{x}{1-q})) \\
&\geq \mu_1(sx_0) \wedge \mu_1(\frac{x}{1-q}) \\
&\geq n_1 \wedge l \wedge \tilde{\rho}(\frac{x}{t(1-q)}) \\
&\geq n_1 \wedge \rho(\frac{x}{m})
\end{aligned}$$

if $0 < m < t(1-q)$. Therefore

$$\sigma = n_1 \wedge (m\rho) \leq -x_0 + \mu_1 \text{ and } \sigma(0) = n_1 > n.$$

Now, by Definition 7.4.4, N_0 the collection of all neighbourhoods of 0 is defined in in the following way:

$$N_0 = \langle B_\rho \rangle = \{\mu \in I^E : \exists \nu \in B_\rho, \mu \geq \nu, \mu(0) = \nu(0)\}.$$

Hence we can state the following:

7.4.7 Corollary

An I-seminormed space (and hence an I-normed space) is an I-fuzzy topological vector space.

Chapter 8

A Fuzzy Hahn-Banach Theorem

8.1 Introduction

This chapter is motivated by the famous Hahn-Banach theorem in classical functional analysis (see [8]):

8.1.1 Theorem (Hahn-Banach)
For E a real vector space, p a seminorm on E, M a linear subspace of E and f a linear functional defined on M such that $\forall m \in M, |f(m)| \leq p(m)$ then there exists a linear functional g on E such that $\forall x \in E, |g(x)| \leq p(x)$ and $g = f$ on M.

This extremely important result has several forms and is, indeed, equivalent to the Axiom of Choice. Katsaras introduced a meaningful idea of a fuzzy seminorm in [27] and thus the stage was set for a fuzzy version of the 'Crown Jewel of Functional Analysis'. Gil Seob Rhie and In Ah Hwang fuzzified the theorem in [48].

8.2 A Fuzzy Version of the Hahn-Banach Theorem

Before we reach the statement and proof of the [48] version of the theorem, it is necessary to establish a few preliminary notions. For this section we will again be considering the vector space E over the field $\mathbb{R}$ of real numbers.
The Hahn-Banach theorem given in [48] does not hold in the general L-fuzzy situation so from now on we shall be working only in the I-fuzzy situation.

The following definition was yielded by Krishna and Sarma in [31] in their discussion on how to generate a fuzzy vector topology from an ordinary vector topology on a vector space.

8.2.1 Lemma ([31])
If ρ is an I-fuzzy seminorm on a linear space E (see Definition 7.4.1), then for each $l \in (0, 1)$,

$$P_l(x) = \bigwedge_{t\rho(x)>l} \{t > 0\} \qquad (\in \mathbb{R}_+)$$

gives an ordinary seminorm on E. This seminorm is called the *induced seminorm.*

Proof.
Let ρ be an I-fuzzy norm, let $l \in (0, 1)$ and $a \in \mathbb{R}$.

(i)

For $x \in E$ we have that $P_l(x) \geq 0$ since the infimum of a collection of positive real numbers is non-negative.

(ii)

if $a \neq 0$:

Choose any $x \in E$. Then
$$\begin{aligned} P_l(ax) &= \textstyle\bigwedge_{t\rho(ax)>l}\{t>0\} \\ &= \textstyle\bigwedge_{(at)\rho(ax)>l}\{at>0\} \\ &= \textstyle\bigwedge_{\frac{a}{a}t\rho(x)>l}\{at>0\} \\ &= \textstyle\bigwedge_{t\rho(x)>l}\{at>0\}. \end{aligned}$$

Now if $a > 0$ then $t > 0$ and hence

$$\begin{aligned} P_l(ax) &= a\textstyle\bigwedge_{t\rho(x)>l}\{t>0\} \\ &= |a|\textstyle\bigwedge_{t\rho(ax)>l}\{t>0\} = |a|P_l(x). \end{aligned}$$

If $a < 0$ then $t < 0$ and thus
$$\begin{aligned} P_l(ax) &= -a\textstyle\bigwedge_{t\rho(x)>l}\{-t : t<0\} \\ &= |a|\textstyle\bigwedge_{t\rho(x)>l}\{-t : t<0\} \\ &= |a|\textstyle\bigwedge_{t\rho(x)>l}\{t : t>0\} \\ &= |a|P_l(x). \end{aligned}$$

if $a = 0$:

$|0|P_l(x) = 0P_l(x) = 0$ and
$$\begin{aligned} P_l(0x) &= \textstyle\bigwedge_{t\rho(0)>l}\{t>0\} \\ &= \textstyle\bigwedge_{\rho(0)>l}\{t>0\} \\ &= \textstyle\bigwedge_{1>l}\{t>0\} \qquad \text{(since } \rho \text{ is absorbing)} \\ &= 0. \end{aligned}$$

(iii)

Let $x, y \in E$. Define
$$A(x,l) = \{t : t > 0, t\rho(x) > l\}.$$
We shall now show that $A(x,l) + A(y,l) \subseteq A(x+y,l)$:

Choose $t \in A(x,l)$ and $s \in A(y,l)$. Then we have

$$t\rho(x) > l \text{ and } s\rho(y) > l.$$

Also
$$\begin{aligned} (t+s)\rho(x+y) = \rho(\frac{x+y}{t+s}) &= \rho((\frac{t}{t+s})(\frac{x}{t}) + (\frac{s}{t+s})(\frac{y}{s})) \\ &\geq \rho(\frac{x}{t}) \wedge \rho(\frac{y}{s}) \end{aligned}$$

(from the convexity of ρ).

Thus
$$[(t+s)\rho](x+y) \geq t\rho(x) \wedge s\rho(y) > l.$$
Hence $(t+s) \in A(x+y,l)$. i.e.

$$A(x,l) + A(y,l) \subseteq A(x+y,l).$$

Therefore we have
$$\bigwedge\{A(x,l) + A(y,l)\} \geq \bigwedge A(x+y,l),$$
i.e.
$$\bigwedge A(x,l) + \bigwedge A(y,l) \geq \bigwedge A(x+y,l),$$
which is precisely the triangle inequality:

$$P_l(x+y) \leq P_l(x) + P_l(y).$$

Unless otherwise indicated, the rest of this section is from the work of Gil Seob Rhie and In Ah Hwang, [48].

8.2.2 Lemma

The function $P : E \longrightarrow \mathbb{R}_+$ defined by

$$P(x) = \bigwedge_{l\in(0,1)} \{P_l(x)\}$$

is a seminorm on E.

Proof.

(i)

$P(x) \geq 0$ since $\forall l \in (0, 1)$, $P_l(x) \geq 0$. (see previous lemma)

(ii)

For $a \in \mathbb{R}$ and $x \in E$ we have
$$\begin{aligned} P(ax) &= \textstyle\bigwedge_{l\in(0,1)}\{P_l(ax)\} \\ &= \textstyle\bigwedge_{l\in(0,1)}\{|a|P_l(x)\} \\ &= |a| \textstyle\bigwedge_{l\in(0,1)}\{P_l(x)\} \\ &= |a|P(x). \end{aligned}$$

(iii)

We will now show that $\forall x, y \in E$,

$$P(x + y) \leq P(x) + P(y).$$

Let $x, y \in E$. Since $\{P_l\}$ is increasing in l, we have that $\forall x \in E$,

$$P(x) = \bigwedge_{l\in(0,1)} \{P_l(x)\} = \lim_{l\longrightarrow 0} P_l(x).$$

Thus

$$\begin{aligned} P(x + y) &= \textstyle\bigwedge_{l\in(0,1)}\{P_l(x + y)\} \\ &\leq \textstyle\bigwedge_{l\in(0,1)}\{P_l(x) + P_l(y)\} \\ &= \lim_{l\longrightarrow 0} P_l(x) + \lim_{l\longrightarrow 0} P_l(y) \\ &= P(x) + P(y). \end{aligned}$$

8.2.3 Theorem

Let ρ_1, ρ_2 be I-fuzzy seminorms and let P_l^1, P_l^2 be induced ordinary seminorms, respectively. If $\forall x \in E, \rho_1(x) \leq \rho_2(x)$ then $\forall x \in E, \forall l \in (0, 1)$,

$$P_l^1(x) \geq P_l^2(x).$$

Proof.

Since $\forall x \in E, \rho_1(x) \leq \rho_2(x)$ we have from Lemma 7.2.3 (2) that $\forall t \in \mathbb{R}, \forall x \in E, t\rho_1(x) \leq t\rho_2(x)$. Let $l \in (0, 1)$ and $x \in E$ be fixed. Since $t\rho_1(x) > l$ implies $t\rho_2(x) > l$,

$$\{t > 0 : t\rho_1(x) > l\} \subseteq \{s > 0 : s\rho_2(x) > l\}.$$

Hence, $\bigwedge_{t\rho_1(x)>l}\{t > 0\} \geq \bigwedge_{s\rho_2(x)>l}\{s > 0\}$, equivalently $P_l^1(x) \geq P_l^2(x)$. This completes the proof.

8.2.4 Definition (The $*$-property)

Let ρ be an I-seminorm on a linear space E. ρ is said to have the *$*$-property* if for every $x \in E$,

$$\rho(x) = \bigwedge_{0<t<1} \rho(tx).$$

8.2.5 Lemma

Let ρ be an I-seminorm on a linear space E with the $*$-property. If $x_0 \in E$ and $\rho(x_0) < l < 1$, then $P_l(x_0) > 1$.

Proof.

Consider $t \in \mathbb{R}$ such that $0 < t < 1$. Because ρ is balanced we have $t\rho(x_0) \leq \rho(x_0) < l$. This implies

$$t, 1 \notin \{t : t > 0, t\rho(x_0) > l\}$$

and because $1 > t$ we have that $t \neq \bigwedge_{t\rho(x_0)>l}\{t > 0\}$.

Thus $P_l(x_0) = \bigwedge_{t\rho(x_0)>l}\{t > 0\} \geq 1$, we will now show that $P_l(x_0) \neq 1$. For this, let $P_l(x_0) = 1$. then $\forall t > 1, t\rho(x_0) > l$. Since ρ has the $*$-property,

$$\begin{aligned} \rho(x_0) &= \textstyle\bigwedge_{0<s<1}\{\rho(sx_0)\} \\ &= \textstyle\bigwedge_{0<s<1}\{\frac{1}{s}\rho(x_0)\} \\ &= \textstyle\bigwedge_{t>1}\{t\rho(x_0)\} \qquad \text{(since } 0 < s < 1 \Leftrightarrow t = \tfrac{1}{s} > 1\text{)} \\ &\geq l \end{aligned}$$

which contradicts the fact $l > \rho(x_0)$. Therefore $P_l(x_0) > 1$. This completes the proof.

8.2.6 Theorem

Let ρ_1 and ρ_2 be two fuzzy seminorms on a linear space E and ρ_2 have the $*$-property. If $\forall l \in (0, 1), \forall x \in E, P_l^1(x) \geq P_l^2$, then $\forall x \in E, \rho_1(x) \leq \rho_2(x)$.

Proof.

Suppose that there exists a $y \in E$ such that $\rho_2(y) < \rho_1(y)$. Let l be such that $\rho_2(y) \leq l \leq \rho_1(y))$. Then we have

$$P_l^1(y) = \bigwedge_{t\rho_1(y)>l}\{t > 0\} \leq 1$$

since if we choose $t \in \mathbb{R}$ such that $t > 1$ then we have that $0 < \frac{1}{t} < 1$ and because ρ_1 is balanced we have $t\rho_1(y) = \rho((\frac{1}{t})y) \geq \rho_1(y)$ and hence $t \neq \bigwedge_{t\rho_1(y)>l}\{t > 0\}$.

We also have

$$P_l^2(y) = \bigwedge_{t\rho_2(y)>l}\{t > 0\} \geq 1.$$

since if we choose t such that $0 < t < 1$ then beacause ρ_2 is balanced we have $t\rho_2(y) \leq \rho_2(y) \leq l$ and hence $t, 1 \notin \{t > 0 : t\rho_2(y) > l\}$. Thus $t \neq \bigwedge_{t\rho_2(y)>l}\{t > 0\}$.

Since $P_l^2(y) > 1$ by Lemma 8.2.5, $P_l^2(y) > P_l^1(y)$ which contradicts the fact that $\forall l \in (0, 1), \forall x \in E$,

$$P_l^1(x) \geq P_l^2(x).$$

Therefore $\forall x \in E, \rho_1(x) \leq \rho_2(x)$. This completes the proof.

We are now in a position to state and prove Gil Seob Rhie and In Ah Hwangs' fuzzification of an analytical form of the Hahn-Banach Theorem.

8.2.7 Theorem

Let E be a linear space over $\mathbb{R}$, let ρ be an fuzzy seminorm on E, and let $M \subseteq E$ be a linear subspace. If f is a linear functional on M such that $1_{B_f} \geq \rho$ on M, then there exists a linear functional g on E such that

(1) $\forall x \in M, f(x) = g(x)$;
(2) $1_{B_g} \geq \rho$ on E;

where $B_f = \{x \in M : |f(x)| \leq 1\}, B_g = \{x \in E : |g(x)| \leq 1\}$.

Proof.
Let $1_{B_f} = \rho_1$. Then $\forall x \in M, l \in (0,1)$,

$$\begin{aligned}
P_l^1(x) &= \textstyle\bigwedge_{t\rho_1(x)>l}\{t > 0\} \\
&= \textstyle\bigwedge_{\rho_1(\frac{x}{t})>l}\{t > 0\} \\
&= \textstyle\bigwedge_{\rho_1(\frac{x}{t})=1}\{t > 0\} && \text{(as } \rho_1 = 1_{B_f}) \\
&= \textstyle\bigwedge_{|f(\frac{x}{t})|\leq 1}\{t > 0\} && \text{(as } \tfrac{x}{t} \in B_f) \\
&= \textstyle\bigwedge_{|f(x)|\leq t}\{t > 0\} \\
&= |f(x)|.
\end{aligned}$$

So, by Theorem 8.2.3, $\forall l \in (0,1), \forall x \in M, |f(x)| \leq P_l(x)$, where $\forall x \in E$,

$$P_l(x) = \bigwedge_{t\rho(x)>l}\{t > 0\}$$

Now by Lemma 8.2.2 we have $\forall x \in M$,

$$|f(x)| \leq P(x) = \bigwedge_{l\in(0,1)}\{P_l(x)\}.$$

Therefore, by the classical Hahn-Banach Theorem, there exists a linear functional g on E such that

(1) $\forall x \in M, g(x) = f(x)$,
(2) $\forall x \in E, |g(x)| \leq P(x)$.

Let $1_{B_g} = \rho_2$. Then for all $x \in E, a \in (0,1)$,

$$\begin{aligned}
P_a^2(x) &= \textstyle\bigwedge_{s\rho_2(x)>a}\{s > 0\} \\
&= \textstyle\bigwedge_{\rho_2(\frac{x}{s})>a}\{s > 0\} \\
&= \textstyle\bigwedge_{\rho_2(\frac{x}{s})=1}\{s > 0\} && \text{(as } \rho_2 = 1_{B_g}) \\
&= \textstyle\bigwedge_{|g(\frac{x}{s})|\leq 1}\{s > 0\} && \text{(as } \tfrac{x}{s} \in B_g) \\
&= \textstyle\bigwedge_{|g(x)|\leq s}\{s > 0\} \\
&= |g(x)|.
\end{aligned}$$

Thus $\forall a \in (0,1), \forall x \in E$,

$$P_a^2(x) \leq P(x) = \bigwedge_{l\in(0,1)}\{P_l(x)\}$$

and hence $\forall l \in (0,1), \forall x \in E$,

$$P_l^2(x) \leq P_l(x).$$

Since 1_{B_g} has the $*$-property, $1_{B_g} \geq \rho$ by Theorem 8.2.6.

For further reading with regard to the Hahn-Banach Theorem, the reader is referred to [3, 6, 47].

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