INVESTIGATING CONCEPTUAL TEACHING OF WORD PROBLEMS THROUGH VISUALISATION PROCESSES: A CASE OF SELECTED GRADE 9 MATHEMATICS TEACHERS

A thesis submitted in fulfilment of the requirements for the degree of:

MASTERS IN EDUCATION
(MATHEMATICS EDUCATION)

OF

RHODES UNIVERSITY

BY

SSENNYOMO BERNARD JOHN

JULY 2020

ABSTRACT

This study investigated how selected Grade 9 mathematics teachers used visualisation strategies to conceptually teach Pythagoras' theorem word problems as a result of an intervention program. This research project is an integral component of the ViProMaths project whose goal is to research the effective use of visualisation strategies in the mathematics classroom in the Southern African region. This case study of mathematics teachers in the John Taolo Gaetsewe (JTG) District in the Northern Cape Province, used a social constructivist theory. The study is grounded within an interpretive paradigm and used an explanatory sequential mixed methods design. It surveyed eighty-seven mathematics teachers from all the secondary schools in the JTG District, focusing on their experiences when using visualisation approaches to teach Pythagoras' theorem word problems. After the survey, three teachers purposively selected from three different schools in the district took part in the intervention programme. I used a survey questionnaire, classroom observations and interviews to collect the data from the study participants. The data were analysed quantitatively and qualitatively.

The findings from survey data revealed that while teachers in the JTG District acknowledge the importance and value of visualisation strategies in the teaching of Pythagoras' theorem word problems, many of them are either using visuals minimally or not using them at all. Challenges such as lack of resources in schools, time constrains, a lack of support from subject advisers, among others were noted. This thus necessitated a need for an intervention with some teachers in the district, specifically focusing on the use of visualisation tools and skills to teach Pythagoras' theorem word problems for conceptual understanding. Lessons observations showed that all observed teachers used visual models to generate images and used the models to develop mathematical ideas. The teachers used the images to create platforms for classroom discussions. The discussions were driven by questions which teachers asked both for ascertaining learners' prior knowledge and for finding out if they understood what the teachers were teaching. Interviews revealed that teachers' perceptions had changed because they now know how to make and use different models to build on learners' prior knowledge, extend what they are teaching to real life and make sense of Pythagoras' proposition in multiple ways and to establish connections among a rich set of mathematics concepts when teaching Pythagoras' theorem word problems.

This study concludes that the use of visual strategies has the potential of enhancing conceptual teaching of Pythagoras' theorem word problems. The implications for teachers are that learners need to be taught how to create visual representations (both internally and externally) of the relations between objects in a word problem as this will help them in maximising understanding. Furthermore, it is hoped that the results of this study could be used by various stake holders who include inter alia, mathematics subject advisers and teacher training institutions to improve the teaching of Pythagoras' theorem word problems.

AKNOWLEDGEMENTS

I would firstly like to thank the almighty God for making it possible for me to complete this research study. Without you by my side, Father, I would not have come this far. Secondly, I would equally like to extend my profound gratitude to my supervisors Dr Clemence Chikiwa and Professor Marc Schäfer whose guidance and professional support have molded this research study. You are indeed a fountain from which one can drink the luxury of intellectual enthusiasm.

I am also indebted to the following people without whose support I could not have accomplished this project.

- The Northern Cape Head of Department of Education, thank you for permitting me to conduct this research in your province.
- Ms Rabotsho, thank you for your aid in getting the questionnaires to and from the teachers.
- Ms Mafara, I am very grateful for your unconditional support during that time of difficulty.
- My research participants, thank you very much for taking part in this study. This project is a success because of you.
- Mrs Jean Schäfer, thank you for proof reading and editing my work.
- NRF SARChi, thank you so much for the financial assistance.
- My wife Refilwe, and children, Michelle and Rochelle, thank you for bearing my absence from home when I was away for my studies. I also thank you for creating a conducive reading environment for me at home.
- Mr Motseotata Thuto McMillan whose computer skills I could tap into anytime whenever my computer skills let me down.

DEDICATION

I dedicate this thesis to my late parents: my father Lawrence Mugendawala and my mother Nakizaye Teddy. Thank you for instilling in me a culture of hard work and giving me a strong and sound educational background. I will live to take your wish to fruition.

DECLARATION OF ORIGINALITY

I Ssennyomo Bernard John, student number (12s0011), hereby declare that this thesis entitled "Investigating conceptual teaching of word problems through visualisation processes: A case of selected Grade 9 mathematics teachers" is my own work and a product of my research. It has not been submitted in any form to another institution. Where I have drawn on the ideas of people from other publications or other sources, I have fully acknowledged these in accordance with Rhodes University, Education Department reference guide.

Ssennyomo Bernard John Date: 13 July, 2020

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ACRONYMS USED IN THIS STUDY

ANA	Annual National Assessment
CAPS	Curriculum and Assessment Policy Statement
DBE	Department of Basic Education
DOE	Department of Education
ELL	English Language Learner
FET	Further Education and Training
HOTS	Higher-Order Thinking Skills
ICT	Information and Communication Technology
JTG	John Taolo Gaetsewe District
LoLT	Language of Learning and Teaching
MKO	More knowledgeable other
MPS	Mathematical Problem Solving
MTBL	Mother Tongue-Based language
MWP	Mathematics word problem
MWPS	Mathematical word problem solving
NCS	National Curriculum Statement
NCTM	National Council of Teachers of Mathematics
NRC	National Research Council
NSC	National Senior Certificate
PGCE	Post Graduate Certificate in Education
PISA	Programme for International Student Assessment
SACMEQ	Southern and Eastern Africa Consortium for Monitoring Education Quality
SAS	Side-Angle-Side
SP	Senior Phase
TIMSS	Trends in International Mathematics and Science Study
ViPProS	Visualisation Processes for Problem Solving

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CHAPTER ONE

CONTEXT OF THE RESEARCH

1.1 INTRODUCTION

The purpose of this study was to investigate how selected mathematics teachers used visualisation strategies to teach Grade 9 mathematics word problems in the context of Pythagoras' theorem for conceptual understanding as a result of an intervention programme. The chapter begins with the research context and background in which I discuss how this study came to be, and its relation to visualisation in mathematics education. This is followed by an outline of the research goals and research questions that guided this study. The chapter further provides a summary of the methodology followed, data analysis, the theoretical framework and the significance of the study. The limitations of this study are also listed and the chapter concludes with a short outline of the structure of the thesis providing a hint of what is discussed in each chapter.

1.2 BACKGROUND OF THE STUDY

This study focused on Grade 9 mathematics teachers from the JTG district in the Northern Cape Province. Prior to this study's intervention programme, it also surveyed eighty-seven other teachers from all the secondary schools in the same district focusing on how they were teaching Pythagoras' theorem word problems. According to the South African Department of Basic Education (DBE) (2011), the National Curriculum Statement (NCS) grades R-12 encourages a learner-centred approach to learning. Central to this is that learning in the classroom should be collaborative, active and critical with the teacher facilitating this process. Therefore, the mathematics curriculum is premised on a constructive perspective which is key to this study.

In the mathematics curriculum, a number of reasons necessitate the inclusion of word problems. One of these is that such problems mirror real life situations (van Garderen & Montague, 2003; Ahmad, Tarmizi & Nawawi, 2010). It is also argued by the DBE (2011) that mathematical word problem solving (MWPS) enables us to understand the world around us and teaches us to think creatively. While MWPS has many benefits, literature reveals that many learners all over the world express difficulties in handling mathematics word problems (MWPs) (Fatmanissa & Kusnandi, 2017). International studies such as Trends in International Mathematics and Science Study (TIMSS) can be used as a reference in determining learners' achievement in solving mathematics test items that measure higher order thinking skills (HOTS) — in this case, MWPs (Hadi, Retnawati, Munadi, Apino & Wulanadari, 2018). As with other lower performing countries, only one-third of South African Grade 9 learners that participated in TIMSS 2015 achieved a mathematics score above the minimum level of competency (400 points).

Solving MWPs was identified as one of the major areas where learners faced great difficulty (Reddy et al., 2016). Similar results from the Southern and Eastern Africa Consortium for monitoring Education Quality (SACMEQ III) project revealed that although there was an increase in the number of learners reaching acceptable mathematics skills (Level 4 and above) from 24.1% in 2000 to 30.9% in 2007 in South Africa, the proportion who reached this level is still alarmingly low at 30.9% (Moloi & Chetty, 2010). Moloi and Chetty (2010) further noted that no province in South Africa had more than 5% of learners reaching Level 8 (the abstract problem solving level) in both years. It is therefore evident from the SACMEQ III project that problem solving tasks (MWPs in this case) that require application of skills associated with higher SACMEQ levels still remain a challenge. Visualisation processes have however been found to aid understanding, reasoning and deeper conceptual understanding of mathematical concepts (Konyalioglu, Konyalioglu & Isik, 2008). In South Africa, one of the aims of the NCS for Grades R-12 is to produce learners that are able to solve problems efficiently and communicate using visual, symbolic and/or language skills (DBE, 2011). Visualisation is thus considered a key factor in mathematics teaching and learning in South Africa.

Reports that include the NSC diagnostic reports of 2017 and 2018 note that while the number of grade 12 candidates who correctly answered routine questions increased, performance would further be improved if attention was given to areas such as exposing learners to complex and problem solving type questions (from earlier grades) which comprise, among other questions, of word problems that are contextualised (based on everyday life situations) (DBE, 2017; 2018). South Africa's mathematics Curriculum and Assessment Policy Statement (CAPS) allocates 30% of the whole curriculum to problem solving (DBE, 2011). With the emphasis given to problem solving in the curriculum by the DBE, matching it with what is happening at implementation level that is in the classroom, teaching of mathematics word problems (MWPs) needs to be researched.

Having taught high school mathematics for many years now, I have observed that learners struggle with MWPS across secondary school grades. This is due to a couple of reasons, one of which is learners' insufficient opportunities to engage in the exploration of word problems. This is supported by Sembiring, Hadi, and Dolk (2008) who assert that some teachers are less innovative than others and their teaching style makes it difficult for learners to learn mathematics with understanding. From the Annual National Assessment (ANA) diagnostic report of 2014, the use of ineffective teaching methods was also cited as one of the key causes of the challenges experienced by learners in certain mathematics topics that include MWPs (DBE 2014). I thus consider this topic as one of the areas that requires urgent attention. This study therefore argues that there is an urgent need to explore the pedagogical approaches used by teachers in the teaching of MWPs in this case. It is expected that this research will assist in gaining a sense of the

challenges caused by these methods. I hence submit that there is a need to research effective pedagogies that can enhance mathematics teaching in the South African classroom. Visualisation processes are considered herewith as some of the approaches that can enhance conceptual understanding in MWPS. This study thus focused on understanding the teaching of word problems on Pythagoras' theorem using visual approaches and was guided by the following research goals and questions:

1.3 RESEARCH GOALS

The overall goal of this study was to explore how visualisation strategies can be used by selected mathematics teachers participating in an intervention programme to teach Pythagoras' theorem word problems for conceptual understanding to Grade 9 learners.

1.4 RESEARCH QUESTIONS

1. How do grade 9 mathematics teachers in a selected district teach Pythagoras theorem and word problems in this context prior to an intervention programme?
2. How do selected teachers use visuals to enhance conceptual teaching of Pythagoras' theorem word problems after participating in an intervention programme?
3. What are the selected teachers' perceptions and experiences in using visual models to conceptually teach word problems in the context of Pythagoras theorem as a result of participating in an intervention programme?

1.5 RESEARCH METHODOOGY

The study is located within an interpretive paradigm. According to Maree (2016), interpretivists believe that reality is socially constructed and for this reason studying people in their social contexts or natural environment provides greater opportunity to understand the perceptions they have of their own activities. Therefore, the onus of the researcher in the interpretive paradigm is to comprehend, articulate and demystify social reality through the eyes of different participants (Cohen, Manion, & Morrison, 2011). The study fundamentally made use of two methods. The first was a survey administered to all Senior Phase and Further Education and Training (FET) teachers in the JTG district, whilst the second method consisted of a case study of three selected Grade 9 mathematics teachers that participated in the intervention programme.

An explanatory sequential mixed methods design was used in this study. Data were collected through a survey questionnaire, lesson observations and stimulated recall interviews. This study which followed an interventionist approach in the JTG District, Northern Cape Province made use of visual models, approaches and strategies that offered teachers concrete demonstrations of, for example, how squares and

areas are involved in proving Pythagoras' theorem and solving word problems. In this study, I explored three models (two area models and scaled-up versions of an original right-angled triangle) to teach Pythagoras' theorem word problems.

Eighty-seven teachers responded to a survey questionnaire that was administered in the district to establish how and to what extent teachers were using visual approaches when teaching Pythagoras' theorem word problems. The survey was followed by an intervention programme that involved three workshops with teachers in the district. The intervention focused on how to use visual models to solve Pythagoras' theorem word problems, followed by creating an awareness of promoting conceptual understanding in the teaching of mathematics through visualisation processes. The programme took place in three cycles all coinciding within the second and third school terms of 2019.

After the workshops, three teachers were purposively selected and were observed presenting their lessons on Pythagoras' theorem to Grade 9 classes using the three visual models. Teaching was organised in three cycles with each cycle related to one of the models. Lesson observations were video-recorded, one lesson per cycle in each teacher's class. My focus was on how the teachers make use of the various resources provided in the intervention to develop visualisation and promote conceptual understanding when teaching word problems on Pythagoras' theorem.

Each video-taped lesson was followed by a reflective one-on-one stimulated recall interview (after each cycle) with the teacher to establish the participant's views and experiences on how and whether the use of visual models enhanced the teaching of MWPs on Pythagoras' theorem. At the end of all the cycles, a focus group interview was conducted with two teachers. The interviews were audio-recorded. The teachers collectively reflected on how visuals were used and whether they facilitated the teaching of MWPs.

1.6 DATA ANALYSIS

The data analysis process consisted of three stages. In stage 1, data from survey questionnaires were first analysed quantitatively using descriptive statistics. The data was later qualitatively analysed using the analytical tool developed from Kilpatrick et al.'s (2001) conceptual understanding indicators. My interest was to ascertain how teachers in the district teach Pythagoras' theorem and problem solving prior to the intervention.

In stage 2, I qualitatively analysed the data from lesson observations, with lesson analysis in this stage focusing on two parts: firstly, the focus was on the teaching of Pythagoras' theorem using the three visual models. The focus here was on how selected teachers used visuals to teach Pythagoras' theorem word

problems taking note of visualisation processes indicators adapted from Kossyln (1996) as well as those emerging from the lesson observations' data. Secondly, the focus was on how the use of visual models to teach Pythagoras' theorem word problems enhances learners' conceptual understanding. The focus was on how selected teachers made use of various visual resources to teach word problems in the context of Pythagoras' theorem taking note of Kilpatrick et al.'s (2001) conceptual understanding indicators.

In stage 3, a thematic analysis approach, defined as a method of identifying, analysing and reporting patterns within data (Braun & Clark, 2006) was used to identify and understand how the selected teachers made use of various visual resources provided in the intervention to develop conceptual understanding during their mathematical word problem teaching.

1.7 SIGNIFICANCE OF THE STUDY

In the South African education curriculum context, this research has the ability of informing curriculum developers and mathematics education researchers about incorporating visualisation approaches into the design of policy guidelines. This project did not only contribute to my knowledge about visualisation in mathematics education, but it also aided me as a mathematics educator to expand my knowledge on how to use effective instructional methods (such as visualisation strategies) in teaching mathematics word problems on Pythagoras' theorem.

It is hoped that the selected mathematics teachers will be empowered with different approaches involved in conceptually teaching word problems on Pythagoras' theorem using visualisation. Additionally, the results of this study could aid the Department of Higher Education and Training to integrate visual approaches in its teacher training programmes so as to adequately prepare teacher trainees to embrace the visual world in handling MWPs.

Lastly, the findings of this study may help authors of mathematics textbooks to include the use of visualisation (such as the area models and scaled-up version of a right-angled triangle) in their textbooks in order to enhance learners' conceptual understanding of Pythagoras' theorem word problems.

1.8 LIMITATIONS

The challenges I faced in my study involved getting the questionnaires to and from the teachers. Some teachers kept on postponing the retrieval date of the questionnaires. The fact that I am a full-time teacher made it difficult for me to distribute the questionnaires to all the schools in the district. However, I met with the mathematics subject adviser who informed me of a forthcoming workshop in the next few days. We took advantage of this workshop and got teachers to fill in the questionnaires, which I retrieved before

the workshop commenced. Some of the questionnaires were partly completed and in some, the teachers did not elaborate their answer.

The exploratory nature of my study does not allow the findings to be generalised. The study initially had three participants from three different schools who were purposively and conveniently selected. One teacher withdrew from the study and the number of participants reduced to two. This study was conducted over a short period of time focusing on a specific area of mathematics which is MWPS on Pythagoras' theorem. Therefore, the time to collect sufficient data was limited. Ideally, more time should have been spent on training teachers and exposing them to the full potential of the use of visualisation strategies to teach Pythagoras' theorem word problems. The outcomes could have been more comprehensive if more teachers used these three visual models (the two area models and the scaled-up versions of an original right-angled triangle) for a longer period of time.

In addition, I met a challenge of interviewing teachers after they had taught their lessons. The teachers always came up with excuses and this adversely affected my interview plan. However, despite this study's limitations, useful insights on the use of visualisation strategies in the teaching of Pythagoras' theorem word problems have been provided by this study.

1.9 OPERATIONAL DEFINITIONS

In this section, I define the key concepts as used in my study.

Word problems are mathematical exercises where information on the problem is presented as text rather than in mathematical notation (Jupri & Drijver, 2016).

Visualisation is the "ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings" (Arcavi, 2003, p. 217). The study focused on internal and external visualisation in the teaching of MWPs in order to enhance leaners' conceptual understanding. These are defined below as used in my study.

Internal visualisation, mental imagery or visualising is the ability, the process of creating, interpreting, using and reflecting upon pictures, images, diagrams in our minds (Arcavi, 2003; Kosslyn, 1996, Kosslyn, Thompson & Ganis, 2006). External visualisation are nonverbal physical representations of text content either constructed by a learner or a teacher (De Koning and van der Schoot, 2013).

Conceptual understanding as described by Kilpatrick et al. (2001) is an integrated and functional grasp of mathematical ideas.

1.10 THESIS OUTLINE

This thesis comprises of five chapters.

1.10.1 Chapter One (Context of the study)

This chapter provides a brief introduction of the thesis, the background to this study as well as operational definitions as used in this study.

1.10.2 Chapter Two (Literature review)

This chapter discusses the conceptual framework and literature that informs and shapes the analysis and interpretations of the results of this project. It begins by defining word problems. This is followed by discussing the reasons for teaching word problems in school mathematics, language and mathematics word problems, difficulty in teaching word problems in mathematics, and MWPS models. Thereafter, it presents the concept of visualisation, discusses visualisation processes, the role of visualisation in mathematics and the challenges in using visualisation. Pythagoras' theorem and visual models as visualisation tools for teaching this concept are also presented. The chapter concludes with a discussion of conceptual understanding and the social constructivism theory that underpins this study.

1.10.3 Chapter Three (Methodology)

This chapter provides an account of how the research was designed and carried out. It discusses the research orientation, research methods employed, research design, data collection tools and sampling techniques used in this study. It further describes issues pertaining to data analysis, validity and ethics.

1.10.4 Chapter Four (Data presentation, Analysis and Discussion)

The chapter presents and discusses the results of this study. It begins by presenting and analysing the data from the three research instruments (survey questionnaires, lesson observations, and interviews — stimulated recall and focus group) so as to produce findings in response to the study's three research questions above. The chapter then concludes with a summary of the research findings from the three research instruments. The analytical tools that were used to analyse the data that was adapted from the work of Kossyln (1996), and Kilpatrick et al. (2001), were also presented in this chapter.

1.10.5 Chapter Five (Conclusions and recommendations)

Chapter Five concludes the study by presenting a summary of the dominant themes, drawing conclusions from the findings and making recommendations and suggestions for future research. The chapter ends with my personal reflections as a result of this study.

CHAPTER TWO

LITERATURE REVIEW

2.1 INTRODUCTION

The aim of this chapter is to provide a critical review of the literature that is relevant to this study. The overall purpose of this study was to investigate how, as a result of an intervention program, selected mathematics teachers used visual approaches to teach Grade 9 word problems in the context of Pythagoras' theorem, with the aim of improving conceptual understanding. Cohen et al. (2011) articulate that in a broader perspective, the purpose of a literature review is to make researchers aware of the existing findings in a given field of study so that they do not rediscover knowledge that has already been reported. Additionally, a literature review can be used to identify lacunas in the field of research the researcher is engaged with, and he is then able to work towards closing them.

I begin this chapter by defining word problems. The chapter also discusses the reasons for teaching word problems in school mathematics, language and MWPs, difficulty in teaching MWPs, and MWPS models. It then presents the concept of visualisation, discusses visualisation processes and the challenges in using visualisation. Pythagoras' theorem and visual models for teaching this concept are also presented. The chapter concludes with a discussion of conceptual understanding and the social constructivist theory that underpins this study.

2.2 WORD PROBLEMS

""...Learning to solve word problems prepares students to use mathematics in real life. Teaching students to think logically about word problems is at the core of the professional responsibility of mathematics educators."" (Knifong & Burton, 1985, p. 13). With reference to solving mathematical word problems, teachers need to be aware that solving these problems follows a certain process, which they need to explicitly teach to learners.

2.2.1 Some definition of word problems

In South Africa's curriculum, word problems in mathematics are taught at all levels of school education. The definition and/or conceptualisation of MWPs is not always clear-cut. Word problems are tasks that involve mathematics concepts being set in a situation, often through the use of a "shallow story context" (Jonassen, 2003, p. 267). Word problems are also defined as story problems that require learners to relate the known and unknown (Ahmad, Tarmizi & Nawawi, 2010). For Pungut and Shahrill (2014), word problems are mathematics questions expressed in sentences and require precise understanding of the questions in order to know what mathematical concepts are needed to solve them. According to van Garderen and Montague (2003), word problems are challenging tasks set in realistic contexts that require

understanding, analysing, and interpreting. They are not simply computational tasks embedded in words, but they also require the selection of approaches that can lead one to a logical answer (Fatmanissa & Kusnandi, 2017). The phrase 'word problem' as will be used in this study will refer to any mathematical exercise where information on the problem is presented as text rather than in mathematical notation (Boonen, Van der Schoot, Van Wesel, De Vries & Jolles, 2013).

2.2.2 Why teach word problems in school mathematics

Word problems have always been an integral component of mathematics education and this is evidenced by their presence in early 4000-year-old Egyptian, Greek and Roman manuscripts (Verschaffel, Greer & De Corte, 2000). They are part of high-stakes assessments like the NSC examinations in South Africa and several international tests (such as the Programme for International Student Assessment (PISA) and TIMSS, which rely heavily on word problems to measure mathematics achievement across countries (Fatmanissa & Kusnandi, 2017). In the PISA mathematics framework, areas such as solving problems set in authentic contexts and formulating situations mathematically are expressed through word problems, and in TIMSS, solving problems in context is among the fields assessed (Fatmanissa & Kusnandi, 2017). Word problems are also found in the NSC examinations in South Africa.

MWPS is one of the crucial areas of problem solving that incorporates real life problems and applications (Ahmad, Tarmizi & Nawawi, 2010), thereby preparing learners to use mathematics in real life (Knifong & Burton, 1985). Zhang and Xin (2012) indicate that in real life, problem features are often ill-defined. That is, some problems may include extra or irrelevant information while others may need multiple steps to solve them. Zhang and Xin (2012) therefore argue that it is imperative to teach real life problems as this helps learners to gain competence in daily life and work. Equally, Jourdain and Sharma (2016) assert that using MWPs in classrooms enables learners to gain a deeper understanding of specific concepts through being applied in a real-life context.

Furthermore, in South Africa, one of the aims of the NCS for grades R-12 is to produce learners that are able to demonstrate an understanding of the world as a set of related systems, by recognising that MWPS contexts do not exist in isolation (DBE, 2011). Therefore, MWPS is largely acknowledged by the South African curriculum as an area which enables learners to understand the world around them and teaches them to think creatively.

Since MWPS requires learners to apply and to integrate many mathematical concepts and skills during the decision making and problem solving process (Tarzimah & Thamby, 2010), mathematical problem solvers are characterised as persons that are:

- flexible and fluent thinkers.

- confident in their use of knowledge and processes.
- willing to take on a challenge and persevere in their quest to make sense of a situation and solve a problem.
- curious, seek patterns and connections, and are reflective in their thinking (Rigelman, 2007, p.308).

Concurring with Anderson (2009), Rigelman (2007) explicates that being a good mathematical word problem solver leads to added merits outside the school environment. This because MWPS enables learners to identify whole or partial information from daily life problems and use strategies to solve these (Fatmanissa & Kusnandi, 2017). Hsiano, Lin, Chen, and Peng (2018) further urge that non-routine MWPS tasks allow learners to see the meaning and relevancy of what they are learning, which in turn facilitates the transfer of contextual knowledge to authentic situations.

It can be observed from literature that engaging learners in MWPS can aid them in relating what they are learning in the classroom to what they may do in their lives in the real world, thus making learning real and meaningful.

2.2.3 Language and mathematics word problems

In the schools where this study was conducted, English is the Language of Learning and Teaching (LoLT) and it is the second language for both the teachers and learners that participated in this research. This does not mean that these learners were at a disadvantage because English is not their first language. In fact, results from Yonson's (2017) study which was conducted in Philippines to ascertain the difference in performance of Grade 4 pupils when MWPs were presented in English and the Mother Tongue-Based language (MTBL) revealed that there was no significant difference in whether word problems were expressed in English language or MTBL. He also noted that the registered mean for the two groups were below proficiency level which may imply that both groups experienced difficulties in solving word problems regardless of whether the items were expressed in the English or in the MTBL. Findings further indicated that learners from the two groups were able to answer easier items and both groups found difficulty in problems that required a deeper understanding. Similarly, Bernado's (2002) study which sought to answer how language affects the bilingual learners' ability to model the structure of word problems in arithmetic ascertained that learners were equally successful in modelling the structure of problems in Filipino and in English whether their first language was Filipino or English. They were just as likely to succeed with the easy problems in both languages and equally likely to have more trouble with difficult problems in both languages. Thus, it could be concluded that language may have assisted in

comprehending word problems. Yonson (2017) however argues that although learners may be able to read and understand the MWP, they may lack the necessary skill in solving the problem.

MWPs are an entity of both mathematics and language. Bernardo (2002) articulates that "word problems have a clear linguistic component as the problem elements are embedded within a text, yet the quantitative elements of the problem can be conceived in abstract symbolic but non-linguistic forms" (p.284). Therefore, in dealing with these problems, teachers need to comprehend them not only as a part of mathematical test or task, but also as a language object that should be addressed from a perspective other than simply mathematical operations (Fatmanissa & Kusnandi, 2017). Since language plays an imperative role in understanding word problems in mathematics (Yonson, 2017), the mathematical language involved in these problems must be understood in order for one to find the correct solution to the problem (Wulandari, Lukito & Khabibah, 2018). Literature, however, reveals that certain groups of learners (English language learners (ELLs), learners with low and average mathematics levels, among others) fall behind when language is involved in MWPS (Abedi & Lord, 2001). Abedi and Lord (2001) add that modifying language demands in MWPs resulted in all these groups benefiting. Research therefore, attributes learners' challenges in handling MWPs to the deleterious effects of the linguistic processing required in solving these problems. Some of the linguistic challenges that word problems cause are presented in the following section.

2.2.4 Difficulty in teaching mathematical word problems

The human brain processes numbers in different areas depending on whether they are presented in symbolic and verbal form (Sousa, 2008). We encounter numbers in word form on a daily basis and the brain has to convert these to number form before they can be processed. This implies that an extra level of processing is required in solving MWPs. MWPs are challenging for many learners (Wulandari, Lukito & Khabibah, 2018; Kashefi, Alias, Kahar, Buhari & Zakaria, 2015; DBE, 2014; Csikós, Szitanyi & Kelemen, 2011) and many of them perform poorly on these problems than on comparable number problems (Fatmanissa & Kusnandi, 2017; Bernardo, 2002). Several studies reported that this discrepancy in performance suggests that mathematical skills do not appear to be the issue, but other contributing factors, including text comprehension problems, highlighting the difficulties predominately caused by linguistic aspects. Linguistic difficulties have variations which include: difficulties in comprehending what is given in the text; defining vocabulary and decoding words; identifying irrelevant information; construing the meaning of number operator symbol; identifying and understanding keywords; and understanding the written language context (Jourdain & Sharma, 2016; Fatmanissa & Kusnandi, 2017; Wulandari et al., 2018); thus placing extra cognitive demands on the learners (Meyer, 2014; Jourdain & Sharma, 2016;) during the problem solving process.

Jupri and Drijvers's (2016) research supports these findings. They attribute students' difficulties in handling MWPs to the mathematization perspective (see Section 2.2.5.6), which they categorised into horizontal and vertical mathematization. They define mathematization as the activity of transforming a realistic problem into a mathematical problem (that is horizontal mathematization), as well as solving the problem expressed in a model and interpreting the solution (that is vertical mathematization). Jupri and Drijvers's (2016) study was in the form of a teaching experiment that investigated fifty-one 7th grade Indonesian learners' difficulties in solving MWPs. Results indicated that the main obstacle encountered by these learners when dealing with word problems in a real life context, concerned horizontal mathematization. That is, most of them faced difficulty in understanding word problems, which was mainly caused by reading difficulties. This led to them making mistakes in formulating mathematical models such as equations, schematics, or diagrams. I concur with these authors and in my experience, I have noted that in solving word problems in the context of Pythagoras' theorem, learners just apply the formula without an understanding of what the problem is asking. An example is a problem such as: A ladder standing against a wall reaches a height of 12 m up the wall and has its foot 5 m away from it, calculate the length of the ladder. Csikós et al. (2011) however, accentuate that successful MWPS requires the understanding of relevant textual information and the ability to visualise the data. In this study, I adopted Konyalioglu et al.'s (2012) argument that the use of visualisation in mathematics problem solving (MWPS in this case) may support learners' comprehension and enable them to come up with multiple solution opportunities during the mathematization process. However, it is important to note that teachers need to be aware that visualisation is not self-explanatory: learners need to be explicitly instructed when it comes to mathematics problem solving situations (Presmeg, 2014). Since word problems are not given in a plain mathematical expression, they require complex steps to solve (Fatmanissa & Kusnandi, 2017). It is therefore, necessary for the teacher to teach and model to learners how to do this through the use of appropriate MWPS models.

2.2.5 Problem solving models

There are quite a number of problem solving processes that researchers have come up with and some of them are discussed below. The models that informed and were used in this study are: Polya's (1957) Mathematical Problem Solving process, Krulik and Rudnik's (1987) Mathematical Problem Solving Model, Reusser's (1996) Situation Model, Lucangeli, Tressoldi, and Cendron's (1998) Problem Solving Model, De Lange's (2006) Mathematical Problem Solving Framework, and Abdullah, Zakaria and Halim's (2012) strategic thinking approach. Many classification schemes of problem solving processes are derived from Polya's conception of mathematics problem solving (Wu & Adams, 2006). I will

therefore start by discussing Polya's work and then I will discuss in detail the other five models that informed this study in latter sections.

2.2.5.1 Polya's problem solving process

In his book "*How to solve it*", Polya (1957) presented four hierarchical principals that can be used in aiding learners to solve MWPs. The four phases are: i) understanding the problem; ii) devising a plan; iii) implementing the plan, and iv) looking back. Succinctly, understanding the problem requires the problem solver to read the problem and identify its principal parts, namely the unknown, the data and the condition. Devising a plan requires learners to come up with a strategy for solving the problem. In order to achieve this, they should be able to find the connection between the data and unknown. Implementing the plan requires learners to solve the problem using their plan and come up with a solution. Looking back requires learners to reconsider and examine the result and the path that led to it (Polya, 1957). However, when handling a MWP, a problem solver will realise that the four steps are not necessarily carried out in a linear manner. ''In practice, the phases are muddled up and are carried out in parallel, each new discovery tending to modify the overall plan" (Polya, 1973, p. *xix).* Below is a problem solving process chart illustrating Polya's four phases and the processes that take place in each phase (Florida DOE, 2010).

Figure 2.1: The four problem- solving steps (Florida DOE) (2010)

The four above-mentioned principals have become the framework often recommended for teaching and assessing skills of problem solving (ibid, 2010). The stage involving understanding the problem is often the most overlooked phase in the problem solving process (Florida DOE, 2010) yet this is essentially a higher form of step one in which a problem solver identifies what the problem is and represents it in a way that is easier to understand (Krulik & Rudnick, 1987). From figure 2.1 above, step one reveals numerous phases that can be used in understanding the problem during the problem solving process. For example, visualisation is acknowledged as a key strategy in this phase as it aids learners to obtain a clearer picture of the problem requirements. This was a key strategy in this study. Kashefi et al. (2015) also brings to our attention that when compared to novice mathematicians, experts frequently use visual

representations in building understanding of the problem stage and in planning a solution. Therefore, the use of visual representation is considered as a general technique in the mathematician's tool box (Polya, 1957).

The Florida DOE (2010) further emphasises strategies that can be used by teachers in supporting learners through the first problem solving step. These include:

- **Frayer vocabulary model**: is a concept map which enables learners to make relational connections with vocabulary words. The model entails identifying the concept/vocabulary word, defining the word in their own words, listing characteristics of the word, and listing or drawing pictures of examples and non-examples of the word.
- **Mnemonic devices**: are strategies that learners and teachers can create to aid learners recollect content. They are memory aids in which specific words are used to remember a concept. For example, letter strategies that include acronyms such as "BODMAS" are commonly used to help learners recall the order of operations in mathematics.
- **Graphic organisers**: are diagrammatic illustrations designed to assist learners in representing patterns, interpreting data, and analysing information relevant to problem solving.
- **Paraphrasing**: is a strategy designed to help learners in restating the mathematics problem in their own words, therefore strengthening their comprehension of the problem. It requires the problem solver to read the problem, underline or highlight key terms, restate the problem in his own words, and then write a numerical sentence.
- **Visualisation:** is the practice of creating pictorial representations of mathematical problems. Learners visualise and then draw the problem, thus obtaining a clearer understanding of what the problem is asking. It requires the teacher to have the learners read the problem, underline important images in the problem, draw a visual representation of the problem and then write a numerical sentence.
- **Cooperative learning**: requires learners to work in groups for a purpose assigned by the teacher. These activities allow learners who differed in achievement, gender, race, and/or ethnicity to work together and learn from each other.

The Florida DOE (2010) further accentuates that the teacher should decide which strategies to use, based on the problem solver's difficulties in comprehending the problem. For instance, if the learner understands the meaning of all the words in the problem, he does not need a vocabulary strategy, but if he cannot restate the problem, teaching him to paraphrase would be beneficial.

2.2.5.2 Krulik and Rudnik's problem solving model

Krulik and Rudnik (1987) explicate a five phase problem solving process which includes: (1) Reading and thinking, (2) Analysing and planning, (3) Selecting a strategy, (4) Solving the problem, (5) Reviewing and extending (confirming the answer). The first step, reading, is when one identifies the problem. He or she does this by noting keywords, asking oneself what is being asked in the problem, or restating the problem in the language that he can understand more easily. The second step, analysing and planning, is when the problem solver looks for patterns or attempts to identify the concept or principal at play within the problem. This is essentially a higher form of step one in which the problem solver identifies what the problem is and represents it in a way that is easier to understand. In this step, however, the problem solver is really asking, ""What is this problem like?"" He or she is connecting the new problem to prior knowledge. The problem solver might draw a diagram of what the situation would look like.

The third step, selecting a strategy is where one draws a conclusion or makes a hypothesis about how to solve the problem based on what he or she found in the first two steps. The fourth step is to solve the problem. Once the method has been selected, the learner applies it to the problem. The final step, reviewing and extending, is where the problem solver verifies his or her answer and looks for variations in the method of solving the problem.

2.2.5.3 Reusser's situation model

Reusser's (1996) model is an extension of Kintsch and Greeno's (1985) model. Kintsch and Greeno proposed a two-step procedural model for solving mathematical word problems. This model requires the problem solver to directly translate the text into an arithmetic problem model which could then be used to solve the word problem. Reusser (1996) however, felt that Kintsch and Greeno's model entailed jumping directly from the text base to the mathematical equation without understanding the situation described by a word problem. It can therefore be argued that Kintsch and Greeno's model may be appropriate for competent problem solvers or learners who have a deep understanding of various word problem types but is not an ideal model for low achieving learners.

Reusser (1996) added an intermediary step to Kintsch and Greeno's model, the non-mathematical situation model. This entails constructing a mental model of the situation described by the text, which can assist the problem solver in reawakening the mathematical knowledge that is required in finding the solution to the problem. Reusser stressed the need to give more emphasis to linguistic elements than just mathematical elements when handling MWPs. His model also puts more emphasis on the problem's

context (unlike the above models) and the imperative of comprehending the problem in its context — this was the case in my study.

2.2.5.4 Lucangeli, Tressoldi, and Cendron's problem solving model

Comprehension of relevant information embedded in the text and the capacity to have a good visual representation of the data are part of the five MWPS steps (comprehension of relevant information embedded in the text; the capacity to have a good visual representation of the data; the capacity to recognise the deep structure of the problem; the capacity to order correctly the steps to arrive at the solution and a good capacity to evaluate the procedure utilised in the solution) proposed by Lucangeli et al. (1998). Literature revealing visualising as an essential comprehension approach highlights the imperative of learners having vivid mental images of the situation described in the word problem. The quality of the constructed mental representation is crucial in understanding and solving the problem (De Koning & van der Schoot, 2013) because it is this model which unifies the information drawn from the text and creates a relationship between the data in the problem and the unknown data (Lucangeli et al., 1998). It can therefore be argued that the better the mental representation a problem solver creates, the more proficient he or she can be in generating a mathematical model and in handling the MWP. However, concurring with extant research, I argue that the stage entailing creating a mental representation is generally left for learners to perform on their own (Gardner, 1985) despite many of them lacking the training associated with this skill (Gates, 2018; Presmeg, 2006).

2.2.5.5 De Lange's problem solving framework

According to De Lange (2006), the mathematization process as carried out by a problem solver has a cyclic character as shown in Figure 2.2. He proposed a four step model that a problem solver goes through when solving mathematical word problems:

- Firstly, given a meaningful problem situated in real life, the problem solver starts the mathematization process by understanding the problem and identifying the relevant mathematical concepts within it.
- Secondly, based on the identified mathematical concepts, the problem solver trims away the irrelevant elements that exist in reality and formulates the problem into a mathematical model to describe the relevant relations involved. This model can be in form of an equation, schema, or diagram.
- Thirdly, the mathematical problem included in the model is solved and the problem solver reflects on the solution process.

- Lastly, the problem solver evaluates and interprets the mathematical solution in relation to the original, realistic situation. In other words, the question that arises is: What is the meaning of the mathematical solution in terms of the real world? Answering this question will require the learner to make connections to real life hence achieving conceptual understanding.

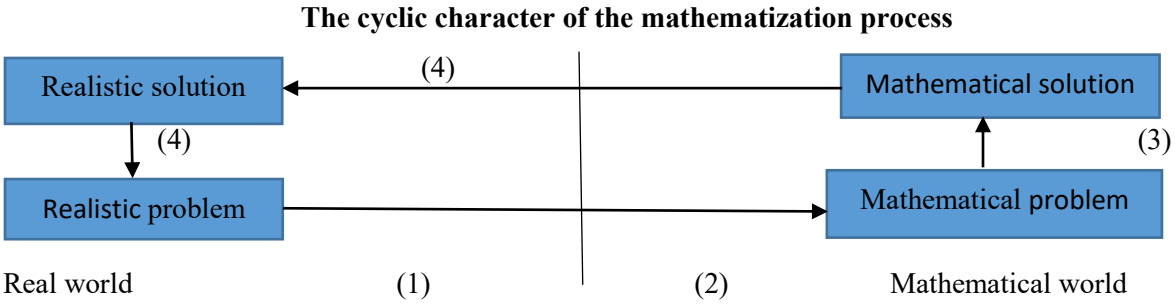


Figure 2.2: The mathematization cycle (De Lange, 2006)

The first two steps, according to Jupri and Drijvers (2016), refer to horizontal mathematization where a realistic problem is transformed into a mathematical problem. Step 3 describes vertical mathematization because it takes place within the symbolic mathematical world. The fourth step again concerns horizontal mathematization because it involves interpreting the mathematical solution in terms of a realistic solution. If the interpretation of the realistic solution in terms of the original realistic problem requires the learner to, for instance, verify all the conditions given in the problem using the obtained answer, then, vertical mathematization is involved (Jupri & Drijvers, 2016).

Jupri and Drijvers (2016) noted that one of the difficulties faced by learners when dealing with MWPs concerned formulating mathematical models evidenced by errors in formulating equations, schemas or diagrams. Jupri and Drijvers' (2016) model however does not indicate training learners to use these mathematical representations (visual ones in this case). I therefore, concur with Presmeg (2014) and argue that teachers need to be aware that visualisation is not self-explanatory; learners need to be taught how to create visual representations of the relations between objects in a problem and discouraging them from including irrelevant pictorial details. Jupri and Drijvers (2016) further highlighted that learners' difficulties in this mathematization subcategory is as a result of the prevailing conventional teaching approaches (heavily reliant on textbooks) where learners tend be exposed to more routine bare mathematical tasks than word problems, thus making it difficult for them to acquire adequate mathematization skills.

2.2.5.6 Abdullah, Zakaria and Halim's strategic thinking approach

Abdullah, Zakaria and Halim (2012) proposed an approach of thinking strategies through visual representation in MWPS. Their model consists of the following strategic thinking steps: (1) understanding

the problem, (2) planning the strategies, (3) implementing the strategies, and (4) checking the solution. These steps are aimed at facilitating learners' understanding and ability to build a process model in solving the given MWP in order to make sure that the plan is suitable to derive the correct answer. Figure 2.2 portrays Abdullah et al.'s (2012) MWPS approach and provides a brief description of what takes place in each stage.

<table>
<tr><th colspan="2">Steps</th><th>Activities Implementation Guide</th></tr>
<tr><td rowspan="2">Planning</td><td>1. Understanding the Problem</td><td>1.1 Reading and understanding the problem
Reading the given problem repeatedly until it is understood.
Underlining crucial information/focusing on the important points.
Identifying what is given and what is being sought.
1.2 Linking the problem with previous knowledge
Have you encountered this problem before?
How to solve it?
1.3 Picturing the problem with your own words or building your own visual representation.
Either in the form of figures/models/sketches to help solve complex problems. These visual representations aid in representing the interconnected values in the given problem.</td></tr>
<tr><td colspan="3">* Self-reflection: Have I understood the problem and am I ready to proceed to the next step?</td></tr>
<tr><td rowspan="2">Monitoring</td><td>2. Planning the Strategies</td><td>2.1 Planning and considering a few strategies that will be implemented
What are some of the suitable strategies?
2.2 Translating the information given into visual representation figures
Drawing and building figures/models.</td></tr>
<tr><td>3. Implementing the Strategies</td><td>3.1 Conducting calculations
Identifying the operations involved.
Always remembering the sequence of steps.
Determining the right time to proceed with the steps.
Choosing the next steps.
Verifying the strategies used to solve the problem.
Executing all processes involved.</td></tr>
<tr><td colspan="3">* Self-reflection: Have I known the methods to overcome the hindrance and identify probable mistakes?</td></tr>
<tr><td rowspan="2">Evaluation</td><td>4. Checking the Solution</td><td>4.1 Measuring the accuracy of procedures used
Rechecking the crucial information identified.
Considering other logical solutions.
4.2 Determining the efficiency of the planning and implementation carried out
Rechecking calculation.
Rereading the question and asking oneself whether I have answered it.
Determining whether the strategies and calculations are correct.</td></tr>
<tr><td colspan="3">* Self-reflection: Have I used the correct solution strategy and answered the question given?</td></tr>
</table>

Figure 2.3: Strategic thinking approach (Abdullah et al., 2012)

It is evident from the problem solving models that the use of visualisation is at the very heart of the MWPS process and the next section discusses what is known about visualisation in detail.

2.3 VISUALISATION IN MATHEMATICS

"As biological and as socio-cultural beings, we are encouraged and aspire to 'see' not only what comes 'within sight', but also what we are un able to see" (Arcavi, 2003, p.215-216). Therefore, "visualisation offers a method of seeing the unseen" (ibid., p. 216). For instance, visual representations of mathematical concepts can assist the problem solver in seeing and understanding the abstract features of these concepts. Arcavi (2003) also argues that during mathematical problem solving, visualisation enables a problem solver to engage with concepts and meanings which could be easily bypassed by the algorithmic solution of the problem.

2.3.1 Some definitions of visualisation

In the mathematics discipline, the terms representation and visualisation are used interchangeably. Visualisation (representation) is defined in different ways by different mathematics researchers. Zimmermann and Cunningham (1991) define mathematical visualisation as the process that entails forming images (mentally, or with pencil and paper, or with the aid of technology) and using these images effectively to aid mathematical discovery and understanding. Pape and Tchoshanov (2001) refer to visualisation as an act of externalising an internal, mental abstraction. Davis, Young and Mclaughlin (1982 referenced in Smith, 2003, p.266) refer to visualisation as a representation that describes "a combination of something written on paper, something existing in the form of physical objects and a carefully constructed arrangement of an idea in one's mind." For Presmeg (2006), visualisation is "the process of constructing and transforming both visual mental imagery and all the inscriptions of spatial nature that maybe implicated in doing mathematics" (p. 206). Therefore, visualisation is used to reason in mathematics and to think logically (Presmeg, 2006). Owen and Clements (1998) refer to visualisation as visual imagery that is seen to play a crucial role in establishing the meaning of a problem, choosing between different approaches of problem solving and influencing cognitive constructions of the problem solver.

Van Garderen (2006, p.496) brings to our attention that "several researchers examining the relationship between visualisation and mathematical problem solving performance have found either a weak or no relationship" in using visualisation to solve mathematics problems. An explanation for these inconsistent findings has to do with lack of consensus on the definitions used by various scholars (Van Garderen, 2006). For the purpose of this study, Arcavi's (2003) definition was key in supporting this project because

it acknowledges visualisation as a both a product and process in mathematics teaching and learning. Arcavi (2003) defines visualisation as

> the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings (p. 217).

According to Rosken and Rolka (2006), Arcavi's definition stresses that visualisation can be a powerful tool in mathematics learning as it facilitates exploration of mathematical problems (MWPs on Pythagoras' theorem in this case) and gives meaning to mathematical concepts and the relationship between them. They further enunciate that "visualisation allows for reducing complexity when dealing with a multitude of information" (p. 458). In my experience in the classroom, pictures speak more than words for those who prefer to think visually. This study with its intervention investigated how and whether the use of visualisation processes to teaching word problems in the context of Pythagoras' theorem enhanced learners' conceptual understanding.

2.3.2 Visualisation processes in mathematics

Various mathematics education scholars have categorised visualisation processes into external and internal visualisation (Zimmermann & Cunningham, 1991; Arcavi, 2003; Presmeg, 2006; De Koning & van der Schoot, 2013; Kashefi et al., 2015). While with external visualisation, images are formed with pen and paper or with technological tools, with internal visualisation processes images are constructed and processing is done mentally. The images from both processes are used to facilitate mathematics discovery and understanding.

Mathematics learners perceive visual representations as a useful tool and occasionally attempt to use them in solving MWPs (Stylianou, 2001; Pape & Tchoshanov, 2001; Premeg, 2006). However, they may face challenges in using these representations due to a lack of training associated with the skill (Gates, 2018; Presmeg, 2006). Mathematics educators need to be able to help the visualisers in their classes to overcome the difficulties and exploit the strengths associated with visualisation (Stylianou, 2001). Gates (2018), Hegarty and Kozhevnikov (1999) and Presmeg (2014) accentuate that teachers need to be aware that visualisation is not self-explanatory; learners need to be taught how to create visual representations of the relations between objects in a problem and discouraging them from including irrelevant pictorial details. This will thus aid them in maximising understanding. In support of this argument, Ho and Lowrie (2014) expound that classes with visual-spatial learners need to be explicitly instructed when it comes to MWPS situations. Van Garderen, Scheuermann and Poch (2014) further assert that teaching a problem solver to draw a diagram without educating him to think or getting him to imagine about the problem may

make the visual tool look useless. Therefore, a teacher plays a crucial role in developing the ability of the learner in representing a word problem (Presmeg, 1986).

Central to this study was how visualisation processes can enhance conceptual teaching of Pythagoras' theorem word problems. Through the use of these visual approaches, the constructivist teacher promotes learning that is meaningful with learners actively and constructively involved in their learning. Although all representational systems are crucial for the development of mathematical understanding (Presmeg, 1986; Pape & Tchoshanov, 2001) and various representation systems can be used to solve MWPs, the focus of this study was mainly on external and internal visualisation in the teaching of MWPs in order to enhance leaners' conceptual understanding.

2.3.2.1 Internal visualisation

Internal visualisation, mental imagery or visualising is the ability, the process of creating, interpreting, using and reflecting upon pictures, images, diagrams in our minds (Arcavi, 2003; Kosslyn, 1996, Kosslyn, Thompson & Ganis, 2006), and is an effective approach for honing comprehension (Csikós et al., 2011; De Koning & van der Schoot, 2013). According to Kashefi et al. (2015), internal representations are cognitive schemata or mathematical ideas or pictures developed by learners through their experience. The Singapore Ministry of Education (2001) echoes that visualising an object or situation entails "mentally manipulating various alternatives for solving a problem related to a situation or object without benefit of concrete manipulatives" (p. 51). Transferring this to a Pythagorean context, I argue that using visual models as visualisation tools to teach Pythagoras' theorem may offer learners an opportunity of creating various pictures of this concept in their minds. Therefore, when solving word problems in this context, learners can visualise different options, which they can mentally manipulate in handling these problems.

Kintsch and Greeno (1985) write that in order to solve word problems, the problem solver must first construct an internal representation of the problem, and build a mental model of the problem situation. Zimmermann and Cunningham (1991) and Bernardo (2002) concurr with Kintsch and Greeno (1985) and contend that problem solvers initially use mental images to assist them in understanding and making sense of mathematical ideas and concepts before they can solve the problem. "Since mathematical concepts and relations are often based on visual mental representations attached to verbal information, the ability to generate, retain, and manipulate abstract images is obviously important in MWPS" (Csikós et al.,2011, pp. 49-50). Therefore, successful MWPS requires the understanding of relevant textual information and the ability to visualise the data. Kosslyn (1996) proposed four classes of imagery abilities that facilitate visualisation approaches. According to Kosslyn (1996), the four cognitive processes, which

include image generation, image inspection, image transformation and/or image use, facilitate mathematical conceptual understanding and are hierarchical. That is, they are carried out in a linear manner as explained below:

- **Image generation** occurs when one constructs a picture in one's mind. The process entails mentally manipulating images/pictures in one's mind. Since a picture in one's mind cannot be seen by anybody, Jonassen and Henning (1999) contend that a model or picture is needed to externalise that representation. This implies that internal and external visualisation processes are interconnected. Image generation is viewed as the most crucial cognitive step in Kossyln's model because the other three processes rely on the image generated in this phase. In drawing images during the lesson, a teacher can recall a previously constructed or seen image. In this case the stored information is activated to guide this process. The type of image drawn will affect learners' understanding of the content being dealt with.
- **Image inspection** entails analysing the image in order to provide a detailed description about it. Since the image depicts spatial relations within the portion of an object or shape, the process of image inspection requires learners and the teacher to examine the image in order for them to interpret patterns and/or properties in that image. The process enables them to make links between the visual image and abstract mathematical concepts as they scrutinise patterns in the image.
- **Image transformation** involves re-arranging spatial representations in order to produce a new shape(s) that may be related or not. This may require transformations such as rotating or translating the image(s) in order to produce new shapes. In this case, the teacher's onus is to ensure that learners experience the different shapes in order for them to develop a deeper understanding of these shapes. In ensuring this, the teacher may for instance ask questions such as these: *What shape has been formed? How do you know that it is that particular shape?* In answering the latter question, learners can draw from the properties of a particular shape which they have experienced.
- **Image use** occurs when the image is used to emphasise and develop the appropriate properties of a particular shape or answer question about an image.

The imagery abilities by Kosslyn (1996) are considered in this study as visualisation processes' indicators. In this study, these indicators formed an integral part of my first analytical framework discussed in the data analysis section of the next chapter — see Section 3.8. This tool was used to analyse the data from lesson observations in Chapter Four.

De Koning and van der Schoot (2013) echo that creating a mental representation of the situation described in a word problem helps the readers to better understand the text, and when learners receive instruction on how to construct mental images during reading, the chances are higher that accurate mental representations describing the situations defined in the text will be constructed. Hibbing and Rankin-Erickson (2003) however, warned that most often visualising in the classrooms is done by teachers who engage in modelling their personal images for learners to observe, with hardly any emphasis on encouraging learners to create their own mental pictures. This in turn makes it difficult for learners to visualise text content themselves, thus creating difficulties in understanding. I concur with Hibbing and Rankin-Erickson (2003), and argue that from my teaching experience, I have observed that creating appropriate mental image processes is generally left up to the problem solvers and as a result of this, many of them fail to construct suitable representations that may assist them in solving the problem.

Visualising also aids the process of understanding during the reading by connecting information in the text and the problem solver's mental representation (Sadoski & Paivio, 2004). However, De Koning and van der Schoot (2013) argue that it is crucial to keep in mind that the final goal of instructing learners how to visualise text situations is to aid them towards the automatic construction of internal representations of events described in the text.

Solving word problems requires an adequate level of reading skills (Csikós et al., 2011). Readers who lack the ability to create internal images when reading often experience text comprehension problems (Hibbing and Rankin-Erickson, 2003). They attribute this inability to difficulty in decoding words, little background knowledge or limited vocabulary, rather than reading skills. However, Hibbing and Rankin-Erickson (2003) argue that if learners are unable to develop images because all their mental energy is devoted to decoding words or their personal experiences have limited their vocabulary and background knowledge, external visual images can be used to develop understanding.

2.3.2.2 External visualisation

De Koning and van der Schoot (2013) define external visualisation as nonverbal physical representations of text content either constructed by a learner or a teacher. According to Janvier, Girardon and Morand (1993), "representations such as numerals, algebraic equations, graphs, tables, diagrams, and charts are external manifestations of mathematical concepts that act as stimuli on the senses" (p. 81) and these aid one in comprehending the problem text (Zahner & Corter, 2007). However, this study was particularly interested in Pythagoras' theorem MWPS and the use of external representations that convey information in spatial terms, such as diagram and spatially organised displays, rather than in external inscriptions that use formal natural language, such as formulas and calculations. The former helps the problem solvers in

abstracting the problem text for important elements and aids them in building a problem schema to solve the problem (Zahner & Corter, 2010).

For Konyalioglu et al. (2012), visualisation is a bridge built between 'the world of experiments' and 'the world of thinking and reasoning'. They further articulate that visualisation presented in the form of drawings, shapes and concrete models ensure a stronger conceptual perception as drawings about abstract concepts lead to mental interpretation. Similarly, Polya (1957) and Arcavi (2003) concur that visual tools such as diagrams are valuable for teaching as these support a discovery approach to problem solving. This study was particularly interested in the teacher's ability to draw a diagram (or create and/or use other external visual representations, such as visual models) to represent a mathematical concept or problem and to use the diagram (or model) to achieve understanding, and as an aid in solving Pythagoras' theorem word problems.

Diezmann and English (2001), define a diagram as "a visual representation that displays information in a spatial layout" (p. 77). They are a way of creating a visible platform to aid the problem solver in self-monitoring, examining progress and increasing motivation during the problem solving process (van Garderen & Montague, 2003). Therefore, external visual representations (for example diagrams, and concrete models) that learners use or construct to enhance their understanding automatically generate a big picture in their minds to aid them in revealing the solution to the problem (Rosken & Rolka, 2006). Similarly, Pape and Tchoshanov (2001) argue that the development of learners' mathematical understanding and representational thinking requires a combination of multiple representations as well as the interaction of both internal and external representations. This implies that internal and external visualisation cannot be used in isolation.

2.3.3 Interplay between internal and external visualisation processes

According to Pape and Tchoshanov (2001), the development of a learner's representational thinking is a two-sided process that entails an interaction of internalisation of external representations and externalisation of mental images (see Figure 2.4). They further highlight that this interaction often takes place within social interaction and this was key for this study.

Furthermore, basing on Piaget and Inhelder (1971), Presmeg (2006) argues that "the position is taken that when a person creates a spatial arrangement, there's a visual image in the person's mind, guiding this creation" (p. 206). That is, some form of mental imagery must initially occur for an external representation to occur. Likewise, Van Meter, Aleksic, Schwartz and Garner (2006) and De Koning and van der Schoot (2013) stress that internal and external visualisation processes do not operate in isolation but are interconnected. It is clear from literature that there is an interaction between internal and external

representations in developing a learner's understanding of mathematical concepts. An example illustrating the interplay between internal and external representations is that of a four-year old child playing with numerous toys (Pape & Tchoshanov, 2001). As the child sets out the toys to play, she names the first as *one,* the second as *two,* and etcetera. The authors highlight that "these number words may simply be words the child has learned to utter as she counts each object in a series of objects" but she has no clue that "these words are symbols for the position of toys in the series the child is enumerating" (p.119). That is, in the beginning, the child may not understand that the last number named is the total number of toys. However, as the child grows older (at around five or six years of age), she begins to understand that the last number stated in this game is the number of toys that are in the set and finally there is a numeral that represents the number of elements in the set. Once learned, the numeral and its name (for example, *5* and *five*) become the external representations that are conventions for the internal abstraction, the number of elements in the set. Therefore, the number name, *five,* and the numeral, *5,* are external representations that act to stimulate an image, the internal representation, of a set of five objects (ibid.).

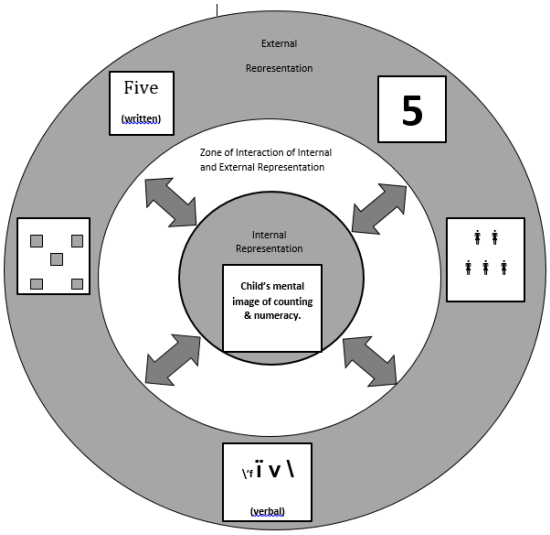


Figure 2.4: The interplay between internal and external representation (Pape & Tchoshanov, 2001)

2.3.4 Role of visualisation in mathematics

Gates (2018) brings to our attention that school mathematics depends heavily on linguistic and textual representation and communication at the expense of visuo-spatial representation. He further stresses that although numerous studies have indicated that visualisation plays a vital role in mathematics teaching and learning, many schools fail to embrace the importance of the visual nature of communication and its implications for learning mathematics.

Arcavi (2003) views visualisation as a powerful tool that plays three crucial roles in mathematical problem solving: Firstly, visualisation can act as a support and illustration of essentially symbolic results and possibly provide proof in its own right. Secondly, visualisation provides a possible way of resolving conflict between (correct) symbolic solutions and (incorrect) intuitions. Thirdly, visualisation enables a problem solver to engage with concepts and meanings which may easily be bypassed by the symbolic solution to the problem. That is, visualisation makes the invisible visible. Therefore, visualisation is central to understanding and reasoning.

Zimmerman and Cunningham (1991) concur with Arcavi's third role and argue that visualisation enables us to see what was formerly unseen in data. Their suggestion is significant in mathematics, as in my experience, many mathematics concepts are abstract to learners. External visual representations can assist in making abstract relationships and/or concepts concrete and clear for learners to comprehend (Zahner & Corter, 2010), thereby supporting learners' reasoning as they transit between physical models and symbolic representations (Gates, 2018). Correspondingly, Duval (1999) argues that visualisation is needed in mathematics because it externally displays organisation of relations and aids learners in forming images of mathematical ideas that are complex. That is, visualisation aids in linking abstract mathematics to learners' concrete experiences.

Furthermore, Zimmerman and Cunningham (1991) maintain that visualisation supplies depth and meaning to understanding, serving as a reliable guide to problem solving, and inspiring creative discoveries. However, in order to achieve this kind of understanding, they propose that visualisation cannot be isolated from the rest of mathematics. That is, symbolic, numeric and visual representation of ideas must be linked. Similarly, Duval (1991) stresses that there is no understanding without visualisation. Therefore, visual representation plays an epistemological and pedagogical role in mathematical teaching and learning.

Konyalioglu et al. (2012) note that the use of the visualisation method may aid learners to look at the problem in different ways and develop different thinking styles for problem solving. They further note that the teacher's use of the visual method particularly at the beginning of the lesson turns learners' passive into active behavior and increases their participation. Jourdain and Sharma (2016) also contend that using tools such as visual aids and manipulatives can assist in contextualising mathematics.

The use of particular modes of representation lead to the development of the mathematical abilities of learners and the improvement of their advanced problem solving and reasoning skills (Brenner et al., 1997; Presmeg, 1997; 1999; Pape and Tchoshanov, 2001)). For example, visual representations of the word problem such as diagrams serve as a medium to aid the learners in comprehending and solving the

problem (Zahner & Corter, 2007; Uesaka, Manolo & Ichikawa, 2007; Abdullah et al., 2012). Likewise results from van Garderen, Scheuermann and Jackson's (2012) study reveal that a diagram used in strategic ways in problem solving is positively correlated to higher non-routine word problem solving performance. Robinson (2002, p.1) further stresses that when utilised in supportive ways with text, visuals can aid to:

- represent the text, providing additional nonverbal memory prompts;
- organise and provide structure and form to text;
- interpret otherwise complex text; and
- transform text into pictorial images that can be stored more efficiently.

Furthermore, findings from Kashefi et al.'s (2015) review of literature of the twenty-six studies on visualisation in problem solving revealed that some learners far prefer to use the visual method in solving non-routine word problems and for non-novel problems, they tended to use non-visual methods. Learners with high spatial ability tend to use their ability to manipulate two and three dimensions in their minds while solving problems. Spatial ability is the ability to mentally manipulate objects (Boonen et al., 2013). Kashefi et al. (2015) however, note that non-visual learners look at the visualisation method as not giving them any benefits because it required them to know how to completely construct a diagram or picture in solving the problem. Thus, even with complicated problems, they resorted to the use of the algebraic method in processing the solution. A visual method according to Presmeg (1986) is one that involves a visual image, with or without a diagram, as an essential part of the method of solution. On the other hand, a non-visual method does not rely on a visual image.

Ho (2010) identifies several roles that visualisation plays as learners use it to solve mathematical problems:

- To **understand** the problem — the problem visually enables learners to understand how the elements in the problem relate to each other.
- To **simplify** the problem — visualisation allows students to identify a simpler version of the problem, solving the problem and then formalising the understanding of the given problem and identifying a method that works for all such problems.
- To see **connections** to a related problem — this involves relating the given problem to previous problem solving experiences.
- To cater for individual **learning styles** — each learner has his or her own preferences when it comes to the use of visual representations when solving problems.

- As a **substitute** for computation — the answer to the problem can be obtained directly from the visual representation itself without the need for computation.
- As a tool to **check** the solution — the visual representation may be used to check for the reasonableness of the answer obtained.
- To **transform** the problem into a mathematical form — mathematical forms may be obtained from the visual representation to solve the problem.

Presmeg (1997) asserts that the use of visual representation facilitates learners' mathematical conceptual development. For instance, using visual models as visualisation tools to teach Pythagoras' theorem and word problems offers learners an opportunity of visualising the process, thus aiding them in understanding this concept. Additionally, Hsiao, Lin, Chen and Peng (2018) echo that learning with visual representation results in higher learning performance and lower cognitive load than learning without them. Visual representations can relieve some of the limited working memory resources that are used to establish relationships between abstract mathematical concepts and their underlying functions, thus problem solvers can acquire a deeper understanding and hone their ability in problem solving (Hsiao et al., 2018). Similarly, Pape and Tchoshanov (2001) assert that external representations embody crucial relationships presented in a story problem. Therefore, they lighten the problem solver's cognitive load and serve to organise his further work on the problem.

Furthermore, Larkin and Simon referenced in Crapo, Waise, Wallace and Willemain (2000, p. 220) stress that "visualisation extends working memory by using the massively parallel architecture of the visual system to make an external representation function as an effective part of working memory". Therefore, visual representations can reduce cognitive load during engagement (Gates, 2018). Zahner and Corter (2010) also postulate that learning with external visual representations can assist in supporting memory as well as offloading working memory load.

Research has revealed that the construction of drawings has been found to be helpful in the comprehension and learning from content area text. Konyolioglu et al. (2012) stress that drawing as one of the visualisation forms, assists in improving the learner's thinking ability. Problem solvers who engage in drawings are more likely to understand new concepts in relation to one another (Van Meter, 2001), hence acquiring more knowledge gain. Van Meter et al. (2006), De Koning and van der Schoot (2013) further point out that as problem solvers engage in the process of creating a drawing, they integrate their prior knowledge with the to-be-learned content which facilitates a deeper understanding of text content. This in turn aids in offloading a reader's working memory load as cognitive processing demands are divided over verbal and visual information processing channels, consistent with the dual coding theory

(Sasdoski & Paivio, 2013). It is therefore evident from the extant literature that constructing a diagram is a commonly suggested approach in solving MWPs.

Constructing drawings has also been found to be useful in facilitating MWPS, but only when an appropriate representation is used. Results from Hegarty and Kozhevnikov's (1999) and Zahner and Corter's (2010) studies revealed that the use of schematic, unlike non-schematic or pictorial representations, was associated with success in MPS. These findings were consistent with van Garderen et al.'s (2012) research. The latter encoded crucial information described within the problem while the former included details irrelevant to solving a problem. For instance, when asked to find the length of the straight path on which Y number of trees are planted, every 6 feet apart, a schematic diagram may involve a number line containing Y number of stars with numbers indicating the distances between them, while the pictorial drawing may entail real trees positioned at an equal distance, plus extra irrelevant details. Furthermore, analyses in use of pictorial and schematic drawings revealed that gifted students often used more schematic representations than students with learning disabilities and average achievers, and this led them into solving more MWPs correctly (van Garderen, 2006).

While Hegarty and Kozhevnikov's (1999) study had limitations such as absence of female participants, results from studies with broader sample selections also revealed that pictorial drawings were negatively associated with success in MPS, with schematic imagery positively related to success (van Garderen & Montague, 2003; Ahmad et al., 2010; Zahner & Corter, 2010). It is thus evident from extant research that schematic unlike pictorial imagery is related to success in MPS.

Pape and Tchoshanov (2001) assert that use of multiple representations (e.g. concrete, visual and abstract) result in the level of learners' interaction automatically going up, leading to an improvement in their conceptual understanding and representational thinking. That is, learners are more eager to exchange ideas using different representations, thus learning from one another's solution approach. Conversely, low levels of learner interaction result in the use of a single representation, which in turn compromises their conceptual understanding and representational thinking.

Although both the National Council of Teachers of Mathematics (NCTM) (2000) and the National Research Council (NRC) (1996) call for problem solvers to be able to flexibly utilise various forms of representation in investigating and communicating about real life situations, problem solvers experience great challenges in developing fluency in using representational forms in mathematics (Pape & Tchoshanov, 2001).

2.3.5 Challenges in using visualisation

Despite the positive recommendations of visualisation, literature reveals that this topic has its difficulties. Arcavi (2003) classifies the challenges around visualisation into cultural, cognitive and social:

Firstly, cultural difficulties refer to the beliefs and values pertaining to what constitutes mathematics. That is, it is about what is legitimate or acceptable, and what not (Arcavi, 2003). He further stresses that within the mathematics community, we are taught to look askance at proofs that make crucial use of diagrams, graphs, or other non-linguistic forms of representation and this disdain is passed on to learners by us. Presmeg (1997) calls this attitude 'devaluation of visualisation' because it leaves little room for classroom practices to incorporate and value visualisation as an integral part of doing mathematics.

Secondly, cognitive difficulties refer to whether the visual is easier or more difficult to work with. Thinking visually can place higher demands on cognition than thinking algorithmically. This is because visualisation may not always have procedurally 'safe' routines to rely on as may be the case with formal symbolic approaches. In such situations, it may be rejected by learners and teachers as well on grounds that it is too risky. Furthermore, learners often experience difficulty translating back and forth between visual and analytic representation (Arcavi, 2003).

Thirdly, from a sociological point of view, many teachers prefer algorithms and procedures to visualisation in the teaching and learning of mathematics (Arcavi, 2003). This is because analytic presentations are sequential in nature and many teachers view them as more pedagogically appropriate and efficient. In support of the sociological difficulty around visualisation as proposed by Arcavi (2003), Dreyfus (1991) and Konyolioglu et al. (2012) also acknowledge that the use of visualisation as a pedagogical approach is still undermined by many mathematics teachers. It is this undermining of visual reasoning as an acceptable practice of mathematics that inspired me to investigate how the use of visual approaches can enhance the teaching of mathematics word problem in the context of Pythagoras' theorem.

In their study which involved twenty-two ninth graders and their mathematics teacher in Turkey, Konyalioglu et al. (2012) ascertained that although visualisation had a positive effect at the preliminary stage of teaching mathematical concepts, learners were determined to abstain from using it in problem solving. They also note that in solving problems asked in the applied achievement test, many learners did not use the visual expressions that the teacher used at the time of instruction, but instead used the analytical method and ended up incorrectly answering most of the questions with this method. The

reasons as to why the visualisation method was ignored by learners in problem solving although the approach was used by the teacher during the instruction period, are explained in Figure 2.5 below.

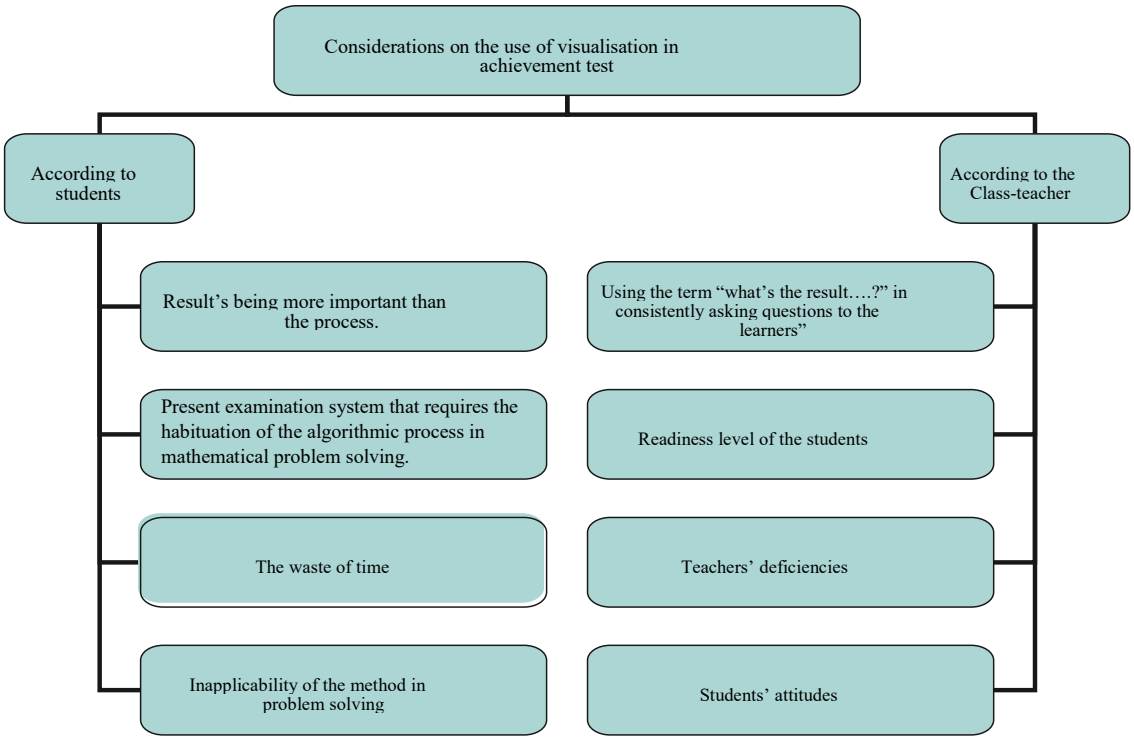


Figure 2.5: Learners' considerations over not using the visualisation method when solving problems (Knyolioglu et al., 2012)

Gates (2018) brings to our consideration that less attention is paid to visual forms in teaching and this has resulted in learners misusing, or not drawing on visual models. This is evident in Van Garderen et al.'s (2014) study that focused on diagram use to solve word problems. Results from their study reveal that low achieving learners did not choose to use a diagram because they did not see diagrams as part of doing mathematics. That is, they could not figure out the structural relationships between the diagram and the targeted concept. They conclude "reasoning with a diagram is a difficult process that learners may need more time and experience to develop" (Van Garderen et al., 2014, p.147).

2.4 PYTHAGORAS' THEOREM

Pythagoras' theorem is a theorem relating the squares of the lengths of the sides of a right-angled triangle (Quadrat, Lasserre, & Hiriart- Urruty, 2001). Newton (2010) asserts that since conceptual learning rather than procedural learning leads to a greater understanding of the content, Pythagoras' theorem is too vital in mathematics to be taught procedurally. He further stresses that not teaching mathematics concepts such as Pythagoras' theorem may have detrimental effects on one's overall comprehension of topics such as

trigonometry. In my experience, I have always taught this theorem from the textbook, giving learners the equation $a^2 + b^2 = c^2$, which I expected them to memorise then apply. I have since come to realise that learning procedures and proofs (Pythagoras' theorem in this case) without a good understanding of why they are important, leaves learners ill- equipped to use their knowledge in later life. It is therefore necessary to pay attention to Roscoe's (2014) call that providing learners with meaningful open-ended experiences where proof is pursued, is a goal that all mathematics teachers should strive towards. Pythagoras' theorem therefore has the ability to afford an excellent opportunity for mathematics teachers to engage their learners in the proof making process. This study, which followed an interventionist approach in the JTG District, Northern Cape Province, South Africa, made use of visual models, approaches and strategies that offered teachers concrete demonstrations of, for example, how squares and areas are involved in proving Pythagoras' theorem and solving MWPs in this context.

2.4.1 Visual models for teaching Pythagoras' theorem and word problems

This section presents a discussion of visual models as visualisation tools used in teaching Pythagoras' theorem and MWPs. It also presents and discusses how these models can be used to teach this concept.

Visual models are reflections of abstract concepts and symbols in the physical world (Konyalioglu et al., 2012). That is, they are visualisation tools used to represent abstract concepts in a constructive way, leading to mental interpretation of images in learners' minds. Konyalioglu et al. (2008) further articulate that visualisation in the form of visualisation tools (for example drawings, shapes and concrete models) as tools for visualisation of abstract mathematical concepts can be a powerful approach in providing learners with different ways of thinking about mathematics. By using these visualisation tools, many mathematics concepts become clear and concrete for learners to understand (ibid.*)*. In the same spirit, Jencks and Peck (1972) argue that the use of visual models in solving MWPs facilitates learners' comprehension and creates solution finding opportunities.

Dunham (1994) cites an interesting book —*"The Pythagorean Proposition"* — by an early 20th century professor Elisha Scott Loomis. This book, republished by NCTM in 1968, has a collection of 367 proofs of the Pythagorean theorem. This implies that numerous visual models for teaching the Pythagorean theorem are found in literature and some of these include: a "visual aid" of the Pythagorean theorem dating to Euclid (c. 300 BCE), dynamic movement and superposition (or shearing a square) proving the Pythagorean theorem (Eve, 2002), copies and rearrangement of congruent right-angled triangles such that left over areas are equal (Gardner, 1984, p. 154), equal areas achieved through clever dissections presented in Adams' proof of the Pythagorean theorem (Loomis, 1968), scaled-up versions of a right-angled triangle (Roscoe, 2014) and a manipulative puzzle proof of the Pythagorean theorem

(www.mrseteachesmath.blogspot.com). The models that are used in this study are: a "visual aid" of the Pythagorean theorem dating back to Euclid (c. 300 BCE), scaled-up versions of a right-angled triangle (Roscoe, 2014), and copies and rearrangement of congruent right-angled triangles such that left over areas are equal (Gardner, 1984, p. 154). The reason as to why I chose to work with these models is because of a sense of universality in each of these proofs, unlike in the other three visual aids I left out. I workshopped teachers about the above three models (the two area models and the scaled-up versions of an original right-angled triangle) on how to teach Pythagoras' theorem and MWPs in this context.

2.4.1.1 Area models: The two area models used areas of squares to establish a relationship between the lengths of the sides of each respective right-angled triangle. Squares with lengths equal to that of a respective side in each triangle were drawn. In the first model (see Figure 2.6), squares drawn on the three sides of a right-angled triangle were partitioned into unit squares and these were examined to ascertain whether a relationship existed between the sides of the triangle (Euclid referenced in Roscoe, 2014, p. 177).

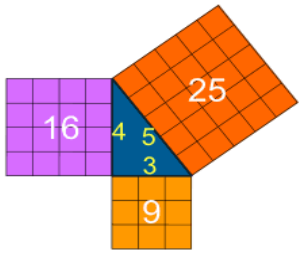


Figure 2.6: A "visual aid" of the Pythagorean theorem dating to Euclid (c. 300 BCE)

In the second model (see Figure 2.7), the demonstration of equal areas was achieved through copies and rearrangements of congruent right-angled triangles (Gardener, 1984, p. 154). The four congruent right-angled triangles were arranged to form two squares: the outer and the inner square. Pythagoras' theorem proof had much to do with the inner square. This square was formed by the hypotenuses of the four congruent right-angled triangles. This square was removed and two of the four congruent triangles were rotated to form two rectangles as seen in Figure 2.7 below. The empty spaces which were created after rotation and translation were filled up with two squares: the small and big square with dimensions congruent to the lengths of the shorter and longer legs of the congruent triangles respectively. The areas of the three squares that filled up the empty spaces were then inspected to establish a relationship.

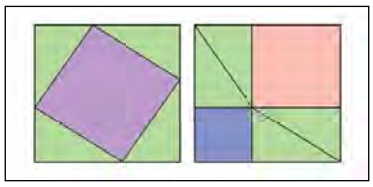


Figure 2.7: Proof by rearrangement: leftover areas must be equal (Gardner 1984, p. 154)

2.4.1.2 Scaled-up versions of a right-angled triangle: The scaled-up versions of an original right-angled triangle approach entails scaling a right-angled triangle with sides ***a***, ***b*** and ***c*** three times (Roscoe, 2014). That is, one triangle is scaled by length ***a***; another by length ***b***; and the third one by length ***c***. The three scaled-up versions of the original triangle put together create new shapes whose properties and/or congruency criteria are used to make sense of Pythagoras' proposition. For example, in Figure 2.8 below, putting together the three scaled triangles produced an isosceles triangle. This visual proof thus led to uncommon proofs of Pythagoras' theorem and rich mathematical connections among topics — similar triangles, ratios, the scale factor, geometry, basic algebra, and so on.

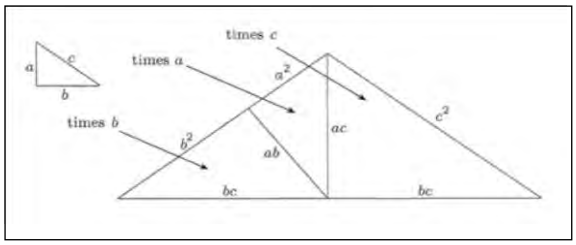


Figure 2.8: Putting together scaled-up versions of a right triangle give birth to an isosceles triangle (Roscoe, 2014)

The models above are crucial and necessary for visually and conceptually aiding the teaching of Pythagoras' theorem and applying it to solve MWPs in this case. I therefore conducted a study underpinned by a ViPProS programme that used visual aspects (when working with the three models) in understanding the problem stage, during finding the solution, and /or when presenting the solution of the problem. The motive behind this was to develop learners' conceptual understanding and sense making in MWPS, and these are part of the strands of mathematical proficiency as identified by Kilpatrick, Swafford and Findell (2001).

2.5 THE STRANDS OF MATHEMATICAL PROFICIENCY

Kilpatrick et al., (2001) refer to mathematical proficiency as the cognitive changes that teachers should promote in learners so that they can be successful in learning mathematics. Kilpatrick et al. (2001) further underscore that ''recognising that no term captures completely all aspects of expertise, competence, knowledge, and facility in mathematics, we have chosen mathematical proficiency to capture what we believe is necessary for anyone to learn mathematic successfully'' (p. 5). According to these authors, mathematical proficiency is made up of five strands which include:

- Conceptual understanding,
- Procedural fluency
- Strategic competence

- Adaptive reasoning
- Productive competence (Kilpatrick et al., 2001).

Although this study focused on teaching for conceptual understanding of Pythagoras' theorem word problems, it is crucial to understand the difference between conceptual understanding and adaptive reasoning and acknowledge that these two strands are inseparable.

2.5.1 Conceptual understanding and adaptive reasoning

Conceptual understanding and adaptive reasoning are part of the five interwoven and interdependent mathematical proficiency strands (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive competence) identified by Kilpatrick et al. (2001). According to Kilpatrick et al. (2001), conceptual understanding is the comprehension of mathematical concepts, operations and relations while adaptive reasoning is the capacity to think logically about relationships among concepts and situations, thus making it the glue that holds everything together. One uses adaptive reasoning to navigate through the many concepts, solution approaches etcetera to see that they all fit together in some way and make sense. Kilpatrick et al. (2001) further acknowledge the significance of interweaving these two strands in the teaching and learning of mathematics. They contend that conceptual understanding provides representations that serve as a source of adaptive reasoning. That is, with the help of representation-building experiences, learners can demonstrate sophisticated reasoning abilities. However, for the purpose and scope of this study, teaching proficiency with respect to using visualisation processes was only analysed against one strand (conceptual understanding) due to its necessity in the development of learners' mathematical proficiency.

2.5.2 Conceptual understanding

Wilson and Heid (2011) described conceptual understanding "as the 'knowing why' of mathematical proficiency." For Konyalioglu et al., (2008), conceptual knowledge is described as mathematical concepts and the relationship between each other. According to Kilpatrick et al., (2001), conceptual understanding is described as an integrated and functional grasp of mathematical ideas. Kilpatrick et al., (2001) further stress that learners with conceptual understanding know more than isolated facts but are able to learn by connecting new ideas to what they already know. For instance, when teaching Pythagoras' theorem, learners can relate this concept to similar triangles, scale factors, measurement and geometry because they have met these concepts in their prior learning. Therefore, teachers should endeavour to promote such in mathematics teaching. Konyalioglu et al., (2008, p.246) also assert that learners with conceptual knowledge see a mathematical task as a group of related concepts and ideas. That is, they search for the structure of the problem as small pieces that fit together rather than looking at the type of problem in

order to identify superficial clues and produce algorithms. Therefore, with conceptual understanding, there is promotion of connection making resulting in learners gaining confidence, which acts as a stepping stone for them to move to another level of understanding (Kilpatrick et al., 2001).

Knowledge which is learned with understanding provides a basis for generating new knowledge and for solving new and unfamiliar problems (Kilpatrick et al., 2001). They further state that "when students have acquired conceptual understanding in an area of mathematics, they can see connections among concepts and procedures and can give arguments to explain why some facts are consequences of others" (p. 119). This seems to align with what Hiebert and Carpenter (1992) expound "that we understand something if we see how it is related or connected to other things we know" (p. 4).

Kanyalioglu et al., (2008) and Makina (2010) acknowledge the use of visualisation as one of the cornerstones of teaching for conceptual understanding. The authors contend that a visual approach to teaching and learning makes many abstract mathematical concepts visually clear for learners to understand. They further stress that the use of a visual approach can be helpful to learners' conceptual learning by providing an alternative new method of viewing the problem (Konyalioglu et al., 2008) and also developing problem solving thinking styles (Konyalioglu et al., 2012) thus facilitating meaningful learning. Research findings from the scholarly works of Presmeg (1986) and Arcavi (2003) also revealed that when constructing meaningful conceptual ideas, learners use different visual approaches. According to Kilpatrick et al. (2001), being able to represent mathematical situations in diverse ways and knowing how different representations can be useful for different purposes is one of the significant pointers of conceptual understanding. The different representations may include several visualisations such as graphs, diagrams, pictures, models, tables and so on.

Kilpatrick et al. (2001) presents four key conceptual understanding indicators necessary for this study and these are discussed below.

2.5.2.1 Building on prior knowledge

This concept refers to the teacher's ability to make links between the mathematics he is teaching and what learners already know. The indicator requires that learners learn mathematics with understanding and build new knowledge from prior experiences. Pape (2004) argues that learners who are successful in solving mathematical problems tend to create visual representations of the word problems and these representations are able to facilitate them in recalling previous mathematical knowledge to solve the problems. Visual representations also aid the learners in integrating the recalled information with new information presented in the word problems (Abdullah et al., 2012). Harvey (2011) further stresses that learners learn better when they relate what they are being taught to their prior knowledge. A learner can

imagine his previous experience or knowledge, impose it on the problem and use it to derive a solution (Arcavi, 2003). Therefore, knowledge that is learned with understanding can be used by learners to generate new knowledge and to solve unfamiliar problems (MWPs in this case) under a teacher's guidance (constructivism).

2.5.2.2 Representing mathematical concepts in different ways

This refers to the teacher's ability to represent the same mathematical concepts presented in a particular word problem in diverse ways and explore the relationships with learners. This in turn enables learners to have a deeper understanding of the underlying concepts and ideas of mathematics, thus deepening their understanding. The NCTM (2000) and the NRC (1996) underscore the use of multiple representations in the teaching of mathematics as these support learners in investigating and communicating about real world phenomena. Similarly, Pape and Tchoshanov (2001) assert that the use of multiple representations facilitate learners' mathematical conceptual development.

2.5.2.3 Connecting concepts and ideas in mathematics

This refers to the teacher's ability to discuss the similarities and differences of the various representations and explore and connect these ideas to other mathematical domains. This in turn facilitates learners' mathematical conceptual development (Brunner et al., 1997) because the degree of learners' conceptual understanding is related to the richness and the extent of the connections they make. The NSC diagnostic report of 2017 also states that mathematics cannot be studied in compartments and it is expected that candidates must be able to apply knowledge from one section to another section of work (DBE, 2017, p.164). The greater the number of appropriate connections one has to a network of ideas, the better the understanding. When learners learn mathematics with understanding, the connected information is more likely to be retained over time. It provides a web of ideas so that if one wants to recall something, reflection on ideas that are related can usually eventually lead one to the desired idea. Capraro and Foffrion (2006) also assert that ""connections make mathematics meaningful, memorable, and powerful"" (p. 162). I concur with the authors and argue that having an understanding of concepts and how they correlate with each other aids the teacher in planning lessons that are meaningful to learners. The visual approach to teaching has the capability to generate such avenues so that learners can engage in making connections among concepts.

2.5.2.4 Extending mathematical concepts to real life contexts

This concept requires the teachers to relate what they are teaching to new areas or daily life contexts, thereby enabling learners to see the practical applications of this knowledge. This may require the teacher to give plenty of examples related to the learners' daily lives, thus making them aware of the utilitarian

value of mathematics (Gafoor & Sarabi, 2015). Connecting mathematics to the outside world helps learners to make sense of what they are learning and to solve real world problems — in this case MWPs.

The conceptual understanding pointers discussed in this section form an integral part of my second analytical framework discussed in the data analysis section of Chapter Three — see Section 3.8. The analytical tool will be used to analyse the data in Chapter Four.

With evidence from literature identifying textual representation understanding as a challenge for many learners in South Africa and elsewhere, there are calls for a need to investigate a reading strategy to support learners in handling MWPs. Research reveals visualisation as an effective method that may mediate teaching and learning in order to yield a change in the difficulty (Van Meter et al., 2006; Kashefi et al., 2015). Visualisation is acknowledged as a corner stone of teaching for conceptual understanding (Konyolioglu et al., 2008). Use of visual aids as visualisation tools enables learners to exhibit conceptual understanding of mathematical concepts. By utilising visual models on Pythagoras' theorem, learners are able to make sense of the visual and symbolic relationships that exist between the sides of a right-angled triangle and also to solve Pythagoras' theorem word problems in multiple ways with the help of these models. I therefore assert that conceptual understanding and visualisation align well with the social constructivist perspective that I present in the section that follows. At the heart of a social constructivist perspective is socially constructing knowledge which is supported by the use of visual aids (in this case), acting as a bridge between the known and unknown among other conceptual understanding indicators.

2.6 THEORETICAL FRAMEWORK: SOCIAL CONSTRUCTIVISM

"The human mind can know only what the human mind has made" (von Glasersfeld, 1995, p.21). This implies that knowledge is not passively received but actively built by the learner.

2.6.1 Social constructivism

"Constructivism is not a theory about teaching… it is a theory about knowledge and learning …the theory defines knowledge as temporary, developmental, socially and culturally mediated, and thus, non-objective" (Brooks & Brooks, 1993, p. vii). The role of the community (teachers and other learners) in this knowledge construction process is to provide the setting, pose the challenges, and offer the support that will encourage mathematical construction (Davis, Maher, & Noddings, 1990, p. 3).

This research study was informed by Vygotsky's (1962) social constructivist perspective that aimed at epistemologically empowering teachers as social agents during workshops, after which they had to implement their learning. In a social constructivist perspective, the process of knowing has at its root social interaction (Von Glasersfeld, 1995) seen by Vygotsky (1962) as an integral part of learning. That

is, learning always originates in some form of social participation that becomes internalised to change our mental functioning. In the same vein, Ernest (2010) argues that learning and knowledge construction take place in the social arena, in the "space between people, even if its end products are appropriated and internalised by those persons individually" (p. 40). On the social plane, the learner interacts with the more knowledgeable other (MKO) who may be a teacher, an adult, a coach or his or her peer(s) (Vygotsky, 1978). This social process of interacting with the MKO extends the learner's range and allows more complex actions to be performed. In my study, the MKOs were the teacher, more able peers and visual tools. Vygotsky also believed that internalising human tools (for example, speech and writing) can lead to developing one's higher level thinking skills (Vygotsky, 1978). In this study, the use of speech and writing in combination with visual aids as visualisation tools aided learners in expressing their mathematical skills and thinking at a higher level.

Based on Vygotsky's work, Ernest (2010) argues that social constructivism regards learners and the realm of the social as indissolubly interconnected, emphasising that it is through social interaction mediated by tools, that human learning is born. Further, drawing from a Vygotskian perspective, Powell and Kalina (2009) state that language usage in the classroom is the most important cognitive tool for social mediation; it can be used as a tool to mediate social interaction and collaboration, and for learners to express their learned knowledge. Yonson (2017) also asserts that language plays a vital role in the understanding of MWPs. In this study, language and visualisation approaches were the tools used to support learners in their meaning making processes. Throughout the lessons, the visuals used were in line with the teachers' verbal explanations and the visual language used was in line with the mathematical language and terminology.

Vygotsky (1962) asserts that direct teaching of concepts is impossible and fruitless. He further argues that ""a teacher who tries to do this usually achieves nothing but empty verbalism, a parrot-like repetition of words by the child, stimulating a knowledge of the corresponding concepts but actually covering up a vacuum"" (Vygotsky, 1962, p. 83). Gates (2018) shares the same sentiment as Vygotsky and argues for learning through physical engagement as this encapsulates a visual element through which the learner monitors, evaluates and hypothesises the world. In the case of my study, visual aids were used to encourage learners to actively and interactively explore the concept of Pythagoras' theorem during their knowledge construction process, with the teacher facilitating this discovery learning process.

2.6.2 Social constructivism and Visualisation

Pape and Tchoshanov (2001) assert that visual representation is inherently a social activity. They underscore that when asking learners to represent data in a visual representation, the representation should

not be seen as an end result, but instead as a tool for discussion to aid them in establishing a relationship or forming a justification in a social context. Therefore, representational thinking is considered by Pape and Tchoshanov (2001) as the ability of a learner to interpret, construct and communicate effectively, with both internal and external representation, individually and within a social situation.

Arcavi (2003) sees visualisation as a process of knowledge construction that promotes understanding because when constructing meaningful conceptual ideas, learners use various visual approaches. He argues for the use of visual tools in teaching because these support a discovery approach to MPS. Visualisation also increases collaboration among learners in group work (Klerkx, Verbert & Duval, 2014) and this is key in the development of an individual's knowledge (von Glasersfeld, 1996). Malabar and Pountney (2002) further contend that the integration of constructivism and visualisation can aid learners to reform their conceptual structures and develop higher-order skills. As learners draw from or mentally form images, they construct their own ideas and concepts (Arcavi, 2003) thus achieving a deeper understanding. Using visual approaches in this study encouraged learners to become immersed in the construction of their knowledge.

Within a social constructivist perspective, learners are actively involved in constructing and reconstructing their knowledge (Orton, 1992). Pape and Tchoshanov (2001) add that ""from a constructivist perspective, the necessary mapping between the concrete materials and the arithmetic algorithm (procedure) requires intensive social co-construction of meanings"" (p. 122). Teachers and learners co-construct their understanding of the steps of the mathematical operation while manipulating the materials (ibid). Similarly, Yackel (2001) highlights that ""learners construct their own meaning from the words or visual images that they see or hear"" (p. 41). I therefore argue that the use of visuals in this study enabled learners to meaningfully construct their knowledge under their teacher's guidance.

Vygotsky (1978) emphasises that learning should take place in a context that is meaningful (that is, one in which learners can interact with one another in an environment that is authentic) and it should not be separated from the learning and knowledge that learners have developed in the physical world. Learners regularly bring to school with them experiences, which are in the form of knowledge that is unstructured. Teachers thus need to guide and establish teaching and learning milieus that transform this knowledge into structured information. Ernest (1991) recommends that this can be realised through teaching and learning environments that support and encourage active construction of mathematical knowledge instead of it being received from the teacher as a final product. It can be argued that the employment of visualisation strategies in my research study vividly promotes Ernest's (1991) recommendation.

Bruner (1960) asserts that learners construct new mathematical concepts or ideas based on their prior knowledge. Hyde and Bizar (1989) also mention that ""we now see the students as actively constructing meaning, connecting new information to existing knowledge structures, and creating new relationships among structures"" (p. 89). Jones and Brade- Araje (2002) add that constructivist teachers are typically acutely aware of the role of prior knowledge in learners' learning and recognise that they are not blank slates to be filled with knowledge. Instead learners draw on their prior learning to construct new understandings (Ernest, 2010) with teachers acting as guides and facilitators (Powell & Kalina, 2009), creating opportunities for learners to construct new knowledge from authentic experiences (Bruner, 1960). Visuals that learners are already exposed to form a repertoire of prior knowledge tools that a constructivist teacher can build on in enhancing conceptual understanding. However, I have noted from my experience that some teachers hardly use visual aids in their mathematics lessons. They rely instead on procedures which learners master without an understanding of the concepts being taught. I thus agree with Konyalioglu et al. (2008), who hold the view that there is a need for teachers to incorporate visually oriented activities that allow learners to make sense of the mathematical concepts in ways that are active and exploratory; hence the need for using visualisation as an approach for introducing the experimental aspect of mathematics (Konyalioglu et al., 2012), which was the case in my study.

Social constructivism was key to this study because the intervention that was undertaken centred on the use of visualisation teaching strategies that inherently entail active collaborative engagement and learning of mathematical concepts. The classroom community was characterised by learners working collaboratively and cooperatively in small groups with problem scaffolding mediated by the teacher, more able peers and visual tools. This enabled learners to handle more sophisticated tasks in contrast to when they were on their own (Wertsch, 1985). During the lesson, the teacher used ""strategic questioning"" (Rowland, Graham & Berry, 1997), asking probing questions, which guided learners to reach the target concept without it having to be provided by him (Malabar & Pountney, 2002). He used visuals and visual activities to create an opportunity for discussion where learners raised questions and shared their groups' observations with the whole class. Silver (1996) contends that engaging learners in activities that are in line with a social constructivist perspective does not only afford them an opportunity of learning mathematical skills and procedures, but also in articulating and justifying their thinking and discussing their observations. This in turn encourages them to think critically about the material they are learning, hence constructing personal meaning. This study thus aligned well with social constructivism as the emphasis was on the shift from knowledge as a product to be transferred by the teacher to knowing as a process of discovery on the part of the learners (Jones & Brade- Araje, 2002), with the teacher guiding this discovery process through the use of visuals in ways that were interactive.

2.7 CONCLUSION

The purpose of this chapter was to present a critical review of the literature that is relevant to this study. I began by defining word problems. This was followed by a discussion of the reasons for teaching these problems in school mathematics, language and mathematics word problems, difficulty in teaching word problems in mathematics, and MWPS models. Thereafter, I presented the concept of visualisation, discussed visualisation processes, the role of visualisation in mathematics and the challenges in using visualisation. Pythagoras' theorem and visual models for teaching this concept were also presented. The chapter concludes with a discussion of conceptual understanding and the social constructivism theory that underpins this study. In the next chapter, I present the methodology that I employed in my research study.

CHAPTER THREE

METHODOLOGY

3.1 INTRODUCTION

The chapter describes the research methodology that underpins this study's research process. I used a mixed methods design that afforded me an opportunity of collecting and analysing quantitative and qualitative data, but utilising a case study as my main research method. I present the instruments that I employed in collecting data to answer my research questions. My research project used the following data collection tools: survey questionnaires, classroom observations and interviews. It also discusses the selection criteria of participants and how I analysed the data. The chapter concludes with an articulation of validity issues and ethical aspects used to enhance the quality of this research study.

3.2 RESEARCH GOAL AND QUESTIONS

The focus of this study was to investigate how selected Grade 9 mathematics teachers used visual approaches to teach word problems in the context of Pythagoras' theorem for conceptual understanding as a result of an intervention programme. This was guided by the following research questions;

1 How do Grade 9 mathematics teachers in a selected district teach Pythagoras theorem and word problems in this context prior to an intervention programme?

2 How do selected teachers use visuals to enhance conceptual teaching of word problems that involve Pythagoras' theorem after participating in an intervention program?

3 What are the selected teachers' perceptions and experiences in using visual models to conceptually teach word problems on Pythagoras theorem as a result of participating in an intervention program?

3.3 RESEARCH ORIENTATION

The way in which ones sees the world influences the way in which one researches the world (Bertram & Christiansen, 2014).

Meanings are not expressed directly, but are embedded by the person in the physical, linguistic and enacted artifacts that they generate (Schwartz-Shea & Yanow, 2012). Pring (2000) argues that a researcher needs to know how participants comprehend and interpret situations. It is because of this reason that researchers talk of the subjective meanings of those whom they are researching: the different understandings and interpretations that participants bring with them to the situation (Pring, 2000). In this study, I explored the subjective understandings and interpretations concerning the use of visualisation processes in the teaching of Pythagoras' theorem word problems, which are the experiences of the three selected teachers. This study was therefore located in an interpretive paradigm.

Clark and Creswell (2008) refer to a paradigm as ""a conceptual model of a person's world view, complete with the assumptions that are associated with that view"" (p. 73). Maree (2006) further adds that a paradigm addresses fundamental assumptions taken on faith, and these include:

- **Ontology**, referring to beliefs about the nature of reality.
- **Epistemology**, referring to the nature of knowledge of that reality. It seeks to comprehend the nature of the relationship between the knower (researcher) and known (researched).
- **Methodology**, referring to the instruments the researcher uses to obtain the desired reality (knowledge and understanding).

In a similar manner, Bertram and Christiansen (2014) stress that a research paradigm is a representation of a particular world view that defines for the researcher who holds this view, what is acceptable to research and how this should be done. They further stress that working within a particular paradigm determines choices such as the following:

- What kind of questions are supposed to be asked
- What can be observed and investigated
- How to collect data
- How to interpret the findings (p. 22).

My research study employed both qualitative and quantitative methods of data collection. An explanatory sequential mixed methods design (Creswell & Plano Clark, 2011) was employed in collecting and analysing data in the interpretive paradigm in which my study was oriented.

3.3.1 Interpretive paradigm

The study is grounded within an interpretive paradigm. Creswell (2003) asserts that researchers in the interpretive paradigm discover reality through participants' views and experiences of the situation being studied. Equally, Maree (2006) writes that interpretivists believe that reality is socially constructed and for this reason studying people in their social contexts or natural environment provides a greater opportunity to understand the perceptions they have of their own activities. Therefore, the onus of the researcher in the interpretive paradigm is to comprehend, articulate and demystify social reality through the eyes of different participants (Cohen et al., 2011).

Furthermore, Bertram and Christiansen (2014) remark that data from the interpretive approaches to research must be authentic and reflect participants' experiences. This was ideal in my study as this research approach accorded me an appropriate platform to make sense of how study participants used visualisation approaches in their mathematics lessons and what their views of such experiences were.

Therefore, a qualitative approach to analysing this data (arising from participating mathematics teachers' classroom observations and interviews) was employed. Mackenzie and Knipe (2006) underscore that the interpretivist researcher is most likely to depend on qualitative data collection methods and analysis or a combination of both quantitative and qualitative methods (mixed methods). My study employed a mixed methods research design as articulated next.

3.3.2 Mixed methods research approach

The mixed methods design is defined by Creswell (2014) as an approach to inquiry involving collecting, analysing and mixing or integrating quantitative (closed-ended) and qualitative (open-ended) data in a research study. He adds that the blending of data provides a stronger comprehension of the research problem or question than using either of these methods alone. Similarly, Tashakkori and Teddlie (2003) assert that utilising the mixed methods research approach aids the investigator to gain complete understanding and comprehension of results, discover new perspectives, or develop new measurement tools.

Mixed methods researchers collect and analyse both qualitative and quantitative data in response to the research question. According to Creswell (2014), quantitative research is a research approach for testing objectives and theories by examining the relationship among variables, so that data can be analysed using statistical procedures. On the other hand, Maree (2016) writes that qualitative research is an inquiry process of understanding, where an investigator develops a complex, holistic picture, analyses words, reports detailed views of participants, and conducts the study in a natural setting.

This study employed an "explanatory sequential two-phase mixed methods design" (Creswell, 2014). An explanatory sequential design as defined by Creswell (2014) is a mixed methods strategy in which the researcher first conducts quantitative research, analyses the results and then builds on the results in order to articulate them in more detail with qualitative research. The design involves a two-phase project in which the researcher collects quantitative data in the first phase, analyses the results and then utilises them to plan (or build into) the second qualitative phase. The reason for choosing this design was to use the qualitative data to explain in more detail the initial survey results (Creswell, 2014). The explanatory sequential mixed methods design seems to be more appealing in quantitatively oriented fields because the study begins (and perhaps is driven) by the quantitative phase of the study (ibid). This was the case with my study, which employed a survey as the first method. Creswell (2014) further writes that a typical procedure when using the explanatory sequential approach may entail collecting survey data in the first phase, analysing the data, and then following up with qualitative interview data to aid in explaining the survey responses. The above-mentioned was typical of my study where I collected data in two separate

phases. I began by collecting and separately analysing qualitative (QUAL) and quantitative (quan) data from the survey questionnaire responses with qualitative data being dominant. This was then followed by the collection and analysis of qualitative (QUAL) data from the interviews and lesson observations during the intervention as seen in Figure 3.1 below.

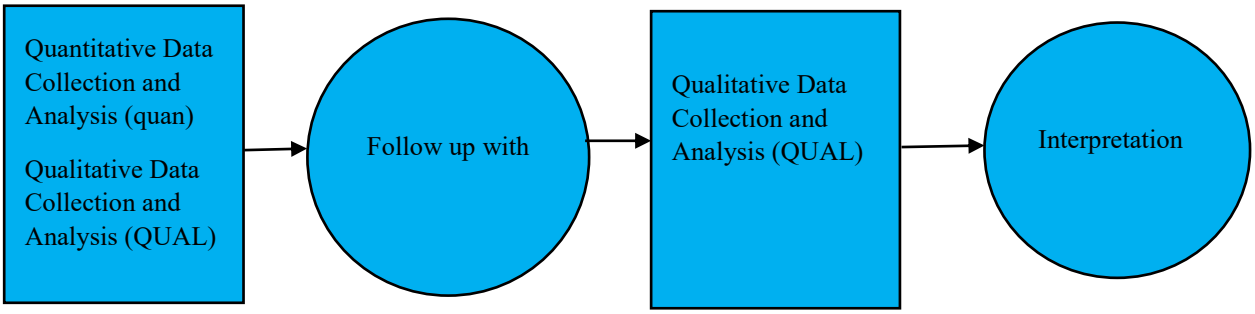


Figure 3.1: Visual diagram of a sequential explanatory mixed methods design to explain findings (Creswell, 2014)

In an explanatory sequential mixed methods design, descriptive statistics are used in analysing quantitative data, whilst a thematic analysis is used to analyse qualitative data, with the two approaches integrated during analysis. In this study, quantitative and qualitative data was obtained from the survey questionnaire administered to Grade 9 mathematics teachers in the JTG District, focusing on their experiences relating to how and to what extent they use visual approaches to teach Pythagoras' theorem word problems. Qualitative data was obtained from interviews and classroom video recorded observations with three mathematics teachers purposefully selected from three different schools in the district. The reason as to why my qualitative sample are the individuals from the initial quantitative sample is because the intent of this design was to follow up the results from the survey and explore them in more depth (Creswell, 2014).

The explanatory sequential mixed methods approach accorded me an opportunity to analyse quantitative and qualitative databases separately (as seen in Figure 3.1). This approach therefore, aided me in explaining survey responses with qualitative follow up data collection and analysis from the interviews (Creswell, 2014), thus adding to the validity and reliability of my research project.

Within an explanatory sequential mixed methods design, the quantitative results provide a general picture of the research problem whilst the qualitative results explain, refine or extend the general picture (Creswell and Plano Clark, 2011). Qualitatively and quantitatively analysing the survey questionnaire responses using thematic analysis and statistical techniques respectively, gave me a wide overview of the teachers' experiences relating to how and to what extent they use visual approaches to teach Pythagoras' theorem word problems to Grade 9 learners in the district. This data informed the drafting of this study's

intervention programme. Teachers' perceptions and experiences of using visual approaches after taking part in an intervention programme were analysed using qualitative approaches. Qualitative findings from the interviews helped to explain the findings from the survey (Maree, 2016).

3.4 RESEARCH METHODS

The study fundamentally made use of two methods. The first method was a survey of eighty-seven Senior Phase and FET teachers in the district, whilst the second one consisted of a case study of three selected Grade 9 teachers that participated in a ViPProS intervention programme.

3.4.1 Survey

A survey was the first method of this study. According to Cohen et al. (2011), surveys are a means of gathering data at a particular point in time with the intention of describing the nature of existing conditions, or identifying standards against which existing conditions can be compared. They further assert that a survey gathers standardised information on a one short-term basis from a large number of people and makes generalisations about the targets of focus. Similarly, Creswell (2014) also writes that with a survey, there is rapid turnaround in data correction.

The survey questionnaire consisted of two parts (A and B) as shown in Appendix 6. Part A was a general section with questions that sought the teacher's profile. Part B consisted of questions that were designed to ascertain how teachers in the district taught Pythagoras' theorem word problems prior to an intervention programme. The survey helped me in finding out what visual tools teachers use, how and to what extent they use them. This information was then used to inform the intervention programme.

3.4.2 Case study

This research followed a case study approach with three selected Grade 9 mathematics teachers from three township secondary schools located in the JTG District of the Northern Cape Province, South Africa ― see Table 3.1 for the school profiles. The unit of analysis was twofold: firstly, to identify and understand how the selected teachers made use of various visual resources provided in the intervention to develop conceptual understanding during their teaching of word problems on Pythagoras' theorem. Secondly, to identify these same teachers' experiences and perceptions on their use of visual models to teach Pythagoras' theorem word problems.

Bertram and Christiansen (2014) define a case study as ""a systematic and in depth study of one particular case in its context, where the case maybe a person (such as a teacher, a learner, a principal or a parent), a group of people (such as a family or a class of learners), a school, a community, or an organisation"" (p. 42). Rule and John (2011) add that a case study is well suited to ""make sense of

feelings, experiences, social situations, or phenomena as they occur in the real world, and therefore study them in their natural setting"" (p. 60). Equally, Cohen et al. (2011) assert that with a case study, the researcher aims at capturing the reality of the participants' experiences of and thoughts about a particular situation. This was the case in this study as I wanted to gain an understanding of selected teachers' perceptions and experiences in using visual models to conceptually teach Pythagoras' theorem word problems as a result of participating in an intervention programme.

This study used an "intrinsic case study" (Maree, 2020). An intrinsic case study as defined by Maree (2020) is a study undertaken in order to better understand a particular case in question. He further argues that this type of case study is not undertaken primarily because it represents other cases or because it illustrates a particular feature or problem, but because in all its particularity and ordinariness, the case itself is of interest. This research project was an in-depth study of a single case of three selected Grade 9 mathematics teachers engaging in a ViPProS intervention programme and implementing it in their lessons. Therefore the close collaboration between the researcher and the participants enabled participants to share their stories (Maree, 2020).

Yin (1994) brings to our attention that a case study is a preferred approach when "how" or "why" questions are being posed about events and situations which the investigator has little or no control. In the second research question for example, I wanted to ascertain how selected Grade 9 mathematics teachers used visual approaches to teach Pythagoras' theorem word problems for conceptual understanding as a result of an intervention programme. I had less control over how the selected teachers would react during and after their engagement with these visualisation strategies. Therefore I had little control over the variables that were involved in this study.

Case studies recognise and accept that there are numerous variables operating in a single case, and therefore, to catch the implications of these variables usually requires more than one data collection tool (Cohen et al., 2011). They add that case studies can blend quantitative and qualitative data making them a prototypical instance of mixed methods research. The above-mentioned is typical of this case study research, which blended quantitative and qualitative data that was collected using various data sources that included a survey questionnaire, classroom observations and interviews.

Table 3.1: School profiles

School name	Type of school	District	Location
School 1	Technical and commercial high school	JTG	Township
School 2	Technical and commercial high school	JTG	Township
School 3	High school	JTG	Township

This study was not focusing on township schools and the fact that I only have this type of schools was because their teachers were the ones who met the selection criterion and willing to take part in my study.

3.5 POPULATION AND SAMPLING

Population as used in research refers to the total number of people, groups or organisations that could be included in a study while sampling entails making decisions about which people, settings , events or behaviours to include in the study. During sampling, the researcher needs to decide on how many individuals, groups or objects (such as schools) to observe. Furthermore, Mertens (2005) asserts that researchers typically select samples with the goal of identifying information-rich cases that will allow them to study a case in-depth when working within the interpretive paradigm. The population of this study comprised of the Senior and FET Phase mathematics teachers of the Northern Cape Province. In the South African context, the Senior Phase comprises Grades 7 – 9 while Grades 10 – 12 constitute the FET Phase. The study had two samples. That is, the quantitative sample (which was a survey of Senior and FET Phase mathematics teachers in the JTG district of the Northern Cape Province) and the qualitative sample (which comprised of the three study participants from different schools and the four Grade 9 resource teachers identified by the district's Senior Phase subject adviser).

3.5.1 Senior and Further Education and Training (FET) Phase mathematics teachers

This study was carried out in the JTG District of the Northern Cape Province. I chose this district because it is where I teach. One hundred-and-fifty survey questionnaires were sent out to all Senior Phase and FET teachers in the district. Eighty-seven teachers managed to respond to this questionnaire and they generated data that helped me to answer the first research question.

3.5.2 Resource teachers for the intervention

The second group of teachers I required were the four resource teachers whose role was to help me in drafting an intervention programme, which I called Visualisation Processes for Problem Solving (ViPProS) intervention programe. That is, their intention was not to provide any data for this research, but

to help in shaping the ViPProS programme. I had three meetings on different days with these teachers after working hours. Each two-hour meeting was to design a tentative ViPProS programme. They shared their experiences and expertise with me in teaching MWPS in the context of Pythagoras' theorem using visualisation strategies.

The study focused on the Senior Phase. There was therefore no need to engage the FET subject advisor. Since the Senior Phase mathematics subject adviser knows the teachers in the district, I collaborated with her to assist me in identifying the four resource teachers that would help me in designing and drafting a ViPProS programme for this study. These teachers were purposefully selected. According to Cohen et al. (2011), in purposive sampling, ""a researcher hand-picks the cases to be included in the sample on the basis of their judgement of their typicality or possession of certain features being sought"" (p. 156). The sampling was purposive because I liaised with the subject adviser on the criteria (to use), which she combined with her knowledge of teachers in the district to select these resource teachers. I was particularly interested in teachers from well-resourced schools that regularly use visual approaches in their mathematics lessons. For the purpose of this study, well-resourced schools are schools whose teachers not only have access to teaching aids, but also often use them in their lessons. I acknowledge that I cannot make the assumption that teachers from these schools are excellent teachers in using visualisation strategies to teach word problems in the context of Pythagoras' theorem. I thus preferred to work with teachers that make fascinating use of the manipulatives available at their schools in creating visually stimulating mathematics learning milieus. In confirming that the selected teachers met this criterion, the subject adviser had to check with the teachers' mathematics HODs in their respective schools. Furthermore, I worked with experienced teachers from well-resourced schools who were willing to participate and who were also easily accessible. Table 3.2 below presents the profiles of resource teachers.

Table 3.2: Profiles of resource teachers

Teachers' names	Qualifications	Area of specialisation	Years of experience	Gender
Teacher 1	Bachelor of Science and Mathematics statistics and PGCE (SP &FET)	Mathematics	8	Female
Teacher 2	Bachelor of science with Education	Mathematics and Physical sciences	20	Male
Teacher 3	B Ed Honours in mathematics Education	Mathematics	18	Female
Teacher 4	Bachelor of Education mathematics Education (SP & FET)	Mathematics	30	Female

3.5.3 Research participants

The third group of teachers I required were research participants, and these provided data for this study by participating in the intervention programme. Three participants from the JTG District in the Northern Cape were also purposefully selected for this study. Since one of the criteria for selecting study participants for this research study entailed convenience of access to the teachers, I had to go in person and talk to each potential participant about my study. During our chats, I asked each teacher how s/he teaches word problems in the context of Pythagoras' theorem. From this conversation, I was then able to identify teachers who met my broad selection criteria:

i) Teachers' willingness to participate in this study.

i) Convenience of access to the teachers, that is, their schools had to be close to mine.

ii) Teachers teaching in the JTG District in the Northern Cape, South Africa.

iii) Mathematics teachers with some experience in using visualisation processes in their teaching.

iv) Teachers with at least three years of teaching experience at Senior Phase (Grades 7, 8 or 9). This is because participants should have sufficient knowledge of Pythagoras' theorem as it is required to be taught at Senior Phase level.

v) They had to be teaching Grade 9 since this study was targeting mathematical content that is for this grade as stipulated by the CAPS.

The profiles of these teachers are presented in Table 3.3 below.

Table 3.3: Profiles of teacher participants

Teachers' names	Qualifications	Area of specialisation	Experience teaching mathematics	Number of lessons taught	Gender
Teacher A	Bachelor of Science Honours in Mathematics plus PGCE	Mathematics and Physical science	5 years	1	Female
Teacher B	Bachelor Science in Mathematics and PGCE	Mathematics	23 years	3	Female
Teacher C	Bachelor of Education	Mathematics and Economics	30 years	3	Female

This study was not focusing on gender issues and the fact that I have females only was because they were the ones who met the criteria and were willing to participate.

3.6 RESEARCH DESIGN

A research design is defined as "a plan or strategy that moves from the underlying philosophical assumptions to specifying the selection of participants, the data-gathering methods to be used and the data

analysis to be done" (Maree, 2020,, p.80). This study's research design entailed five phases that were interlinked. Below is a description of these phases of the data generation process.

3.6.1 Phase 1: Survey questionnaire piloting, distributing and selection of resource teachers

In this phase, I started by piloting the survey questionnaire (see Appendix 6) with two mathematics teachers at my school. The teachers' responses enabled me to check for ambiguities in questions and also to see if the questionnaire was effective in generating data that is appropriate for this study. The teachers indicated to me that there were questions (3, 6, 7, 8, and 9) which they did not understand because of unfamiliar vocabulary that was used. The feedback from the pilot exercise offered me an opportunity of making the necessary changes in the questionnaire by including the definition of visualisation processes and visual models so as to make this tool suitable for my study (see Appendix 6). After this piloting exercise, survey questionnaires were disseminated to the one hundred-and-fifty Senior Phase and FET mathematics teachers in the district with the help of the mathematics subject adviser. At first only twelve questionnaires were returned. Fortunately, the subject adviser informed me of a mathematics workshop which was to take place a few days later. Teachers were sent reminders on their WhatsApp forums to come with their questionnaires, but most of them had lost them. The subject advisers gave me twenty minutes for the teachers who had lost their questionnaires to fill in new ones. I managed to collect seventy-five questionnaires that day. The questionnaire aimed at ascertaining how these teachers in the JTG District taught Pythagoras theorem and word problems in this context prior to my planned intervention. It is in this phase that the four resource teachers (see criteria in section **3.5.2**) were selected to assist in drafting an intervention programme for this study.

3.6.2 Phase 2: Analysing of the survey data

It is in this phase that I analysed all eighty-seven respondents' questionnaires. Data from this questionnaire was analysed quantitatively using descriptive statistics (tables, pie charts and graphs) and qualitatively looking at both the emerging themes and themes informed by literature. The data analysed in this phase helped me to answer Question 1. The results of the analysis of the survey questionnaire together with experiences of the resource teachers on how to use visualisation processes when teaching, were used to draft the intervention programme.

3.6.3 Phase 3: Planning of a teaching programme and workshops with study participants

It is in this phase that I invited six teachers to an awareness workshop in which we discussed the ViPProS programme that was designed and drafted in Phase 2. Integral to this workshop's content was creating an awareness of conceptual understanding and how visualisation strategies could aid the development of conceptual understanding. After this workshop, based on their willingness and active participation during

workshops, I invited three teachers to volunteer to participate in this study. I then conducted three ViPProS workshops with these teachers (that ran for two hours each) where I formally introduced them to some of the approaches to teaching Pythagoras' theorem using visuals (that is the two area models and scaled-up versions of an original right-angled triangle) as articulated in Section 2.4.1.

I workshopped teachers on how to use each visual model to solve MWPs on Pythagoras' theorem as well as creating an awareness of promoting conceptual understanding in the teaching of mathematics through visualisation processes. During the workshops, we examined how the teaching of Pythagoras' theorem with the aid of the mentioned models could assist in honing conceptual understanding. The selected teachers were not mere recipients but active participants in the on-going discussions. After the discussions, we further collectively developed four lesson plans with one lesson plan per an area model and two lesson plans for the scaled-up versions of an original right-angled triangle ― see Appendix 8a to 8d. These contained word problems that the teachers used during their teaching in conjunction with the models. The participating teachers each received visual models prepared by me that they used during workshops and their lessons, and which, as a gesture of gratitude they took home after the study. Below is a summary of the lessons that were taught and observed.

Table 3.4: Teacher A, Lesson 1

Teacher	**Details of the lesson planned**	**Visualisation activities**
TA	Subject: Mathematics Grade: 9 Time allocated: 40 minutes Topic: Pythagoras' theorem ― using Euclid's area model This lesson focused on the use of the area model to establish a relationship between the sides of a right-angled triangle and how to use it to solve MWPs on Pythagoras' theorem.	• The teacher used a manipulative and a drawing of the area model in establishing the relationship between the sides of a right-angled triangle. • The triangles in each model had different dimensions, but the areas of the squares drawn on their sides were related in each case as in Pythagoras' theorem. • Used the area model to solve the word problem.

Table 3.5: Teacher B, Lessons 1– 3

Teacher	Details of the lesson planned	Visualisation activities
TB	**Lesson 1** Subject: Mathematics Grade: 9 Time allocated: 40 minutes Topic: Pythagoras' theorem — using Euclid's area model. The teacher demonstrated how to use the area model to represent Pythagoras' theorem and how to use it to solve word problems.	• Reminded learners that the theorem is only applicable to right-angled triangles. She drew examples from learners' home backgrounds in trying to relate the theorem to real life. • Pasted two manipulatives of the area model with right-angled triangles of different dimensions. • She engaged learners in counting unit squares in each square as a way of establishing their dimensions and/or areas. • They used these areas in each model to establish a relationship between the sides of a right-angled triangle. • Looked at real life problems and encouraged learners to use the area model and also work in their groups.
	Lesson 2 Subject: Mathematics Grade: 9 Time allocated: 40 minutes Topic: Pythagoras' theorem — proof by rearrangement. The teacher presented a lesson on how to use the area model to establish the relationship between the sides of a right-angled triangle and to solve word problems.	• Assembled four congruent triangles on a board to produce two squares. • Pasted a square to cover up the remaining space and asked learners for the dimensions and the area of this square. • Removed the square and transformed the two triangles to produce rectangles; she filled the remaining spaces with two squares and asked learners for the areas of these squares. • The two squares were removed and pasted next to the first square**.** Learners were asked to establish a relationship between the areas of these three squares. • Wrote a word problem and discussed it with learners. They jointly came up with a sketch as they interpreted the problem. • Encouraged learners to use manipulatives of the area model to solve this problem in their groups
	Lesson 3 Subject: Mathematics Grade: 9 Time allocated: 80 minutes Topic: Pythagoras' theorem — using scaled-up versions of a right-angled triangle The focus in this lesson was to make sense of Pythagoras' theorem using various shapes built by putting together the three cut out scaled triangles	• Asked learners to draw diagrams that they used in understanding the two, word problems in the previous lesson which was not recorded. • Learners used the three scaled-up versions of their original triangle they had created in the previous lesson to build different shapes (isosceles triangle, kite, square, rectangle, and parallelogram). • The teacher used her manipulatives to put learners' shapes up on the board; these formed a basis for a whole class discussion as they tried to verify whether Pythagoras' proposition as stated in the word problems was valid. • Images developed from the same transformation process in both questions were presented concurrently.

Table 3.6: Teacher C, Lessons 1– 3

Teacher	Details of the lesson planned	Visualisation activities
TC	**Lesson 1** Subject: Mathematics Grade: 9 Time allocated: 40 minutes Topic: Pythagoras' theorem — using Euclid's area model The focus of this lesson was to use this model to establish a relationship between the sides of a right-angled triangle and how to use it to solve word problems.	• Asked what Pythagoras' theorem was and where it is used. • Three area models were used: one was a manipulative and the other two were drawn by learners. She asked all groups to draw a right-angled triangle with the same dimensions except for one group, which was told to draw a non-right-angled triangle with similar dimensions. • She engaged them in counting the unit squares in each square so as to establish their areas, which were to be used to develop their knowledge of the Pythagorean concept. • Gave learners a word problem to work on in their groups encouraging them to use visual approaches.
	Lesson 2 Subject: Mathematics Grade: 9 Time allocated: 40 minutes Topic: Pythagoras' theorem — proof by rearrangement (left-over spaces must be equal) The teacher looked at how to use this area model to establish a relationship between the sides of a right-angled triangle and to solve word problems.	• Checked learners' knowledge of quadrilaterals. • Pasted a manipulative of a right-angled triangle on the board and labeled its sides *a, b,* and *c.* She pasted the remaining three triangles such that the four congruent right-angled triangles formed two squares. • Filled up the empty space with a square and asked learners for its dimensions and area. She removed the square and pasted it aside. The two right-angled triangles were transformed into rectangles. Two squares were then placed in the left-over spaces; learners were asked for the dimensions and areas of these squares. • They were asked to establish a relationship between the three squares that filled up the remaining spaces. • She wrote a word problem and encouraged learners to use manipulatives of this area model as they worked in their groups.
	Lesson 3 Subject: Mathematics Grade: 9 Time allocated: 80 minutes Topic: Pythagoras' theorem — using scaled-up versions of a right-angled triangle. The focus of this lesson was to build various shapes by putting together the three cut out scaled triangles, and using these shapes to assist the teachers and their learners in making sense of the Pythagoras' proposition.	• Asked learners to draw sketches of the scalene and isosceles triangles they developed when interpreting the word problems in the previous lesson. • Asked learners to put together their three accurate scaled triangles of the original triangle so as to produce new shapes. • She used her manipulatives of the three scaled-up right-angled triangles to build these shapes on the board • She used the shapes to create a rich and meaningful classroom dialogue that enabled the class to verify whether the statements ($a^2 + b^2 = c^2$) alluded to in the word problems were correct.

3.6.4 Phase 4: Pilot study

The first lesson in Cycle 1 was a pilot done by the selected individual study participants with a Grade 9 class in their schools not taking part in the intervention. The pilot's aim was to check if the data collection tools worked. This lesson was video-taped. The video-taping at this stage was also a pilot that enabled me to determine the effectiveness of the recordings before the actual implementation of the teaching. I, together with the teacher analysed the piloted lesson to clear any challenges that may have risen in the pilot. As we watched the video during the interview, I realised that I had missed out on some of the aspects that the teacher was articulating because I focused mostly on what the learners were doing. With this experience, I had to shift my attention from the learners to the teacher in the coming lessons because my study was about teaching, not learning.

3.6.5 Phase 5: Implementation of intervention programme

The programme took place in three cycles, all coinciding within term 2 and 3 of 2019. Data collection followed from the three lessons after the pilot exercise. A diagram showing what took place in each cycle is illustrated in Figure 3.2 below, with Cycle 3 following a similar pattern.

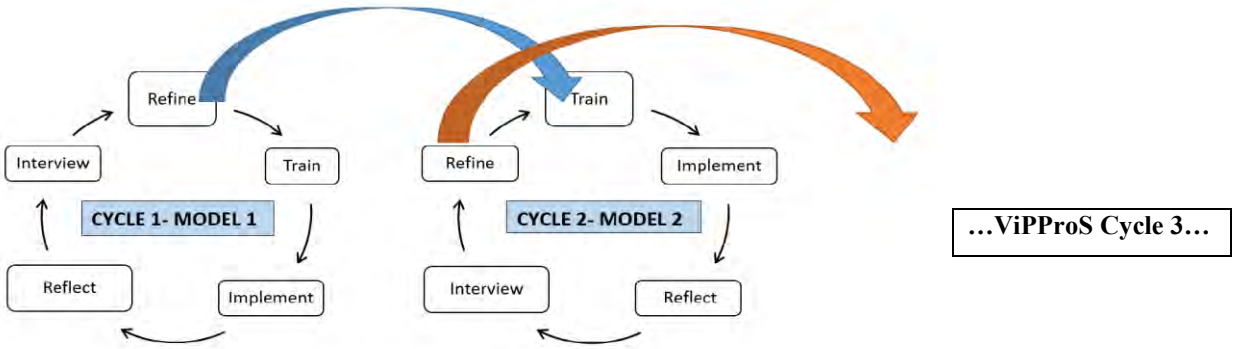


Figure 3.2: Cycles of the ViPProS programme: the empirical field

Each cycle focused on a particular model with cycle 1 and 2 containing one lesson each and cycle 3 containing two lessons. I took the role of "non-participant observer" (Bertram & Christiansen, 2014), observing teachers presenting their lessons on Pythagoras' theorem to their Grade 9 classes using the three visual models. My study used video-taping for observation purposes. I video recorded one lesson per cycle in each teacher's class. Therefore, seven classroom observations were conducted (see Section 3.6.3). My focus was on how the teachers made use of the various resources provided in the intervention to develop visualisation and promote conceptual understanding when teaching Pythagoras' theorem word problems. This assisted me to answer Research Question Two. The collection of data for this study lasted for at least two months.

3.6.6 Phase 6: Interviews

Each video-taped lesson was followed by a reflective one-on-one stimulated recall interview (after each cycle) with the teacher to establish the participant's views and experiences on how and whether the use of visual models enhanced the teaching of Pythagoras' theorem word problems. Furthermore, at the end of all cycles, a focus group interview was conducted with the two teachers. The interviews were audio-recorded. The teachers collectively reflected on how visuals were used and whether they facilitated the teaching of MWPs. All this data assisted me in answering Question 3.

3.7 DATA COLLECTION INSTRUMENTS

I used the following data collection instruments in my case study research: survey questionnaires, observations and interviews (stimulated recall and focus group interviews). One of the reasons for utilising these three instruments together was to make up for the deficiencies in each one, a process called triangulation (Bertram & Christiansen, 2014) and to put more distinct data forward.

3.7.1 Survey questionnaire

A survey questionnaire as seen in Appendix 6 was distributed to all Senior Phase and FET mathematics teachers in my district with the aid of the mathematics subject advisers. I managed to receive eighty-seven questionnaire responses. The aim of using this questionnaire was to obtain data about how teachers in the district taught Pythagoras' theorem and word problems in this context prior to an intervention programme. The questionnaire enabled me to collect different categories of data. These included background information, teaching practices and the use of visuals. For example, in the category of teacher's use of visuals during teaching, I was looking for information on teachers' experiences relating to what extent they were using visuals to extend what they were teaching to real life and build on learners' prior knowledge among others. This data was analysed and used to inform the drafting of the intervention programme.

3.7.2 Observations

'Observation' refers to ""a systematic process of collecting data that relies on a researcher's ability to gather data through his/her senses without questioning or communicating with participants"" (Anthanasou et al., 2018, p. 91). Cohen et al. (2011) hold the view that:

> observation offers the investigator the opportunity to gather 'live' data from naturally occurring social settings; allowing the researcher to look directly at what is taking place *in situ* rather than relying on second–hand accounts, p. 456.

Similarly, Bertram and Christiansen (2014) aver that observation enables the researcher to report on things she has witnessed and recorded herself as opposed to the things other people have to tell her. This

blends well with my study in which I video recorded seven lessons of the three participating teachers (see Section 3.6.3 above) as they taught using visual models as visualisation tools. I was particularly looking for data on how the selected teachers used visuals to enhance learners' conceptual understanding of Pythagoras' theorem word problems. During the lesson, I was positioned in an unnoticeable but yet strategic position, which enabled me to video record the whole lesson. In observing these teachers' actions with respect to using visual models, the analysis of observation specifically focused on visualisation processes as propounded by Kossyln (1996) — see Table 3.8. In observing evidence of conceptual teaching, the analytical instrument as portrayed in Table 3.9 was employed.

3.7.3 Stimulated recall interviews

Stimulated recall interviews entail video recording the teacher while engaging in instruction, then a recall session is conducted after the recording has taken place in order for the teacher to reflect on his/her teaching (Stough, 2001). I conducted a one-on-one stimulated recall interview after every lesson taught by a participating teacher during the reflective workshops. The teacher and I discussed the video-taped teaching sequence as we watched the recorded lesson. The teachers were free to stop the video recording at strategic intervals to reflect on their teaching using visual approaches. I also paused the video tape at any time I wished and asked open-ended questions like: *What were you trying to accomplish at this point? Why did you utilise visuals here? How did the use of visual approaches at this point enhance conceptual understanding?* The open-ended questioning approach accorded teachers an opportunity to provide as much information as they could and it also afforded me an opportunity to probe for clarity. These interviews were guided by the themes discussed in the analytical framework in Table 3.9. The questions that I was asking were directly related to how participants used visuals in their lessons. My Interest was in eliciting teachers' perceptions and experiences on how and whether the use of visual models enhanced the teaching of Pythagoras' theorem and word problems. Each participative reflective session was audio taped for analysis purposes.

3.7.4 Focus group interviews

These are a form of group interview that capitalise on communication between participants in order to generate data (Kitzinger, 1995). With a focus group approach, group processes can aid participants to explore and clarify their views in ways that would less easily be accessible in a one-to-one interview (ibid.). At the end of the third cycle, I met with the two teachers to conduct a focus group interview which was recorded. The teachers collectively reflected on whether the use of visual models enhanced the teaching of word problems in the context of Pythagoras' theorem. If it did enhance teaching, how? If not, I then probed for reasons and what they perceived as possible solutions.

Table 3.7: A summary of the data gathering techniques

Stage	Tools	Data generated	Purpose
Stage 1	Survey questionnaires	Qualitative and quantitative data	To ascertain teachers' experiences relating to how and to what extent they use visual approaches to teach Pythagoras' theorem word problems.
Stage 2	Observations	Qualitative data	To identify and understand how the selected teachers made use of various visual resources provided in the intervention to develop conceptual understanding during their teaching of Pythagoras' theorem word problems.
Stage 3	Interviews	Qualitative data	To identify these participating teachers' experiences and perceptions on their use of visual models to teach Pythagoras' theorem word problems.

3.8 DATA ANALYSIS

Analysis is a close or systematic study, or the separation of a whole into its parts, for the purpose of study (Bertram & Christiansen, 2014). Data analysis is therefore a ""process of making sense out of data"" (Merriam, 1998, p. 178). This study employed an explanatory sequential mixed methods design where I collected data in two separate phases: I began by collecting and analysing quantitative and qualitative data from the survey questionnaires and this was followed by the collection and analysis of qualitative data from lesson observations and interviews during the intervention as shown in Table 3.7. The data analysis process consisted of three stages as seen in Table 3.7.

Stage 1 Analysis

In Stage 1, I analysed survey questionnaire data both quantitatively and qualitatively. Descriptive statistics were used to analyse quantitative data from questionnaire responses. The results were presented using tables, pie charts, graphs, and percentages. However, most of the data collected from the survey were qualitative and these were therefore qualitatively analysed. In this case it was the data collected from teachers' experiences relating to how they taught Pythagoras' theorem word problems prior to the intervention

Stage 2 Analysis

In Stage 2, I analysed the data from lesson observations qualitatively. In this stage, the lesson analysis focused on two parts: firstly, the focus was on the teaching of Pythagoras' theorem using the three visual models. The focus here was on how selected teachers used visuals to teach Pythagoras' theorem word problems taking note of the dominant themes such as image generation, image inspection, image transformation and image use as shown in the analytical tool A, discussed in Table 3.8. However, interestingly there were other visualisation processes that emerged from the lesson observations' data.

These themes include: visual language (Debes, 1969), gestures, generalisation and symbolic inscriptions (Presmeg, 1991). In identifying image inspection for example, I was particularly interested in how the generated images were examined by the teacher and learners in establishing a relationship between the sides of a right-angled triangle and/or in solving word problems. The verbatim interview of Teacher A below is an indication of image inspection. She said, "*...at first we were actually counting in order to find the size of a square and also they could find area of a square, now they could see that to find hypotenuse side, it is not just adding the other two sides, but you need to add the areas of the squares on the two sides in order for you to find area of the square on the hypotenuse.*"

Secondly, the focus was on how the use of visual models to teach Pythagoras' theorem word problems enhances learners' conceptual understanding. The focus was on how selected teachers build on learners' prior knowledge, represent mathematical concepts in different ways, extend mathematics to real life contexts and connect ideas and concepts in mathematics as seen in the analytical instrument B**,** discussed in Table 3.9. For instance, in extending mathematics to real life, I was particularly interested in how teachers effectively used visuals to show the relevancy of what they were teaching to learners' everyday life situations. Here is an observation from Teacher A's lesson "*... for those of you who did not get it, let me use this broom as my ladder. This ladder is leaning against a wall and it forms a right triangle-angled triangle as you can see. So this one is the wall and the wall from there to where the ladder meets it has a height of 8m and down there they say it is 6m. This broom is our ladder we are supposed to find its length.*"

Table 3.8: Analytical tool A — Visualisation processes indicators (adopted from Kossyln, 1996)

Visualisation processes pointers	**Teacher' activities in explaining each indicator in relation to each model**
Image generation	The teacher spontaneously creates an image to develop the mathematical concept. It is this image that forms the basis from which the teacher will then manipulate certain features in order to further demonstrate or develop the mathematical concept.
Image inspection	There is evidence of the teacher examining the visual image in order to establish patterns in the image, which are discussed by the entire class. These patterns are demonstrated by manipulating the image.
Image transformation	The teacher re-arranges spatial representations (images) in order to produce new shapes (related or not) that can aid in demonstrating certain properties and/or criteria. This may be evident through transformations such as scaling, rotating and translating shapes.
Image use	The teacher uses the visual image(s) to emphasise and develop a particular concept and/or relationship. For example, how does the teacher articulate that the sum of the squares of the two legs of a right-angled triangle is equal to the square of the hypotenuse? How does the teacher for instance make use of a kite to justify that Pythagoras' proposition holds true? How does the teacher manipulate features of a square to show that it is a special type of kite?

Table 3.9: Analytical tool B — Conceptual understanding indicators (adapted from Kilpatrick et. al., 2001)

Conceptual understanding pointers	Description
Building on prior knowledge	The teacher's ability to make links between the mathematics she is teaching and what learners already know. This requires that learners learn mathematics with understanding and build new knowledge from prior experiences.
Multiple representation	The teacher's ability to represent the same mathematical concepts presented in a particular word problem in diverse ways and explore the relationships with the learners.
Extending mathematics to real life contexts	Teachers relate what they are teaching to new areas or daily life contexts, thereby enabling learners to see the practical applications of this knowledge. The teacher connects mathematics to the outside world to help learners to make sense of what they are learning and to solve real world problems — in this case MWPs. The teacher utilises everyday shapes and articulates them in relation to the used model, drawing examples from the learners' context.
Making connections among concepts and ideas in mathematics	The teacher uses visuals to establish connections among concepts and to establish relations. The teacher uses visuals to explore relationships among concepts and to make connections. He encourages learners to explore links among concepts. The teacher uses visuals in a way that allows learners to connect new concepts to what they already know.

Stage 3 Analysis

To analyse the data in this stage, I adopted a thematic analytic approach for qualitative data generated from interviews. Thematic analysis is defined by Braun and Clark (2006) as a method of identifying, analysing, and reporting patterns within data. Thematic analysis is an essentialist method for my research because it reports experiences, meanings, and the reality of study participants. Additionally, this approach of analysis examines the ways in which events (interviews), realities, meanings and experiences are the effects of a range of discourses operating within society (Braun & Clark, 2006). Thematic analysis entails searching across a data set to find recurring patterns of meaning (ibid.). This blends well with this study's interview analysis because after transcribing interview data, I engaged in repeatedly reading this data for any patterns related to selected teachers' experiences and perceptions on their use of visual models to teach Pythagoras' theorem word problems.

I approached this data with specific questions in mind (connecting mathematics to prior knowledge, connecting mathematics to the real world, connecting concepts and ideas in mathematics and use of multiple representations) which I coded. I used different colours to highlight the texts in the transcripts that were speaking to these themes. However, as I matched the codes up with the data extracts, certain patterns emerged during the analysis. The patterns that emerged are teachers':

- appreciation of the use of visuals
- beliefs in the use of visuals

- participation in the use of visuals
- impressions of the use of visuals
- insights into the use of visuals
- challenges in using visuals.

3.9 VALIDITY

Cohen et al. (2011) contend that ""validity is an important key to effective research. If a piece of research is invalid then it is worthless"" (p. 179). In enriching validity, I used more than one data source for triangulation purposes. Triangulation is the collection of data from various data sources to see if the data collected from one source contradicts or confirms the data from another source (Betram et al., 2014; Creswell, 2014). Combining different data gathering methods enables the researcher to obtain a better substantive picture of reality (Berg, 2007).

This study aligns well with the above descriptions as I used multiple data generating methods to validate the data collected. The data was collected using a survey questionnaire, lesson observations, stimulated recall interviews and focus group interviews. Using various data collection methods in one study helped me to identify common and different themes that arose from my study. I triangulated the interview questions in the interview schedule by asking the same questions in different ways. This helped me to check if the respondents answered the same questions in a similar manner.

Stimulated recall interviews were used to assist in validating the video tapings as I watched each recorded lesson with the teacher. These entailed asking probing questions in trying to help the teachers to recollect their actions on the use of visual aids in their lessons. Engaging in the probing approach also helped me to validate the data from interviews.

Creswell and Miller (2000) define member checking as the process of taking back data and interpretations to study participants in order to confirm the credibility of the information and narrative account. Recorded interviews were transcribed and given back to each teacher to examine and comment on whether he thought it was an accurate reflection of what he said. This member checking or respondent validation allowed me to correct errors of fact. It also offered respondents an opportunity to verify whether what I had transcribed was a true reflection of what they had said. In ensuring the validity of the survey questionnaire, analytical tool, interview and observation schedule, I had my supervisor review them in order to check for any validity threats.

Furthermore, in ensuring credibility, I used mechanical means to record data (Bertram & Christiansen, 2014). For example, I used video and audio recording devices respectively to record lesson observations

and verbatim interviews. I believe that technology contributed to the credibility of this study's data because transcripts are more accurate than the researcher having to jot down notes during interviews or lesson observations.

Lastly, the participants had to pilot a lesson with a Grade 9 class not taking part in the study. This assisted in rectifying any mistakes in the data collection tools. In maintaining questionnaire validity, since this instrument was researcher-made, it passed through necessary procedures to establish its validity. Firstly, I presented the questionnaire to my supervisor for his recommendations and suggestions. The supervisor's input for the improvement of the questionnaire was taken into consideration by the researcher. Secondly, I piloted the survey questionnaire with two mathematics teachers at my school before disseminating it to mathematics teachers in the district. This assisted in pointing out question ambiguities.

3.10 ETHICS

According to Bertram et al. (2014), ethics is a vital consideration in the research process, particularly research which entails humans and animals. The research employed ethical principles stipulated by Rhodes University Ethics Standard Committee (RUESC) such as respect and dignity, transparency and honesty, accountability and responsibility as well as integrity and academic professionalism. These are articulated in detail in the subsections that follow.

3.10.1 Respect and dignity

Before getting the participants' consent to take part in this project, I communicated the objective of the research fully and what was to be expected of them during the study. Participants were informed that they were free to withdraw from the study at any time. They were further assured that the data collected would remain confidential between me and my supervisor, unless I obtained their consent to do otherwise. In the case of the participants requesting the results of the study, I will make them available to them. Pseudonyms were used in the final write-up to protect the identity of participants and the research site. For example, codes such as **TA**, **TB**, and **TC** were used to distinguish between study participants while codes ranging from **Tr1** to **Tr87** were allocated to the teachers' questionnaire responses.

3.10.2 Transparency and honesty

In order to conduct this project, I firstly applied and obtained ethical clearance (see Appendix 1) from RUESC, a committee that is responsible for ethical approval at the university. Secondly, I applied for permission from the Head of the Northern Cape Department of Education (see Appendix 2) to allow me conduct research at schools. I then applied for permission from the school principals where I intended to conduct my study (see Appendix 5). I also applied for permission from the Senior Phase mathematics

subject adviser to assist me in selecting resource teachers for this study (see Appendix 3). Furthermore, each participant's transcribed interviews were given back to him to check and comment whether he thinks it was a true reflection of what he said.

3.10.3 Accountability and responsibility

The data collected was safely secured on my laptop's hard drive (for the duration of this study) which is password protected in order to prevent unauthorised access to it. Participants being my fellow colleagues, I maintained integrity at all times, not compromising Rhodes University's ethical standards. Being at the same professional level as my study participants, positionality issues did not have any effect on this research study. The participants knew me because we have always engaged in a community of practice on a fortnightly basis to improve the teaching of certain mathematics topics. My presence in these teachers' classrooms did not therefore cause them any discomfort. All teachers shared their experiences with ease during interview sessions.

3.10.4 Integrity and academic professionalism

This thesis was my own work and where I drew on ideas of other people, I acknowledged them according to the referencing guide of Rhodes University. The collected data for this research study are presented the way they are without any manipulations.

3.11 CONCLUSION

The research paradigm that underpinned my study and guided my design and process was discussed in this chapter. I also articulated how the six phases that I used in my research design enabled me to answer my research questions. Survey questionnaires, observations and interviews which were the dominant data collection instruments were also described. I further discussed the data analysis and the sampling process of my study. The chapter concluded with a presentation of the validity and ethical aspects of the study. The chapter that follows presents data collected from the above-mentioned data collection instruments.

CHAPTER FOUR

DATA PRESENTATION, ANALYSIS AND DISCUSSION

4.1 INTRODUCTION

The goal of this study was to investigate how, as a result of an intervention programme, selected mathematics teachers used visual approaches for conceptual understanding, to teach Grade 9 word problems in the context of Pythagoras' theorem. The data analysed in this chapter was obtained from survey questionnaires, lesson observations, and interviews (stimulated recall and focus group). The chapter begins with a presentation and analysis of survey questionnaire data on how, prior to an intervention programme, teachers in the JTG District taught Pythagoras' theorem and word problems. The chapter continues with an overview of the seven lessons taught by the selected teachers (see Section 3.6.3). This is followed by a presentation, analysis and discussion of data from observations of all the participants' lessons, taking note of the evident visualisation processes' dominant themes. Furthermore, Kilpatrick et al.'s (2001) conceptual understanding indicators (see Table 3.9) are discussed jointly in all the lessons observed. Research results from the interviews with study participants are discussed, taking note of these emerging themes: **teacher's appreciation of the use of visuals, teacher's participation in the use of visuals; teacher's impression of the use of visuals; teachers' insights into the use of visuals; and teacher's challenges in using visuals**. The chapter then concludes with a summary of the research findings from the three research instruments.

In this chapter, data is presented as authentically as possible. The study participants' real work, words and responses in the survey questionnaire, lesson observations and interviews have been utilised to support the presented data and these are coded as shown Table 4.1 below.

Table 4.1: Teacher coding

Codes	Meaning
Tr1	Teacher 1 in the survey
TAV1	Teacher A video transcript lesson 1
TBV2	Teacher B video transcript lesson 2
TCV3	Teacher C video transcript lesson 3
TAI1	Teacher A interview lesson 1
TBI2	Teacher B interview 2
TCI3	Teacher C interview 3
TBFGI	Teacher B focus group interview
TCFGI	Teacher C focus group interview

4.2 THE SURVEY QUESTIONNAIRE

Data from survey questionnaire responses of the eighty-seven teachers were first analysed quantitatively using descriptive statistics. The data was also later qualitatively analysed using the analytical tool developed from Kilpatrick et- al.'s (2001) indicators of conceptual understanding. My interest was to ascertain how teachers in the district taught Pythagoras' theorem and word problems prior to the intervention. Sticking to the ethical agreements, each teacher's questionnaire response was allocated a particular code; the codes ranged from **Tr1** to **Tr87** as explained in Chapter Three. The teachers' responses are presented and analysed in two parts: background information and emerging themes.

4.2.1 Background information

This section provides the context in which the study was done.

4.2.1.1 Teacher qualifications and experience

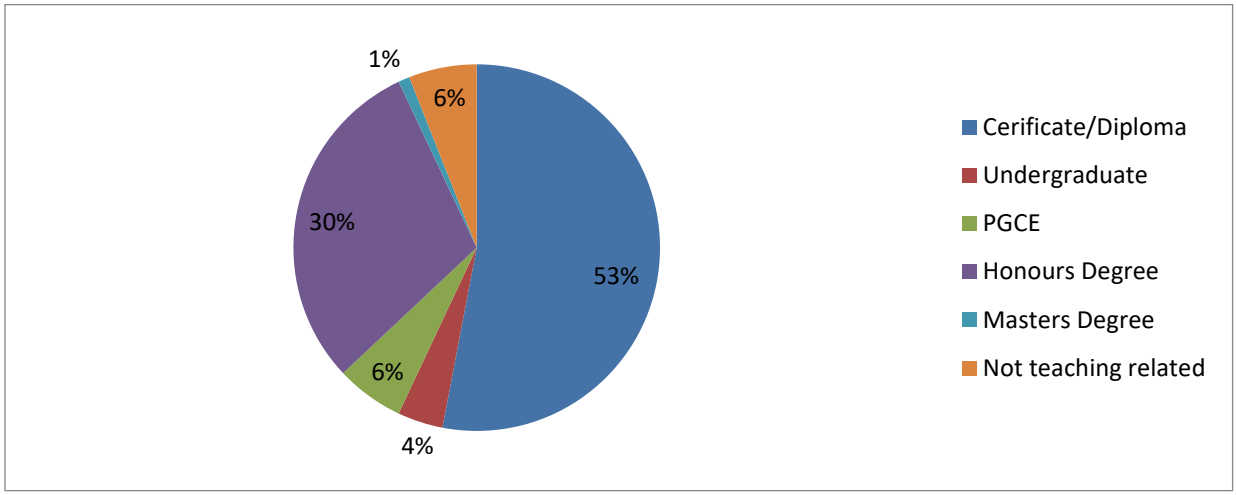


Figure 4.1: Qualifications of participants

Most of the teachers who participated in this survey were qualified to teach the grades and levels at which they were teaching. I noted that 41% of the surveyed teachers held at least a teaching degree related to mathematics and 6% of them possessed qualifications which were not related to teaching — see Figure 4.1. However, the majority of teachers (53%) were either diploma or certificate holders. Since the majority of teacher training institutions in South Africa are now offering BEd degrees or PGCEs, it revealed to me that the majority of teachers were trained more than fifteen years ago because the government shut down teacher training colleges between 1994 and 2000. The study thus noted that the implications of this are that if these teachers are not involved in in-service training programmes, they are less familiar with new trends and latest teaching strategies that have gained recognition in the last five or so years, such as the use of various visualisation approaches.

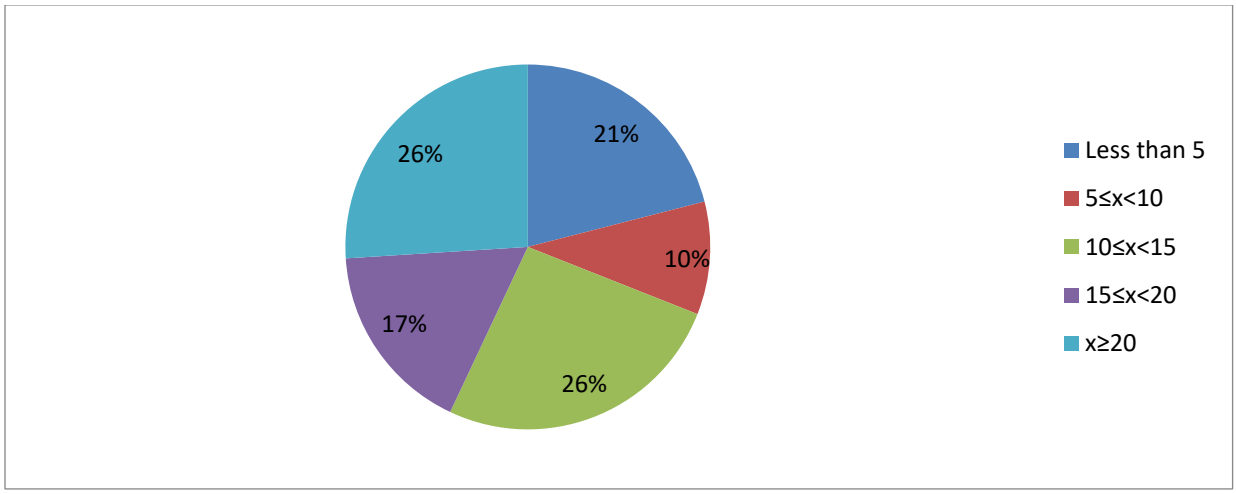


Figure 4.2: Teaching experience of participants

Of the surveyed teachers, 79% had attained five years or more of teaching experience (see Figure 4.2). Thus, this survey dealt mostly with teachers who had had substantial mathematics teaching experience at school level. It was thus my assumption that since the majority of teachers had already acquired much experience in teaching mathematics, they had also tested and tried various visual methods of teaching Pythagoras' theorem.

4.2.1.2 Teacher Training

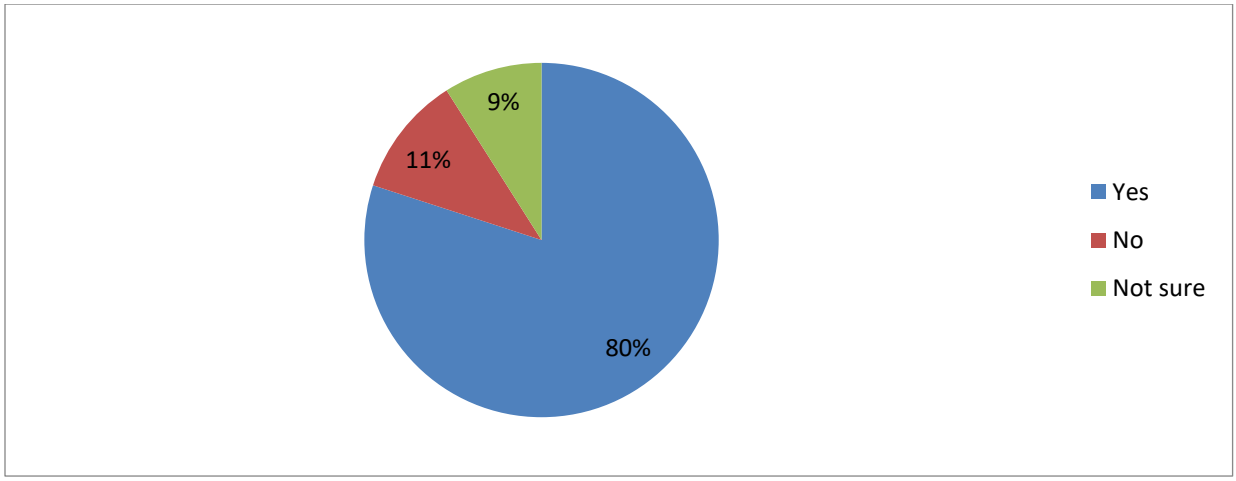


Figure 4.3: Teachers trained to use visuals or not

In this study, 80% (see Figure 4.3) of all participants indicated that use of visuals for teaching mathematics was part of their curriculum when they trained as teachers. Some respondents even cited examples of how they were trained to use visual models to teach Pythagoras' theorem. For example, **Tr19**

said "*We were trained using areas of squares, semi-circles and regular pentagons on the sides of a right-angled triangle in establishing the relationship between the three sides."* Of the surveyed sample, 11% of the respondents indicated that they were not trained to use visuals while 9% were not sure if this was taught during their training (see Figure 4.3). While this raised my hopes that I would find insightful ideas from these teachers' post college/university experiences, the same teachers highlighted that they were not using some of these strategies for various reasons. These included time constraints, lack of resources in their schools and syllabi related constraints among others. One teacher **[Tr32]** said that "*I was trained to use visuals but the training used limited references to practical scenarios*." Yet another one **[Tr58]** mentioned "*I was trained before the advent of white boards, smartboards, laptops and various computer software that is now available*." Such responses indicated to me that while the training was originally done, very few in-service programs later supported these teachers on how to use visual strategies in mathematics classes. During interviews, Teacher B explained her experiences after the workshops: "*While I remember being taught how to use manipulatives when I was at college, it was not on Pythagoras' theorem. Thus this workshop has helped me on how to use visuals now on Pythagoras' theorem and to solve word problems*" ***[TBFGI]***. Such responses from Teacher B indicated to me that some teachers benefited from the intervention.

4.2.1.3 Teacher use of visual models

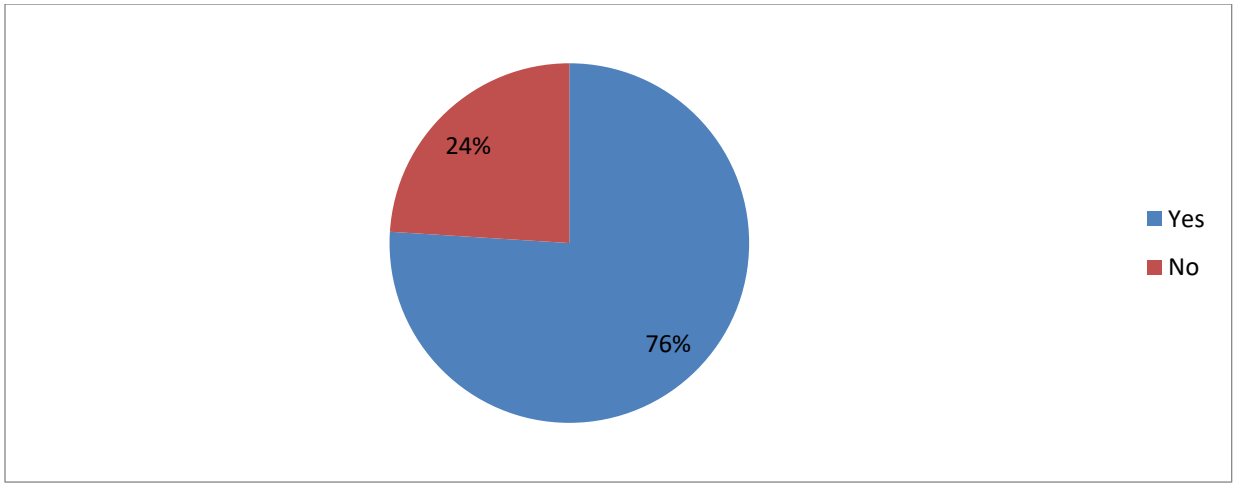


Figure 4.4: Teachers using visual models or not

In this survey, 24% of all the participants indicated that they were not using visual models in their lessons (see Figure 4.4). These teachers articulated how they taught Pythagoras' theorem, highlighting factors that prohibited them from using visuals aids (see Excerpt 1). Of the surveyed teachers, sixty-six (76%) of the respondents indicated that they use visuals in the teaching of Pythagoras' theorem (see Figure 4.4). Some

respondents even cited examples of visuals they use and how they use them to assist learners in understanding ─ see Section 4.2.1.4 and 4.2.1.5 respectively.

Excerpt 1: Some of the teachers' responses

Teacher code	Teacher responses	
	How I teach Pythagoras' theorem	**Prohibiting factors**
Tr27	*The old fashioned way. Drawing a right triangle on the board with a missing side then apply the formula.*	*Lack of electricity in our school*
Tr53	*Its conceptualisation of formula and application of formula to solve problems.*	*Time consuming and syllabi to be finished.*
Tr79	*I only use the given information to substitute and find the missing information.*	*I was never shown how.*
Tr81	*I draw a scale triangle*	*Lack of resources like teaching aids, ICTs and projector.*
Tr86	*Stating the formula, doing a few problems and giving them similar problems to try out.*	*There is always pressure on the completion of the syllabus. It is also difficult to use.*

It can be urged from the above excerpt that when teaching the Pythagorean concept, teachers in the district largely emphasise analytical approaches at the expense of visualisation strategies. Factors which include *inter alia*, time constraints, pressure on completion of the syllabus and a lack of resources were some of the factors highlighted by these teachers that prohibit them from using visualisation approaches.

4.2.1.4 Teaching aids used by teachers in the JTG District

This survey ascertained various visual aids used by the teachers in the district. The types of visual models used by 76% of the teachers that indicated use of teaching aids (see Figure 4.4) are portrayed in Table 4.2.

Table 4.2: Visual models used by teachers

Teaching aids	Frequency (%)
Diagrams, e.g. of right-angled triangles and squares.	38
Concrete objects e.g., electric pole supported by a wire, corners of buildings, floor of a classroom, sticks or straws and strings used to construct a right triangle.	9
Models of right-angled triangles and squares made from charts, square grids, wire, and on posters.	17
Information Communication Technology (ICT) e.g. projecting videos, GeoGebra software.	2
Pictures of right-angled triangles made from building blocks.	5
N/A	29

Table 4.2 highlights various teaching aids used by teachers in teaching Pythagoras' theorem and word problems. Most of the teachers (38%) preferred to use diagrams compared to other visual aids, indicating that diagrams do not require a lot of preparation. ICT was used by the lowest percentage of teachers, yet we live in the Fourth Industrial Revolution where technology is expected to be at the heart of the teaching/learning process.

Furthermore, of the 76% of the teachers who said that they use visuals (see Figure 4.4), about a third of them (29%) could not indicate what visuals they use and interestingly they also went on to answer a question on why they do not use visuals which was meant for the other group (24%) that indicated not using visuals. The prohibiting factors that were mentioned included challenges such as time constraints, the DBE's strict adherence to pacesetters, lack of support from subject advisers and lack of suitable equipment in schools among others. For example, one teacher **[Tr77]** said "*There's a lack of resources and time to go out and explore."* The factors that prohibited these teachers (29%) from using visuals were similar to those of the teachers (24%) who indicated not using visuals in Figure 4.4. While these teachers know the importance of using visuals, they always have excuses for not using them — hence there was a need for an intervention programme to show them how visuals save time and how they can be made from the available resources.

4.2.1.5 How teachers use visuals when teaching Pythagoras' theorem

This section focused on teachers' views and experiences on how they use visual material to assist learners to understand Pythagoras' theorem and word problems. The teaching aids that the surveyed teachers indicated using during teaching were diagrams, concrete objects, models, ICT and pictures (see Table 4.2). What was common to these teachers' experiences was the use of a visual proof of Pythagoras' theorem that dated back to Euclid. Excerpt two below portrays some of the teachers' experiences of how they use visualisation strategies to assist learners in understanding.

Excerpt 2: Some of the teachers' responses

Code	Teacher responses
Tr16	*Real life example like the electricity pole being supported by a wire which acts as the hypotenuse. I usually draw them to scale, then I draw squares on the side of this right- angled triangle and compare the areas.*
Tr19	*By using areas of squares, circles and regular pentagon on the sides of a right- angled triangle.*
Tr21	*I make square models with different measurements out of a square grid and put them on the board in a way that will form a right-angled triangle. Learners count the number of unit squares in each square and find the relationship between the number of squares in a bigger square and the total number of squares in two small squares.*
Tr29	*I use a projector to show videos such as water filling small squares and pouring into a bigger square on the hypotenuse.*
Tr34	*I put learners in small groups and ask different groups to draw right-angled triangles and others scalene triangles. I then ask them to draw squares with unit squares on the sides of each triangle. After, the number of unit squares on each side will be counted and recorded. The results will show that if you add the results on two sides, it is equal to the other side. From this learners are able to see on which type of triangle the theorem applies.*

From their experiences of using visual aids when teaching (see excerpt two), it can be reasoned that most teachers in the district are fond of using a visual proof of Pythagoras' theorem that dated back to Euclid. There was thus a need for an intervention to expose some teachers to other interesting visual models that can be used to teach Pythagoras' theorem word problems. After the focus group interview, Teacher C said "*I now know to use different visual aids in teaching Pythagoras' theorem word problems unlike before where I relied on the one from the textbook which has a triangle with squares drawn on its sides*" ***[TCFGI]***. Teacher C's experiences indicated to me that some teachers had profited from the workshop especially on how to teach Pythagoras' theorem word problems using different visual models.

4.2.2 Emerging themes

This section provides a detailed description of the themes that emerged from the surveyed teachers' responses relating to how they teach Pythagoras' theorem and word problems prior to an intervention.

4.2.2.1 Perceptions relating to teacher practices

Some teacher perceptions relating to their practice emerged from this study. Figure 4.5 shows that 21% of the surveyed teachers considered a good mathematics teacher as one who uses visual aids in every lesson. In this same district, 59% believed that good mathematics teachers are characterised by sequential and logical teaching. It is also evident in Figure 4.5 that 73% of the surveyed teachers regard a good teacher as one who emphasises formulae and procedures. This information indicated to me that most teachers in the JTG District still emphasise rote and algorithmic approaches at the expense of other approaches such as visualisation strategies.

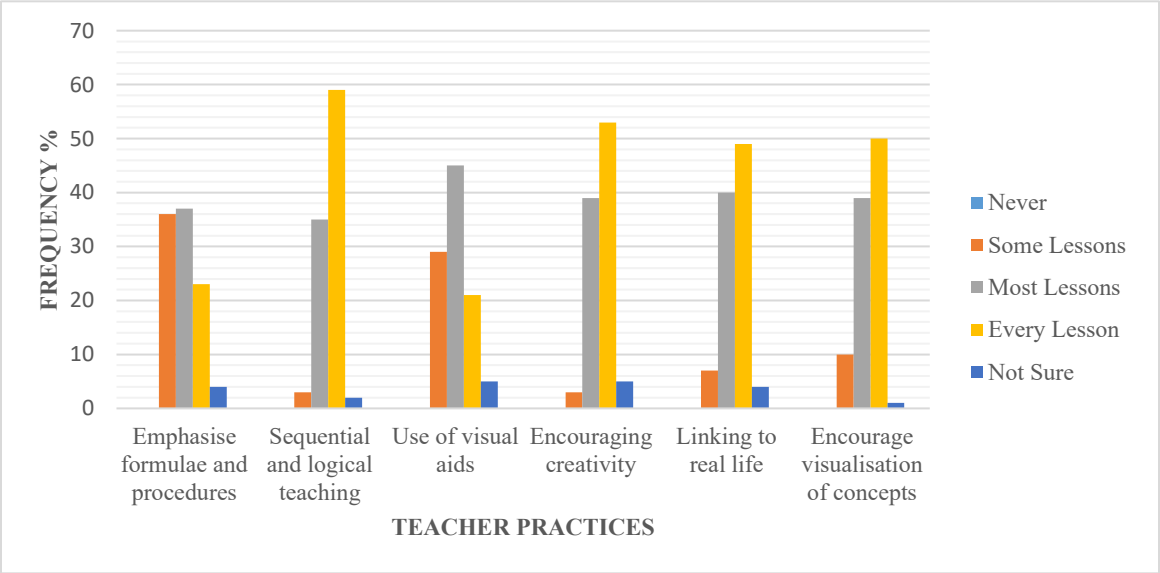


Figure 4.5: Good mathematics teacher practices

I also noted that 74% of the surveyed teachers (see Figure 4.5) were of the opinion that a good teacher uses visuals in some or most lessons but not necessarily in every lesson. This was so despite 50% saying that good teachers should encourage visualisation of concepts in every lesson. While teachers are keen to use visual tools and strategies in their teaching, it appears that they experience some constraints that may need to be identified and addressed. One teacher **[Tr77]** for instance said "*There is limited access to resources for visualisation and time to go out and explore prohibit me from using visuals in the teaching of Pythagoras' theorem*". These same teachers will thus need to be assisted to encourage visualisation of concepts in each of their lessons especially to enhance conceptual teaching. This prompted me to carry out an intervention programme to assist some of the teachers in this district. After the second workshop, Teacher C said "*My thinking has shifted on Pythagoras' theorem because I now have more insights as to how I can incorporate some visuals from my learners' home backgrounds to improve my teaching*." *[TCI2]* Such experiences as indicated by Teacher C revealed to me that some teachers had benefited from this workshop, especially on how to use visuals to teach Pythagoras theorem.

4.2.2.2 Experiences of using visuals during mathematics teaching

Surveyed teachers responded in varied ways as to how they use visuals during mathematics teaching. Figure 4.6 shows that on average, 72% of the teachers who responded to the questionnaires indicated that they use visual aids when referring to real life situations in some or most lessons, while only 16% indicated that they use visuals in every lesson. For example, one of the surveyed teachers, **Tr63**, said "*In extending mathematics to everyday life, I use real life examples to assist my learners understand and it works very well*." During interviews, Teacher A indicated, "*I used visuals in solving a real life problem of the ladder from their context; many learners were participating well in the lesson and they were able to see that Pythagoras' theorem is not in isolation but applicable in real life.*" ***[TAI1]*** In the interview Teacher B also said "*I try to find related aspects outside the classroom that may illustrate a concept. It is not always easy but doable*." ***[TBI1]***

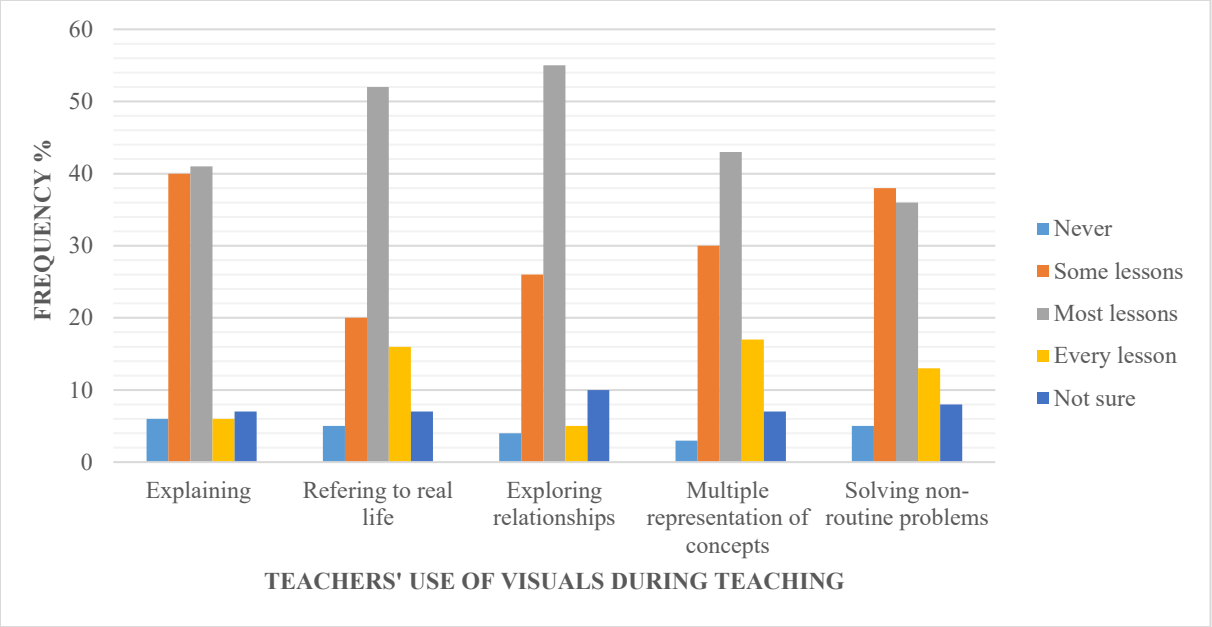


Figure 4.6: Frequency of a teacher's use of visuals during the lesson

Making connections to everyday lives and picking those aspects from outside the classroom that exemplify the concept was evident in teachers' responses and lessons that I observed. Kilpatrick et al. (2001) posit that when teachers are able to connect mathematics to the outside world to help learners to make sense of what they are learning and to solve real world problems, conceptual understanding is promoted. Also, use of these everyday examples help teachers to make links between the mathematics, learners' prior knowledge and experiences, which are necessary elements for enhancing conceptual understanding.

I also noted that 55% of the teachers who responded to the survey questionnaire indicated that they use visuals in most of their lessons when exploring relationships while only 5% indicated that they use visuals in every lesson. For example, in illustrating the Pythagorean concept, one of the surveyed teachers, **Tr34** said *"I draw squares on the two shorter sides of the right-angled triangle and a bigger square on the longer side. I ask learners to calculate the areas and compare. This helps them to understand the concept of squaring in the formula where it comes from."* Thus, this teacher was referring to her experiences with the area method. This was observed in Teacher C's lesson, when she pasted a right-angled triangle with squares drawn on respective sides and asked learners to establish a relationship between the sides. During interviews, Teacher C said "*Like I did in the video, I took a right-angled triangle pasted it on the chalkboard with squares that fit on the respective sides, learners were able to count the number of squares in each square and by doing so they were able to establish a relationship between the sides.*" ***[TCI1]*** This highlighted to me that in demonstrating Pythagoras' theorem, this teacher linked Pythagoras'

concept to area. Teacher **C** used visuals to establish connections between the concept of area and Pythagoras' theorem. This competence in making such mathematical links and connections to other domains of mathematics is one of Kilpatrick et al.'s (2001) significant pointers for conceptual understanding.

Furthermore, in this study, 40% of the teachers surveyed indicated that they use visuals to explain concepts in some lessons, while 41% said they use visuals in most of their lessons. Only 6% of the surveyed teachers indicated that they use visuals in every lesson to articulate concepts. For example, one of the surveyed teachers, **Tr40** said "*Visual aids help in explaining the concept of Pythagoras and they reduce lengthy verbal explanations by the teacher as the teacher is more of a guide as learners work by themselves. Therefore, visuals help to reduce teacher-talk in the classroom.''* During interviews, Teacher A mentioned that "*I have realised that visuals have many advantages in my lessons especially when explaining concepts like Pythagoras' theorem where I was able to tap into learners' everyday prior knowledge of similar situations to show connections*." ***[TAI1]*** The intervention helped Teacher A to realise the importance of visuals when explaining concepts. Similar experiences were reiterated by Teacher B in the interview: "*Visuals helped me talk less while my learners were kept busy by the manipulatives, and my role was to facilitate them. Visuals inform of diagrams also helped me when we were dealing with word problems. I will use them more from now onwards*." ***[TBI3]***

It is also evident in Figure 4.6 that 73% of the surveyed teachers indicated that they use visual aids in some or most lessons when representing mathematics concepts in diverse ways while 17% indicated that they use visuals in every lesson. One of the surveyed teachers, **Tr29**, said *"I use illustrative diagrams and audio-visual aids from the internet and other sources when illustrating Pythagoras's theorem and word problems in Grade 9.*" After finding the area of the square on the missing side of the triangle, Teacher B said "*I counted the squares at the base, and also used the concept of area of a square in identifying the length of the missing side and also used the concept of square root.*" ***[TBI1]*** Similar experiences were revealed by Teacher C: "… *in verifying whether Pythagoras' proposition claimed in the word problem was valid, I used different shapes (isosceles triangle, kite, square, rectangle, and parallelogram) and each shape led us to Pythagoras' proposition. It was interesting for the parallelogram proof, I used congruency and the property of sides of this shape, but each approach led us to Pythagoras' theorem. This helped my learners understand better.*" ***[TCI3]*** Konyalioglu et al. (2012) argue that the use of visualisation in mathematics problem solving may support a problem solver's comprehension and ability to come up with multiple solution opportunities during the mathematization process.

Although visualisation has a high potential to aid teachers in promoting conceptual understanding during teaching, teachers in the surveyed district currently incorporate visual strategies sparingly. This can be

seen in Figure 4.6 where most of them rarely use visuals in every lesson as this category had one of the lowest percentages. Such hesitance to use visuals indicated to me that there may be some prohibiting factors which need to be identified in order to support teachers in integrating visualisation strategies in their lessons.

4.2.3 Summary of the survey questionnaire

While teachers in the JTG District acknowledge the importance and value of visualisation approaches in the teaching of Pythagoras' theorem and word problems, many of them are either using visuals minimally or not using them at all. Challenges such as lack of resources in schools, time constrains, the DBE's strict adherence to pacesetters and lack of support from subject advisers among others were noted as reasons why many teachers were not using visualisation strategies to extend what they were teaching to real life, make connections among concepts in mathematics, build on learners' prior learning, and represent a concept in multiple ways.

4.3 OVERVIEW OF ALL THE LESSONS

All the seven lessons were on Pythagoras' theorem at Grade 9 level. Teacher A taught only one lesson whilst Teacher B and C each taught all the three lessons (see Section 3.6.3).

Lesson 1 and 2 focused on the use of the two area models to establish a relationship between the sides of a right-angled triangle and to solve Pythagoras' theorem word problems.

Lesson 3 focused on sense making of the Pythagorean proposition when solving the two word problems. It required putting together the three cut-out scaled right-angled triangles of the original right-angled triangle in order to form new shapes, and using these shapes to make sense of Pythagoras' equation.

Teachers were interviewed after each and every lesson and a focus group interview was conducted with the two teachers after all the lessons were taught.

4.3.1 Research results from lesson observations

In this section, I discuss the data from observations of all seven lessons of the three teachers. Data was collected from lesson observations to answer the second research question. The purpose of these observations was to investigate how selected teachers presented their lessons using visualisation approaches to enhance learners' understanding of Pythagoras' theorem word problems. The data are discussed against the analytical tools in Section 3.8 (see Tables 3.8 and 3.9).

The lesson observations' data are presented in the following section according to themes. I start off by presenting visualisation processes followed by conceptual understanding pointers.

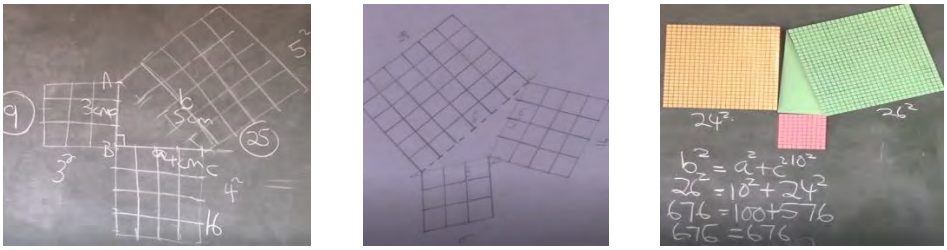
4.3.1.1 Visualisation processes indicators

In this section, I present the categories of visualisation processes that facilitated mathematical conceptual understanding. Some of these themes were informed by literature and these include: image generation, image inspection, image transformation and image use. The other themes emerged from the data. The themes that emerged are: visual language (Debes, 1969), gestures, generalisation and symbolic inscription (Presmeg, 1991). I will start by presenting the themes that were informed by theory, followed by the emerging ones.

- ***Image generation***

From the lesson observations, all teachers used manipulatives to generate images during their lesson presentations, and used these manipulatives to develop the mathematics idea to hand. All teachers were able to dynamically manipulate their initial images to support their articulations. This resonates with Arcavi's (2003) visualisation definition in Section 2.3.1.

The common practice for Teacher A and B in Lesson 1 was to use both drawings and manipulatives of the area model which they pasted on the board. However, Teacher B only pasted manipulatives of this model. All the teachers used images that they generated to explain the relationship between the sides of their triangles. Teacher A and B only used right-angled triangles in their presentation. However, Teacher C's approach was slightly different as seen in Figure 4.7 below.



(a) **(b)** **(c)**

Figure 4.7 (a, b, c): Areas of the squares on the legs of a right-angled triangle

Teacher C asked all groups to draw accurate congruent right-angled triangles (see Figure 4.7 (a)), except for one group which she asked to draw a non-right-triangle (see Figure 4.7 (b)), but with congruent dimensions as for the other groups. Learners were then asked to draw squares with lengths congruent to that of the respective side of a triangle and subdivide them into unit squares in order to investigate whether a relationship existed between the three squares ― see extract below.

> *TC: …so how can you connect the squares drawn on the legs of the triangle with the square drawn on the hypotenuse (longest side)? How are the sides related? Referring to Figure 4.7 (a) and (b).*

L: The relationship here is that, $9 + 16 = 25$ *(see Figure 4.7(a)), but in the non-right-angled triangle,* $9 + 16 \neq 36$ *(see Figure 4.7 (b)).*

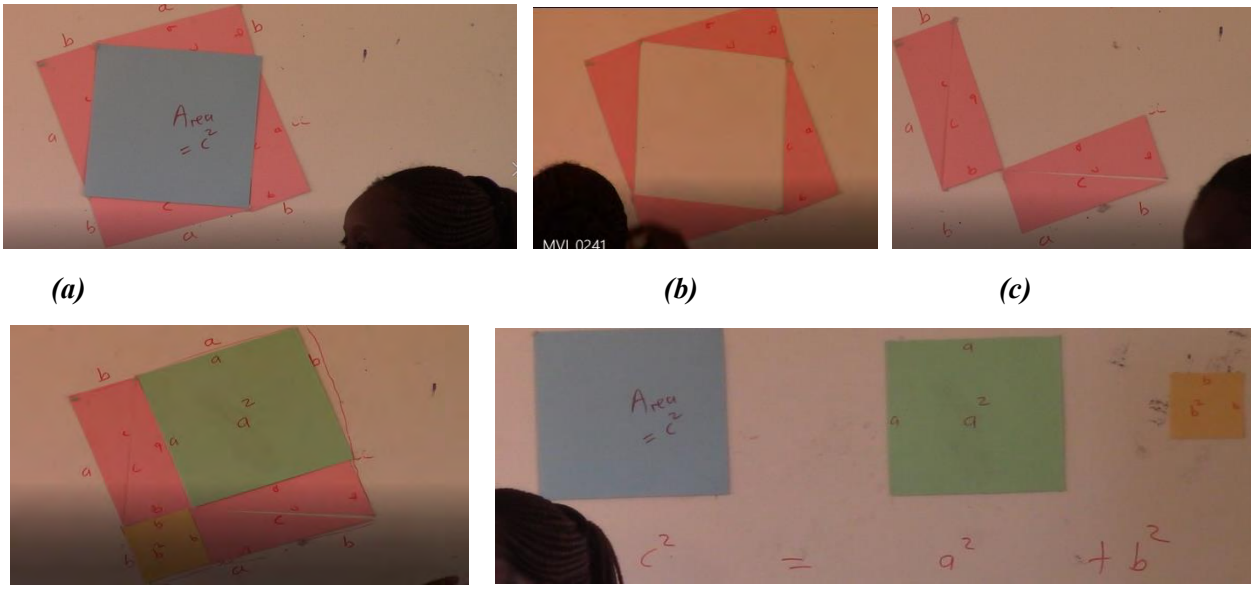
TC: Is the sum of 25 squares the same with all of you who drew a right-angled triangle?

Ls: Yes mam

TC: Ok, that sum of 25 illustrates Pythagoras' theorem ***[TCV1]***

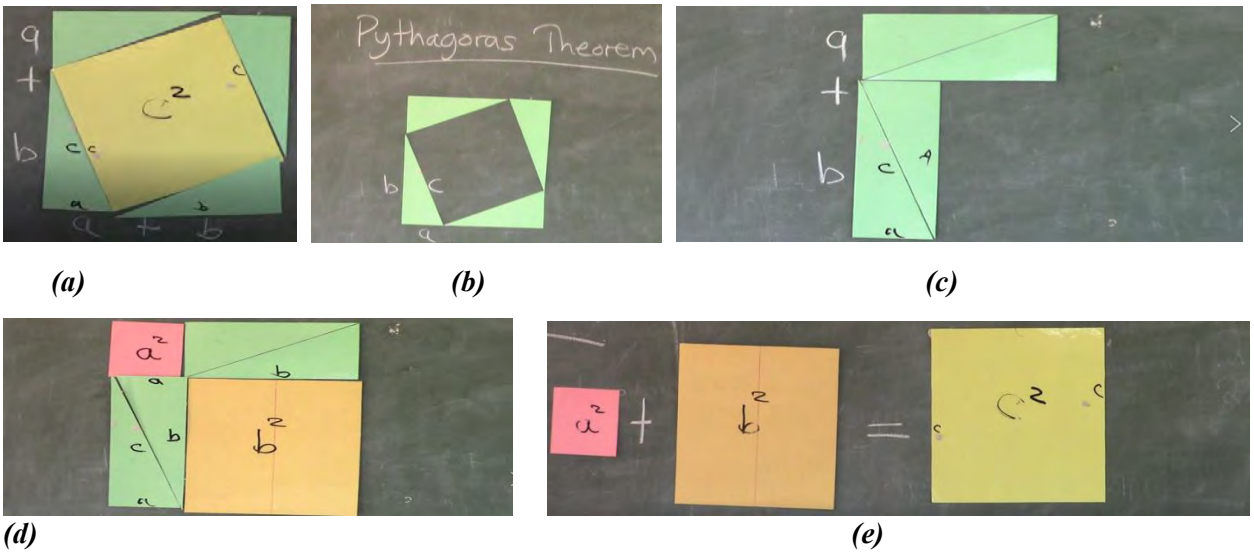
Polya (1957) and Arcavi (2003) assert that visual tools such as diagrams are valuable for teaching, as these support a discovery approach to mathematics problem solving. It can be deduced from the extract that it was after according learners an opportunity to generate area models with different types of triangles that learners were able to figure out that a relationship between the squares drawn on the sides of a triangle only exists with right-angled triangles. This I consider as having enhanced learners' understanding of the pertinent concept. During the interview, Teacher C said "*I let them draw a right-angled triangle with squares divided into unit squares on each side of the triangle. I also had a control group who drew a non-right-angled triangle also with squares divided into unit squares on each side of the triangle. They all counted the number of unit squares in each square. After I asked them for the relationship between the three squares. The ones who drew a right-angled triangle discovered that the square on the hypotenuse was equal to the sum of the squares of the other two sides unlike the ones who drew a non-right-angled triangle."* ***[TCI1]***

In Lesson 2, teachers B and C put together the manipulatives of the four congruent triangles to generate a square (see Figure 4.8 (a) and 4.9 (a) respectively). It is this square that was dynamically manipulated to create left-over space which was then filled with squares of dimensions congruent to the respective sides of the triangles — see Figure 4.8(d) and 4.9(d) respectively. The teachers then used the three squares that filled up the left-over spaces to articulate the relationship between the sides of the right-angled triangle as seen in Figure 4.8(e) and 4.9(e) respectively.



(a) ***(b)*** ***(c)*** ***(d)*** ***(e)***

Figure 4.8 (a, b, c, d, e): Establishing the relationship between the sides of a right-angled triangle by Teacher B

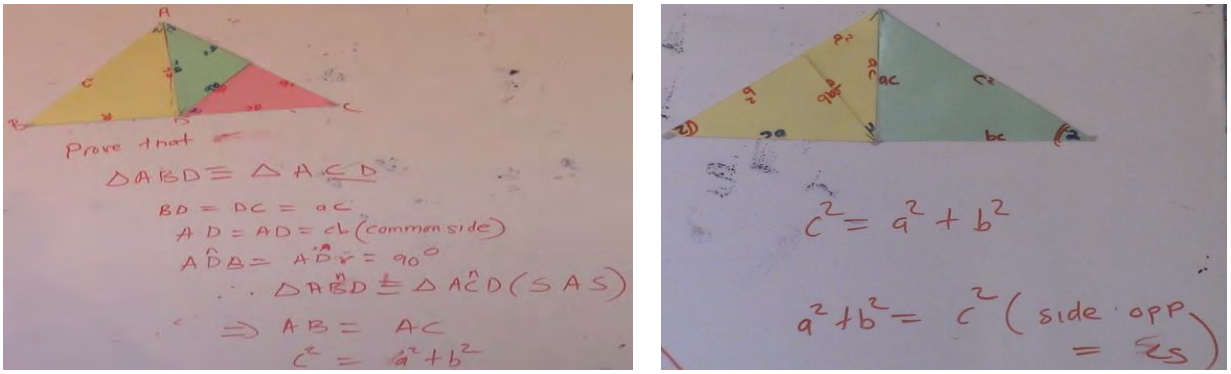


(a) ***(b)*** ***(c)*** ***(d)*** ***(e)***

Figure 4.9 (a, b, c, d, e): Establishing the relationship between the sides of a right-angled triangle by Teacher C

In Lesson 3, both teachers used the three scaled versions of the original right-angled triangles to generate different shapes during their lesson presentation. They were able to dynamically manipulate the initial image of an isosceles triangle to generate different shapes. Figures 4.10 to 4.13 portray the shapes generated by Teacher B while Figure 4.14 to Figure 4.17 depicts the shapes created by Teacher C. The teachers used these shapes to verify whether Pythagoras' proposition stated in the two word problems (see Appendix 8c) was valid. Teacher B for example indicated that "*In trying to promote learners'*

understanding when using these visuals, learners had to put their three enlarged triangles together to form different shapes then use their properties and/or congruency to make sense of Pythagoras' theorem in the word problems." ***[TBI3]***



(a) ***(b)***

Figure 4.10 (a, b): Discovering an isosceles triangle in both word problems

(a) ***(b)***

Figure 4.11 (a, b): Rotating the largest triangle in the isosceles triangle produces a square (a) and a kite (b)

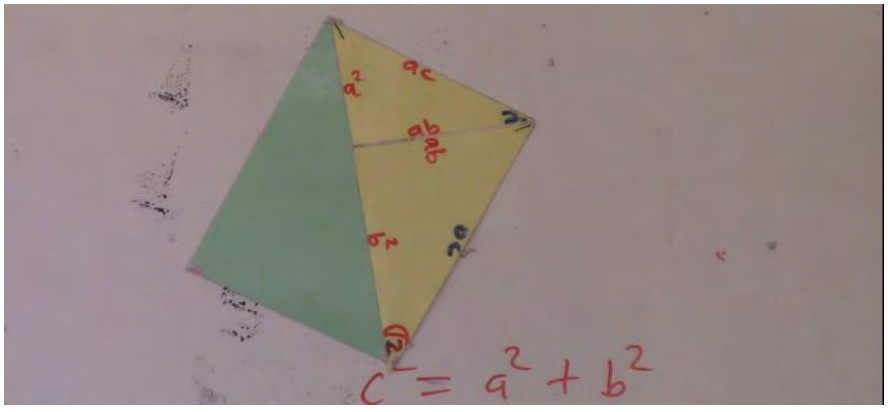
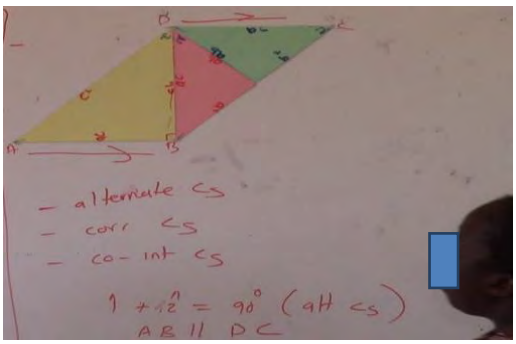
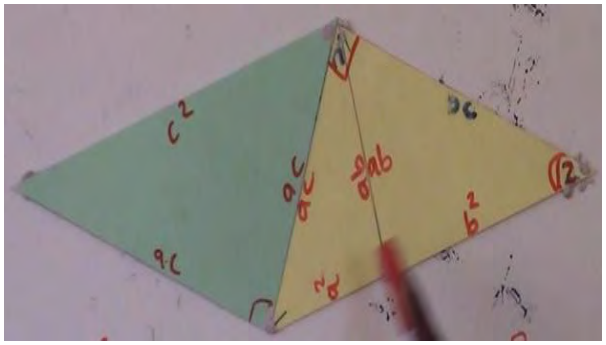


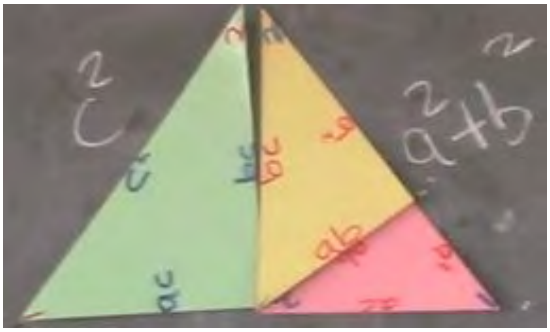
Figure 4.12: Flipping the largest triangle in the kite arrangements produced a rectangle

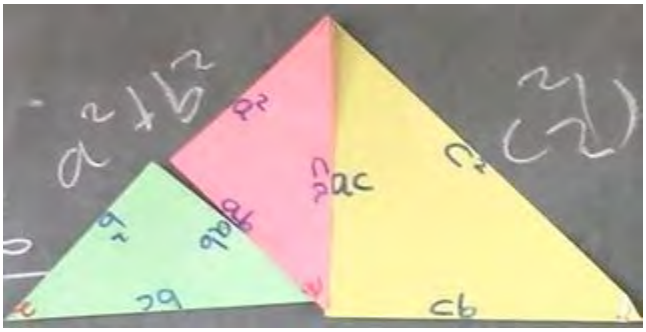




(a) ***(b)***

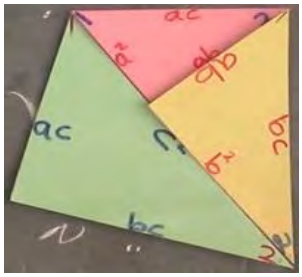
Figure 4.13 (a, b): Translating the flipped triangle in the square and rectangle arrangements gave birth to a parallelogram





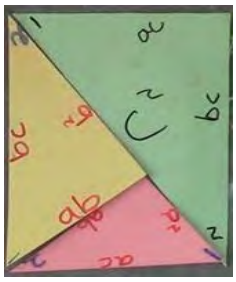
(a) ***(b)***

Figure 4.14 (a, b): Discovering an isosceles triangle (a) and (b) in both word problems

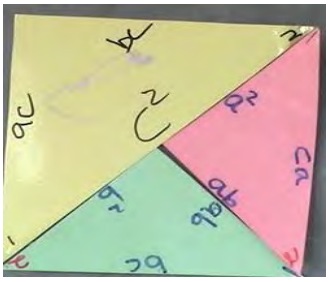


(a) ***(b)***

Figure 4.15 (a, b): Rotating the largest triangle of an isosceles triangle produces a kite (when original triangle was scalene) and a square (when original triangle was isosceles)



(a)



(b)

Figure 4.16 (a, b): Flipping the largest triangle in the kite and square arrangements produces a rectangle and square respectively

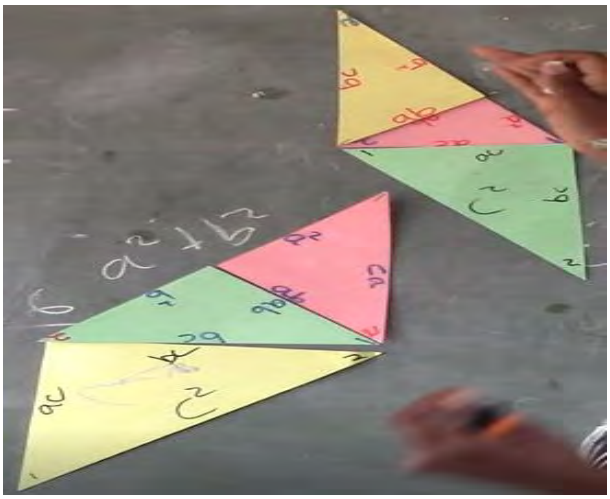
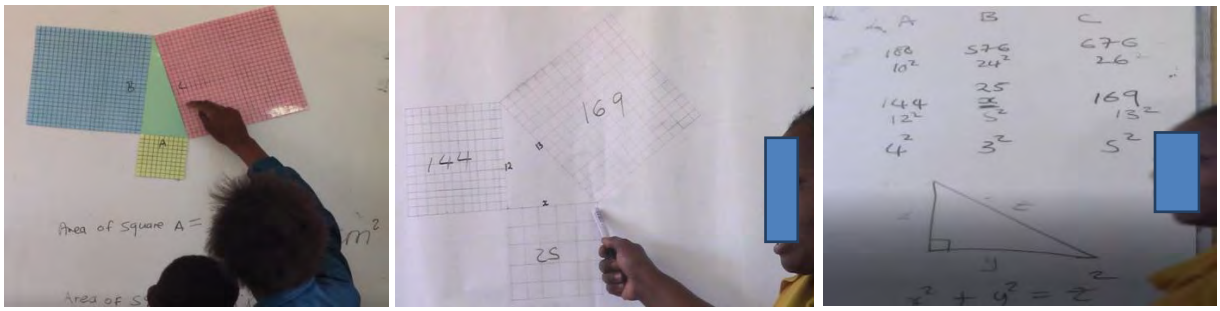


Figure 4.17: Translating the flipped triangle in the rectangle and square arrangements produced a parallelogram.

- ***Image inspection***

Visual representations are a way of creating a visible platform to aid the problem solver in self-monitoring, examining progress and increasing motivation during the problem solving process (van Garderen & Montague, 2003). In each lesson, images were generated using visual models and/or drawings. The generated images were displayed on the board and examined by both the teacher and learners in order to make sense of Pythagoras' theorem. The images created a platform for a classroom discussion. In Lesson 1 for example, area models with right-angled triangles of different dimensions were generated and inspected by Teacher B and the learners in establishing a generalisation of the relationship that exists between the sides of any right-angled triangle — see Figure 4.18. During the inspection process, the teacher created a table to record the areas of squares in the models. This created a platform where the teacher and learners could inspect the areas of squares in each model by counting the unit squares or applying the concept of area as seen in Figure 4.18 (a) and (b).



(a) ***(b)*** ***(c)***

Figure 4.18 (a, b, c): Relationship between the areas of squares drawn on the sides of a right triangle

The teacher and learners were able to inspect the areas of the squares in each model, but the Pythagorean concept was observed in all the models used. This I consider to have enhanced learners' understanding of Pythagoras' theorem. Image inspection was also evident in the teacher's statement when she said "*I asked learners to come and count the squares and to find the area of those squares aa then I tried to draw a table which has the area of square A, B and C. I wanted them to establish a relationship which exists between the squares in each model. Learners could actually see that when you add the areas of the two squares you get the area of the big square."* ***[TBI1]*** Such interview responses and lesson observations indicated to me that using visuals did not only stimulate learners' active participation during the lesson, but also enhanced their understanding. Rosken and Rolka (2006) accentuate that learning mathematics through visualisation can be a powerful tool to explore mathematical problems and give meaning to mathematical concepts and relationships between them.

In Lesson 1, Teacher C and her learners inspected the areas of squares in the different models as seen in Figure 4.7 above. With the help of visuals, the teacher was able to write the area of each square next to it. This made it easy for learners to examine whether a relationship existed between the three squares in each model. Learners could visually see that in any right-angled triangle, the sum of the area of the squares drawn on its legs equals to the area of the square on the hypotenuse (see Figure 4.7 (a) and 4.7 (c)), unlike in a non-right-angled triangle as shown in Figure 4.7 (b). This is in agreement with what Kanyolioglu et al., (2012) and Kashefi et al., (2015) stress — that drawing is one of the visualisation forms that assists in improving the learner's thinking ability.

After Teacher B and C arranging and transforming their four congruent triangles in Lesson 2, empty spaces were created and filled up with squares — see Figure 4.8 and 4.9 respectively. The teacher and learners then examined the dimensions and areas of these squares and used them in establishing the relationship between the sides of a right-angled triangle as seen in Figure 4.8 (e) and 4.9 (e) respectively. Below is an excerpt showing Teacher C's approach:

TC: …remember this empty space occupied by these two squares with area $\boldsymbol{a^2}$ *and* $\boldsymbol{b^2}$ *(referring to Figure 4.9 (d)) is where our first square which filled our first empty space was (referring to Figure 4.9 (a)) because our original square with sides* $a + b$ *has not changed. Now can you tell me the relationship between the areas of the two squares and the square with an area of* $\boldsymbol{C^2}$*?*

Ls: $a^2 + b^2 = c^2$.

TC: Yes, if we add the area of these two squares (referring to Figure 4.9 (d)) they cover up the empty space that was covered by this one (referring to Figure 4.9 (a)) and that is Pythagoras' theorem ***[TCV2]***

It can be deduced from the above extract that affording learners an opportunity to inspect the visual aids under the teacher's guidance enabled them arrive at the relationship between the sides of a right-angled triangle. This I regard as having honed their understanding. Yackel (2001) stresses that learners construct their own meaning from the words or visual images that they see or hear.

In Lesson 3, both teachers generated various images by putting together the three scaled-up versions of the original right-angled triangles as seen in Figures 4.10 to 4.17. The generated images were then displayed on the board and inspected. During the inspection process, both teachers and their learners examined each of the created images to ascertain whether the statement ($a^2 + b^2 = c^2$) in the word problems was valid. For the kite, square and rectangle arrangements both teachers and their learners inspected the two triangles on either side of the diagonal in ascertaining whether they were congruent. Through the teachers' discussions with their learners, the triangles were deemed congruent by the SAS (side-angle-side) postulate for congruency hence their argument of congruent diagonals (see Figure 4.11, 4.12, 4.15 and 4.16). In the parallelogram arrangement both teachers relied on congruency and the property of sides of a parallelogram while in the isosceles triangle arrangement, Teacher C only relied on the property of sides of an isosceles triangle while Teacher B used both congruency and the property of sides. The extract below and Figure 4.10 show Teacher B's approach in inspecting the isosceles triangle:

TB: So we have our isosceles triangle there (referring to Figure 4.10 (a)). Now we are going to apply congruency to prove that $\Delta ABD \equiv \Delta ACD$. *What can you say about BD and DC?*

Ls: $BD = DC = ac$

TB: Let us look at AD and AD, what can you say about AD and AD?

Ls: $AD = AD = cb$ *(common side)*

TB: What else can be said? Can you see $\angle ADB$*, what is its size?*

Ls: $90°$

TB: Remember we established that $angle\ 1 + angle\ 2 =$*…. (Referring to* $\angle ADC$)

Ls: $90°$

TB: So we are saying that $\angle ADB = \angle ADC = 90°$. From that information, what can we say about our triangle? We now therefore, conclude that $\Delta ADB \equiv \Delta ACD$. What condition did we use?

Ls: SAS

TB: Therefore, $\Delta ABD \equiv \Delta ACD$ …. (SAS). So what does it imply?

Ls: $AB = AC$

TB: But on our diagram what is AB and AC?

Ls:$AB = c^2$ and $AC = a^2 + b^2$

TB: $Therefore,\ a^2 + b^2 = c^2$, so the statement is correct. Also for you who answered question one with an original scalene right triangle (referring to Figure 4.10 (b)), the same congruency criterion is still applicable. But let me use another way of proving that these two sides are equal. As you see in this triangle (referring to Figure 4.10 (b)), if we go to Euclidian geometry, we are saying this angle 2 is equal to this angle 2. It implies this side $c^2$is equal to …

Ls: $a^2 + b^2$

TB: What reason do you write in Euclidean geometry to support that the two sides are equal?

Ls: Sides opposite equal angles

TB: So besides using congruency, we can use Euclidean geometry ***[TBV3]***

It can be reasoned from the extract that the use of visuals created an environment where the teacher and learners jointly examined the images of an isosceles triangle. This enabled them to make sense of Pythagoras' proposition in different ways. I thus consider this as having enhanced learners' understanding of Pythagoras' concept through the isosceles triangle arrangement. Pape and Tchoshanov (2001) assert that the use of multiple representations facilitate learners' mathematical conceptual development.

- ***Image transformation and image use***

In all the teachers' lessons taught, image transformation and image use occurred simultaneously, except in Lesson 1 where there was no evidence of image transformation. For example, in Lesson 1, after establishing the relationship between the three squares drawn on the sides of a right-angled triangle, Teacher A altered the square on the small leg by removing the unit squares and dimensions, as shown in the extract and Figure 4.19 below. She used this as an activity to create a discussion and test learners' understanding of Pythagoras' proposition.

TA: I have a square of length 5 drawn on the longest side and a square of length 4 drawn on the longer leg. A square of sides 5 you know and can see area is 25 and a square of sides 4 you see area is 16. Now can you tell me the area of the square on the shortest leg and how do you find it?

Ls: 9 which is $25 - 16$.

TA: Why are you subtracting?

Ls: When you add 9 plus 16 you get 25.

TA: Yes, you can see here on the diagram that if we add the 16 squares here and 9 squares here will give us 25. So if it is 9, you can partition the small square into 9 unit squares and count the squares at the base. This will help you to find the length of the missing side. Since this is a square, you can also ask yourself, which number do I multiply by itself to get 9?

Ls: three ***[TAV1]***

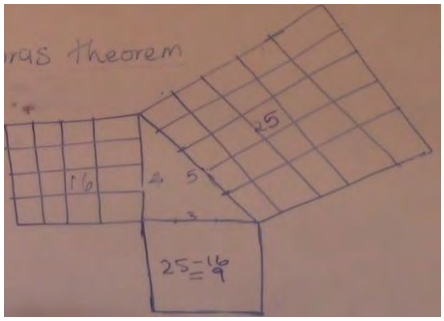


Figure 4.19: Establishing the length of the missing side

From the extract, the teacher made use of the visual aid to check learners' understanding of the use of Pythagoras' proposition and also to enhance their understanding. Learners made use of the visual to justify their thinking. Since the majority of learners in this study were low achieving learners, my findings contradict those of Van Garderen et al. (2014) who argue that low achieving learners could not figure out the structural relationships between the diagram and the targeted concept. Furthermore, it was interesting to note that when learners were asked to find the area of the square on the shorter leg, most of them processed their answer mentally. This indicated to me that learners had already developed a picture in their minds of the relationship between the areas of squares drawn on the sides of a right-angled triangle. It is this picture that they mentally manipulated to arrive at the answer. In the same spirit, Rosken and Rolka (2006) assert that external visual representations that learners construct or use to enhance their understanding, automatically generate a big picture in the mind to aid them in unearthing the solution to the problem.

In Lesson 2, after arranging the four congruent triangles into a square, Teacher B and C filled up the empty space with a square (see Figures 4.8 (a) and 4.9 (a)). This square was removed and pasted somewhere on the board. The teachers rotated the two congruent triangles and the process produced rectangles, as seen in Figure 4.8 (c) and 4.9 (c). They then translated one of the two triangles to a side and the process created empty spaces which they filled up with two squares as shown in Figure 4.8 (d) and 4.9 (d). It is these three squares that filled up the left-over spaces, which the teachers used to demonstrate and justify the relationship between the sides of a right-angled triangle as seen in Figure 4.8 (e) and 4.9 (e).

During the third lesson, Teacher B and C begun by putting the three scaled-up versions of the original right-angled triangle together and formed isosceles triangles as seen in Figure 4.10 and 4.14 respectively.

It is these isosceles triangles that formed the bedrock for other shapes. Rotating the larger triangle in the isosceles triangle about the vertex containing the altitude of this triangle, led to the birth of a kite (when the original triangle was scalene as in the case of Question 1) and a square (when the original triangle was isosceles as in the case of Question 2) as shown in Figure 4.11 and 4.15 above. Flipping the largest triangle in the kite and aligning its hypotenuse with the length a^2 and b^2 in the two small triangles gave birth to a rectangle as seen in Figure 4.12 and 4.16 (a). Translating the flipped triangle in the rectangle and square arrangement gave birth to a parallelogram — see Figure 4.13 and 4.17. The teacher then used congruency and/or the property of sides of shape to demonstrate and justify that Pythagoras' proposition ($a^2 + b^2 = c^2$) as alluded to in the two word problems (see Appendix 8d) held true.

- ***Visual language***

A person who is visually literate is one who is in a position of reading and writing visual language, that is, s/he should be able to appropriately decode and encode visual messages (Debes, 1969). Throughout the lessons, the visuals used were in line with the teachers' verbal explanations and the visual language used was in line with the mathematical language and terminology. For example, in lesson one, Teacher A used the area model to explain the relationship between the sides of a right-angled triangle. She provided a written explanation for learners to see it more visually and conceptualise it — see Figure 4.20 below.

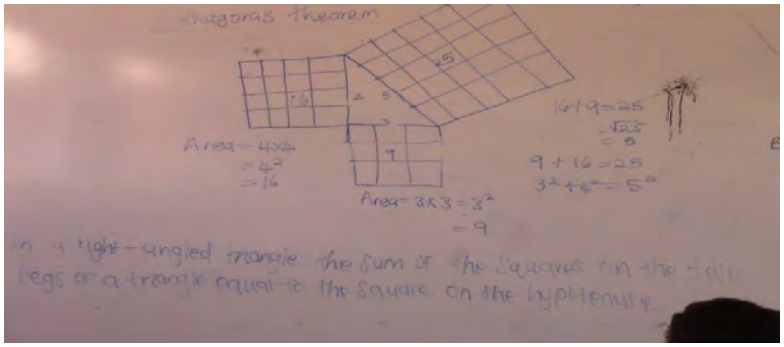


Figure 4.20: Relationship between the three sides of a right-angled triangle

In Lesson 2, Teacher C demonstrated proper use of the language of mathematics. For example, turning (rotating), anti-clockwise, clockwise, and translating, and her verbal articulations were in line with the visuals she utilised in the lesson — see extract below and Figure 4.9.

> *TC: …remember I have my empty space here (referring to Figure 4.9 (b)), but I want to turn this triangle in an anti-clockwise direction to here so that I form a new shape. I also turn this triangle, I rotate it in a clockwise direction, in the direction of the arms of a clock so I have…*
>
> *Ls: Two rectangles mam (referring to Figure 4.9 (c)).*
>
> *TC: Now I am going to translate. I am going to slide one rectangle to the left and move it over here a little beat and fill the empty spaces created with two squares (referring to Figure 4.9 (d))* ***[TCV2]****.*

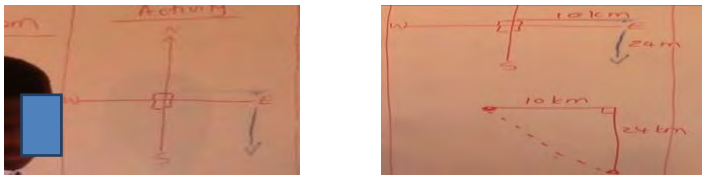
Also common in Lesson 3 was the use of mathematical language — similarity by both teachers. The three scaled-up versions of the original right-angled triangles that the teachers and learners used were in line with the mathematical language that was used. Following this up in the interview, Teacher B said "*...it was more of similarity because the three scaled triangles are in proportion. But when we proved we used congruency. You know learners confuse the difference between similarity and congruency. Similarity is the three triangles we used. When we looked at congruency, it is the theorems which we were using to prove.*" ***[TBI3]*** Drawing from a Vygotskian perspective, Powell and Kalina (2009) assert that language usage in the classroom is the most important tool for social mediation; it can be used as a tool to mediate social interaction and collaboration, and for learners to express their learned knowledge.

- ***Gestures***

Gestures were used in conjunction with the visuals in the teachers' lessons. In Lesson 1 for instance, Teacher A used a broom from the learners' classroom to aid in interpreting and understanding the word problem — see Figure 4.21 below. She kept of moving her right hand in trying to illustrate the situation in the word problem.

Figure 4.21: Use of a visual from learners' context

Gestures were also evident in both teachers' second and third lessons when interpreting and understanding the word problems. However, Teacher B's approach was more appealing than that of Teacher C in lesson 2. She guided learners in coming up with an appropriate diagram. The teacher did this by engaging the whole class in a discussion through asking questions as a way of monitoring their attention and checking their understanding. She kept on moving her fingers as a way of illustrating John's movements- see Figure 4.22 and extract below.



(a) ***(b)***

Figure 4.22 (a, b): Diagram generated in understanding the word problem

> *TC: ...for us to answer that classwork which I have written on the board, I want us to jointly come up with a sketch first before we use the manipulative I gave you. The question is saying*

John drove his car 10km due east, will he be going to my left or right side (teacher asks while pointing)?

Ls: Your right hand side mam.

TC: When he turns from east to South, is he going up or down? (Moving her finger up and down)

Ls: Down mam

TC: Come and demonstrate

L: Demonstrates as can be seen in Figure 4.22 (a) ***[TBV2]***

From the extract, it can be argued that the teacher engaging learners through asking probing questions and using gestures assisted them in understanding the word problem, thereby coming up with an appropriate diagram — see Figure 4.22. Zahner and Corter (2010) assert that diagrams help problem solvers in abstracting the problem text for important elements, and aid them in building a problem schema to solve the problem.

- ***Generalisation***

Teachers used visualisation to promote mathematics abstraction and generalisation in their lessons. For example, in Lesson 1, after establishing the relationship between the sides of a right-angled triangle with the help of visuals (see Figure 4.18), Teacher B drew a right-angled triangle (see Figure 4.18 (c)) and asked the learners "*Can you come up with a generalisation of the relationship which exists between the three sides, x, y, and z?"* ***[TBV1]*** Furthermore, when presenting her solution to the word problem in Lesson 1, Teacher B worked out the problem algorithmically as can be seen in Figure 4. 23 below.

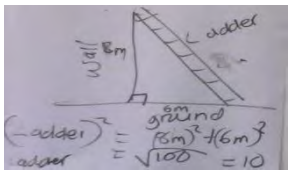


Figure 4.23: Solution to the word problem

Following this up in the interview, I noted that Teacher B said "*When I solved the word problem, I applied the formula because we started the lesson by doing the proof using models which led us to Pythagoras' equation."* ***[TBI1]*** In Lesson 3, Teacher C also applied the analytical approach in calculating the length of the wire in the first word problem — see Figure 4.24 below.

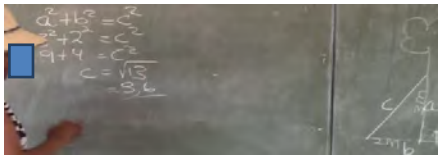


Figure 4.24: Finding the length of the missing side

During the interview, Teacher C said, *"…in finding the length of a wire I used the equation of Pythagoras because learners already had a picture of where this equation comes from when using the manipulatives of the first two lessons."* ***[TCI3]***

It can be deduced from lesson observations and the interview responses that these teachers assumed that learners had developed a picture of Pythagoras' proposition in their mind after the visual proofs. This therefore implies that the teachers' aim of using visuals in some of their lessons was to achieve generalisations, after which visual thinking was to be dispersed after serving its initial purpose (Presmeg, 1991).

- ***Symbolic inscriptions***

External visual representations can aid in making abstract relationships and/or concepts concrete and clear for learners to comprehend (Zahner & Corter, 2010), thereby supporting learners' reasoning as they transit between physical models and symbolic representations (Gates, 2018). In all their lessons, the teachers assisted learners in making connections between visual and symbolic inscriptions of the mathematical concept they were dealing with. In Lesson 1 and 2, teachers used areas of squares in the area models in establishing a symbolic relationship between the sides of a right-angled triangle — see Figures 4.7 (a) and (c); 4.8; 4.9; 4.18 and 4.19. In Lesson 3, both teachers created various shapes which they used to aid learners in making sense of Pythagoras' equation — see Figures 4.10 to 4.17.

4.3.1.2 Teaching for conceptual understanding of Pythagoras' theorem

In this section, I present the conceptual understanding indicators that I noted in the teachers' lessons.

- ***Prior knowledge***

Prior knowledge acts as a bridge between learners' prior understanding and new information (Kilpatrick et al., 2001). From all the teachers' lessons observed, visuals were used as a bridge to move learners from the known to the unknown. After generating an image, teachers posed questions that required learners to reflect on their prior learning. Building on prior knowledge was, for example evident in Teacher A's first lesson when she was establishing a relationship between the sides of a right-angled triangle — see Figure 4.20 and the extract below.

> *TA: …you see on each side of the triangle there is a square drawn there. We want to see if we can establish a relationship between the sides of the triangle. How do we find the area of a square?*
>
> *Ls: side squared*
>
> *TA: You are right, but in this case you can also count the unit squares in each square* ***[TAV1]***

In her teaching, as indicated by the extract, the teacher made use of visuals in seeking learners' prior understanding of the concept of the area of a square which was to be used in establishing the relationship

between the sides of a right-angled triangle. She did this by asking questions that required learners to reflect on their previous learning. Building on learners' prior knowledge was also evidenced in the interview when the teacher said "*In establishing a relationship between the sides of a right-angled triangle, I used area of a square because learners do area in Grade 8.*" ***[TAI1]***

In lesson 2, Teacher C referred to learners' prior knowledge by engaging the whole class in a discussion during the establishment of the relationship between the sides of a right-angled triangle as seen in the extract below:

> *TC: …in order to achieve our goal of establishing the relationship between the sides of our right-angled triangles, we are now going to do some transformations, do you remember in Grade 8, we learnt about transformations. We could change shapes, we could move them around in different ways but they stayed the same. Anyone who remembers what those transformations were?*
>
> *Ls: (Few learners answered) translation, reflection, and rotation.*
>
> *TC: You are right, but in this lesson we will only focus on rotation and translation* ***[TCV2]***

Building on learners' prior knowledge was also evident in Teacher C's second lesson when she referred to learners' prior understanding of the compass directions in interpreting and understanding the word problem as seen in Figure 4.25 and the extract below. The word problem used in the classroom is: John drove his car due east for 10 km. He then turned south and drove 24 km. How far was he from where he started?

(a) ***(b)***

Figure 4.25(a, b): Images created in understanding the word problem

> *TC: I am sure you did social sciences in Grade 8. How do we call this (referring to Figure 4.25(b))?*
>
> *Ls: Compass direction*
>
> *TC: True, can someone come and draw a diagram showing John's movement.*
>
> *L: Draws a sketch as seen in Figure 4.25 (a) above.*
>
> *TC: That is correct, remember a compass direction has…?*
>
> *Ls: North, south, west and east [referring to Figure 4.25 (b) above].*
>
> *TC: Ok, so he started from here, he went there so I can represent it with this right-angled triangle (referring to Figure 4.25 (b)). Going east 10km and here south 24km* ***[TCV2]***

It is evident from excerpts from Teacher C' second lesson that she kept on referring learners to what they had learned in the previous grades, through asking questions. Learners then used this knowledge as a basis during the teaching of Pythagoras' theorem and the solving of word problems. This implies that the teacher built from what learners already knew. This I consider as having made it easy for learners to learn with understanding. In support, Jones and Brade-Araje (2002) argue that constructivist teachers are acutely aware of the role of prior knowledge in learners' learning and recognise that they are not blank slates to be filled with knowledge. Instead learners draw on their prior learning to construct new understandings (Ernest, 2010) with teachers acting as guides and facilitators (Powell & Kalina, 2009), creating teaching and learning milieus (visual ones in this case) that support and encourage active construction of mathematical knowledge (Ernest, 1991).

In Lesson 3, after generating a particular shape, Teacher B begun by checking learners' prior knowledge of the properties of that shape in order to move learners from the known to the unknown — see extract below.

> *TB: ...for those who had an original isosceles triangle, what shape did you generate after rotating your big triangle in your isosceles triangle (referring to Figure 4.10 (a)).*
>
> *Ls: Square (see Figure 4.11 (a)).*
>
> *TB: Can you remind us the properties of a square?*
>
> *Ls: All sides are equal, all angles are right-angles, the diagonals bisect at right-angles.*
>
> *TB: You are right, all these properties we can get them from this square (referring to Figure 4.11 (a)). What about those who had an original scalene triangle, what shape did you get after rotating the big triangle in your isosceles triangle (referring to Figure 4.10 (b))?*
>
> *Ls: Kite (see Figure 4.11 (b)).*
>
> *TB: Can you tell us the properties of a kite*
>
> *Ls: Adjacent sides are equal, longer diagonal bisects opposite angles, and longer diagonal bisects the short one at right angles*
>
> *TB: Now we need to make is a parallelogram. What are the properties of a parallelogram?*
>
> *Ls: Two opposite sides are equal and parallel*
>
> *TB: (Referring to Figure 4.13 (a)), this one is* ***bc****, this one is* ***ac,*** *but from the start* $\boldsymbol{a = b}$*. So it means* ***bc*** *is equal to* ***ac****. But we are having a challenge, we do not know whether* $a^2 + b^2 = c^2$*. Remember in a parallelogram we have to prove parallel lines. What are the conditions for parallel lines?*
>
> *Ls: (A few learners) Alternate angles are equal, Co-interior angles add up to* $180°$ *and Corresponding angles are equal* ***[TBV3]***

From the extract above, it can been seen that Teacher B kept on referring to learners' prior learning. This was common during sense making of Pythagoras' theorem. After building a particular shape, the teacher

engaged the whole class in a discussion in order to check on learners' understanding of the properties of that shape. It is these properties that she used in making sense of Pythagoras' equation. When interviewed, Teacher B said, "*Working with visual aids in this lesson I was able to identify learners' prior knowledge of properties of angles and shapes which I used as a basis in helping my learners to make sense of Pythagoras' theorem in the two word problems. This indeed enhanced learners' understanding than before because I made use of what they knew in proving that* $a^2 + b^2 = c^2$*." [**TBI3**]* Such interview responses and the data from video recorded lesson observation indicated to me that the intervention was beneficial to this teacher.

- ***Connecting mathematics to real life***

Teacher A believed that visuals assisted her in connecting Pythagoras' theorem to real life, by using a broom from the learners' context in making sense of the word problem — see extract below and Figure 4.21.

> *For those of you who did not get it, let me use this broom as my ladder. This ladder is leaning against a wall and it forms a right-angled triangle as you can see. So this one is the wall and the wall from there to where the ladder meets it has a height of 8m and down there they say it is 6m. This broom is our ladder we are supposed to find its length **[TAV1]***

When I followed up this in the interview, Teacher A said, "… *the word problem we dealt with was a real life situation and learners could see that this theorem is applicable beyond classroom boundaries. I also used a broom to illustrate the ladder situation, it helped learners to see that there is a right-angled triangle in that situation." **[TAI1]***

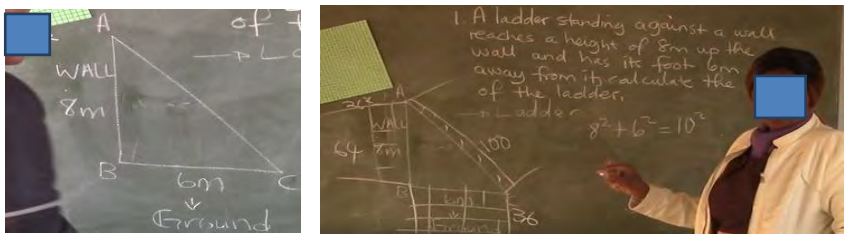
Teacher B also used visual examples from learners' school context and/or home background when showing them the relevancy of Pythagoras' theorem in real life during lesson one. These include: the school flag pole being perpendicular to the ground and an electric pole which was across the main road anchored by a slanting wire. During the interview, Teacher B said, "*...I referred to situations from within and outside their school context which relate well to this theorem such as a flag pole being perpendicular to the ground and an electric pole supported by a slanting wire." **[TBI1]***

It can be deduced from the above lesson observations and interview remarks (concerning Lesson one) that Teacher A and B attempted to relate what they were teaching to real life by drawing visuals from learners' schools and/or home backgrounds in trying to assist them in making sense of the word problem and Pythagoras' theorem. I thus consider this as having enhanced learners' understanding. This seems to be in line with what Jourdain and Sharma (2016) contend — that using tools such as visual aids and manipulatives can assist in contextualising mathematics.

Furthermore, in all their lessons, teachers extended what they were teaching to real life contexts through the use of word problems from learners' daily life situations — see lesson plans in Appendix 8 (a) to 8 (d). In the interview for example, Teacher C said "*I gave a real life problem and they know people drive following certain directions… where John drove his car east then south which represented a right triangle. I asked learners to use the four congruent triangles in each group to solve the problem as I guided them.*" ***[TCI2]*** The data from the interview and lesson observation indicated to me that learners could see the relevancy of Pythagoras' theorem outside the classroom. This I believe to have helped them in making sense of what they were learning. Kilpatrick et al. (2001) underscores the need for teachers to relate what they are teaching to new areas or daily life contexts, thereby enabling learners to see the practical application of this knowledge.

- ***Connecting ideas and concepts in mathematics***

Competence in making mathematical links and connections to other domains of mathematics is one of Kilpatrick et al.'s (2001) significant pointers for conceptual understanding. Throughout their lessons, teachers generated images which they used in connecting related mathematics concepts. In Lesson 1, for instance, Teacher C asked learners to draw a right-angled triangle with squares divided into unit squares on each side of the triangle. She then asked them to count the number of unit squares in each square after which they had to establish a relationship between the three squares. After counting the unit squares in each square, learners noticed that the number of unit squares in a particular square was equivalent to the area of that square. Teacher C also drew from the concept of area and exponents when solving the word problem — see Figure 4.26 (b) below.



(a) ***(b)***

Figure 4.26 (a, b): Solution to the word problem

During interviews, Teacher C also said "… *from the drawing we came up with areas*, *from the areas we now have exponents there, then we are adding (algebra) in order to come up with area of the square of the missing side.*" ***[TCI1]*** It can be deduced from the lesson observation and interview remark that the generated image was used to make conceptual links between mathematics concepts (that is Pythagoras' theorem, area, exponents, and algebra), and this I consider as having honed learners' understanding. The teacher's solution in Figure 4.26 (b) also indicated to me that she based this on the assumption that

learners had developed a picture of the area model in their minds, and that is why she did not draw complete squares on the sides of the triangle.

Additionally, it was interesting to note that after drawing a sketch to the word problem (see in Figure 4.26 (a)), the teacher asked the learner some probing questions as can be seen in the extract below.

TC: Why did you decide to use 8m that side?

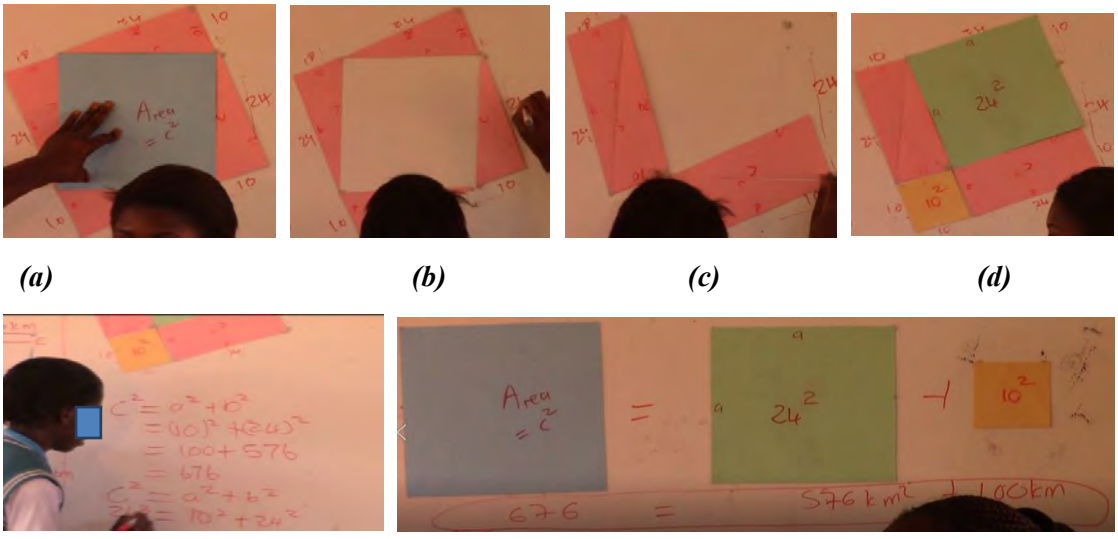
L: Because the question says height of the ladder 8m up the wall

TC: And then 6m?

L: Is the distance of the ladder away from the wall ***[TCV1]***

Based on the extract, it can be argued that drawing a sketch assisted the learner in understanding the problem. This seems to be in line with Csikós et al.'s (2011) argument that "successful problem solving requires the comprehension of relevant textual information and the ability to visualise the data" (p. 49).

In Lesson 2, both teachers used visuals in making connections between related mathematics concepts. However, Teacher B brought out these connections differently when handling the word problem with her learners. After working in their groups, the teacher invited one of the learners to illustrate their group's solution to the classmates. The teacher then used the learner's images as a basis for a whole class discussion — see Figure 4.27 and the extract below.



(a) ***(b)*** ***(c)*** ***(d)***

e) ***(f)***

Figure 4.27 (a, b, c, d, e, f): Learner illustrates to classmates her group's solution to the word problem

TB: What are the dimensions and area of the square that fills the empty space (see Figure 4.27 (a))?

Ls: c by c; area is c^2

TB: Take out the blue square and paste it somewhere on the board. Do another model. After rotating and translating the triangles (referring to Figure 4.27 (b)), what shapes are formed?

Ls: Rectangles (see Figure 4.27 (c))

TB: That is correct, fill up your empty spaces. As you can see we have got two squares. Tell me the dimensions and areas of the squares that fill up the empty spaces created (referring to Figure 4.27 (d)).

L: This one is 10 by 10; $Area = 10^2$*; and this one is 24 by 24;* $Area = 24^2$.

TB: So now what can we say in terms of the empty space covered by the three squares?

L: The green and yellow squares (see Figure 4.27 (d)) fill up the same area covered by the blue square (see Figure 4.27 (a)) because the square I started with did not change, it is still 34 by 34.

TB: So what is the area of the blue square?

Ls: 676

TB: That is correct, remember we are going back to a square. We are having a square which is having an area of 676, what would the lengths of its sides be?

Ls: 26

TB: How do you get that?

Ls: (Some learners) $\sqrt{676} = 26$*; (other learners)* $26 \times 26 = 676$ ***[TBV2]***

Her presentation in the extract reveals that visuals assisted the teacher in connecting Pythagoras' theorem to related mathematics concepts such as area, algebra and transformations like rotation and translation. This I believe enhanced learners' conceptual understanding. During the interviews, Teacher B mentioned that "*I have realised that using visuals to teach Pythagoras' theorem and word problems helped me to link several concepts like transformations, congruent triangles, properties of squares and rectangles, area of a square, exponents and algebra all which were not possible with the analytical method I used before. This was really interesting."* ***[TBI2]*** This indicated to me that the intervention helped the teacher to see the benefits of using visuals in teaching Pythagoras' theorem and word problems. On the same note, Duval (1999) argues that visualisation is needed in mathematics because it externally displays organisation of relations and aids learners in forming images of mathematical ideas that are complex.

In Lesson 3, both teachers used the three scaled versions of their original right-angled triangle in building various shapes (see Figures 4.10 to 4.17) which they used in making sense of Pythagoras' proposition in the two word problems. In so doing, they were able to make connections among a rich set of mathematics concepts. After learners had generated their images under the teacher's guidance, the teacher used her manipulatives to put these images up onto the board because her visuals were clear, visible and readable. These images created a platform for a whole class discussion. Learners were not mere recipients but active participants. What was common in Teacher B's and C's lessons was the discovery of an isosceles

triangle when solving the two word problems — see Figures 4.10 and 4.14 respectively. Teacher B's proof relied on both the two congruent triangles on either side of the altitude of the isosceles triangle in her argument for congruent sides ($\boldsymbol{a^2 + b^2}$ and $\boldsymbol{c^2}$) and the property of the sides of an isosceles triangle, because the two base angles in the composite triangles were equal. Teacher C's proof hinged on the latter. Also common in Teacher B's and C's lessons was the discovery of a kite (see Figures 4.11 (b) and 4.15 (a) respectively) and square (see Figures 4.11 (a) and 4.15 (b) respectively) when the original right-angled triangle was scalene (in the case of Question 1) and isosceles (in the case of Question 2), respectively. The kite and square proofs both relied on congruent triangles on either side of the diagonals in order to arrive at the desired equality of lengths $\boldsymbol{a^2 + b^2}$ and $\boldsymbol{c^2}$. These triangles were found to be congruent by SAS congruency criterion. Other properties of squares and kites were reinforced by Teacher B in the square and kite arrangements. She also reinforced that a square is a special type of kite. This is what Teacher B said, "*...all properties of a square can be seen in this shape (referring to Figure 4.11 (a)). A square is a special kite. Why do I say that? If we look at the properties of a kite (referring to Figure 4.11 (b)): adjacent sides are equal (that is,*$ac = ac; and\ bc = bc$*), the diagonal bisects opposite angles, longer diagonal bisects shorter diagonal at right angles. That is why we classify a square as a kite because it shares kite properties."* ***[TBV3]***

Furthermore, of interest was the discovery of a rectangle by Teachers B and C in Lesson 3 — see Figures 4.12 and 4.16 (a) respectively. They both flipped the largest of the three triangles in their kite arrangement and aligned its longest side $\mathbf{c^2}$ with the lengths $\mathbf{a^2}$ and $\mathbf{b^2}$ in the two smaller triangles and the process generated a rectangle. Like the square and kite arrangements, an argument for congruent triangles on either side of the diagonal was used in arriving at congruency (SAS congruency) and the desired equality of lengths $\mathbf{a^2 + b^2}$ and $\mathbf{c^2}$. Unlike Teacher B, Teacher C reinforced other properties of rectangles in the rectangle arrangement and that a square is a special type of rectangle — as seen in the extract below.

> *TC: ...for you who had an original isosceles triangle, what shape did you get after flipping the largest triangle in your square?*
>
> *Ls: Still a square mam (see Figure 4.16 (b)).*
>
> *TC: Yes, but is a square not a rectangle?*
>
> *Ls: No mam*
>
> *TC: Remember, you said during the introduction that in a rectangle opposite sides are equal; are opposite sides not equal this square? Referring to Figure 4.16 (b).*
>
> *Ls: They are mam*
>
> *TC: Are the interior angles not* $90°$ *here in the rectangle (referring to Figure 4.16 (a)).*
>
> *Ls: They are mam*

TC: Therefore, a square is a special type of a rectangle because it has all properties of a rectangle. So all squares are rectangles but not all rectangles are squares.

In the extract, visuals were used by the teacher in connecting the properties of a square to those of a rectangle. It can be argued that this enabled learners to visually see that a square is a special type of a rectangle as it possessed all rectangle properties. The NSC diagnostic report of 2017 underscores that mathematics cannot be studied in compartments and it is expected that the candidates must be able to apply knowledge from one section to another section of work (DBE, 2017).

Lastly, also common was the discovery of a parallelogram by Teachers B and C — see Figures 4.13 and 4.17 respectively. They translated the flipped triangle in the rectangle and square arrangements and aligned one of its legs with a congruent side, producing a parallelogram. The teachers relied on congruency of triangles and the property of parallelograms which states that opposite sides are congruent and parallel, in order to arrive at the desired equality of side lengths $\mathbf{a^2 + b^2}$ and $\mathbf{c^2}$. However, Teacher C used the parallelogram arrangement to reinforce the properties of parallelograms, unlike Teacher B. The extract below is an observation from Teacher C's lesson:

TC: So we can see that we already proved from the rectangle arrangement that the big triangle is already congruent by SAS congruency to the other two triangles combined (referring to Figure 4.17). So, meaning that corresponding angles are equal as you can see. So now how do we know that it is a parallelogram? Angle 1 in the big triangle and angle 1 in the other combined triangles alternate and are equal. Therefore, $a^2 + b^2 \,//\, c^2$*.This implies that* $a^2 + b^2 = c^2$*. Therefore the statement in the word problem is right. What else can you see in this shape?*

Ls: Opposite angles are equal; diagonal bisect opposite angles ***[TCV3]***

It can be noted from the interview responses and observations of Lesson 3 that Teachers B and C developed different shapes which they used in making sense of Pythagoras' proposition. In the process, they were able to establish connections among a rich set of mathematics concepts such as similar triangles, congruency, properties of shapes (isosceles triangles, kites, squares, rectangles and parallelogram), angle properties (alternating, corresponding and co-interior angles), transformation (rotating, flipping, and translating), and basic algebra. This I consider as having enhanced learners' understanding. In the interview, Teacher C said, "*...working with visuals we managed to build various shapes which we used in proving the theorem of Pythagoras. This helped us to link several concepts within the same lesson. This was my first time to see such connections since I have taught this theorem. My learners were also fully engaged during the lesson as they wanted to discover new shapes."* ***[TCI3]*** Such remarks pointed out to me that the intervention assisted the teacher in appreciating how powerful visuals can be in linking concepts and also affording learners an opportunity of constructing their knowledge under the teacher's guidance. Hiebert and Carpenter (1992) expound "that we understand something if we see how it is related or connected to other things we know" (p. 4).

- ***Multiple representations***

Kilpatrick et al. (2001) alluded to the fact that being able to represent mathematical situations in diverse ways and knowing how different representations can be useful for different purposes is one of the significant pointers of conceptual understanding. There is evident from lesson observations that different representations were used when solving the word problems. For example, in Lesson 1, Teacher C's approaches to solving the word problem show both calculations and visual representations — see Figure 4.26. That is, the teacher used symbolism, a diagram and explanations when solving this problem and this seems to have provided learners with different solution opportunities from which to choose. Following this up in the interview, Teacher C said, "*...using this model afforded my learners an opportunity of solving the word problem in different ways and this helped them to understand better than before."**[TCI2]*** Such interview responses and data from the video recorded lesson are an indication that the teacher benefited from this intervention. Konyalioglu et al., (2012) notes that the use of the visualisation method may aid learners to look at the problem in different ways and develop different thinking styles for problem solving.

In Teacher B's second lesson, dynamically manipulating her initial image in Figure 4.27 (a) as she illustrated her group's solution to the classmates, the learner was able to generate new images as seen in Figure 4.27 (c) and Figure 4.27 (d). It is at this stage (referring to Figure 4.27 (e)) that the learner applied Pythagoras' proposition and symbolically solved the word problem (see Figure 4.27 (e)). The learners' solution at this stage indicated to me that the visuals which the teacher used in proving Pythagoras' theorem (see Figure 4.8) assisted her in creating a picture in her mind and it is this picture which she drew from in solving the problem (referring to Figure 4.27 (e)). The teacher kept on engaging the learner(s) as seen in Figure 4.27 (f) above and the extract below.

> *TC: Can you take those two squares out (referring to Figure 4.27(d)) and put them there next to that square there (referring to the square which filled up the space in Figure 4.27 (a)).What relationship exists between the areas of these three squares (referring to Figure 4.27 (f))?*
>
> *Ls: The area of the big square is equal to the areas of those two squares.*
>
> *TC: Why?*
>
> *L: The two small squares (referring to Figure 4.27 (d)) occupy the same space that was occupied by the big square (referring to Figure 4.27 (a)) because the original square has not changed*
>
> *TC: That is correct, that is Pythagoras' concept that we are talking about **[TCV2]***

Makina (2010) asserts that visualisation is a very imperative corner stone in "teaching for understanding" in mathematics because it aids the teacher to facilitate the lesson, as it creates a platform where learners are more engaged with visual images. The extract reveals that the teacher used the visual to engage the class by asking questions which assisted her in checking for learners' understanding. The extract further

reveals to me that learners used the visual to justify their thinking and also to make sense of the Pythagorean concept.

In Lesson 3, both teachers developed different visual images (isosceles triangle, kite, square, rectangle, and parallelogram), each of which they used in making sense of Pythagoras' proposition in the two word problems — see Figures 4.10 to 4.17. In some shapes (square, kite, and rectangle), sense making of this proposition only relied on congruency of the two triangles on either side of the diagonal in argument for congruent diagonals, while in others (isosceles triangle and parallelogram) the proof relied on both congruency and the property of the sides of a shape. In making sense of Pythagoras' proposition using an isosceles triangle arrangement, Teacher B used both congruency and the property of sides of an isosceles triangle, while Teacher C only relied on the latter. In the parallelogram arrangement, both teachers made sense of Pythagoras' proposition using both congruency and the property of sides of a parallelogram. During the interview, Teacher C said, "*...using manipulatives we made different shapes which we manipulated in finding out in different ways if Pythagoras' proposition as stated in the word problems was true. This really worked well and learners enjoyed it.*" ***[TCI3]*** Teacher B also indicated that "*this activity was so enticing because of the various ways that got us to Pythagoras' proposition. Putting together the three triangles we generated an isosceles triangle, kite, square, rectangle and parallelogram. My learners were actively involved in building these shapes and this helped them to understand better.*" ***[TBI3]*** It can be deduced from the lesson observations and interview statements and remarks that the exploratory nature of this activity afforded learners an opportunity to build different shapes which were then discussed by the whole class in making sense of Pythagoras' proposition. This I consider as having enhanced learners' understanding because they were exposed to different ways of making sense of Pythagoras' theorem using different shapes.

4.3.2 Summary of the observations of the teachers' lessons

It was noted in the teachers' observed lessons that the use of visualisation processes in their illustrations and explanations indeed assisted most of the learners in developing an understanding of Pythagoras' theorem and how to solve word problems, compared to how they were taught before.

It was observed that during their lessons, all teachers used visual models to generate images and used the visual models to develop the pertaining mathematical idea. The generated images were displayed on the board and these created a platform for discussion during the lessons. The discussions were driven by questions which teachers asked in probing learners' prior knowledge and finding out if they understood what they were teaching. These discussions were fascinating and one could really see that learners comprehended what was being taught. It was also noted that the teachers dynamically manipulated their

initial images to support their explanations. Therefore, the teachers' explanations were linked to the visuals used.

The two teachers taught all their lessons using the three models (that is, the two area models and the scaled-up versions of a right-angled triangle) with each lesson focusing on a particular model. However, one teacher only taught one lesson using the area model that dates back to Euclid. In my observation, I noted that although models were developed and provided to the teachers by the researcher prior to the lessons, all the teachers pasted them onto the board during the lessons and examined them jointly with learners. To me, this was ideal because the process captured learners' attention and got them actively engaged in the lesson. Learners could see how images are created, inspected, transformed and used in making sense of Pythagoras' theorem and in understanding and solving word problems. During the process, learners were exposed to a rich set of related mathematics concepts, all because of according them an opportunity of engaging with the Pythagorean concept visually.

It was also noted that when working with the area model that dated back to Euclid, the two teachers and their learners drew diagrams and also pasted onto the board the manipulatives of this model, while the other teacher only worked with the manipulatives of the area model. However, in my observation I noted that teachers who drew diagrams could not finish the planned content unlike the teacher who worked with manipulatives only.

All Kossyln's (1996) visualisation processes (image generation, image inspection, image transformation and use) dominated in all the lessons that the teachers taught except in Lesson 1 where image transformation did not occur. It was also interesting to note that there were other visualisation processes that emerged from the lesson observations' data. These included: visual language, gestures, generalisation and symbolic inscription. However, generalisation was not that common in the teachers' lessons.

Furthermore, teaching for conceptual understanding was dominated by the use of multiple representations, building on learners' prior knowledge and connecting concepts and ideas in mathematics. Apart from using word problems from real life in all their lessons, the indicator of extending mathematics to real life was not especially visible in the teachers' lessons. It was only evident in Lesson one when one of the teachers used a broom from the learners' classroom in contextualising the word problem and also when the other teacher drew examples from the learners' school and/or home backgrounds in showing them the relevancy of the theorem in real life.

4.4 RESEARCH RESULTS FROM INTERVIEWS WITH THE PARTICIPANTS

In this study, data was also collected from stimulated recall and focus group interviews to answer Research question 3. I conducted three one-on-one stimulated recall interviews with each of the two

participating teachers and one one-on-one stimulated recall interview with the teacher who dropped out. A focus group interview with the two participants was conducted after all lessons were taught. The purpose of the eight interviews was to ascertain selected teachers' perceptions and experiences on whether the use of visual models enhanced conceptual teaching of word problems on Pythagoras' theorem after participating in an intervention programme. The interview data is presented in the following section according to themes. Some of the themes were informed by literature. These comprise: connecting mathematics to prior knowledge, connecting mathematics to the real world, connecting concepts and ideas in mathematics and use of multiple representations. Others themes emerged from the data. The themes that emerged are: **teachers' appreciation of the use of visuals; teachers' beliefs in the use of visuals; teachers' participation in the use of visuals; teachers' impressions of the use of visuals; teachers' insights into the use of visuals; and teachers' challenges in using visuals.**

4.4.1 Teachers' perceptions and experiences

This section provides a detailed description of the themes that emerged from the selected teachers' perceptions and experiences when using visualisation strategies to conceptually teach word problems in the context of Pythagoras' theorem.

4.4.1.1 Teachers' appreciation of the use of visuals

The teachers recognised the importance of using visual approaches in teaching word problems. They stressed that visuals aided learners in understanding what the problem required and helped to offload working memory load. They also noticed that visuals accommodate learners with different learning abilities. They further underscored that visuals did not only help learners in understanding Pythagoras' concept but also to develop an understanding of a rich set of mathematics concepts connected to this theorem, all which was a result of affording learners an opportunity to engage with visualisation strategies. This indicated to me that because of the intervention, these participating teachers realised that learners understood better when visuals are used. Teacher A mentioned that:

> *The diagram helps them to see what the problem is asking, in that way it is easy to answer the question unlike the way I used to teach where I would just go straight to the formula which one is the hypotenuse from this information and substitute without even learners understanding what the problem is asking. But with visuals learners can see which information is needed and which one is not necessary thereby understanding the problem and using them to manipulate it. Visuals also assisted learners in creating pictures in their minds which they could always use when solving word problems* ***[TAI1]***

Teacher B mentioned that:

> *…visuals were very helpful in explaining Pythagoras' theorem and solving word problems because by using them I was able to build from learners' prior learning and this enabled me to connect many mathematics concepts. In Lesson 3 for example, learners were actively involved in building different shapes which they used in solving the word problems in different ways. This*

helped them in not only making sense of Pythagoras' theorem but also in connecting various mathematics concepts. This really helped my learners to see that mathematics concepts are not learned in isolation but interconnected. I also observed that visuals in form of diagrams helped my learners in understanding and solving the word problems ***[TBFGI]****.*

Teacher C indicated that

> *I have noticed that visuals have a lot of advantages such as creating pictures in learners' minds which they can always go back to when they have forgotten. It is not easy to forget what you have done touching with your hands and seeing with your eyes. It is easy to forget what you are told. Working with visuals involved more than one sense unlike in the theory method it is just hearing sense. So visual aids accommodate different learners. Those who learn by hearing, seeing and touching. It also accommodates the slow learners. I realised that using visuals gives learners a chance to revise because it builds on previously learned concepts. Visuals enable learners to translate the word problems into something real which they can touch and manipulate in order to solve the problem. They also assist in offloading the verbal component to the visual component of the brain which has always been over used in my lessons before* ***[TCFGI]****.*

From the verbatim interview above, teachers largely acknowledge and appreciate the benefit of using visuals in their lessons. In reference to Teacher B's verbatim interview [TBFGI], the teacher instructed learners to draw sketches of the word problems after which she provided them with graph paper to accurately draw the triangles and scale them three times in order to produce three scaled triangles. The teacher guided learners in putting these manipulatives together and they managed to create different shapes as seen in Figure 4.10; 4.11; 4.12 and 4.13 above. She used her manipulatives to put these shapes up on the board and used the shapes to generate a whole class discussion. They made use of congruency of triangles on either side of the diagonal and/or the property of sides of a particular shape in solving the word problems. As they manipulated these shapes, a rich set of underlying mathematics concepts such as similarity, transformations (like rotation, flipping and translation), properties of shapes and angle properties were brought to the surface, which was as a result of affording learners an opportunity to engage with visuals in solving word problems. This indicated to me that encouraging learners to draw diagrams when interpreting word problems and encouraging them to manipulate these diagrams when solving these problems did not only enable them to understand the word problems but also to solve them. It also afforded them an opportunity of solving these problems in multiple ways and visually experiencing a rich set of mathematics concepts. This I thus consider as having enhanced their understanding. Research has shown that the use of a visual approach can be helpful to learners' conceptual learning by providing an alternative new method of viewing the problem (Konyalioglu et al., 2008) and also developing problem solving thinking styles (Konyalioglu et al., 2012), thus facilitating meaningful learning.

External visual representations that learners use or construct to enhance their understanding automatically generate a big picture in their minds to aid them in discovering the solution to the problem (Rosken & Rolka, 2006). Teacher C's response " … *that use of visuals helps in offloading the verbal component of*

the brain" indicated to me that the teacher acknowledges that the use of visuals in explaining mathematics concepts helped in creating pictures in learners' minds which they could draw from in solving word problems either mentally or by creating an external representation. This was evident in Teacher B's second lesson when a learner was presenting her group's solution to her classmates. The learner used an external visual representation in solving the word problem up to a certain stage (see Figures 4.27 (a) to 4.27 (d)) after which she applied symbolism (see Figure 4.27 (e)). This meant that the learner had developed a picture in her mind when the teacher was visually proving this theorem, which she harnessed in solving the word problem when she got to stage 4.27 (d), thus offloading her working memory load. Similarly, Van Meter et al. (2006) and De Koning and van der Schoot (2013) highlight that as problem solvers engage in the process of creating a drawing, they integrate their prior knowledge with the to- be-learned content which facilitates a deeper understanding of text content. This in turn aids in offloading a reader's working memory load as cognitive processing demands are divided over verbal and visual information processing channels, consistent with the dual coding theory (Sasdoski & Paivio, 2013). It can also be deduced from Teacher C's remarks that it was after participating in this intervention programme that the teachers realised that visual learners were accommodated in her lessons. This indicated to me that because of the intervention, the participating teacher realised that visuals accommodate learners with different learning abilities. Ho (2010) stresses that use of visualisation caters for individual learning styles because each learner has his or her own preferences when it comes to the use of visual representations when solving problems.

Lastly, all teachers agreed unanimously that visuals assist in contextualising word problems and this enhances learners' understanding and solving of these problems. In agreement, Yonson (2017) asserts that learners show an interest in solving word problems using drawings, diagrams or manipulatives as these aid them in picturing out what the problem is asking.

4.4.1.2 Teachers' beliefs in the use of visuals

Teachers indicated that they believe that visual models have the potential of encouraging learners to work hard and also to enhance their understanding. This is what Teacher B had to say:

> *...I have used visuals for only three lessons but I believe that if I continuously use them they are going to lay a good foundation for my learners in terms of understanding and this will help them to learn new topics with ease such as trigonometry, analytical geometry, and measurement, among others which are related to this theorem as they progress to higher grades* ***[TBFGI]***

Teacher C concurred as she said:

> *I feel that if I use it more it will encourage my students to work hard and also to be competent in solving real life problems. From my experience most of my learners fail to answer word problems*

because they don't even bother to come up with a diagram. So now if they can visualise and come up with a right diagram, it means they will minimise mistakes ***[TCFGI]****.*

Teachers feel that continuously using visuals will not only encourage their learners to work had, but they will also become competent real life word problem solvers. They also believe that learners will find it easy to learn new topics (such as trigonometry, analytical geometry, etcetera) related to Pythagoras' theorem because they will be in a position to use this knowledge as a stepping stone to learn the new content. I believe this will enhance their understanding.

4.4.1.3 Teachers' participation in the use of visual models

All teachers supported the use of these visual aids in the teaching of Pythagoras' theorem word problems because they could see that visuals helped learners to understand better, unlike the way they taught these problems in the past, without visuals. Teacher A said that:

> *...at first we were actually counting the small squares to find the size of each square and also they could find the area of each square. Now they could see that to find the square on the hypotenuse it is not just adding the other two sides, but you need to add the areas of the two squares. This helped them to understand better than I taught before. In finding the length of the missing side, when asked to find the area of the square drawn on that side, most learners processed their answers mentally* ***[TAI1]****.*

Teacher B indicated that:

> *...when using model two, I arranged four congruent triangles and asked learners for the new shape and why they say so. We then identified the area of the middle square and put it aside. I rotated the triangles to form another shape and learners figured out that it was a rectangle. I then translated one of the two rectangles and filled the empty spaces with two other squares. I also asked learners for the areas of the squares. After engaging in a discussion, learners could see that the last two squares filled up the same space we started with because the original square had not changed thus establishing the relationship between the areas of the three squares. I then gave them a word problem and encouraged them to use the visuals I had provided each group. When presenting the solution for her group the learner arranged the manipulatives up to a certain stage then applied the formula* ***[TBI2]****.*

Teacher C mentioned that:

> *If we look at the last model, learners translated the word problems into right-angled triangles which they accurately drew and scaled three times to produce three scaled triangles. Using the three triangles, learners made different shapes and I used my triangles to put up the shapes learners had built on the board. These shapes formed a basis for a discussion with the whole class in examining whether Pythagoras' theorem as stated in the two word problems was valid. In the isosceles triangle and parallelogram, we used properties of sides of shapes in the kite, square and rectangle arrangement we used congruency but each shape's proof led us to Pythagoras' proposition. Learners were able to see a number of mathematics concepts like similarity, congruency, transformation and algebra that are involved when we used these manipulatives unlike when I taught using the telling method. I don't think they can forget this* ***[TCI3]****.*

From the interview responses, it can be deduced that the way visual models were used, helped learners in understanding what was taught. Teachers used the models to build on learners' prior learning, make sense of Pythagoras' proposition in multiple ways, and to establish connections between a rich set of mathematics concepts (Kilpatrick et al. 2001). It is also evident from the responses above that visuals assisted in bringing to the surface underlying concepts that were involved when working with Pythagoras' concept. In agreement, Arcavi (2003) alluded to the fact that visualisation enables a problem solver to engage with concepts and meanings which may be easily bypassed by the symbolic solution of the problem.

Teacher A and B's interview responses above indicated to me that it was after these teachers visually proving the theorem of Pythagoras with their learners that these learners developed pictures in their minds which they harnessed in visualising and solving the word problems — see Figure 4.19 and 4.27 (d) respectively.

4.4.1.4 Teachers' impressions of the use of visuals

In this section, I looked at the responses for the teachers' impressions of the visuals they worked with. The responses came from two teachers because one teacher had dropped out after working with only one model. The two teachers taught word problems on Pythagoras' theorem using three models (that is, the two area models and the scaled-up versions of a right-angled triangle). They acknowledged that all these models have the wherewithal to enhance learners' understanding and solving of Pythagoras' theorem word problems. They however highlighted that they preferred the last model (scaled-up versions of a right-angled triangle) because it had a rich set of mathematics connections which the other models did not have, and that it made learners curious to discover more shapes. Teacher B pointed out that:

> *All the models we looked at were helpful in promoting understanding but I preferred the last one because learners enjoyed it. It was more of doing and it is like they were playing a game. Because no one told them that you should put this one like this, it was them trying to come up with the shapes in their groups under my guidance. They came up with isosceles triangle, square, kite, rectangle and parallelogram. It was very interesting because each shape they manipulated still led them to Pythagoras' proposition. It indeed helped them understand better. It had so many concepts involved when compared to the first two models* ***[TBFGI]****.*

Similarly Teacher C concurred with the statement by Teacher B as she mentioned that:

> *The last model was the most interesting one to me and my learners. They got more involved when using the model right from drawing, enlarging, cutting, they were doing a lot of things there. They enjoyed making different shapes to come up with the same conclusion. We revised a lot of topics with this model unlike in the first ones* ***[TCFGI]****.*

The teachers' preference of using this model was because of the multiple ways of making sense of Pythagoras' theorem and also its richness in mathematics concepts connected to this theorem. It can be

deduced from the teachers' transcribed interviews that it was through using this model that the teachers were accorded an opportunity to build on various mathematics concepts. This I consider as having assisted learners in making sense of Pythagoras' proposition as they endeavored to verify whether the theorem was valid in the two word problems. This concurs with Harvey's (2011) argument that learners learn better when they relate what they are being taught to their prior knowledge

4.4.1.5 Teachers' challenges in using visuals

Teachers indicated that time was their biggest challenge especially when drawing was involved. They also observed that some of their learners struggled to build shapes in their attempt to make sense of Pythagoras' theorem. Despite those challenges, teachers realised that after participating in this intervention programme, there was a shift in learners' involvement and the teacher's role during lessons. Below is what the teachers said in support of this. Teacher A:

> *...working with visuals I realised that I talked less because I was not giving out information. We were actually discussing what we all saw. Learners were paying attention because they wanted to see me perform whatever I was performing. As you can see in the video some could come and count how many small squares filled up a particular square, even some could come and draw the diagram. Then mine was to guide, so I talked less as learners did more of the talking. This was not common with the telling method I always used. I also realised that having to draw consumed a lot of our time, but if I had several of this model for each group I think it was going to be more helpful because I would just guide them as they manipulate, so it would save time for drawing* ***[TAI1]****.*

Teacher B be concurred as she mentioned:

> *...I noticed that learners' level of focus increased when teaching with visuals. Visuals captured learners' attention, they were actively involved in constructing their knowledge in their groups as I supported them. Therefore, I talked less because most of the doing was done by the learners most of the times which was uncommon with the lecture method. Some learners were failing to build shapes but they were motivated to keep trying because it was practical* ***[TBI3]****.*

Teacher C also said:

> *When using visuals, learners were very excited compared to the telling method I was fond of using. They were really involved in it and they were enjoying what they were doing and that only tells you that this can stay in their memories better than formula. Most of the doing was done by learners as I facilitated them, this enabled me to talk less unlike in the telling method. In lesson one, we spent a lot of time on drawing and we could not finish the planned content. If I had more models like this one which you gave us during the workshops it was going to be much quicker that they would assemble them just like I did on the board rather than letting them draw. Lesson 2 and 3 were a bit quicker compared to the first one because we had enough models for each group prior to the lesson. Theirs was to manipulate them* ***[TCFGI]****.*

Visualisation increases collaboration among learners in group work (Klerkx, Verbert & Duval, 2014) and this key in the development of an individual's knowledge (von Glasersfeld, 1996). From their experiences of using visuals in teaching, teachers indicated that visuals got learners actively involved as learner-to-

learner interaction increased in the lessons. Teachers noted that their role shifted from them being the centre of attention to that of being a facilitator in helping learners to construct their own knowledge. They also observed that some of their learners struggled with building shapes as they endeavored to make sense of Pythagoras' theorem in the two word problems. However, they observed that the same learners were motivated by the exploratory nature of the activity. Lastly, all teachers agreed that having to draw models is time consuming. They however suggested that if they are provided with several of these models, they would be enthused to adopt visualisation approaches in their mathematics lessons as previously made visuals will minimise on the time to be spent on drawing.

4.4.1.6 Teachers' insights into the use of visual aids

Teachers indicated that prior to the intervention they hardly taught word problems on Pythagoras' theorem and mathematics concepts in general, using visualisation approaches. They however acknowledged that after participating in this intervention they were empowered on how to teach Pythagoras' theorem word problems differently. This is what Teacher A had to say:

> *This workshop was very helpful because before me I have never when explaining Pythagoras' theorem and even when solving word problems, I have never used those things, it was for the first time. I was always using formula honestly, but now I know how to make and use different models to improve teaching of this theorem* ***[TAI1]****.*

On the same note Teacher B indicated that:

> *Before I used to have a challenge of resources and that is why I relied on formula in teaching Pythagoras' theorem but now I have more insights on how I can teach word problems on Pythagoras' theorem differently with different teaching aids which I did not know* ***[TBFGI]****.*

Teacher C on the other hand supported these ideas as she purported that:

> *My thinking has shifted on Pythagoras' theorem because I now have more insights as to how I can incorporate some visuals from learners' home background to improve my teaching unlike before when I taught theoretically* ***[TCFGI]****.*

Such experiences as indicated by the teachers revealed to me that these teachers had benefited from this intervention programme, especially on how to use visuals to teach Pythagoras' theorem word problems differently.

4.5 SUMMARY OF THE EMERGENT THEMES FROM INTERVIEW RESPONSES

The purpose of the interviews was to answer Research Question 3:

What are the selected teachers' perceptions and experiences in using visual models to conceptually teach word problems on Pythagoras' theorem as a result of participating in an intervention programme?

Analysis of the interview responses revealed that visual aids are at the very heart of teaching word problems on Pythagoras' theorem. Teachers indicated that using visuals accorded learners an opportunity to see wat they were doing and/or to manipulate the images, thus enabling learners to create pictures in their minds which they used in solving word problems.

All the teachers observed that when teaching Pythagoras' theorem word problems, visuals helped them in building on learners' prior learning, extending what they were teaching to real life and in bringing to surface underlying concepts. This enables learners to visually experience a rich set of mathematics concepts connected to this theorem. They further noted that visuals enabled learners to solve word problems in multiple ways, thus enhancing their understanding. The teachers also stated that visuals helped in honing learners' thinking ability as some were able to manipulate and solve word problems mentally.

Additionally, teachers noted that as they manipulated them, the visuals got learners actively involved in their groups. They realised that this afforded them a chance of talking less as their role shifted from them being the centre of knowledge to that of a facilitator in aiding learners to construct their own knowledge. To them, this helped learners to understand Pythagoras' theorem better than how they taught it before.

Furthermore, the teachers asserted that teaching using visuals promoted a balance in the use of the senses because working with visual aids encompassed different senses, accommodating learners with different learning abilities. On top of this, they pointed out that using visuals helped in offloading working memory load as cognitive processing demands are divided over verbal and visual information processing channels (Sadoski & Paivio, 2013).

Lastly, the teachers highlighted that although visual aids were very helpful in enhancing learners' understanding when teaching Pythagoras' theorem word problems, teaching using these aids requires a lot of time especially when the models have to be drawn. They noted that in lessons where the teacher and learners had to draw the models, they could not complete what they had planned to cover in that time. Drawing from their lesson experiences where models which they used were prepared prior to the lesson, they suggested that teachers should be provided with enough models for learners to manipulate in their groups instead of having to draw them during the lesson. In that way it would help to save time and also encourage teachers to use them in improving their teaching.

4.6 CONCLUSION

In this chapter I have presented and discussed the data collected from survey questionnaires, lesson observations, and interviews (both stimulated recall and focus group interviews). Data from each research instrument has been discussed separately and concluded with a brief summary. The data arising from the lesson observations was based on visualisation processes. The data arising from the survey questionnaires and interviews was based on the perceptions and experiences of the participating teachers. Some of the themes used to categorise and discuss the data from survey questionnaires, lesson observations and interviews emerged from the data while others were informed by theory.

Survey questionnaire responses revealed that most teachers in the JTG district still emphasise rote and algorithmic approaches at the expense of other approaches such as visualisation strategies. Challenges such as a lack of resources in schools, time constraints, the DBE's strict adherence to pacesetters, and lack of support from subject advisers were noted as the reasons why many teachers were not using visualisation strategies to extend what they were teaching to real life, make connections among concepts in mathematics, build on learners' prior knowledge and represent a concept in multiple ways.

Such survey questionnaire responses necessitated a need for an intervention programme to support some teachers on how to optimally use visual tools and strategies in mathematics classes. These teachers were therefore observed after the intervention programme and the observations are presented below.

Teachers' lesson observations revealed that all the teachers used visual aids to create images during their lessons, and used visual aids to develop the mathematical idea at hand. All study participants were able to manipulate the images they generated to support their explanations as they solved the word problems. The visuals used were in line with teachers' verbal explanations and the visual language used was also in line with the mathematical language and terminology. The teachers were observed using gestures in conjunction with visual aids in their lesson presentations. The teachers assisted learners in making connections between visual and symbolic inscriptions of the pertaining mathematical concept they were dealing with.

It was also noted that some teachers used visuals to promote mathematics abstraction and generalisation, particularly in the first lesson. Furthermore, it was noted that all the teachers pasted their models up on the board during the lessons and examined them jointly with learners. This was ideal because the learners could see what was being done and the concepts involved as how the visuals were manipulated. This got learners actively engaged in the lesson and one could see that they were understanding what the teacher was teaching.

Lastly, Kilpatrick et al.'s (2001) conceptual understanding indicators (use of multiple representations, building on learners' prior knowledge, and connecting concepts and ideas in mathematics) dominated in all the lessons that the teachers taught. Apart from using word problems from real life in all their lessons, the indicator of extending mathematics to real life was not very visible in the teacher's lessons. It was only evident in Lesson 1 when one of the teachers used a broom from the classroom in contextualising the word problem and also when another teacher drew examples from the learners' school and/or home backgrounds in showing them the relevancy of the theorem in real life.

Interview findings provide strong evidence to support data from survey questionnaires and lesson observations. It was revealed from interviews that the teachers made use of visuals to assist them in demonstrating different mathematics ideas, linking them to other mathematical constructs and to real life situations. For example, when demonstrating Pythagoras' theorem and solving word problems, teachers made use of visuals to build from what learners already knew. They were also able to link concepts like area, transformations, congruency, similarity, Euclidean geometry, *inter alia* which enabled learners to see the connections, thus they learned with understanding.

Teachers also noted that working with visuals got learners actively involved in the lessons as learner-to-learner interaction increased. They realised that their role shifted from that of being the centre of knowledge to that of a facilitator aiding learners to construct their own knowledge.

Additionally, it was observed from interview responses that the teachers acknowledged that working with visuals helped learners in offloading their working memory load, thereby increasing their chances of understanding and solving word problems.

Lastly, it was noted from the interviews that teachers realised that after participating in this intervention they were empowered on how to create and use visuals to teach Pythagoras' theorem and word problems, in contrast to when they predominantly relied on formula.

CHAPTER FIVE

CONCLUSIONS AND RECOMMENDATIONS

5.1 INTRODUCTION

The focus of this study was to investigate how, as a result of an intervention program, selected mathematics teachers used visualisation approaches for conceptual understanding to teach Grade 9 mathematics word problems in the context of Pythagoras' theorem. The chapter concludes my research study in which I present the following: a summary of the findings, outline the contribution of the study, recommendations to scholars of mathematics education and policy-makers, limitations of the study and suggestions for further research. Finally I present my personal reflections as a result of this study.

5.2 SUMMARY OF FINDINGS

Research Question One

How do grade 9 mathematics teachers in a selected district teach Pythagoras theorem and word problems in this context prior to an intervention programme?

Survey questionnaires revealed that while teachers in the JTG District acknowledge the importance and value of visualisation approaches in the teaching of Pythagoras' theorem and word problems, many of them are either using visuals minimally or not using them at all. Challenges such as lack of resources in schools, time constrains, the DBE's strict adherence to pacesetters, a lack of support from subject advisers, among others, were noted as reasons why many teachers were not using visualisation strategies to extend what they were teaching to real life, make connections among concepts in mathematics, build on learners' prior learning and represent concepts in multiple ways.

The teachers who indicated using visual aids gave examples of visuals they used when teaching Pythagoras' theorem word problems. These include: diagrams, concrete objects (such as an electric pole anchored by a wire, sticks or straws and strings used to construct a right triangle), models, ICT and pictures. What was common to these teachers' experiences was the use of a visual proof of Pythagoras' theorem that dates back to Euclid. Most of these teachers preferred to use diagrams compared to other visual aids indicating that diagrams do not require a lot of preparation. Furthermore, it was interesting to note that some teachers who indicated using visuals, gave similar reasons for not using visuals as those given by the teachers who preferred teaching symbolically.

Research Question Two

How do selected teachers use visuals to enhance conceptual teaching of word problems that involve the Pythagoras theorem after participating in an intervention programme?

The teachers drew diagrams and also arranged manipulatives of the visual models on the board in order to generate images. These images were used to develop the mathematics idea at hand — not necessarily using strict definitions of the mathematics idea and/or real calculations. Images were generated throughout the lesson presentations and during the process of image generation, the teachers used manipulatives of the visual models to systematically create images. The generated images were then inspected jointly by teachers and learners in establishing a relationship between the sides of a right-angled triangle and solving word problems. The teachers used the images to create platforms for classroom discussions. The discussions were driven by questions which teachers asked both for ascertaining learners' prior knowledge and for finding out if they understood what the teachers were teaching. Therefore, there was a connection between the visuals that were displayed on the board and the discussions of the visuals between teachers and learners.

Pythagoras' proposition was illustrated through image transformations only in the teachers' last two lessons. The transformation of images (through rotation, translation and flipping) gave birth to new images which the teachers used to create discussions with their learners. These transformations captured learners' interest visually, and got them actively engaged in the lessons. One could see that learners were not only enjoying, but also understanding what the teacher was teaching.

Throughout the lesson observations, teachers did not only demonstrate the visual relationship between the right-angled triangle and its sides, but also the symbolic one. The teachers assisted learners in making connections between visual and symbolic inscriptions of the mathematical concept they were dealing with. This process helped learners in creating pictures in their minds, which they used in understanding and solving the word problems in different ways — thus enhancing their understanding.

Furthermore, it was noted that the visuals used were in line with teachers' verbal explanations and the visual language used was also in line with the mathematical language and terminology. For example, when making sense of Pythagoras' theorem and solving word problems, teachers demonstrated proper use of mathematical language such as rotating, clockwise, anticlockwise, translating, similar triangles, among others and their verbal explanations were in line with the visuals used.

The teachers also used gestures in conjunction with the visuals in their lesson presentations. These were common when interpreting and understanding the word problems. The teachers kept moving their hands

in trying to illustrate the situation in the word problem. This I consider as having assisted learners in understanding the word problem.

Additionally, some teachers used visualisation to promote mathematics abstraction and generalisation. Their aim was to achieve generalisation after which visual thinking was to be dispersed after serving its initial purpose in some of their lessons. However, this was only common in one teacher's first lesson, when she solved the word problem symbolically.

Lastly, it was observed that teachers used visuals to connect Pythagoras' theorem to real life. They used real life problems and everyday examples in enhancing learners' understanding of the theorem's application in real life. During the lesson, one teacher used an example of an electric pole anchored by a wire, forming a right-angled triangle. This was just across from the school and learners could see it. When solving the word problem, another teacher used a broom from the classroom in contextualising the word problem, which I consider as having enabled learners to understands and solve the problem.

Research Question Three

What are the selected teachers' perceptions and experiences in using visual models to conceptually teach word problems on Pythagoras theorem as a result of participating in an intervention programme?

Teachers indicated that teaching using visuals got learners actively involved in the lessons as learner-to-learner interaction increased. They noticed that their role shifted from that of being the centre of knowledge to that of being a facilitator aiding learners to construct their own knowledge.

Teachers highlighted that visuals enabled them to build on learners' prior learning, make sense of Pythagoras' proposition in multiple ways and to establish connections among a rich set of mathematics concepts. When verifying whether Pythagoras' proposition held true in the two word problems, teachers and learners built different accurate shapes (isosceles triangle, square, kite, rectangle and parallelogram) which they used in creating a whole classroom discussion. They indicated that the proof in some shapes (isosceles triangle and parallelogram) relied on the property of sides of a particular shape while in other shapes (kite, square and rectangle) it relied on congruency of triangles on either side of the diagonal. They added that during this sense making process of the Pythagorean proposition, learners experienced a rich set of mathematics concepts (similarity, congruency, angle properties, and properties of shapes, transformations, congruency, and algebra) connected to Pythagoras' theorem thus achieving a deeper understanding. The teachers also observed that visuals helped them in extending what they were teaching

to real life and this helped learners in seeing that Pythagoras' theorem is not detached from real life — in contrast to how they had been taught before.

They indicated that visuals enabled learners to translate the word problem into something real which they could touch and manipulate in order to get to the answer. They added that visuals boost learners' interest in learning because they are practical. This helps them in seeing and manipulating what they are being taught thus enhancing their understanding.

Furthermore, the teachers pointed out that visuals help learners in creating permanent pictures in their minds which they can always reflect on to assist them in solving word problems. They highlighted that when solving word problems, some learners drew an external representation, some used the formula and others manipulated the problems mentally, but all arrived at the same answer. They therefore argued that teaching using visuals enables learners in solving a word problem in different ways, thus enhancing their thinking ability.

Additionally, teachers mentioned that teaching word problems on Pythagoras' theorem using visuals helps in offloading the working memory load as cognitive processing demands are divided over verbal and visual information processing channels. They also indicated that working with visuals promoted the use of different senses because a variety of senses are involved when one teaches using visual models, thus accommodating learners with different learning abilities.

The teachers indicated that even though the three models were useful in teaching word problems on Pythagoras' theorem, they preferred the last model (scaled-up versions of a right-angled triangle) because it had a rich set of mathematics connections that other models did not have. They suggested that this model can be very helpful in teaching Euclidean geometry at Grade 10.

Lastly, teachers asserted that although visual aids are acutely relevant in enhancing learners' understanding when teaching Pythagoras' theorem word problems, they were challenged by time when the drawing of models was involved. Considering their lesson experiences where they taught using pre-made models, the teachers suggest that teachers should be provided with enough models of this type. This would help to address this challenge and also encourage teachers to use models to improve their teaching. They also suggest that teachers should be empowered on how to make and use these models; this would help to combat the problem of shortage of resources. They acknowledged that the workshops were very powerful because they now know how to make and use different models in teaching Pythagoras' theorem and word problems in this context.

5.3 CONTRIBUTION OF THE STUDY

In the South African education curriculum context, this research has the ability of providing curriculum developers and mathematics education researchers with meaningful information and recommendations on incorporating visualisation into the design of policy guidelines. This project did not only contribute to my knowledge about visualisation in mathematics education, but it also aided me as a mathematics educator to expand my knowledge on how to use effective instructional methods in teaching mathematics word problems on Pythagoras' theorem.

It is hoped that the participants will be empowered with the different approaches required in conceptually teaching word problems on Pythagoras' theorem using visualisation. Additionally, the results of this study could help the Department of Higher Education and Training to integrate visual approaches in teacher training programmes so as to adequately prepare teacher trainees to embrace the visual world in handling mathematics word problems.

Lastly, the findings of this study may help authors of mathematics textbooks to include the use of visualisation in their textbooks in order to enhance learners' conceptual understanding of mathematics word problems.

5.4 RECOMMENDATIONS

I am acutely aware that although the findings of this study cannot be generalised to the entire South African education system, they can contribute to those who wish to use visualisation as an instructional tool for conceptual understanding in teaching Pythagoras' theorem word problems. Based on the findings of this study, I would like to make the following recommendations. After being workshopped, mathematics subject advisers should be encouraged to provide workshops on how to use visual models (that is, scaled-up versions of a right-angled triangle and area models) as visualisation tools with special emphasis on aspects that promote learners' conceptual understanding of mathematics word problems such as:

- Visualisation processes
- Connecting mathematics to prior knowledge
- Connecting mathematics to the real world
- Connecting concepts and ideas in mathematics
- Use of multiple representations

Alternatively, curriculum developers should consider including and making visuals such as the area models and scaled-up versions of the original right-angled triangle model compulsory in teaching Pythagoras' theorem word problems. This is because the findings of this study show that the use of such

models as visualisation tools helps learners in understanding and solving word problems. Therefore, the use of such models has to be emphasised in official documents.

Additionally, universities training teachers should ensure that teacher trainees are well prepared in creating and using visualisation strategies (both external and internal) to teach word problems. During their training, emphasis should be accorded to the role that visualisation plays in mathematics education.

From my experience, most South African schools are well-resourced with different materials (like charts) that teachers can harness to build different visual models (such as scaled-up versions of the original right-angled triangle and the area models) that they can use in their teaching of Pythagoras' theorem word problems if they are trained. This can help address the challenge of resources. On this note, all the visuals that teachers used in this study were made from charts by the researcher.

Teachers should make and/or be provided with enough of these models so that learners can simply manipulate them in their groups instead of having to draw them during the lesson. This will help to minimise the time spent on drawing. Also, in using these models, teachers should afford learners an opportunity of exploring them as they facilitate this process. With such an approach, learners will be in a position to create pictures in their minds which they can reflect on and use to solve Pythagoras' theorem word problems (either mentally and/or externally). This will in turn help learners to reduce their working memory load because their cognitive processing demands will be divided over verbal and visual information processing channels as they solve the word problems.

If these visual models are to be optimally used, teachers should familiarise themselves with them prior to lessons; otherwise they will be frustrated in assembling their manipulatives to generate images, which can guide them in establishing relationships and/or solving word problems. They should also consider using these models in creating a platform for discussion, ensuring that their verbal explanations are in line with the visuals used, and that the visual language used is in line with the mathematical language and terminology.

Furthermore, the use of visualisation strategies help learners in solving Pythagoras' theorem word problems accurately. Therefore, teachers need to train their learners on how to use these strategies and also encourage them to use such strategies when solving word problems. This will in turn visually expose learners to a rich set of mathematics concepts connected to Pythagoras' theorem, which could easily be bypassed if learners applied the formula straight away.

It is also imperative for teachers to make their learners aware that there is a process to be followed when solving word problems and this entails: understanding the problem and identifying the relevant

mathematical concepts within it; formulating the problem into a mathematical model to describe the relevant relations involved; solving the mathematical problem included in the model and reflecting on the solution process; and evaluating and interpreting the mathematical solution in relation to the original realistic situation. This process also needs to be explicitly taught to the learners.

Common to most mathematics textbooks is the area model dating back to Euclid. Therefore, mathematics textbook writers should consider incorporating more visualisation in their books by creating space for more fascinating visual models (such as the scaled-up versions of the right-angled triangle and the other area model — proof by rearrangement) that are not only focusing on the details of the image but also on visualisation processes. This will assist learners in analysing and creating concrete meaning out of them.

5.5 LIMITATIONS

The challenges I faced in my study involved getting the questionnaires to and from the teachers. Some teachers kept on postponing the retrieval date of the questionnaires. The fact that I am a full-time teacher made it difficult for me to distribute the questionnaires to all the schools in my district. However, I met with the mathematics subject advisers who informed me of a workshop that was forthcoming in the next few days. I took advantage of this workshop and got teachers to fill in the questionnaires, which I retrieved before the workshop commenced. Furthermore, some of the questionnaires were partly completed and in some, the teachers did not elaborate on their answers.

The exploratory nature of my study does not allow the findings to be generalised. The study initially had three participants from three different schools who were purposively and conveniently selected. One teacher withdrew from the study and the number of main participants was reduced to two. My study was conducted over a short period of time focusing on a specific area of mathematics which is mathematics word problem solving on Pythagoras' theorem. Therefore, the time to procure sufficient data was limited. Ideally, more time should have been spent on training teachers and exposing them to the full potential of the use of visualisation approaches to teach word problems on Pythagoras' theorem. The outcomes could have been more comprehensive if more teachers used these three visual models for a longer period of time.

In addition, I faced a challenge in interviewing teachers after they had taught their lessons. The teachers always came up with excuses and this adversely affected my interview plan. However, despite this study's limitations, useful insights on the use of visualisation strategies in the teaching of word problems on Pythagoras' theorem have been provided by this study.

5.6 SUGGESTIONS FOR FUTURE RESEARCH

Since this study was only done on a small-scale, I recommend the following areas of research:

- Conducting research with more research sites and a bigger sample size in order for the findings to be generalisable to a wider context.
- Conducting research on the same topic, but expanded to include learners, curriculum developers and authors of textbooks.
- Conducting research on the role of gestures in mathematics word problem solving.
- Conducting research on learners' views and experiences on the use of visualisation processes in learning word problems on Pythagoras' theorem for conceptual understanding.
- Conducting research on word problems in other areas of mathematics to ascertain if visualisation strategies will also yield positive results.

5.7 PERSONAL REFLECTIONS

My research journey has been full of trials and challenges. There are moments where I felt like giving up but I agreed with Winston Churchill who once said "The journey of life is to move from failure to failure without loss of enthusiasm."

This was my first experience in engaging with proper academic research. This process required me to spend lots of time on sifting through and reading numerous articles, journals, and textbooks in order to find nexuses between extant research and my own study. These readings did not only expand my knowledge and understanding of the topic I was researching, but also improved my academic writing skills.

Furthermore, this study afforded me an opportunity to reflect on how I teach mathematics word problems on Pythagoras' theorem to my learners. This has really been an eye opener for me. I will therefore support and encourage other teachers to embrace the visual world not only in teaching word problems on Pythagoras' theorem, but mathematics topics in general.

5.8 CONCLUSION

It is evident from this study that visualisation strategies are imperative in the teaching of word problems in the Pythagorean context. It is noted that teaching Pythagoras' theorem using visual models as visualisation tools helps learners to create permanent pictures in the minds which they use to understand and solve these problems.

In this chapter, I presented a summary of the key findings, contribution of the study, limitations, recommendations, and suggestions for future research. My personal reflections concluded this research study.

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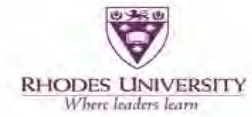
APPENDICES

Appendix 1: Ethical clearance



Human Ethics subcommittee
Rhodes University Ethical Standards Committee
PO Box 94, Grahamstown, 6140, South Africa
t: +27 (0) 46 603 8055
f: +27 (0) 46 603 8822
e: ethics-committee@ru.ac.za

www.ru.ac.za/research/research/ethics
NHREC Registration no. REC-241114-045

27 May 2019

John SSENNYOMO

Email: g12S0011@campus.ru.ac.za

Dear John SSENNYOMO

Re: Med Proposal; Ssennyomo Bernard John 12s0011**,** Med Proposal; Ssennyomo Bernard John 12s0011 **(**0305**,** Mar**,** 2019**)**

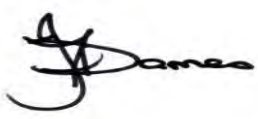
Principal Investigator: Dr Clemence Chikiwa

Collaborators: Mr Bernard John Ssennyomo, Prof Marc Schäfer,

This letter confirms that the above research proposal has been reviewed and **APPROVED** by the Rhodes University Ethical Standards Committee (RUESC) – Human Ethics (HE) sub-committee.

Approval has been granted for 1 year. An annual progress report will be required in order to renew approval for an additional period. You will receive an email notifying when the annual report is due.

Please ensure that the ethical standards committee is notified should any substantive change(s) be made, for whatever reason, during the research process. This includes changes in investigators. Please also ensure that a brief report is submitted to the ethics committee on completion of the research. The purpose of this report is to indicate whether the research was conducted successfully, if any aspects could not be completed, or if any problems arose that the ethical standards committee should be aware of. If a thesis or dissertation arising from this research is submitted to the library's electronic theses and dissertations (ETD) repository, please notify the committee of the date of submission and/or any reference or cataloguing number allocated. Sincerely



Prof Joanna Dames

Chair: Human Ethics sub-committee, RUESC- HE

Appendix 2: Permission letter from Letter from the Northern Cape Department of Education Head

DEPARTMENT OF EDUCATION

Enquiries: G.D. SIBIYA
Contact No: 053 839 8703
Reference:
Date: 17 April 2019

TO: MR BERNARD JOHN SSENNYOMO

GRANTING OF PERMISSION TO CONDUCT RESEARCH IN SELECTED HIGH SCHOOLS IN THE JOHN TAOLO GAETSEWE DISTRICT IN THE NORTHERN CAPE PROVINCE

The Northern Cape Department of Education has noted your request to conduct research under the topic titled "Investigating conceptual teaching of word problems through visualization processes; A case of selected grade 9 mathematics teachers" in the John Taolo Gaetsewe District: Northern Cape Province.

We regard this research as important in providing the department with data regarding how the use of visual models as visualization tools can be used in the teaching of mathematical word problems in the context of Pythagoras' theorem as well gaining insight into the Grade 9 teachers' perceptions and views of using these models after the implementation of the intervention program and therefore trust that your research findings will be shared with the department in the near future.

We further trust that the selected schools for this research (Batlaredi H/S, Bankhara Bodulong H/S and K.P. Tolo H/S) will give you all the necessary support to complete this task.

Permission to your request is therefore granted on condition that your research does not interfere or disturb teaching and learning, we further take this opportunity to wish you all the best in your studies.

Thank you

MR. G.T. PHARASI
SUPERINTENDENT-GENERAL

Page 1 of 1

Appendix 3: Acceptance letter from the district's Senior Phase mathematics subject adviser

DEPARTMENT OF EDUCATION

Enquiries: KP Rabotsho
Contact No: 053 773 9600
Ref No:
Date: 15 May 2019

TO: MR BERNARD JOHN SSENNYOMO

RE: REQUEST FOR ASSISTANCE IN IDENTIFYING FOUR GRADE 9 TEACHERS THAT WILL HELP IN DESIGNING AND DRAFTING AN INTERVENTION PROGRAM FOR THIS STUDY

I hereby consent to assist Mr. Ssennyomo in identifying four resource teachers that will help him in designing a ViPProS program for the research project he plans to conduct in the Kuruman area (JTG District).

I Support this research and wish you well in your future endeavors.

I would appreciate it if the findings will also be made available to the JTG district Department of Education at the end of the research.

Kind Regards,

Ms K.P Rabotsho
SENIOR PHASE MATHEMATICS
SUBJECT ADVISOR: JTG DISTRIC

Appendix 4: Informed consent from participating teachers

INFORMED CONSENT FORM

Research Project Title:	Investigating conceptual teaching of word problems through visualization processes: A case of selected grade 9 mathematics teachers.
Principal Investigator(s):	SSENNYOMO BERNARD JOHN

Participation Information
• I understand the purpose of the research study and my involvement in it • I understand the risks and benefits of participating in this research study • I understand that I may withdraw from the research study at any stage without any penalty • I understand that participation in this research study is done on a voluntary basis • I understand that while information gained during the study may be published, I will remain anonymous and no reference will be made to me by name or student number • I understand that data collection requirements particular to this research, e.g. observation results, video recording and interviews may be used • I understand and agree that the interviews will be recorded electronically • I understand that I will be given the opportunity to read and comment on the transcribed interview notes • I confirm that I am not participating in this study for financial gain

Information Explanation
The above information was explained to me by: Ssennyomo Bernard John
The above information was explained to me in English and I am in command of this language:

Voluntary Consent	
I, [redacted] hereby voluntarily consent to participate in the above-mentioned research.	
Signature: CMB	Date: 10 /06 / 2019

Investigator Declaration	
I, **Ssennyomo Bernard John**, declare that I have explained all the participant information to the participant and have truthfully answered all questions asked by the participant.	
Signature: SBJ	Date: 10 / 06 / 2019

RHODES UNIVERSITY

INFORMED CONSENT FORM

Research Project Title:	Investigating conceptual teaching of word problems through visualization processes: A case of selected grade 9 mathematics teachers.
Principal Investigator(s):	SSENNYOMO BERNARD JOHN

Participation Information
• I understand the purpose of the research study and my involvement in it • I understand the risks and benefits of participating in this research study • I understand that I may withdraw from the research study at any stage without any penalty • I understand that participation in this research study is done on a voluntary basis • I understand that while information gained during the study may be published, I will remain anonymous and no reference will be made to me by name or student number • I understand that data collection requirements particular to this research, e.g. observation results, video recording and interviews may be used • I understand and agree that the interviews will be recorded electronically • I understand that I will be given the opportunity to read and comment on the transcribed interview notes • I confirm that I am not participating in this study for financial gain

Information Explanation
The above information was explained to me by: Ssennyomo Bernard John
The above information was explained to me in English and I am in command of this language:

Voluntary Consent		
I, [redacted]	hereby voluntarily consent to participate in the above-mentioned research.	
Signature: [redacted]		Date: 10 / 06 / 2019

Investigator Declaration	
I, **Ssennyomo Bernard John**, declare that I have explained all the participant information to the participant and have truthfully answered all questions asked by the participant.	
Signature: SB	Date: 10 / 06 / 2019

RHODES UNIVERSITY

INFORMED CONSENT FORM

Research Project Title:	Investigating conceptual teaching of word problems through visualization processes: A case of selected grade 9 mathematics teachers.
Principal Investigator(s):	SSENNYOMO BERNARD JOHN

Participation Information

- I understand the purpose of the research study and my involvement in it
- I understand the risks and benefits of participating in this research study
- I understand that I may withdraw from the research study at any stage without any penalty
- I understand that participation in this research study is done on a voluntary basis
- I understand that while information gained during the study may be published, I will remain anonymous and no reference will be made to me by name or student number
- I understand that data collection requirements particular to this research, e.g. observation results, video recording and interviews may be used
- I understand and agree that the interviews will be recorded electronically
- I understand that I will be given the opportunity to read and comment on the transcribed interview notes
- I confirm that I am not participating in this study for financial gain

Information Explanation

The above information was explained to me by: Ssennyomo Bernard John

The above information was explained to me in English and I am in command of this language:

Voluntary Consent

I, [redacted] hereby voluntarily consent to participate in the above-mentioned research.

Signature: [signature] Date: 10 / 06 / 2019

Investigator Declaration

I, **Ssennyomo Bernard John**, declare that I have explained all the participant information to the participant and have truthfully answered all questions asked by the participant.

Signature: [signature] Date: 10 / 06 / 2019

Appendix 5: Approval letter from school principals

Ssennyomo Bernard John

P. O. Box, 328

Mothibistad

30 April 2019

The Principal and SGB Chairperson

[redacted] Technical and Commercial High School

P. O. Box 2629,

Kuruman, 8460

Dear sir/Madam

Re: Request for permission to conduct research at your school

My name is Ssennyomo Bernard John, a mathematics teacher at KP Toto High School in Batlharos. I am currently studying towards a masters' degree in Mathematics Education with Rhodes University. My research interest lies in the use of visual teaching approaches in the mathematics classroom. The research study is entitled *''investigating conceptual teaching of word problems through visualization processes: A case of selected grade 9 mathematics teachers''*. The aim of this study is to analyze how the use of visual approaches to teaching mathematics word problems can enhance conceptual understanding as well as gain insight into the teacher's perceptions and views of using these visual approaches after the implementation of the intervention program.

The research I wish to conduct for my Masters' thesis requires me to observe and video-record mathematics teaching at high school and interview the teacher of the class. I plan to observe the study participant's lessons three times over a period of two months. I assure you that the study will not cause any disturbance to the normal school program, as it will be integrated within the normal schedule of the school. This research will be conducted under the supervision of Dr. Clemence Chikiwa and Professor Marc Schäfer.

I would like to conduct this study at [redacted] High School as one of your teachers [redacted] has already shown an interest in taking part in the study (see attached consent form). Permission to conduct the study has been granted by the Northern Cape Department of Basic Education (see attached letter) and I also attach a letter of provisional ethical approval from the Ethics Committee of the Faculty of Education Rhodes University.

I intend to protect the anonymity of the school and teacher to the maximum possible extent. The name of the school and teacher will not be used in anyway in this research. In any writing and publication arising from this study's results, the school and participant will be referred to by pseudonyms. Information and results of this study may be presented at academic conferences, and published in conference proceedings, national and international journals, and to research funders. In those cases where any information collected through video recording, in this study, is used for conferences, participants' privacy and anonymity will be observed to the utmost.

Upon completion of the study, I undertake to provide you and the participant teachers with access to the research findings. The results of this study will hopefully contribute to improving teaching of Pythagoras' theorem problem solving.

I trust that you will grant me the permission to conduct this study at your school. Thank you for your understanding and cooperation in this regard.

Yours faithfully

Ssennyomo Bernard John

Masters' Degree Student Researcher in Mathematics Education

Rhodes University

Permission to conduct this research at B[redacted]High School has been approved/~~declined~~.

Signature: Principal, B[redacted]igh School

DEPARTMENT OF EDUCATION
B[redacted]IGH SCHOOL
P.O. BOX 2629 KURUMAN 8460
2019 -05- 05
[redacted]@gmail.com
0785347993
NORTHERN CAPE PROVINCE

Ssennyomo Bernard John

P. O. Box, 328

Mothibistad

30 April 2019

The Principal and SGB Chairperson

[redacted] Technical and Commercial High School

Private Bag X108,

Mothibistad, 8474

Dear sir/Madam

Re: Request for permission to conduct research at your school

My name is Ssennyomo Bernard John, a mathematics teacher at KP Toto High School in Batlharos. I am currently studying towards a masters' degree in Mathematics Education with Rhodes University. My research interest lies in the use of visual teaching approaches in the mathematics classroom. The research study is entitled *''investigating conceptual teaching of word problems through visualization processes: A case of selected grade 9 mathematics teachers''*. The aim of this study is to analyze how the use of visual approaches to teaching mathematics word problems can enhance conceptual understanding as well as gain insight into the teacher's perceptions and views of using these visual approaches after the implementation of the intervention program.

The research I wish to conduct for my Masters' thesis requires me to observe and video-record mathematics teaching at high school and interview the teacher of the class. I plan to observe the study participant's lessons three times over a period of two months. I assure you that the study will not cause any disturbance to the normal school program, as it will be integrated within the normal schedule of the school. This research will be conducted under the supervision of Dr. Clemence Chikiwa and Professor Marc Schäfer.

I would like to conduct this study at Baitiredi High School as two of your teachers [redacted] and [redacted] have already shown an interest in taking part in the study (see attached consent form). Permission to conduct the study has been granted by the Northern Cape Department of Basic Education (see attached letter) and I also attach a letter of provisional ethical approval from the Ethics Committee of the Faculty of Education Rhodes University.

I intend to protect the anonymity of the school and teacher to the maximum possible extent. The name of the school and teacher will not be used in anyway in this research. In any writing and publication arising from this study's results, the school and participant will be referred to by pseudonyms. Information and results of this study may be presented at academic conferences, and published in conference proceedings, national and international journals, and to research funders. In those cases where any information collected through video recording, in this study, is used for conferences, participants' privacy and anonymity will be observed to the utmost.

Upon completion of the study, I undertake to provide you and the participant teachers with access to the research findings. The results of this study will hopefully contribute to improving teaching of Pythagoras' theorem problem solving.

I trust that you will grant me the permission to conduct this study at your school. Thank you for your understanding and cooperation in this regard.

Yours faithfully

Ssennyomo Bernard John

Masters' Degree Student Researcher in Mathematics Education

Rhodes University

Permission to conduct this research at B[redacted]igh School has been approved/~~declined~~.

Signature: Principal, B[redacted]High School

Ssennyomo Bernard John
P. O. Box, 328
Mothibistad
30 April 2019

The Principal and SGB Chairperson
[redacted] Technical and Commercial High School
PO Box 328,
Mothibistad, 8474

Dear sir/Madam

Re: Request for permission to conduct research at your school

My name is Ssennyomo Bernard John, a mathematics teacher at KP Toto High School in Batlharos. I am currently studying towards a masters' degree in Mathematics Education with Rhodes University. My research interest lies in the use of visual teaching approaches in the mathematics classroom. The research study is entitled *''investigating conceptual teaching of word problems through visualization processes: A case of selected grade 9 mathematics teachers''*. The aim of this study is to analyze how the use of visual approaches to teaching mathematics word problems can enhance conceptual understanding as well as gain insight into the teacher's perceptions and views of using these visual approaches after the implementation of the intervention program.

The research I wish to conduct for my Masters' thesis requires me to observe and video-record mathematics teaching at high school and interview the teacher of the class. I plan to observe the study participants' lessons three times over a period of two months. I assure you that the study will not cause any disturbance to the normal school program, as it will be integrated within the normal schedule of the school. This research will be conducted under the supervision of Dr. Clemence Chikiwa and Professor Marc Schäfer.

I would like to conduct this study at KP Toto High School as two of your teachers [redacted] and [redacted] have already shown an interest in taking part in the study (see attached consent form). Permission to conduct the study has been granted by the Northern Cape Department of Basic Education (see attached letter) and I also attach a letter of provisional ethical approval from the Ethics Committee of the Faculty of Education Rhodes University.

I intend to protect the anonymity of the school and teachers to the maximum possible extent. The name of the school and teachers will not be used in anyway in this research. In any writing and publication arising from this study's results, the school and participants will be referred to by pseudonyms. Information and results of this study may be presented at academic conferences, and published in conference proceedings, national and international journals, and to research funders. In those cases where any information collected through video recording, in this study, is used for conferences, participants' privacy and anonymity will be observed to the utmost.

Upon completion of the study, I undertake to provide you and the participant teachers with access to the research findings. The results of this study will hopefully contribute to improving teaching of Pythagoras' theorem problem solving.

I trust that you will grant me the permission to conduct this study at your school. Thank you for your understanding and cooperation in this regard.

Yours faithfully

Ssennyomo Bernard John

Masters' Degree Student Researcher in Mathematics Education

Rhodes University

Permission to conduct this research at [redacted] High School has been approved/~~declined~~.

Signature: Principal, KP Toto High School

[redacted] TECHNICAL AND COMMERCIAL HIGH SCHOOL
2019 -05- 10
P.O. BOX 328
[redacted]
NORTHERN CAPE PROVINCE

Appendix 6: Survey questionnaire

RHODES UNIVERSITY
Where leaders learn

SURVEY QUESTIONNAIRE FOR MATHEMATICS TEACHERS IN THE JTG DISTRICT

PART A

General information

Instructions

- Use a cross (X) to select an appropriate response in each question where necessary.

1. How many years of teaching experience do you have in the subject?
2. a) Do you possess any teaching qualifications? Yes ☐ No ☐

   a) If yes, what teaching qualifications do you possess?

Grade 12	Certificate in Education	Diploma in Education	Honours degree in Education	Masters of Education

3. a) During your teacher training, were you trained on how and why to incorporate visual approaches in your lessons? Please elaborate?
4. How would you visually illustrate the concept of Pythagoras' theorem?
5. In your view and experience, a good mathematics teacher's practice is characterised by:

Tick a column in each row	Never	Some lessons	Most lessons	Every lesson
Putting emphasis on formula and procedures				
Teaching in a sequential and logical manner				
Teaching using visual aids				
Encouraging learners to be creative				
Linking what the teacher is teaching to real-life				
Encouraging learners to visualise the mathematical concepts that are being taught				

Part B

The following questions in this section are designed to find out more about how you teach Pythagoras' theorem. Please note that this research study focuses on visualisation and how teachers make use of visual models to teach word problems in the context of Pythagoras' theorem in order to develop learners' understanding. Visualisation processes are categorised into external and internal visualisation. While with external visualisation, images are formed with pen and paper, or with technological tools, with internal visualisation processes, images are constructed and processing is done mentally. Visual models are reflections of abstract concepts in the physical world. They can be in form of diagrams, manipulatives that one uses to facilitate his or her teaching).

6. Do you use visual models in the teaching of Pythagoras theorem? Yes ☐ No ☐

   a) i) If no, please explain how you teach this concept?

   ii) What are the factors that prohibit you from using visual models in teaching Pythagoras' theorem?

   b) i) If yes, what visual models do you use and where do you get them from? Please give examples and elaborate.

   ii) In your own view and experience, how do you use these models to assist learners in understanding Pythagoras' theorem?

7. In your own opinion, how important is the use of visual models in the teaching of the Pythagoras theorem?

8. How often do you use visual aids to assist you in doing the following during your mathematics lessons?

Tick a column in each row	Never	Some lessons	Most lessons	Every lesson
When explaining the reasoning behind a given mathematical concept or idea				
When linking what you are teaching to real life				
When exploring relationships				
When representing a mathematical concept or idea in multiple ways				
When solving non- routine problems				

9. When does the use of visual aids on the teaching of Pythagoras theorem work best in mathematics lessons?

Tick a column in each row	Never	Some lessons	Most lessons	Every lesson
During the introduction of the lesson				
During individual work without a learner getting aid from the teacher				
During individual work with a learner getting support from the teacher				
During the teaching of the whole class with all learners working together as a group				
During small group work without learners getting help from the teacher				
During small group work with learners getting support from the teacher				
No evidence of use of visual materials by the teacher				

10. In your opinion, does the use of teaching aids help in improving the teaching of Pythagoras theorem for conceptual understanding? Please elaborate?

END

Thank you for your time in completing this questionnaire.

Appendix 7: Interview schedule for the study participants

1. During our lessons you have been teaching problem solving in the context of Pythagoras' theorem using visuals. In your view and experience, do you think it is imperative to use visual approaches in teaching word problems on Pythagoras' theorem? Why, please explain your answer?
2. In what ways (if any) did the use of visuals expedite (facilitate) your teaching?
3. How did you use visual approaches to develop learners' conceptual understanding? What did you do with the visuals to promote learners' understanding of the concept(s) being dealt with? Please supply examples and make use of the video- taping to identify where this occurred.
4. How did the use of visual aids assist you in integrating real life situations in your lessons?
5. Did in anyway visuals assist you in making links between concepts during your lesson? If yes, how did you use visuals to assist you in making these links? Please supply examples and make use of the video- taping to identify where this occurred.
6. Did visual aids assist you in building on learner's prior knowledge? If yes, how did visuals assist you in moving learners from the known to the unknown?
7. Did you find related aspects outside the classroom (from learners' context) to assist you in illustrating a concept? Please articulate.
8. In your view, what are the benefits of using visuals in the mathematics lesson that you taught?
9. Considering the visuals that you used in our lessons, which model(s) in particular do you think helped learners in understanding the content that you were dealing with? Please give examples and use the video-taping to identify where this happened.
10. What experiences did you get working with visual aids in the lesson that you taught?
11. What were the challenges that you experienced when using visual approaches in your mathematics lesson? How can these challenges be overcome?
12. What do you see different as compared to the teaching of word problems on Pythagoras' theorem you have been doing as you taught using visuals? (Does a lesson with the use of visuals differ from the one without visuals? If yes, how does it differ? What changed?
13. What are some of the merits and demerits that you noted when using visuals during your teaching?
14. Are you willing to utilise the visual approach again in teaching word problems? Please articulate your response.

15. Do you have any comment about using visuals in the teaching of Pythagoras' theorem and problem solving?

Appendix 8a: Lesson 1

Subject: Mathematics **Date**:

Topic: Pythagoras' theorem — using Euclid's area model

Instructional material: Whiteboard or chalkboard, visual aids (squares, right-angled triangle), worksheets

Objective(s): At the end of the session, learners should be able to:

- Establish a relationship between the sides of a right-angled triangle.
- Use visual approaches to aid them in solving mathematical word problems in the Pythagorean Theorem context.

Stage	Teacher activity	Learner activity
Prior learning	• Draws triangle ABC on the board • Asks learners to name the sides • Asks learners the relationship between the sides of the triangle and its angles. • Asks learners what is the largest angle in a right-angled triangle and the name of the longest side which is always opposite that angle. • Asks learners to state the Theorem of Pythagoras.	Responding to the teacher's questions as a way revealing their learning.
Introduction	• Introduces the lesson by connecting the planned concept to learners' prior learning. • For example, in a right-angled triangle, the largest angle is the right angle, so the longest side is always opposite the right angle and it is called the hypotenuse; the shortest is opposite the smallest angle. • Teacher demonstrates this using manipulatives of a right triangle.	Observing visuals as the teacher does the demonstrations

Content presentation	• Teachers draws a right-angled triangle • Draws squares on the length of each side of the triangle that are of equal length to the respective side. • Divides each square into equal small unit squares. • Asks learners to establish a relationship between the three squares drawn on the sides of a right triangle.	Responding to questions asked by the teacher as well as taking notes.
Assessment of the lesson	• Teacher writes word problems on the whiteboard and asks learners to come up with solutions in their groups, encouraging them to use visual strategies.	Using visualisation approaches to come with solutions
Conclusion	• In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides. • Teachers uses a "visual aid" of the Pythagorean theorem dating to Euclid (c. 300 BCE) in articulating this relationship. • He emphasizes characteristics, examples, non-examples and real life applications of Pythagoreanism. • Uses Frayer's vocabulary model to the demonstrate this.	Observing, contributing and taking notes

Visualisation Processes for Problem Solving (ViPProS) worksheet

Area model 1: A "visual aid" of the Pythagorean Theorem dating to Euclid (c. 300 BCE).

1. A ladder standing against a wall reaches a height of 8m up the wall and has its foot 6m away from it, calculate the length of the ladder.

2. A 17-meter piece of string is stretched from the top of a 15-meter flagpole to the ground. How far is the base of the flagpole from the string?

Appendix 8b: Lesson 2

Subject: Mathematics **Date**:

Topic: Pythagoras' theorem — Proof by rearrangement

Instructional material: Whiteboard or chalkboard, visual aids (squares, right-angled triangle), worksheets

Objective(s): At the end of the session, learners should be able to:

- Establish a relationship between the sides of a right-angled triangle.
- Use visual approaches to aid them in solving mathematical word problems in the Pythagorean Theorem context

Stage	Teacher activity	Learner activity
Prior learning	• Asks learners properties of shapes, for example, squares and rectangles. • Ask learners how to find the area of a square • Asks learners to state the Theorem of Pythagoras.	Responding to the teacher's questions as a way revealing their learning
Introduction	• Introduces the lesson by connecting the planned concept to learners' prior learning. • Teacher demonstrates to learners four right-angled triangles which are congruent. • Teacher does this by visually illustrating to learners that these triangles are equal in size and are of the same shape. That is, their corresponding sides and angles are equal.	Observing visuals as the teacher does the demonstrations.

Content presentation	• Teacher pastes a manipulative of a right-angled triangle on board and the sides are labelled. • The shorter and longer leg are labeled **a,** and **b** respectively. The hypotenuse is labelled **c**. • The teacher pastes the other three triangles such that the four right-angled triangles form two squares of different sizes. • Teacher asks learners to identity the squares and why they think these are squares. The teacher pastes a square in the empty area and asks learners what is the area of that square. She moves the square and pastes it aside. • Teacher rotates two of the triangles to form a new shape-rectangle. S/he asks learners what shape is formed and why they think it is that particular shape. • Teacher then translates one of the new shapes. S/he places two squares (in the left-over spaces) of lengths equivalent to those of the respective legs of the right triangle. • Asks learners to establish a relationship between the three squares which were placed in the left-over areas.	Responding to questions asked by the teacher as well as taking notes.
Assessment of the lesson	• Teacher writes word problems on the whiteboard and asks learners to come up with solutions in their groups, encouraging them to use visual strategies.	Using visualisation approaches to come with solutions

Conclusion	• Teacher refers to the visual aid to articulate that in a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.	Observing, contributing and taking notes

Visualisation Processes for Problem Solving (ViPProS) worksheet

Area model 2: Proof by rearrangement.

1. John drove his car due east for 10 km. He then turned south and drove 24 km. How far was he from where he started?

Appendix 8c: Lesson 3

Subject: Mathematics **Date**:

Topic: Sense making of Pythagoras' proposition using scaled-up versions of a right-angle triangle.

Instructional material: Sheet of centimetre grid paper with a 4 cm square drawn in the upper left corner, centimetre ruler, pair of scissors, crayons or coloured pencils.

Objective(s): At the end of the session, learners should be able to:

- Scale the original triangle (that they developed from the word problem) by ***a, b,*** and ***c*** in order to construct three new triangles of side length, $\boldsymbol{a^2}$, $\boldsymbol{b^2}$, $\boldsymbol{and\ c^2}$ respectively that are similar to the original one.

Stage	Teacher activity	Learner activity
Prior learning	• Asks learners to explain what similar shapes are. • Asks learners to state the Theorem of Pythagoras.	Responding to the teacher's questions as a way revealing their learning

Introduction	• Introduces the lesson by connecting the planned concept to learners' prior learning. • Teacher writes two word problems on the whiteboard and allocates a problem to each group. • This exercise is guided by an activity which learners use to develop three scaled-up right-angled triangles as seen in the worksheet below. • Teacher provides learners with a sheet of cm grid paper with a square that measures 4cm drawn on a side in the upper- left corner.	Reading the problem.
Content presentation	• Teacher asks learners to come up with sketches for their problems in their groups. • Learners are instructed to draw an accurate right-angled triangle of their sketch in the square labelling the hypotenuse ***c*** and the two legs of the triangles ***a,*** and ***b.*** • Teacher asks learners to label the angles in the triangle. • Learners are asked to scale the original triangle by ***a*** in order to construct a new triangle that is similar to the original that has a side length of $\boldsymbol{a^2}$. • Teacher asks them to scale the original triangle by ***b*** in order to construct a new triangle that is similar to the original that has a side length of $\boldsymbol{b^2}$. • Learners are asked to scale the original triangle by ***c*** in order to construct a new triangle that is similar to the original that has a side length of $\boldsymbol{c^2}$. • Teacher asks learners to draw each scaled triangle and label the side lengths algebraically in terms of ***a***, ***b***, and ***c*** on the interior of each triangle. • Learners are asked to cut out each scaled-up triangle and ensure that the labels travel with each figure.	Drawing a sketch Scaling the original triangle three times by the length of its sides (***a, b,*** and ***c***) in order to construct three new triangles that are similar to the original one.

Conclusion	• Teacher concludes the lesson by posing the question; how does each of the three triangles that you have created relate to the original triangle? • This in a way helps the teacher to check if the lesson objective has been achieved.	Responding to the teacher's question

Visualisation Processes for Problem Solving (ViPProS) worksheet

Area model 2: **Scaled-up versions of a right-triangle**.

2. A newly planted tree needs to be staked with a wire. The wire is attached to the trunk 3 feet above the ground, and then anchored to the ground 2 feet from the base of the tree. Peter found out that the square of the length of the wire that is needed to stake the tree is equivalent to the sum of the squares of the other two lengths (i.e., the length from the base of the tree to a point on the ground where the wire is anchored and the length from the base of a tree to the point on the trunk where the wire is attached), verify if he was correct.
3. Sarah marched 2 m east and 2 m north. She found out that square of the distance from where she ended marching to her starting point is equivalent to the sum of the squares of the distances that she marched east and north, verify if she is correct.

Hint: After drawing a sketch, use the activity below to guide you in answering the questions above

Activity

1. You are provided with a sheet of centimetre grid paper with a square that measures 4cm on a side in the upper- left corner.

a) Draw either an isosceles right- triangle (with legs of dimensions 2cm) or a scalene right triangle (with legs of dimensions 3cm and 2cm) in the square
b) Label the hypotenuse of the triangle ***c*** and the two legs of the triangle ***a,*** and ***b***.
c) Use a centimetre ruler to measure the length of the missing side.
d) Label the angles in the triangle

e) Scale the original triangle by $\boldsymbol{a}$ in order to construct a new triangle that is similar to the original that has a side length of $\boldsymbol{a^2}$.

f) Scale the original triangle by $\boldsymbol{b}$ in order to construct a new triangle that is similar to the original that has a side length of $\boldsymbol{b^2}$.

g) Scale the original triangle by $\boldsymbol{c}$ in order to construct a new triangle that is similar to the original that has a side length of $\boldsymbol{c^2}$.

h) Draw each scaled triangle and label the side lengths algebraically in terms of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ on the interior of each triangle.

i) Cut out each scaled-up triangle and ensure that the labels travel with each figure.

j) After cutting out the triangles, use the three manipulative to attempt to prove the theorem.

Appendix 8d: Lesson four

Subject: Mathematics **Date**:

Topic: Sense making of Pythagoras' theorem

Instructional material: Manipulatives of the three cut out scaled right-angled triangles, white board.

Objective(s): At the end of the session, learners should be able to:

- Use visual approaches to aid them in solving mathematical word problems in the Pythagorean Theorem context
- Build various shapes by putting together the three cut out scaled triangles and using these shapes to assist them in making sense of the Pythagorean proposition. This will require them to make use of congruency criteria and/or properties of various shapes in justifying that, $a^2 + b^2 = c^2$ as claimed in the word problems held true.

Stage	Teacher activity	Learner activity
Prior learning	• Teacher asks learners properties of shapes, for example, squares, rectangles, kites and parallelograms. • Learners are asked to state congruency criteria.	Responding to the teacher's questions as a way revealing their learning

Introduction	• Introduces the lesson by connecting the planned concept to learners' prior learning.	Paying attention and taking notes.
Content presentation	• Teacher encourages learners to use the three manipulative to build different shapes and use them in making sense of Pythagoras' theorem.	Building various shapes by putting the three cut out right triangles together
Conclusion	• Teacher refers to the different shapes that have been created in concluding the lesson and also emphasising a sense of universality of this proof.	Observing, contributing and taking notes

Visualisation Processes for Problem Solving (ViPProS) worksheet

Area model 2: **Scaled-up versions of a right-triangle**.

2. A newly planted tree needs to be staked with a wire. The wire is attached to the trunk 3 feet above the ground, and then anchored to the ground 2 feet from the base of the tree. Peter found out that the square of the length of the wire that is needed to stake the tree is equivalent to the sum of the

squares of the other two lengths (i.e., the length from the base of the tree to a point on the ground where the wire is anchored and the length from the base of a tree to the point on the trunk where the wire is attached), verify if he was correct.

3. Sarah marched 2 m east and 2 m north. She found out that square of the distance from where she ended marching to her starting point is equivalent to the sum of the squares of the distances that she marched east and north, verify if she is correct.

Appendix 9: Certificate of Editing

proofEDit

make sure it's correct

Certificate of editing

23 January 2020

The Master's thesis entitled "**Investigating Conceptual Teaching Of Word Problems Through Visualization Processes: A Case Of Selected Grade 9 Mathematics Teachers**" by **Ssennyomo Bernard John** has been edited by Jean Schäfer.

Editing has consisted of the following:

- Checking house style and consistency with academic specifications
- Proofreading the document
- Checking for typos
- Correcting grammar, tenses, punctuation and spelling
- Correcting errors accordingly
- Formatting for ease of reading
- Making comments and recommendations for the student to attend to
- Ensuring consistency in the use of phrases, capitalisation and terminology
- Inserting a Table of Contents
- Some formatting

Jean Schäfer

www.jeanschafer-editor.com
member of the Professional Editor's Guild
(Membership no. SCH011)

Disclaimer: The editor of this thesis by no means takes on the role of a supervisor. Strengthening of the argument and refining of the conceptual framework are discussed and agreed upon with the supervisor.