# Why do learners and teachers experience problems with the concept of zero?

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# ABSTRACT

The controversy around the inclusion of zero in the number system has been widely documented. Influential mathematicians in various ancient cultures did not accept zero as a number. The idea of the empty set was too abstract and they could not conceptualise division by zero. Surprisingly, understanding of the concept is still a matter of concern today. In spite of expansive reports on and recommendations for developing conceptualisation of the concept, learners and teachers still experience problems similar to those that ancient mathematicians struggled with.

The study was initiated by an observation of Grade 7 learners‟ inability to solve the problems $4 \times 0$ and $0 \div 7$ effectively or at all. I investigated why Grade 3 to 6 learners and mathematics teachers on a BEd (in-service) course and an accredited ACE course experience problems with the concept of zero. I was especially interested in the understanding of multiplication and division by zero. I investigated teachers‟ knowledge of zero‟s characteristics as a number, the history of zero and how they teach the concept, in order to support my assumptions. The data production process was performed over a period of two years. It involved a multi-case opportunity sample approach embedded in the empirical field that formed the backdrop of my involvement as mathematics education specialist in schools in the Western and Eastern Cape. The interpretative orientation of the study allowed me to conduct inquiries that served to confirm or challenge my assumptions and enabled me to construct generalisations that depict learners‟ and teachers‟ knowledge construction. The qualitative data analysis informed the presentation and discussion of the findings.

The single most important message conveyed to readers of this study is that the value of zero as a number, its importance in the number system, its properties and its behaviour in calculations, should not be underrated. Teaching of this abstract concept requires competent teachers who are able to mediate understanding in the most effective and innovative manner. Professional development programmes should orchestrate this competence and

curriculum developers and textbook authors should acknowledge the significance of learning and teaching the concept of zero.

I dedicate this thesis

to

My mother, aunt, son, daughter, son-in-law and grandsons:

Mummy, Aunty Vera, Zerick, Haadiya, Theo,

Alexander & Matthew

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# ACRONYMS

| | |
|---|---|
| ACE | Advanced Certificate in Education |
| BEd | Bachelor of Education Degree |
| CAPS | Curriculum and Assessment Policy Statement |
| DBE | Department of Basic Education |
| DoE | Department of Education |
| FET | Further Education and Training Phase |
| FP | Foundation Phase |
| IP | Intermediate Phase |
| NCS | National Curriculum Statement |
| NDoE | National Department of Education |
| RUMEP | Rhodes University Mathematics Education Project |
| SDU | Schools Development Unit |
| SP | Senior Phase |
| UCT | University of Cape Town |
| WCED | Western Cape Education Department |

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# CHAPTER 1
# INTRODUCTION

This chapter provides a general introduction to inform the reader of the content of the study. I offer descriptions of the research problem, the need for performing the study and the purpose for conducting the research. A summary of the theory underpinning the research design is provided as well as the clarification of key concepts related to the study. Lastly, I present the systematized organisation of the thesis.

My involvement in school-based mathematics support and development projects since 2004 raised awareness of difficulties that primary school learners experience with calculations involving zero. My interest in conducting formal research into the problems that learners experience with the concept of zero was sparked in 2007 with my involvement as mathematics facilitator in an in-service rural school-based project in the Western and Southern Cape. I co-facilitated a Grade 7 mathematics lesson where a school psychologist conducted mental calculation speed tests to assess learners‟ basic calculation skills. While assisting with the assessment of the tasks, I was alarmed by the inability of the majority of Grade 7 learners to solve calculations such as $4 \times 0$ and $0 \div 7$ correctly. The school psychologist granted me permission to use the mental test questionnaire in further investigations of learners‟ conceptions of multiplication and division by zero (N. Botha, personal communication, 2007). I conducted a pilot research project in two rural multi-grade project schools in Grade 3 to 6 classrooms.

The understanding and application of the concept of zero as a number and placeholder have been contentious issues over centuries in different cultures. Indian mathematicians invented zero as a placeholder and number in relation to arithmetical operations about 1500 years ago after Babylonian, Roman and Greek mathematicians struggled with the idea of something representing *nothing*. Although the invention of zero may be regarded as the most noteworthy achievement in the history of number, the development of the concept of zero is vague in our mathematics curriculum (Anthony & Walshaw, 2004). Explicit and effective teaching and learning of the concept are uncommon, as is reflected in the remarks by learners and the teacher"s assertion quoted below:

- *You can never multiply or divide by zero because zero is nothing. Each and every number you multiply or divide by zero you get zero. And when you subtract one from zero or zero from one you get one. Because when you subtract something from nothing you get the same number, that number that you subtract. Zero is nothing at all.* (A grade 11 learner).
- *To start with, zero is nothing. If you subtract one from zero it is nothing. If you subtract any number from zero, it stays zero. For example, 0 – 11 = 0. If you have no apples and you take away 11, then there are no apples.* (A grade 5 learner)
- *I have never taught it in detail because I have never given the value of zero much thought. I assume they* [the learners] *know*. (An in-service teacher in the Further Education and Training [FET] Phase).

In this study, I investigate understanding of the concept of zero among primary school learners in grades 3 to 6. I also examine understandings of the concept of zero among teachers in a Bachelor of Education Degree (BEd) and Advanced Certificate in Education (ACE) accredited mathematics courses.

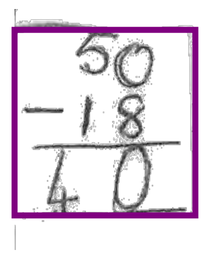
**RATIONALE**

As a mathematics education specialist and facilitator in school-based projects as well as my reading of the literature relevant to the research topic, I formulated the following assumptions:

1. Most learners experience problems with calculations involving zero.
2. Learners have procedural but not conceptual knowledge of the concept of zero.
3. Correct responses in mental calculation tasks do not always reflect conceptual understanding.
4. The teaching and learning of the concept of zero is neglected, overlooked, or regarded as insignificant by teachers, curriculum developers and textbook writers.
5. Teachers contribute to learners" misconceptions of the concept of zero.

6. Most teachers‟ content and pedagogical content knowledge about zero is limited, and the teaching and learning of the concept of zero occurs through the transfer of uninformed rules that teachers have learnt while they were in school.
7. The curriculum, teacher-training programmes, and teaching and learning material do not provide support or enhance teacher knowledge for developing the concept of zero.

Primary school learners and teachers attribute their notion of zero directly to their learning experiences in the classroom. I encountered some of the problems that learners experience with the concept of zero in various classrooms in primary schools before 2007. In 2004, I interviewed Grade 4 learners from different schools to discover why they made errors in calculations involving zero. Teachers‟ influence in learner concept construction of zero as a placeholder is evident in learners‟ responses to subtraction problems involving zero. A Grade 4 learner‟s thinking and reasoning in solving the problem 50 – 18 = ☐ are reflected in the picture showing the calculation and the interview exchange below.




L1: *Naught minus eight is naught. Five minus one is four.*

Z: *So, fifty minus eighteen is equal to forty?*

L1: *Fifty minus eighteen is forty.*

Z: *Are you sure?*

L1: *If I have naught, I can't minus eight. Then naught minus eight is equal to naught.*

Z: *How do you know that?*

L1: *Since Sub A, Miss . . . taught me that, if I have naught, I can't minus eight. Or if you have a number, then you can't minus naught with any number.*

(Jooste, 2004)

As portrayed in the learner‟s assertion above, the application of uninformed rules supplied by the teacher obscured mathematical thinking and reasoning and could well be detrimental to future learning. It is probable that formal arithmetic teaching of the concept of zero will arbitrate "shallow, context-bound rules and procedures" (Semenza, Granà & Girelli, 2006:10). Various researchers caution about the harmful effects that traditional rule-based teaching could have on learning (Orton, 2004; Wood, Cobb & Yackel, 1993). They endorse a constructivist approach to teaching and learning in terms of which learners construct and discover knowledge by actively engaging with the social and physical environment (Lerman, 1989; Clements & Battista, 1990; Kamii & Lewis, 1990; Clements, 1997; Yager, 2000; Steele, 2001; Carpenter, 2003; Vianna & Stetsenko, 2006).

In this study, I compare learners‟ instant responses to calculations with zero in mental calculations to recorded explanations of their understanding of calculations with zero. Over the past few years in the Western Cape, extensive focus was placed on mental mathematics. Educationists at tertiary level and within the Department of the Western Cape (WCED) realised, through the analysis of results reflected in systemic tests conducted in Grade 3 and 6 since 2004, that learners do not know the basic number facts. They lack knowledge of the basic requirements for multiplication and division, addition and subtraction number bonds and properties of numbers, which is the cornerstone of algebraic thinking and reasoning. Many of today‟s adults experienced the drill and recall of number facts in their primary school years, especially the multiplication tables, with speed and accuracy as the objective. This practice resulted in a generation who, although they could solve arithmetic problems effectively and mechanically, the learners experienced fear and anxiety during mathematics lessons (Ebbutt & Askew, 1997). With the implementation of Curriculum 2005, the focus in many mathematics lessons shifted to a decrease in mental and oral arithmetic tasks resulting in a lack of challenges to think about numbers and the development of awareness, decisiveness and alertness in dealing with numbers (Ebbutt & Askew, 1997).

The introduction of the Foundations for Learning Campaign (South Africa. NDoE, 2008) placed emphasis on learner competence concerning the basic number facts. More importantly, learners are expected to develop insight and skill in mental arithmetic that goes further than the recall of number facts. The recently developed Curriculum and Assessment Policy Statement, i.e. CAPS in short (South Africa. DBE, 2010:27, 93, 163) for the

Intermediate Phase, requires that an average of eight hours are spent on mental mathematics per term. Learners should practise mental calculations daily for ten minutes. Harries & Spooner (2000) suggest that mathematics generally involve mental activity and calculations even when written work is involved. Written calculations are representations of mental constructions. Mental calculations are often regarded as calculations that are performed without peripheral assistance or written accounts of thinking processes and entail the recollection of existing information (Ell, 2001). According to Atkinson (1997), learners‟ personal mental images are developed through mental arithmetic, which is more than only speedy recollections of number facts. Anghileri & Johnson & Kouba, et. al. (in Kouba & Franklin, 1995) assert that primary school learners often experience problems with multiplication and division tasks. Kouba & Franklin (1995) question the teaching of multiplication and division through the assessment of facts and algorithms, which learners have to memorize and recall in providing numerical responses without thinking critically about the responses. Ebbutt & Askew (1997) advocate that mental calculations should not only involve memorization and recall of multiplication tables or number facts. Learners should be expected to explain their thinking and reasoning processes.

Research reflects the problems that ancient mathematicians, and teachers and learners in more modern times, have encountered with the perception of zero‟s place in our number system and in calculations in general (Reid, 1956; Wheeler & Feghali, 1983; Gullberg, 1997; Kaplan, 1999; O‟Connor & Robertson, 2000; Anthony & Walshaw, 2004). A literature search on the problems that learners and teachers experience with multiplication and division by zero, yielded a wide range of information emerging from various research projects conducted in institutions abroad (Wheeler & Feghali, 1983; Van den Heuvel-Panhuizen, 2001; Anthony & Walshaw, 2004; Levenson, Tirosh & Tsamir, 2004; Semenza, et. al., 2006; Levenson, Tsamir & Tirosh, 2007; Quinn, Lamberg & Perrin, 2008). The reports assert that primary school learners struggle with conceptualising multiplication and division tasks with zero. Various studies report on the problems that teachers experience with the concept of zero as a number, and division by zero. According to these studies, the limited knowledge and uncertainties of division by zero on the part of teachers could play a role in learners‟ perception of the concept. Some teachers display an unwillingness to acknowledge zero as a number and experience difficulties in computations with zero as a dividend or a divisor (Wheeler & Feghali, 1983; Quinn, et. al., 2008). The literature surveyed did not include any information on the problems that teachers experience with multiplication by zero, or teachers‟ knowledge of the history of zero in the number system. No studies by South African researchers on the problems that teachers and learners have with the concept of zero were found.

The meagre attention given to the concept of zero in our mathematics curriculum is a matter of concern. The development of the concept of zero is explicitly mentioned for the first time in the National Curriculum Statement (NCS) for primary school mathematics in the grade 5 assessment standards. The standards state that learners should be able to "Recognise and represent numbers to describe and compare them: 0 in terms of additive inverses" (South Africa. Department of Education [DoE], 2002:41). Learners in the lower grades are expected to "know number names and read symbols *from* [italics added] 1 to . . ." and "count forwards and backwards" in different intervals "*between* [italics added] 0 and . . ." (South Africa. DoE, 2002:20). The assessment standards in our current primary school curriculum obviously do not require teachers to develop learners‟ concept of zero, especially with regard to acknowledging zero as an important number in the number system, its role in the place value system and in calculations. The South African DoE recently embarked on streamlining the NCS (South Africa. DoE, 2002) in the form of the CAPS (South Africa. DBE, 2010). In the first draft of the CAPS Mathematics – Foundation Phase (Grade R) released in 2010, the concept of zero is introduced in Grade R. The statement declared that, "The teacher points out that zero means „nothing‟ and that counting actually starts at 1" (South Africa. Department of Basic Education [DBE], 2010:6). This conception of zero could be problematic when the teacher later has to use "a height chart ready against the wall to plot each learner‟s height" (South Africa. DBE, 2010:25); counting to measure starts at zero. Introducing the number zero with reference to *nothing* is omitted in the CAPS Foundation Phase, Mathematics Final policy document (South Africa, DBE, 2010). In the new document, it is suggested that the number one is introduced in the first term. The number zero is introduced in the fourth term together with the numbers eight, nine and ten (South Africa. DBE, 2010:37). This does not make sense if numbers are introduced systematically according to cardinal value in the first grade of schooling. No mention is made in the policy document of the development of the concept of zero in Grade 1 although it is suggested that the learners use a number line from 0 to 10 to "describe, order and compare . . . numbers up to 20" in term 2 (South Africa. DBE, 2010:50). Illustrations are provided to indicate the use of number lines for performing addition and subtraction in Grade 1. The clarification of contents state that, "These kind of activities help learners to see where numbers are in relation to one another (South Africa. DBE, 2010:94-99). Zero is irrevocably part of the numbers on the number line but not included in concept development of numbers. Zero is included in the Grade 2 and 3 standards in the CAPS where learners have to identify, recognize read and write number names and symbols as well as "add and subtract multiples of 10 from 0 to 100" (South Africa. DBE, 2010:61 & 74). For Grade 3 it is stated that learners should "rapidly recall" multiples of 10 from 0 to 100 in doing mental addition and subtraction (South Africa. DBE, 2010:76). The inclusion of the concept of zero has thus been considered by curriculum developers although not completely consistent across the Foundation grades. In the IP Final CAPS document issued to publishers in March 2011, Grade 5 and 6 learners are expected to develop understanding of "0 in terms of its additive property" (South Africa, DBE, 2010:13) although no indication of development of the concept of zero is indicated for Grade 4 learners.

Contrasting views exist about the inclusion of the concept of zero in the primary school mathematics curriculum. Researchers claim that learners will not develop a total positive conception of zero prior to their second year of schooling (Inhelder & Piaget, 1969; Oesterle in Anthony & Walshaw, 2004; Wellman & Miller in Semenza, et. al., 2006). According to Inhelder & Piaget (1969), the conception of zero does not develop satisfactorily until learners achieve the stage of formal operations. Oesterle (in Anthony & Walshaw, 2004) claimed that primary school learners should not be engaged in calculations with zero. According to Oesterle (in Anthony & Walshaw, 2004), there is no actual basis for initiating zero facts in basic operations until addition and multiplication of two-digit numbers are introduced. I do not concur with the suggestions of these researchers. In my opinion, young learners could possibly construct meaning of the concept of zero through mediation, which entails constructive learning experiences and effective teaching strategies. This view is supported by Wilcox (2008:204) who engaged her Grade 1 daughter in a game involving a number line, confronting her with the perception of the value of zero and the notion of negative numbers. After successful meaning construction of these concepts, the child exclaimed, "Zero is a hero!" According to Fischer (1980:482-484), children apply procedures or "cognitive structures" to develop understanding, i.e. "a structure for knowing" or a "scheme" in Piagetian terms. They also need coordinated "cognitive skills" however, that are gradually stimulated by the environment to construct meaning (Fischer, 1980:482-484).

It is natural to overlook the existence of zero in "the empty set in concrete situations" (Anthony & Walshaw, 2004:40), but there are mathematical consequences when zero is related to *nothing* and disregarded as a number. It is common practice for learners to exhibit misconceptions regarding zero as a number because they relate the concept of zero to *nothing* (Reid, 1956; Wheeler & Feghali, 1983; Levenson, et. al., 2007). Knowledge of the concept of zero is significant in different areas of mathematics. For example, counting and calculating on a number line, constructing and interpreting simple graphs, reading temperature on a thermometer, using a measuring tape or scale for measuring length or mass accurately, understanding rational numbers (decimals, for example 0,5) and integers, estimation, rounding off and the construction of simple and sine, cosine and tan graphs would not be possible without the number zero. Zero is acknowledged as a number in real life and mathematical discourses in instances where it does not represent a concrete, empty set, and hence, *nothing*. When counting back to extend the whole numbers to include negative numbers the numbers are expressed as, for example three, two, one, zero (not nothing!), minus one, minus two, etc. When reading temperature we do not refer to „nothing degrees" as the freezing point for water but rather to „zero degrees". Zero is often not included in the numbers that young learners explore. This exclusion might contribute to the difficulties that older learners experience with the concept of zero in performing calculations, especially multiplication and division by zero based on traditional algorithms.

Wheeler & Feghali (1983) and Fenema, Carpenter & Loef (1989) assert that the problems that teachers experience with the concept of zero necessitate particular awareness of mathematics educators in professional development programs, learning and teaching materials and curriculum development. Many teachers might not develop the appropriate knowledge or engage in research of the concept on their own. They might also not teach the concept effectively, nor even teach the concept at all if they are not required to. Teaching materials emphasising the development of the concept of zero with regard to its history, meaning and mathematical attributes and uses are uncommon. One textbook series suggested to teachers that knowledge of the properties of zero could be developed most efficiently by engaging learners in practise exercises (Buys, 1989). Various researchers emphasize the fact that teaching of the concept of zero requires competent and resourceful teachers with effective content and pedagogical content knowledge (Shulman, 1986; Cockburn, 1999; Ball, 2003; Kahan, Cooper & Bethea, 2003; Ball, Thames & Phelps, 2008; Quinn, et. al., 2008).

In my opinion, this research study could be a significant contribution to the teaching and learning of mathematics in South African schools. It could enhance the understanding of mathematics educators, curriculum developers, textbook authors and teachers, of the importance of the development, the characteristics, and the use of the concept of zero, so that we can all better assist learners to develop the concept effectively. We need to establish the existing understanding of the concept among teachers and learners, so that we can determine the knowledge required in teaching and learning the concept of zero. Developing understanding of the concept of zero could be a complicated process of abstraction, which possibly necessitates mediation, rather than the mere transmission of uninformed rules that neither teachers nor learners understand. Some mathematics education and training programs probably do not explicitly focus on the development of pre- and in-service teachers‟ pedagogical content knowledge for teaching zero‟s role in the decimal number system and the different uses of zero in mathematical tasks. Learners‟ conceptualization of the concept of zero possibly depends on the depth of teachers‟ conceptualization of the concept. I hope this research will raise awareness in the broader South African mathematics community of the conceptions that learners and teachers have concerning the concept of zero.

**PURPOSE OF THE STUDY**

The goal of the study is to arrive at an understanding of learners‟ and teachers‟ conceptions of zero in multiplication and division tasks. The study intends to establish:

- understanding of primary school (grades three to six) learners‟ conception of multiplication and division by zero and
- teachers‟ understanding of multiplication and division with zero, their knowledge of zero as a number in its own right, and their approach to teaching the concept of zero.

**RESEARCH QUESTION**

The question that originally sparked off this study was: "Why do primary school learners experience problems with the concept of zero?" As the study progressed, additional questions surfaced and it became clear that I also needed to develop understanding of teachers‟ understanding of the concept. I intended to find out how teachers conceptualize multiplication and division by zero, their knowledge of zero as a number and their approach to teaching the concept formed secondary questions of this research. The main question was therefore adapted to: "Why do learners and teachers experience problems with the concept of zero?"

**Subsidiary questions**

- What are the conceptions that learners and teachers construct in multiplication and division problems involving zero?
- What do teachers understand about the attributes of zero as a number?
- What do teachers know about the history of the inclusion of zero in the number system?
- How do teachers teach the concept of zero?

The research question is investigated firstly in terms of Grade 3 and 4 learners‟ conceptualisation of multiplication by zero, because this case involved a classroom intervention that assisted learners in developing an understanding of multiplication by zero. Secondly, I investigate the question concerning Grade 5 and 6 learners‟ conceptualisation of multiplication and division by zero, and thirdly, teachers‟ conceptualisation of multiplication and division by zero. Fourthly, I respond to the question in terms of teachers‟ knowledge of zero as a number, its origin in the history of number and their approach to teaching and learning the concept in their classrooms.

**TERMS AND DEFINITIONS**

I briefly define some of the terminology that is used in the study to ensure that it is clearly understood. Some of the terms are discussed in more detail in Chapters 2 and 3.

| | |
|---|---|
| *The concept of zero*: | Generally refers to the understanding of zero as a number and its function in calculations. |
| *Uninformed rules:* | Algorithms applied to provide solutions to problems involving calculations with zero. Although correct, the rules or generalisations are not informed by conceptual reasoning but were rather transmitted by teachers during teaching. |
| *Conceptual understanding:* | Knowledge construction entailing different but related chunks of mathematical knowledge, skills, values and beliefs. This knowledge could be retrieved, assimilated, accommodated, reconstructed and applied in new concept development to demonstrate understanding. These processes might involve the recall of unrelated memorized facts learnt by rote. Learners who have developed conceptual understanding[1] are however able to connect and restructure |

[1] Hiebert, Carpenter, Fenema, Fuson, Wearne, Murray, Olivier, Human, (1996:8) and Kilpatrick, et. al., 2001:5
& 116) talk about *conceptual understanding* while Star (2005) talks about *conceptual knowledge*. Skemp (1976) refers to conceptual understanding as *relational understanding*.

| | |
|---|---|
| | unrelated concepts so that they make sense (Kilpatrick, Swafford & Findell, 2001). |
| *Procedural understanding:* | Procedures or steps for solving problems could be learnt without conceptual understanding or making sense of the problems and solutions. Effective procedural understanding[2] or fluency, however, entails the application of efficient steps to calculate more easily and smartly, thereby revealing conceptual understanding. Learners who have developed conceptual understanding normally know when procedures are wrong or about to go wrong (Matz, in Olivier, 1989; Kilpatrick, et. al., 2001). |
| *Conception:* | The learner‟s original or existing understanding of a concept through his or her own experiences or the influence of society. Newly constructed concepts are assimilated and accommodated into the existing conception. Whether the existing conceptions are correct or incorrect, the new concept might be understood or misrepresented so that it develops into a misconception – not because the learner does not understand, but rather because the learner understands something else (Davis, 1983; Wilcox, 2008). In this study, the use of the term misconception is used to refer to misrepresentations. Gerald (2006) described misconceptions as often-intelligent generalisations. |
| *Structure:* | The representations or models (often intuitive), i.e. procedures or illustrations, that learners construct and apply to demonstrate conceptual understanding. Learners‟ intuitive structures could reflect understanding or misunderstanding (Mulligan & Mitchelmore, in Ell, 2001). |

[2] Hiebert, et. al. (1996:8) refer to *procedural understanding* as *skills* while Star (2005) talks about *procedural knowledge*. and Skemp (1976) refers to the understanding of mathematical procedures and algorithms as *instrumental understanding*.

*Multiplicative thinking:* Refers to efficient structures that learners apply to illustrate effective conceptualisation of multiplication and division concepts based on previous efficiently constructed counting, addition and subtraction structures. Multiplicative thinkers are able to develop conceptual understanding by connecting basic intuitive structures to properties of operations and numbers such as inverse operations, the commutative, associative and distributive laws to justify solutions (Steffe, in Mulligan & Wright, 2000; Steffe, in Lutovac, 2008).

*Mediation:* Purposeful and well-planned instruction focused specifically on addressing learners" misrepresentation or misunderstanding of new concepts that is in conflict with the existing conception. Active involvement in internalising and using new concepts based on known concepts occurs in the zone of proximal development – the ZPD (refer to p. 41, Chapter 2). The process of mediation necessitates intervention by an innovative and well-informed teacher (Steele, 2001; Bruner, in Wink & Putney, 2002; Clarke, 2002).

**OVERVIEW OF THE RESEARCH METHODOLOGY**

This study was conducted in an interpretive, qualitative paradigm, informed by the theory of constructivism. I attempted to make sense of learners" and teachers" authentic learning experiences and the realities they perceived individually and socially in constructing and demonstrating knowledge of the concept of zero. I observed their interactions and oral and written responses in their own socio-cultural environments. Various research techniques were employed to produce data in multi-cases, which mostly entailed an opportunity sample approach – opportunities provided by school-based and teacher training projects I was involved in. The data analysis process allowed me to interpret, sort and validate the data through inductive reasoning. I recognised commonalities, differences, and patterns for answering the research question and producing generalisations (Connole, 1998; Cohen & Manion, 2000; Babbie, Mouton, Vorster & Prozesky, 2001).

The study was conducted in three stages from 2007 involving multi-case studies, which included teachers and learners in the Western and Southern Cape and the Eastern Cape in South Africa. The data production methods and techniques entailed an individual mental calculation test completed by Grade 3 to 6 learners and a written calculation elaboration task questionnaire completed by Grade 5 and 6 learners in co-operative learning groups. Individual BEd and ACE in-service teachers completed the same written calculation elaboration questionnaire as the grade 5 and 6 learners. The teachers were requested to implement the same written elaboration task in their classrooms and to reflect on the process as an assignment. The results of this assignment are used as additional data. The teachers completed a second questionnaire requiring individual responses concerning knowledge of zero as a number, knowledge of the history of zero and indications of how teachers teach the concept of zero. Photographs were taken during observations of Grade 3 and 4 learners working in co-operative learning groups to develop understanding of multiplication by zero during a classroom intervention. A focus group semi-structured interview was conducted with Grade 5 learners and a focus group unstructured interview involved BEd teachers.

The data analysis in this study involved analysis of responses to the mental calculation tasks completed by Grade 3 to 6 learners and written elaboration tasks completed by Grade 5 and 6 learners and BEd and ACE teachers. I also analysed the questionnaire entailing knowledge and teaching of the concept of zero completed by the BEd and ACE teachers, and transcriptions of the semi- and unstructured focus group interviews conducted with Grade 5 learners and BEd teachers. I transcribed the teacher interviews and posted a transcript of the audio-recorded interview to the teachers so they could check the accuracy of the transcripts. A student transcribed the learner interviews and I checked the transcription for correctness. Through the process of triangulation, I hoped to test my perception that learners and teachers experience problems with the concept of zero in multiplication and division calculation tasks. The data produced from the questionnaires and participant observations were used to validate my original assumption that learners experience problems with the concept of zero. The interviews served as supplementary information to ensure the comprehensive representation and interpretation of data (Bergman, 2008). Emerging patterns were categorised, represented in tables, and described to reflect different categories of learners‟ and teachers‟ conceptions of the concept of zero. I looked for connections between patterns occurring in this study and those reported in the literature.

**ORGANISATION OF THE THESIS**

In this chapter, I present the rationale for and purpose of the study, as well as an outline of it. I include descriptions of the problem statement, information regarding the background of the study, and theory relevant to the study. I also include terms and definitions to resolve ambiguities and ensure a clear understanding of the terminology used in the study.

In Chapter 2, I review the literature applicable to this study to provide insight into the on-going nature of problems experienced with the concept of zero. Gaps were detected in the literature regarding, for instance, teachers‟ understanding of multiplication by zero, their knowledge of the parity of zero and the history of the number system, and the teaching of the concept of zero in the primary school. The literature reviewed offered evidence that the problem addressed in this research study is enduring and prevalent. Awareness of the problem is required because of the implications it has for teaching and learning mathematics.

In Chapter 3, I describe and explain the methodology used in the study. The ontological and epistemological positions I assume, as well as the different empirical fields involved, required a range of research methods and techniques based on a variety of methodological traditions, such as ethnography and case study. Various methodological aspects regarding ethics, reliability, validity and generalisability were considered. As a researcher and participant observer, I was centrally involved in the data production process.

In Chapter 4, the data obtained from learners in the various case studies is presented, analysed and discussed. I combine the presentation and discussion of the results and support the discussion with quantitative data to reflect findings, learner quotes from questionnaires, vignettes from the focus group semi-structured interview, and photographs of learners‟ work obtained during observations. The Grade 3 and 4 learner results are presented and discussed separately from the Grade 5 and 6 results.

In Chapter 5, I present and discuss the results from the teachers. The discussion of the results is supplemented with quantitative data, quotes from questionnaires and vignettes obtained in the BEd focus group unstructured interview. I do not distinguish between the results of the two groups of teachers, but present and discuss the teacher results as a whole. The chapter is divided into different sections depicting results and discussions. First, I present and discuss teachers‟ understanding of multiplication and division with zero. Secondly, I deal with the data concerning teachers‟ general knowledge of the characteristics of zero and the history of zero, and lastly, their approaches to teaching the concept of zero.

In Chapter 6, I report the essential conclusions drawn from both learner and teacher findings concerning multiplication and division by zero. I provide generalisations concerning these concepts and teachers‟ content and pedagogical content knowledge of the concept of zero. I discuss the implications and provide recommendations for the teaching and learning of mathematics relating to the concept of zero. Ideas for further research are presented.

# CHAPTER 2
# LITERATURE REVIEW

## 2.1. INTRODUCTION

Various research studies (Wheeler & Feghali, 1983; Anthony & Walshaw, 2004, for example) have reported on the difficulties that people have encountered with the conceptualisation of zero as a number in its own right, and with calculations involving zero. First, I will show that the problem is longstanding and widespread by considering a range of literature that emphasises the difficulties that learners, teachers and mathematicians generally have experienced with understanding the concept of zero. I draw on the studies of Reid (1956); Wheeler & Feghali (1983); Kaplan (1999); O‟Connor & Robertson (2000); Van den Heuvel-Panhuizen (2001); Anthony & Walshaw (2004); Levenson, et. al. (2004); Semenza, et. al. (2006); Levenson, et. al. (2007) and Quinn, et. al. (2008).

The difficulties experienced with the concept of zero have implications for mathematical learning and requires the review of literature concerning learning theories. I discuss constructivism, the theory fundamental to this study. I consider Piaget‟s cognitive constructivist and Vygotsky‟s socio-constructivist learning theories. I draw on both theories because I believe knowledge construction depends on the individual building on existing knowledge to construct new concepts as well as on the influence of society in assisting learners to obtain new knowledge. I further explore notions of conceptual and procedural knowledge to make sense of learners‟ and teachers‟ understanding of abstract concepts. I define misconceptions, as the construction of conceptions leading to misconceptions are a key focus of both teacher and learner data. I particularly draw on the works of Clements & Battista (1990), Wood, Cobb & Yackel (1993), Steele (2001), Semenza, et. al. (2006), Vianna & Stetsenko, 2006), Clarke (2002), Wink & Putney (2002), Kilpatrick, et al. (2001), Sewell (2002), Ginsburg (1977), Davis (1983), Olivier (1989) & Mulligan & Wright (2000).

I further consider literature relating to the content and pedagogical knowledge that teachers‟ need to acquire in order to teach the concept of zero effectively. I utilise studies performed by Wheeler & Feghali (1983); Levenson, et. al. (2007) & Quinn, et. al. (2008), who have all reported on the difficulties that teachers experience with the concept of zero, with zero‟s properties and division by zero. A search for literature regarding the difficulties that teachers experience with multiplication by zero did not deliver any results. The works of Shulman (1986); Fenema, et. al. (1989); Ball (2003); Kahan, et. al. (2003); Ball, et. al. (2008) are drawn upon for the discussion of the content and pedagogical knowledge that teachers need to teach effectively, and the implications for professional development.

**2.2. THE INCLUSION OF ZERO IN THE NUMBER SYSTEM**

The inclusion of zero into the decimal number system might be regarded as the most remarkable accomplishment in the history of number (Anthony & Walshaw, 2004). Centuries ago, even some of the most influential mathematicians did not use zero as a symbol. They did not have a clear conception of zero‟s mathematical importance. The empty space was represented differently by different cultures. *Nothing* was not regarded as an object or a number, rather "as a condition" (Kaplan, 1999:22). Ancient Greek mathematicians strongly disliked the idea of zero. Archimedes totally disregarded zero. The philosopher Aristotle wanted to have zero banned because it made a mess of computations when he tried to divide by it (O‟Connor & Robertson, 2000).

Progress in the understanding and appreciation of the concept of zero happened over many, many centuries. The view exists that the origin of zero is vague (O‟Connor & Robertson, 2000; Anthony & Walshaw, 2004). Mathematicians are of the opinion that their Hindu colleagues used zero as a number in its own right for the first time. In the sixth century, they invented a symbol to represent a column with zero beads. They developed a need to represent numbers in a consistent manner on a counting board. A dot called *sunya* meaning emptiness was used to fill empty columns (Reid, 1956). The Indian mathematicians employed a place value system using zero to indicate an empty place (O‟Connor & Robertson, 2000) since about 200 AD. Aryabhata used the word "kha for position" in about

500 AD which later became known as zero (O‟Connor & Robertson, 2000:3). In earlier manuscripts a dot for both zero and the unknown *x* were used. The first instance of zero used as a symbol was observed on a stone tablet in 876 AD created by Indian mathematicians. Explicit measurement calculations involving zero were displayed. The numbers 270 and 50 appeared as we use them today; the zero was just a bit smaller and slightly higher than we write it today (O‟Connor & Robertson, 2000). At that stage there was as yet no clarity about the use of zero as a number. By 600 AD Indian mathematicians had invented the number zero and used it in calculations as we do. The Indian mathematician Brahmagupta made an attempt to supply arithmetical rules for zero and negative numbers during the seventh century. It was only one-hundred-and-fifty years later that the Arabs accepted the Indian mathematicians‟ breakthrough and included zero in their system. It was at last realised that performing calculations was much easier when you had *nothing*, i.e. zero, to help you count (O‟Connor & Robertson, 2000). The use of zero as a number came to Europe in the 12$^{th}$ century when Indian numerals spread from the Arab countries. Leonardo de Pisa or Fibonacci, an Italian merchant and mathematician brought new ideas about the development of the number system to Europe. He described the nine Indian symbols as well as the symbol for zero in 1200 AD. He was however not bold enough to regard 0 with the same respect as the other numbers, 1 to 9. He referred to zero as "the sign" but called the other signs numbers (O‟Connor & Robertson, 2000:4).

The complex nature of operations with a zero quantity is reflected in the history of mathematics (Quinn, et. al., 2008). The use of zero in calculations appeared in the books of the Indian mathematicians Brahmagupta, Mahãvira and Bhaskara (O‟Connor & Robertson, 2000). Brahmagupta, who created arithmetical rules for operating with zero in the four basic operations, asserted that any number multiplied by zero results in zero. He struggled with an explanation for dividing by zero just as many people in modern times struggle with the conceptualisation of the calculation. Brahmagupta claimed that:

> A positive or negative number when divided by zero is a fraction with zero as denominator. Zero divided by a negative or positive number is either zero or expressed as a fraction with zero as numerator and the finite quantity as denominator. Zero divided by zero is zero. (O‟Connor & Robertson, 2000:3)

O‟Connor & Robertson (2000) describe Brahmagupta‟s attempt to define operations with zero as brilliant. Brahmagupta was the first person who attempted the extension of numbers and operations to include negative numbers and zero. They claim that Brahmagupta did not actually say anything significant when he suggested that $n$ divided by zero is the same as the fraction $\frac{n}{0}$. They argue that the mathematician had problems with the concept and assert that his statement "zero divided by zero is zero" is incorrect. Mahãvira updated Brahmagupta‟s book 200 years later in 830 AD and declared that, "A number multiplied by zero is zero and a number remains the same when zero is subtracted from it". But Mahãvira‟s attempt to define division was incorrect: "A number remains unchanged when divided by zero" (O‟Connor & Robertson, 2000:4).

Quinn, et. al. (2008:72) claimed that multiplication is sophisticated addition and $6 \times 4$ indicates that 6 fours are added together. If division is considered as sophisticated subtraction then $24 \div 4$ implicates that 4 is subtracted from 24 six times. Mahãvira‟s assertion implied that $24 \div 0 = 24$, thus abandoning the translation to subtraction, which should have led to the question, "How many times can zero be subtracted from 24?" It appeared though that the concept of repeated subtraction linked to division mystified this reasoning.

More than 500 years after Brahmagupta‟s book was written, Bhaskara still found it difficult to explain division by zero:

> A quantity divided by zero becomes a fraction the denominator of which is zero. This fraction is termed an infinite quantity. In this quantity consisting of that which has zero for its divisor, there is no alteration, though many may be inserted or extracted; as no change takes place in the infinite. . . .
>
> (O‟Connor & Robertson, 2000:4)

It was indeed difficult for the Indian mathematicians to acknowledge that division by zero is impossible. Other properties of zero stated by Bhaskara were however accurate: $0^2 = 0$ and $\sqrt{0} = 0$ (O‟Connor & Robertson, 2000:4).

According to Kaplan (1999:73), Bhaskara asserted that a number divided by zero is the same as a fraction with zero as denominator, which is an infinite quantity, $\frac{a}{0} = \infty$ (refer to pp. 30-31). The calculation, $\frac{12}{4} = 3$ has meaning because it is understood as a relationship between numbers. Mathematicians asserted, however, that infinity is not a number (not even a silly one!). We intuitively know that the number 5 is not the number 12. Mahãvira and Brahmagupta argued that, "Any number multiplied by zero is zero". This implies that all numbers are the same if we reason that 5 × 0 = 0 and 12 × 0 = 0. If we assume that division by zero is possible then we have the situation $\frac{5 \times 0}{0} = \frac{12 \times 0}{0}$. Cancelling out the zeroes leads to 5 = 12. It makes sense, then, to say that division by zero is not lawful, and that *a* ÷ 0 is meaningless. Kaplan (1999) concludes that the Indian mathematicians were concerned with principles of mathematics but not with proving them. Because of the Indian mathematicians" invention of the number zero, mathematicians in Italy, Germany, England and France eventually extended their breakthrough to shed light on zero"s behaviour in addition, subtraction, multiplication, and its craziness in division.

Today, almost 1400 years since Indian mathematicians succeeded in inventing the number zero, people still diminish zero"s importance as a number and struggle with the conceptualisation of calculations and number properties involving zero. Renowned mathematicians such as Aristotle and Archimedes, who made highly significant contributions to the study of mathematics, found the concept of zero overwhelming. We should therefore be empathetic towards the problems that learners and teachers today experience with understanding the concept.

## 2.3. THE NON-ACCEPTANCE OF ZERO AS A NUMBER

It became apparent that studies reporting on the difficulties that people experience with the concept of zero relate mostly to their limited conception of zero (Wheeler & Feghali, 1983; Anthony & Walshaw, 2004; Levenson, et. al., 2004; Levenson, et. al., 2007). It is common practice for learners (and teachers) to exhibit misconceptions with regard to the understanding of zero as a number because they undervalue the importance of zero. This mainly happens because of the real life connotation of zero as *nothing*. Connecting zero to *nothing* and disregarding zero as a number could cloud learners‟ concept of calculations with zero and the classification of zero as an even number. Classifying a number as even is to reason that the number must be divisible by two without a remainder. In doing this, a grade 6 learner in the study of Levenson, et. al. (2007:89) disregarded zero as a number by reasoning that, "You don‟t really have anything to divide from . . . So, you just don‟t divide anything . . . You can divide (0) by two, but you don‟t get any answers". The learner‟s reluctance to accept zero as a number that can be divided by 2 to result in 0, did not allow him to connect understanding of 4 as an even number because $4 \div 2 = 2$ and therefore zero is an even number because $0 \div 2 = 0$.

A study of Reys & Gouws (1975) indicated that Grade 4 and 6 learners did not regard zero as a number. Associating zero with *nothing* could be ascribed to the real life interpretation and use of the concept. For example, if you have eight cookies and you eat all eight of them you are left with *nothing*, i.e. eight take away eight leaves *nothing*. The mathematical number sentence $8 - 8 = 0$ is decoded from the real life discourse in which zero relates to *nothing*. Learners often claim that zero "doesn‟t do anything . . . you can add zero, or take away zero, because nothing happens" (Anthony & Walshaw, 2004:40). Grade 1 and 2 learners in the study of Levenson, et. al. (2007:84-85) reasoned that $3 \times 0 = 3$ ". . . because you start with 3 and do nothing". Learners might not regard multiplication by zero as a valid multiplicative condition. Thinking about zero as *nothing* could obscure the understanding of the "deep, complex structure of zero". Arguing that $3 \times 0 = 3$ implies that the law of inverses is defied because $3 \div 0 \neq 3$. The diminishing conception of zero succeeds from primary school to secondary school learners and even pre-service teachers. Learners often struggle with the idea of "dividing nothing into something or something into nothing". Everyday discourse generates a trivial configuration of zero and causes confusion, which successfully averts a profound conceptual understanding of zero as a number (Levenson, et. al., 2007:84-85).

### 2.3.1. Zero is a number in its own right

Zero is integral to the set of whole numbers, which consists of the natural numbers as well as zero. Zero is not regarded as one of the natural numbers although it logically and naturally fits in with these numbers. It replies in a way similar to that of the natural numbers to the question, "How many . . . ?" (Reid, 1956:11). For example, "How many people are there in the room where you are reading this book?" or "How many elephants are there in the room where you are reading this book?" To the first question, the answer could possibly be one or maybe two. Answering the second question would most likely produce a zero. Zero is a number just as two or three is a number. Two is the numeral that represents all sets containing two objects. Zero represents the empty set – a set with no people, no elephants, etc. The possibilities of what could be obtained in a set of two are endless, for example two apples and two birds. The set of zero is different from the set of one or two. Zero represents only one set – the empty set. A set that has zero men, zero elephants or zero flies, is still the same set – the empty set. Zero should be regarded as a number but also as a unique number, different from other numbers in the sense that "zero is the only number which can be divided by every other number, and the only number which can divide no other number" Reid (1956:11).

Kaplan (1999) emphasised zero"s importance in mathematics by asking:

> Why should zero, that O without a figure, as Shakespeare called it, play such a crucial role in shaping the gigantic fabric of expressions that is mathematics?
>
> (Kaplan, 1999:2)

Kaplan (1999:1) raises an argument that provokes reflection on the abstract nature of the concept of a number. He argues that ". . . if there are seven apples in a bowl, exactly what does the seven belong to? Not to any one of the apples taken singly (not even the last one you counted, since you could have arranged them differently), nor to the bowl that contains them, but – to there being just seven of them". According to Kaplan (1999:37), some mathematicians (like Reid, 1956) claim that, ". . . seven is the set of all those sets that contain seven objects. If you eat one of the apples, where has the seven gone? Fled, presumably, to those that still or newly have seven members". Kaplan (1991:37) maintains that the concept of zero is even more abstract because "Names belong to things, but zero belongs to nothing". He classifies zero as a number by reasoning that "[i]t counts the totality of what isn"t there", i.e. the empty set.

Reid (1956) maintains that problems with the concept of zero could be related to the fact that we do not usually think of or engage with zero as a number because we do not usually deal with zero as a number. She further claims that positional calculation is dependent on zero, the symbol, and can be performed effectively without knowledge of zero as a number. Reid suggests a „test" that could assist us in realising that we are more competent in demonstrating knowledge of zero as a symbol (place holder) than demonstrating knowledge of zero as a number symbol. For example, calculations with zero as a placeholder involve 1 + 10 =; 10 – 1 =; 10 × 1 =; 10 × 10; 10 ÷ 1 =, etc. which are easier to explain than calculations with zero as a number. Calculations with zero as a number symbol involve, for example 0 – 1 =; 1 × 0 =; 0 ÷ 1 =; 1 ÷ 0 =, etc. To be classified as a number, numerals should associate and combine with already existing numbers. For zero to be assigned equivalent grading as the existing numerals, its behaviour in the basic operations of addition, subtraction, multiplication and division should be understood. Indian mathematicians first did this around 773 AD (Reid, 1956:6).

### 2.3.2. Disregarding zero as a number

Nowadays, many individual learners and teachers experience difficulties similar to those of ancient mathematicians when they attempt to consider and value zero as a number, with its value connected to "nothingness". People frequently refer to zero as "nothing" and find the idea of "emptiness" difficult to grasp. Words and phrases such as "none", "nothing left", "empty", "all gone", etc. are often used to refer to the content of empty sets. Learners, young and adult, do not think of zero in the same way as the other numbers (Wheeler & Feghali, 1983:147; Anthony & Walshaw, 2004:38). The empty or null set – a set without any elements – is an abstract concept for young learners and encourages them to overlook the existence of zero. They might be able to make sense of the thought that you can have no sweets, no bananas or no money, but the idea of zero sweets, zero bananas or zero money could be experienced as abstract and difficult to make sense of. Having *nothing* is thus more understandable than having zero objects. Early counting experiences do not involve the inclusion of zero in whole numbers, which consist of the natural numbers and zero (Anthony & Walshaw, 2004:38-42).

Although zero is an exception to various mathematical rules (Anthony & Walshaw, 2004), to regard zero as *nothing* is to effectively disregard it as a number. The importance of zero in calculations such as 6 – 6 = 0 and 0 × 6 = 0, for example, should be emphasised to encourage learners to realise that zero is a legitimate solution. Learners mistakenly assume that numbers stop at zero or do not include zero at all, although zero‟s place on the number line is as explicit as positive and negative integers (Quinn, et. al., 2008). Zero has different appearances, i.e. as a number, as a metaphor for desolation or disenchantment, and as "a nothing that is actually something" (Kaplan, 1999:1).

Many teachers do not teach the concept of zero explicitly, and yet the need to teach the concept is frequently apparent in mathematical situations. Both learners and teachers continue to experience problems with the concept, which leads to difficulties in performing arithmetic calculations concerning zero. Associating zero with *nothing* and imposing real life discourse on concepts involving zero lead to misunderstandings. Learners often assert that zero "doesn‟t do anything . . . you can add zero, or take away zero, because nothing happens" (Spitzer; Henry; Gouws & Reys; Hefendehl-Hebeker in Wheeler & Feghali, 1983; Reys & Gouws in Anthony & Walshaw, 2004:40). Reasoning in this manner is an acknowledgement that one can operate with zero, for example 5 + 0 or 5 – 0. Asserting however that *nothing* happens or that zero does not do anything is inaccurate because something does happen – mathematically. If you add 0 to 5 the result is 5; 5 + 0 = 5 and if you take 0 away from 5 the result is 5; 5 – 0 = 5. The learners‟ reasoning implies that the number you add zero to or subtract zero from remains the same – *nothing* happens to the number. This thinking could relate to performing operations on a number line or with physical manipulatives, for example. No *action* is taken when you add or subtract 0 from a number. This type of reasoning could lead to possible difficulties in conceptualising, for example 50 – 18 (p. 3, Chapter 1) or even 0 – 5 = -5, in future algebraic reasoning.

In a study involving 52 pre-service primary school teachers‟ knowledge of zero, it was found that the subjects displayed limited knowledge concerning the concept of zero. Teachers regarded zero as *nothing* when they responded to the question, „What is zero?‟ (Wheeler & Feghali, 1983). Wheeler & Feghali (1983) claimed that the student teachers were not equipped to assist learners in developing understanding of the meaning and use of zero. Educators of mathematics teachers need to pay attention to the lack of students‟ knowledge in this regard.

**2.3.3. Disregarding zero as an even number**

Investigation of the development of understanding of the parity of zero (the property of zero as an even number) could unwrap opportunities for important mathematical concept development, for example knowledge of the number system, counting, number patterns, multiples of numbers, the exploration of multiplication tables, divisibility rules and negative numbers. The parity of zero is not explicitly addressed in any grades in the mathematics curriculum. Counting in odd and even numbers is normally introduced in the Foundation Phase, when learners learn to count in different intervals (South Africa. DoE, 2002:21). The recognition and representation of odd and even numbers are explicitly addressed in Grade 4 (South Africa. DoE, 2002:40; South Africa. DBE, 2010:12). Zero is normally not included in the set of even numbers.

The study of Levenson, et. al. (2007) provides evidence that both learners and teachers are uncertain about zero‟s character as an even number. Evidence of the misconceptions that learners experience with this concept is reflected in the assertion of a grade 6 learner in their study. The learner disregarded zero as an even number and asserted that $0 \div n = 0$ is not a valid operation: "Fourteen is an even number because it is divisible by two but zero is neither an even nor an odd number. The number zero is not divisible by anything so it can‟t be even" Levenson, et. al. (2007:89).

Yet learners are able to construct meaning regarding the parity of zero with appropriate and effective mediation. The Israel National Mathematics Curriculum guidelines (in Levenson, et. al., 2007) include a note to teachers stating that zero is an even number. It states, however, that teachers do not have to address the concept unless it is raised by learners in the classroom. The Principles and Standards for School Mathematics (NCTM) in Levenson, et. al. (2007), on the other hand, includes an account given by a grade 1 learner to highlight young learners" competence in mathematical thinking and reasoning when they are required to justify inferences. The learner supplied an informal proof by contradiction for the argument by asserting that: "If zero were odd, then zero and one would be two odd numbers in a row. But even and odd numbers alternate. So zero must be even" Levenson, et. al. (2007:89).

Ball & Bass (in Levenson, et. al. 2007:86) offered Grade 3 learners" inferences and justifications for the parity of zero, for example "The ones on each side is odd" (so zero is even). The study of Levenson, et. al. (2007) thus indicates that effective construction of the parity of zero is within the zone of proximal development of young learners.

In section 2.3, I have provided evidence in support of arguments that zero is a number in its own right. Relating zero to „nothing" could cause difficulties in conceptualising the concept. It took many years before the significance of zero in the number system was realised. Although the Indian mathematicians made a major contribution to mathematics by considering zero in relation to calculations with other numbers, they had difficulty in conceptualising division by zero. This complicated concept is a prevailing problem in learning mathematics in modern times. Disregarding zero as a number could cloud the conceptualisation of zero as an even number and division and multiplication by zero. Understanding zero as a quantity describing the empty set is imperative for effective conceptualisation of the concept. Development of the concept of zero requires operating with zero as a placeholder and as a number, for example in place value in numbers (1 005), as a number in counting sequences (-2; -1; 0; 1; 2; . . .), and as a number in calculations (25 × 0).

## 2.4. PROBLEMS WITH CALCULATIONS WITH ZERO

In this section, I present evidence from the research literature concerning the difficulties that learners and teachers experience with multiplication and division by zero. This will support the analysis of results and discussion of findings in this study as regards the misconceptions that learners and teachers experience with multiplication and division by zero.

### 2.4.1. Difficulties with multiplication by zero

Learners often provide accurate responses in mental and written tasks concerning multiplication by zero. This is not necessarily an indication that they have a conceptual understanding of the concept. A grade 2 learner in the study of Levenson, et. al. (2004:245) was able to apply repeated addition in multiplication without any problems. The learner however experienced multiplication with zero in a different way, as indicated in the interview exchange below:

> I: And what is 3 times 0?
>
> S: 0
>
> I: Why?
>
> S: Because . . .it doesn"t have a number. If you had one, then it could be different. Because you can"t do 3 times 0. It"s still 0.
>
> I: Why can you do 3 × 2 but you can"t do 3 × 0?
>
> S: Because 0 is a number but it"s . . . it"s nothing. It"s nothing.

It appears that some learners do not even attempt to connect multiplication by zero to their existing knowledge because of their preconception that zero means *nothing* – they do not treat it as a legitimate number.

Rule-based teaching and learning do not facilitate the conceptual understanding of multiplication by zero. Older learners often provide uninformed rules to explain understanding, as indicated in the study of Levenson, et. al. (2004). The authors declare that most Grade 5 and 6 learners in their study were able to solve multiplication by zero problems effectively. They assert that this was an indication that most learners who learnt multiplication in class knew that multiplication by zero results in zero. Learners are able to explain multiplication with natural numbers but cannot conceptualise multiplication by zero. The Grade 5 and 6 learners in their study displayed procedural understanding in stating that $3 \times 0 = 0$ and $0 \times 3 = 0$, but they could not provide conceptual explanations of these concepts. A number of learners repeated the rule "Every number times zero must equal zero" (Levenson, et. al., 2004:244) to explain their understanding. It appears that learners often do not make an effort to explain rules for calculating with zero because they regard the rule as an explanation in itself.

Levenson, et. al. (2004) further claim that it might be expected that learners in higher grades would be conscious of the use of the commutative property to justify that $0 \times 3 = 0$ because $3 \times 0 = 0$. This knowledge however was not demonstrated in their study. Mulligan & Wright (2000) argue that the development and consolidation of repeated addition or subtraction reproductions and sharing representations should be followed by the extension of understanding to include recognition of the concept of commutativity and the application of the inverse relationship between multiplication and division. In our mathematics curriculum (South Africa. DoE, 2002:44; South Africa. DBE, 2010:13), the concepts of the reciprocal relationship between multiplication and division and the commutative property of numbers are introduced in Grade 4. But development of the concept of zero is not an explicit focus in the curriculum. It might therefore be anticipated that our learners would not apply these properties of numbers to justify multiplication by zero, as in the study of Levenson, et. al. (2004). Using the inverse relationship between multiplication and division concerning zero could however be a challenging concept for Grade 4 learners. The calculation $0 \times 3 = 0$ could be justified as $0 \div 3 = 0$. If you consider that $2 \times 3 = 6$ so that $6 \div 2 = 3$ and $6 \div 3 = 2$ and apply the inverse relationship to $3 \times 0 = 0$, it could cause difficulties because $0 \div 3 = 0$ but $0 \div 0 \neq 3$. Zero does not have a multiplicative inverse (Quinn, et. al., 2008). Levenson, et. al. (2007) presented a grade 6 learner with the rule of inverse operations to validate division and multiplication calculations. The learner reasoned accurately that $10 \div 5 = 2$ because $2 \times 5 = 10$. He was however consistent in denying that zero can be divided, even though the inverse multiplication operation provided the proof in his recorded response. It appears that the learner referred to $5 \div 0$, which is impossible:

> 0 ÷ 5 = 0 and 0 × 5 = 0. Zero divided by five is zero. So . . . but like it can"t even divide by five because it has nothing to divide. . . you don"t really have anything to divide from . . . So, you just don"t divide anything . . . You can divide (0) by two, but you don"t get any answers. (Levenson, et. al., 2007:89)

The new CAPS curriculum for the Intermediate Phase (South Africa. DBE, 2010:9, 43, 49) suggests that learners progress from "counting reliably to calculating fluently", that the teaching of multiplication and division should happen concurrently and emphasises learners" understanding that "any division statement can be changed into a multiplication statement". Problems that learners might experience with conceptualising multiplication and division with zero are however not highlighted. The studies of Anghileri; Carpenter, Ansell, Franke, Fenema & Weisbeck; Clark & Kamii; Kouba; Mulligan & Mitchelmore; Steffe (in Mulligan & Wright, 2000:17) have recognised early learners" problem-solving strategies. They have identified the significance of reproduction and representation in the development of effective problem-solving strategies. Analysis of these studies points to the assimilation of counting strategies into processes involving repeated addition and subtraction which are generalised as the dual operations of multiplication and division respectively. Concrete and sensory model strategies are abstractly internalised and reproduced reflecting escalating complexity.

The structure that learners develop from counting strategies to repeated addition in order to demonstrate an understanding of multiplication with natural numbers raises cognitive conflict when it assimilates to multiplication by zero. This notion prevents learners from constructing conceptual understanding of the concept. Davis (1983) maintains that multiplication results in an answer bigger than the two natural numbers multiplied. Multiplying by zero (and 1) is an exception to this understanding. Applying the intuitive structure of repeated addition gives bigger solutions, for example 3 × 5 = 5 + 5 + 5 = 15 and 5 × 3 = 3 + 3 + 3 + 3 + 3 = 15. The problem 3 × 0 probably does not relate to learners" intuitive meaning of multiplication because 3 × 0 = 0 + 0 + 0 = 0 with an answer less than 3. Davis (1983) argues that learners do not develop their conception of multiplication further than their original understanding. He suggested that the conceptualisation of multiplication should be extended to include alternative structures.

The preservation of non-productive structures results in the retention of incorrect notions that hinder the development of productive structures. Incorrect constructions can be retained and inaccurately recovered even after awareness is created of the circumstances under which they are warranted. New and correct structures might become accessible but the original ones continue to exist (Davis, 1983). Knowing how to multiply to get the correct solutions, i.e. procedural understanding, is not adequate knowledge when teachers have to explain and justify the rules for multiplication by zero. Explaining efficiently how an algorithm works requires understanding beyond being able to use it fluently and accurately, i.e. conceptual understanding. Teachers have to know why procedures work and why certain properties are true. They have to know which relationships exist and on what bases they exist. Understanding how a concept can be defined, how mathematical claims are justified and how to scrutinize and justify propositions, are important in illustrating conceptual understanding (Shulman, 1986; Wood, et. al., 1993; Ball, 2003).

### 2.4.2. Difficulties with division by zero

Levenson, et. al. (2007:84-85) note that learners often struggle with division with zero as a dividend and a divisor (refer to p. 31). Everyday discourse interferes with and prevents the construction of insightful conceptual understanding of zero. Quinn, et. al. (2008) assert that learners of all ages find division by zero confusing. Teaching the concept therefore requires teachers with real conceptual insight. Understanding the concept of division by zero is essential for the development of more advanced mathematical concepts such as the connection between multiplication and division and trigonometric concepts. Tsamir, et. al. (in Quinn, et. al., 2008) emphasise that learners have an instinctive belief that each mathematical calculation requires a numerical solution, which acts as a barrier to their coming to understand that division by zero is undefined.

Tsamir, et. al.; Reys and Henry (in Quinn, et. al., 2008) reported on primary and early high school teachers‟ difficulties with the concept of zero, especially with understanding zero as a number and division by zero. Teachers often replied that, for example 6 ÷ 0 = 0, or that it is undefined, without being able to explain why it is so. According to Kahan, et. al. (2003) and Ball & McDiarmid (1989), pre-and in-service teachers‟ understanding of key mathematical ideas and facts is inadequate. Student teachers who majored in mathematics did not perform any better than those who did not qualify as mathematics experts when they had to justify why, for example 7 ÷ 0 is not possible or not allowed (Kahan, et. al., 2003). Although some teachers have a good conceptual understanding of division by zero (Quinn, et al., 2008), those who know that division by zero is undefined are often not able to explain the concept effectively (Levenson, et. al., 2007). The teaching of division by zero is normally circumvented in primary school. High school learners who are aware of the fact that division by zero is impossible or undefined are often unable to justify the concept (Quinn, et. al., 2008). They often hold teachers responsible for conveying the fact without explanation (Henry; Reys in Quinn, et. al., 2008).

The present study was partly inspired by the study of Van den Heuvel-Panhuizen (2001), which intended to investigate learners‟ problem solving skills. The study by Van den Heuvel-Panhuizen (2001) is significant concerning the development of conceptual and procedural understanding of division by zero through constructive learner and teacher discussion and debate. The Tal Team at the Freudenthal Institute in the Utrecht University performed an investigation of the mathematical attitude required in the development of problem solving skills (Van den Heuvel-Panhuizen, 2001:237-240).

The study of Van den Heuvel-Panhuizen (2001) involved grade 6 learners who were presented with the task to calculate problems involving subtraction, multiplication and division with zero and requested to elaborate on their solutions using stories. They had to solve and explain the problems 0 – 1 = □; 1 – 0 = □; 0 × 1 = □; 1 × 0 = □; 0 ÷ 1 = □ and 1 ÷ 0 = □. Learners found the calculation 1 ÷ 0 = □ to be the most challenging as reflected in the quote below.

> I"ve found something illogical in arithmetic. That the problem 1 ÷ 0 has no answer, at least not in grade 6. The teacher said that later you learn that the answer is infinity. But I think that"s illogical. I think that 1 ÷ 0 = 1. Because if I have a cake and invite people round and no one comes then the cake doesn"t have to be cut up and I still have one cake left (ibid.)

The learner"s practical real life model of the concept of sharing linked to the division operation was accurate, but the mathematical reasoning was questionable. She reasoned that 1 ÷ 0 = 1; if she had a cake and had no-one to share it with then she still had one whole cake left. On the contrary, 1 ÷ 1 is also equal to 1 (1 ÷ 1 = 1), so both solutions could not have been accurate. Agreement was reached that 1 ÷ 1 = 1, but no accord could be reached for 1 ÷ 0 = 1. One learner reasoned that 1 ÷ 2 = ½, meaning that you divide by 2 and asking „How much does each one get?" He found it difficult to answer this question for 1 † 0 because there is no one (zero people) which meant the question could not be answered. Another learner presented an alternative concept – subdivision instead of equal sharing. She related 18 ÷ 3 = 6 to how many times 3 m pieces of rope could be cut from 18 m. The learner applied the inquiry to 1 ÷ 0 = ☐ by asking how many times a piece of 0 m could fit into 1 m. The learner claimed that the pieces would be uncountable – it carries on forever; lots of times and reckoned that 1 ÷ 0 = infinity as suggested by the class teacher. Another learner demonstrated conceptualization of the problem with long division and claimed that 1 † 0 = ∞ remainder 1. He argued that, if 1 † 0 = ∞ then 2 † 0 = ∞ which implied that the two statements were equal and the same as 1 000 † 0 = ∞. He regarded this as a crazy situation. A different learner applied the law of inverses. She reasoned that 18 ÷ 3 = 6 and 3 × 6 = 18 which justified the solution. She turned the division problem into a multiplication problem. The learner argued that, if 1 † 0 = ∞ then ∞ × 0 = 1 or 0 × ∞ = 1 and 2 † 0 = ∞ would then result in ∞ × 0 = 2. She regarded this situation as more unwise than the other situations offered and reckoned that 1 ÷ 0 results in no solution. The class was eventually persuaded by this learner"s reasoning, i.e. calculating with ∞ as if it is a whole number did not concur with ordinary mathematical problems. If ∞ + ∞ = ∞ and you apply the inverse law, then ∞ - ∞ = 0. You could therefore conclude that ∞ is not a normal number. You cannot use it in normal arithmetic procedures. After much deliberation, the learners concluded that division by zero results in no answer – it cannot be done. The study of Van den Heuvel-Panhuizen (2001) provides evidence that Grade 6 learners have the ability to construct the meaning of division by zero without the presentation of rules that do not make sense to them. To make sense of division by zero, learners need a strong basis of various mathematical concepts, which they could assimilate to facilitate understanding. This kind of classroom intervention needs knowledgeable teachers who are able to facilitate and mediate the learning process effectively and confidently (refer to pp. 37 & 41-42).

The conceptualisation of division by zero as undefined or not allowed is a difficult concept for primary school learners to grasp. Learning and teaching of the concept is often avoided in primary school (Levenson, et. al., 2007; Quinn, et. al., 2008). Learners could however attempt to make sense of the concept through constructive discussion and debate (Van den Heuvel-Panhuizen, 2001). The study of Van den Heuvel-Panhuizen (2001) is in accordance with the principles of social constructivism (Clarke (2002). The power of discourse in the knowledge construction process, and the development of more complex and sophisticated reasoning while the learners constructed their own sense through constructive debate (Clarke (2002; Hiebert, et. al., 1996), was reflected remarkably. It should however be noted that it was obvious that the learners" studied by Van den Heuvel-Panhuizen (2001) basic mathematical concepts were well developed. They displayed a good sense of multiplication and division concepts and number properties, which allowed them to engage in mathematical discourse at an advanced level. They were allowed to disagree, individual opinions were respected and valued and the teacher"s views were not imposed on their thinking and reasoning.

In section 2.4, I have considered literature concerning difficulties that learners and teachers experienced with multiplication and division by zero. It appears that literature regarding teachers" difficulties with multiplication by zero is non-existent. Evidence was obtained that learners normally make sense of multiplication with natural numbers but are not able to connect this knowledge to multiplication with zero. Older learners often provide rule-based explanations to illustrate understanding of multiplication by zero. The learners do not use advanced concepts such as the commutative property to justify understanding of multiplication by zero. The literature provides evidence that both teachers and learners struggle with sense-making of division by zero. Real life discourse and the belief that calculations require numerical solutions are barriers to making sense of division by zero as a divisor. Teachers who know that division by zero is undefined are not able to display conceptual understanding of the concept. Learning of division by zero requires a progressive learning and teaching environment.

## 2.5. CALCULATIONS WITH ZERO ARE BASED ON (UNINFORMED) RULES

Freire (in Clarke, 2002:95) refers to the traditional approach to learning and teaching as "banking", in terms of which teaching involves the "deposit" of knowledge through talking-and-chalking. Teaching and learning of the concept of zero is often based on the transfer of rules that learners do not make sense of (Levenson, et. al., 2004; Semenza, et. al., 2006). Understanding does not develop when knowledge is separated into parts, i.e. understanding of isolated rules does not lead to understanding of related concepts and mathematics as a whole. Clarke (2002:95) argues that the conceptualisation of "complex systems" cannot be based on the understanding of smaller parts disconnected from the systems.

In mathematics, it often happens that understandings of facts, rules and procedures are disconnected from each other. The „deposit" of the zero multiplication and division rules, which could be successfully recovered from memory or memorised independently of arithmetical context, does not facilitate conceptual understanding. Knowing that division by zero is undefined or multiplication with zero results in zero, for example, is no real basis for an understanding of the complex and abstract thinking and reasoning behind these concepts. Solving calculations with zero typically involves superficial rules and procedures disconnected from conceptual understanding. Incoherent conceptual and procedural knowledge is a primary hindrance to mathematics achievement for learners at all levels of development (Semenza, et. al., 2006). The demonstration of knowledge attained by rote learning does not reflect understanding. The provision of algorithms involving a "sequence of symbols" does not display conceptual understanding (von Glaserfeld, 2011:9).

Levenson, et. al. (2004) note that mathematically and practically based elaborations for multiplication by zero decreased in higher grades. They ascribe the phenomenon to an increase of rule-based explanations, which could contribute to problems that learners experience with the concept of zero. Ball (2003) affirms that a teacher who knows algorithms only procedurally and does not understand them conceptually will not be able to assist learners in developing conceptual understanding. Ball, et al. (2008) suggest that rules stating that a number multiplied by zero is equal to zero should be constructed by learners and not supplied by teachers. Semenza, et. al. (2006) assert that calculations with zero involving isolated rules and procedures without conceptual construction create barriers to the construction of meaning for learners of all ages. Levenson, et. al. (2004) suggest that learners should not be introduced to rules at all before the occurrence of formal learning. Learners should make sense of mathematics by making connections to real life situations and/or base their learning on existing knowledge. Semenza, et. al. (2006) report that the understanding of facts, rules and procedures can be disconnected from each other, from conceptual insight and from related understanding, such as the zero multiplication rule. This rule could be successfully recovered from memory or could be independent of arithmetical context (Semenza, et. al., 2006). Mathematical learning based on inaccessible facts leaves no room for understanding which can be applied flexibly and connect to meaning-making. The demonstration of formal ways to solve problems entails a traditional rather than an inquiry-based teaching and learning practice. In the traditional approach, teachers assume that learners do not have cognitive tools and strategies of their own to solve problems. Interaction in the traditional teaching sense is based on learner responses initiated and evaluated by the teacher (Mehan; Sinclair & Coulthard, in Wood, et. al., 1993; Hiebert, et. al., 1996).

Various researchers, for example Lerman (1989), Clements & Battista (1990), Wood, et. al. (1993) and Vianna & Stetsenko (2006) have explicitly promoted a constructivist perspective on teaching and learning mathematics. The Western Cape Education Department"s (WCED) Literacy and Numeracy Strategy for 2006–2016 (WCED, 2006:4) explicitly endorsed constructivist learning. This progressive alternative to the traditional teaching approach was embarked on to improve the learning and teaching of numeracy and literacy. The WCED"s strategy envisaged that an awareness of the epistemology of learning would promote the development of conceptual instruments that facilitate conceptual development. It was assumed that the development of conceptual tools such as originality, inventive thinking, and innovation would direct teachers to the implementation of the "new educational pedagogy". Policy-makers and curriculum developers in South Africa thus support constructivism as a learning theory, which centralises learners as thoughtful and sense-making human beings.

Literature employed in section 2.5 revealed that learners in higher grades often offer rules to explain understanding of calculations with zero. The fluent or successful recall of rules does however not illustrate conceptual understanding. It appears that the teaching and learning of calculations with zero is mainly based on a traditional teaching and learning approach.

## 2.6. CONSTRUCTIVISM

Constructivism is concerned with cognition, the progression and development of thinking and reasoning as a human action by individuals and between individuals and society. A constructivist perspective assumes that humans actively construct knowledge through their own experiences. Knowledge is not received passively from people in their social environment (Harries & Spooner, 2000; Clements & Battista, 1990; Clarke, 2002; WCED, 2006). Learners need learning environments in which they can flexibly define and solve problems using strategies that allow them to adapt new learning (Hiebert, et. al., 1996). Appleton (in Carpenter, 2003) reasoned that the theory of constructivism is based on a combination of different learning theories and is employed to guide teaching. Piaget and Vygotsky, for example, are constructivist theorists who both advocated and exemplified the "transactional, relational and contextualised" approaches to construing human development as occurring through interaction with the environment (Vianna & Stetsenko, 2006:84). But their theories reflect significant foundational dissimilarities. In Piagetian terms, the achievement of knowledge occurs by adapting existing knowledge to construct better understanding. This is a continuous, individual, internally driven process intended to search for a condition of equilibrium between existing and new knowledge. The focal point in this constructivist perspective is on what learners learn and how they learn. According to the Vygotskian view of constructivism, learners obtain new knowledge through interactive and collaborative participation with peers, teachers and parents; that is, through mediation, an external socio-cultural process. This socio-cultural constructivist perspective focuses on the circumstances for possible learning (Cobb, 1994; Clarke, 2002). The cognitive and socio-cultural constructivist theories of Piaget and Vygotsky are discussed in more depth later in this chapter (refer to pp. 38-42). Clarke (2002:103) describes a range of interconnected and integrated constructivist principles central to teaching and learning. The principles include the promotion of content and processes, the creation of opportunities for action, the connection to learners" levels of understanding, the promotion of guided discovery and the promotion of co-operative learning. Central to all these principles are the development of language and language interaction.

Vico (in von Glasersfeld, 2011:4) maintained that, "The human mind can only know what the human mind has made". The view of this 18th century mathematician is reflected in the constructivist learning theory in which the belief exists that learners are active and enthusiastic constructors of their own knowledge (Clements, et. al., 1990; Yackel, et. al., 1990; Kamii & Lewis, 1990). Their understanding of the world develops through their own convictions, social practices and outlooks on the learning environment. They experience knowledge as multifaceted i.e. the perspective of one person is not necessarily more valid than that of another (Hiebert, et. al., 1996; von Glasersfeld, 2011). In a constructivist learning environment learners are encouraged to construct their own knowledge through meaningful learning experiences. These experiences allow opportunities for discovery and invention in social settings that promote the explanation, negotiation, sharing and evaluation of conceptions (Clements, et. al., 1990; Yager, 2000; Ball, 2003). The constructivist teacher realises that learning by rote and repetition does not necessarily produce effective understanding. To understand mathematics, learners need to know mathematics; know how concepts work and why they work (Shulman, 1986; Ball, 2003). Learning could occur independently of the teacher, and each learner‟s experiences have exceptional and distinctive meaning. Learners‟ conceptions and strategies, even if ineffective or inefficient, should be the foundation for instruction. They should be allowed to develop cognitive structures that are more significant, advanced, sophisticated and abstract than their existing structures. (Hiebert, et. al., 1996; Clements, 1997; Yager, 2000; Boghossian, 2006; WCED, 2006; von Glasersfeld, 2011).

Non-productive constructions could hinder learners‟ development. This often occurs in traditional teaching and goes unnoticed because one-word solutions are accepted by teachers as indicators for conceptual understanding. The constructivist teacher facilitates and builds on learners‟ understanding to motivate construction of more effective conceptual thinking and reasoning to inform learners‟ actions and thinking. For teaching and learning to occur in a constructivist manner requires a highly refined and reflexive teaching approach. The teacher initiates, negotiates and guides mathematical sense-making. Both teachers and learners are committed to reflect on and communicate ideas. This approach requires a fundamental change from traditional instruction and poses a challenge to the practice of learners reproducing teachers‟ methods. It maintains, controversially, that learners‟ experiences should be self-organised in an environment that constitutes actual constructive communication through social interaction. Teachers understand learners‟ existing levels of thinking and their potential constructions. The success of this teaching and learning approach depends on the character, quality and the degree of the teachers‟ own understanding of and beliefs about the mathematics that they teach (Wood, et. al., 1993).

Wood, et. al. (1993) assert that open-ended tasks should replace traditional textbook tasks to encourage various learner solutions. Open-ended tasks assist in the development of underlying conceptual operations rather than standard algorithmic procedures. Kilpatrick, et. al. (2001) however suggest that learners should be engaged in both open-ended and routine tasks involving the application of step-by-step procedures that they make sense of (refer to pp. 43-46). Wood, et. al. (1993) maintain that learners need opportunities to discuss solution strategies; to reflect on their own solutions as they explain them to others. As they listen and try to make sense of others‟ solutions, they reconceptualise their own thinking. This creates opportunities for learning as learners and the teacher negotiate mathematical meaning. Learners are allowed to make links between their own mathematical constructions and the shared meanings of the classroom. The teacher and learners thus interactively constitute a basis for mathematical communication. This constructive practice is reflected in the study of Van den Heuvel-Panhuizen (2001), which highlights the attempts of Grade 6 learners to make sense of division by zero. The author demonstrates the importance of discussion and debate in constructing the meaning of the concept (refer to pp. 30-31).

Wood, et. al. (1993) state that teachers who constrain and limit learner participation in discussions about structures and strategies, have a negative influence on learners‟ opportunities and willingness to express their thinking. The teacher should create a classroom atmosphere that reflects mutual trust. In a classroom that portrays an effective culture of learning, the teacher facilitates learners‟ mathematical thinking through authentic communication. Learners are allowed to express their mathematical thinking confidently, respect is shown for each learner‟s opinion, and teachers are sensitive to the potential mathematical constructions that learners make. Hiebert, et. al. (1996) define a learning environment conducive to the development of understanding in terms of the character of learner activities, the teacher‟s part in the learning process, the social norms practiced in the classroom, the type of available resources and the ease of access to mathematics for all learners. Learners should be involved in challenging problem solving tasks that build on their existing knowledge in an environment where concepts and strategies are shared and appreciated and misunderstandings are used as opportunities for learning.

For the purpose of this study, I adopt the constructivist position of Wood, et. al. (1993). The authors maintain that social interaction between the teacher and learners is essential in creating opportunities for learning. Learners should genuinely communicate mathematical thinking and reasoning while being involved in problem solving and investigative learning experiences.

## 2.7. CONSTRUCTIVIST LEARNING THEORIES

In this section, I discuss the individual constructivist theory of Piaget and the social constructivist theory of Vygotsky. These theories are encompassed in the general theory of constructivist learning and form a basis for informing the analysis and discussion of learner and teacher responses in this study.

### 2.7.1. Piaget's cognitive development theory

The cognitive development theory of Jean Piaget postulates the ability of individual learners to build increasingly more complex records of the world and progressively organise, understand, and adapt to them (Clarke, 2002). Cognitive development theory is concerned with inner mental constructions and continuous, conscious reflection on these constructions (Hiebert, et. al., 1996). Learners actively structure and re-structure knowledge and experiences in meaningful problem solving situations (Harries & Spooner, 2000; Clarke, 2002). They attempt to find matches between the world as they experience it and their understanding of it, constantly confronted as they are with new information. They engage actively in continuous processes of adaptation to the environment to extend their understanding by employing three on-going interacting processes:

- Assimilation occurs when learners are confronted with new knowledge, which they fit into their existing cognitive structures. They assimilate the new knowledge to what they know to extend or adapt their existing knowledge.
- When the new knowledge is in contrast with their existing knowledge, they adjust or reshape the known to accommodate the unknown knowledge – they revise current structures according to new experiences. For example, a learner who knows the symbol 6 will extend the concept to construct meaning of 60.
- Continuous, simultaneous interaction persists between the assimilation and accommodation of known and unknown knowledge. Progressive accommodation

exposes more possibilities for assimilation and progressive assimilation opens up further opportunities for accommodation in escalating cycles. For example, the specific meaning that the symbol 6 has for the learner, is disturbed by the figures 60, 462, 0,6 or $\frac{1}{6}$. The cognitive conflict should be complemented with concept development of place value, decimals and fractions. This dynamic balancing of knowledge is organised across the different records of knowledge in relation to a complete understanding or cognitive structure of the world at any particular time, through the process of equilibration (Harries & Spooner, 2000; Clarke, 2002).

Cognitive conflict is reached when learners are unable to deal with new knowledge that does not fit into their existing structure of cognition. They then have to adjust the entire structure to shift towards a more useful and powerful manner of organising and dealing with their world. New levels of adjustments result from a new range of more complex problem-solving ideas, which occur in different stages. Learners at different stages of development internalize understanding that symbols or numbers are related to various contexts (Harries & Spooner, 2000). Adjustments between known and unknown concepts result in a growing ability to engage with and manipulate concepts. These adjustments are even employed to predict likely outcomes for operating with concepts (Clarke, 2002). For example, learners intuitively apply counting and repeated addition to demonstrate their understanding of multiplication (Davis, 1983), but they cannot in this way make sense of multiplication by zero (Levenson, et. al. 2004). These characterisations of learners" conception of multiplication (with natural numbers) – the existing concept and multiplication by zero, the unknown concept – imply that learners have difficulty in restructuring the existing structure to construct meaning of multiplication by zero. They have problems with establishing a state of equilibration between the new and the existing concept, which results in a state of cognitive conflict.

### 2.7.2. Vygotsky’s social constructivist theory

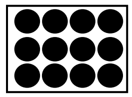
Lev Vygotsky‟s social constructivist theory is based on the understanding that learning occurs from the outside – the social world of the learner. Communication and active engagement in the social environment are requirements for learning (Wood, et. al., 1993; Hiebert, et. al., 1996; Harries & Spooner, 2000). Cognitive development arises through external social relationships, which involve tools for cognitive development – context/culture, language and mediation. Society and the people involved in the lives of learners influence their understanding of their world. The type and quality of the developmental tools guide the pattern and velocity of the development. The Vygotskian theory suggests that learners engage in the construction of meanings that are shared among parents, peers, teachers and others in their specific social context. They progressively develop and adapt new understanding by construction in the space between existing knowledge and confrontations with unfamiliar concepts in social interactions. Vygotsky believed that understandings are social constructions – learning develops with assistance of capable and knowledgeable peers and teachers. The constructions are inseparable from their context and are built up and transferred through interaction between people who exist in a social context with wider historical and cultural meanings. These meanings are not rigid but develop dynamically and are ever changing. Some are common across various social contexts while others are more particular to specific contexts. The process of individual cognitive development also takes place through the same process of social interaction, indicating connections between the nature of knowledge and how cognitive development occurs (Harries & Spooner, 2000; Clarke, 2002; Wink & Putney, 2002).

Vygotsky‟s theory is especially important in understanding language development as an instrument of cognitive development. Language is both a deliverer of understanding and a resource for developing understanding, among people in groups, communities and cultures as well as individually, through mediation. Social constructivist theory is especially applicable to classroom learning and teaching experiences. Mathematical discourse and knowledge are shared to develop conceptual understanding in classrooms. As learners actively communicate their existing mathematical thoughts, the teacher develops understanding of their thinking. The teacher connects learners‟ informal understanding to formal, sophisticated mathematical concepts and language. Opportunities are created for learners to participate actively in learning and teaching experiences that promote discussion, reasoning and debate in order to construct meaning. Meaning cannot be taught in a straightforward manner. Learners can memorise definitions, procedures and rules but they do not or are not able to connect them to previous concept development. For Vygotsky, meaning construction occurs first on a social level and then on an individual level through the generalisation of concepts (Steele, 2001; Wink & Putney, 2002).

Hiebert, et. al. (1996) support reflection and communication as essential cognitive tools for the development of conceptual understanding. Learners who get opportunities to express their ideas and listen to the thinking and reasoning of others are prompt to think about and rethink their own ideas. Learners who are actively involved in the internalisation and use of new concepts mediated by well-informed teachers are allowed opportunities to convert unknown into known concepts. Learning occurs in the zone of proximal development (ZPD) – the critical space in a person"s existing knowledge. It is in this gap that a person or social group could be motivated to construct a new stage of understanding or agreement through proximal (face-to-face) interactions or mediations. Learners are not always able to construct meaning independently, but nevertheless have the potential to make meaning with the assistance of other resourceful individuals or groups in their social contexts. The collaborative learning experiences, i.e. the progressive development of new or adapted constructions that learners experience and interact with in the ZPD through social interaction, lead to AHA!-moments. These constructive experiences motivate future independent and effective learning (Clarke, 2002).

In Vygotskian terms, cognitive development is driven by mediation occurring in the ZPD. Cognitive (individual) and social mediation both aim at the development of novel stages of understanding or agreement. Mediation entails intentional intervention and facilitation without telling but rather through directing and suggesting (Clarke, 2002). The process seldom involves giving information to address such dilemmas and questions that learners might encounter in collaborative learning experiences. During the mediation process, learners are challenged to construct meaning at increasingly advanced levels by challenging, clarifying and explaining ideas. They think more intensely about their understanding to define their thinking more explicitly (Hiebert, et. al., 1996; Clarke, 2002). Well thought-through teaching interventions produce effective mediation processes that assist learners in assimilating new, accurate knowledge with existing knowledge, thereby developing more advanced sophisticated and abstract cognitive structures. The process of mediation requires teachers with profound content and pedagogical knowledge (Ball, 2003). Teachers should guide the learning process through appropriate contexts and questioning techniques to understand what learners know and what content they require to develop effective conceptual understanding (Steele, 2001; Bruner in Wink & Putney, 2002, Clarke, 2002; WCED, 2006). The development of mathematical language – both oral communication and symbolic representation – is fundamental in understanding mathematics. Learners" demonstrations and illustrations, their application of the spoken language and written symbolic equations from and to real life problems, all promote problem-solving skills and allow the teacher to gain insight into their levels of understanding. "To translate information in problems into symbolic representation models is the work that real mathematicians do" (Moyer, 2000:521).

Concerning the theory of social constructivism in relation to the research problem addressed in this study, Davis (1983) maintains that learners intuitively apply the concept of repeated addition to multiplication, for example $3 \times 5 = 5 + 5 + 5$, but experience difficulty when this structure is applied to the conceptualisation of $3 \times 0$. The choice of structure they impose on $3 \times 0$ is mediated by the original structure they impose on multiplication, which is repeated addition. The problem that learners experience with the concept could be related to the teaching and learning of one specific structure, which worked for multiplication with natural numbers. If they were not previously exposed to developing an understanding of multiplication by zero and they are confronted with the problem, it makes sense that they would apply the structure that previously worked for them. They normally are able to demonstrate procedural understanding, i.e. $3 \times 0 = 0$, but are not capable of showing conceptual understanding, i.e. $3 \times 0 = 0 + 0 + 0$ because of a lack of cognitive tools.[3] Concept development with regard to multiplication should therefore be adjusted to an alternative structure that can accommodate multiplication by zero. This process should happen through mediation so as to provide learners with the necessary cognitive tools (Davis, 1983). Teachers could build on learners‟ intuitive structure of repeated addition and allow them to adapt the area model for multiplication by zero. For example:

$3 \times 4 = 3$ groups of 4  and $3 \times 0 = 3$ groups of 0 

Solving calculations with zero typically involves shallow rules and procedures disconnected from conceptual understanding (Ball, 2003; Levenson, et. al., 2004; Semenza, et. al., 2006). Incoherent conceptual and procedural knowledge is considered as a primary hindrance to mathematics achievement in learners at all levels of development. In section 2.8, I discuss the notion of conceptual and procedural understanding.

**2.8. CONCEPTUAL AND PROCEDURAL UNDERSTANDING**

There is a need for an exploration of the notions of conceptual and procedural understanding to develop insight into the data analysis in this study regarding the conceptualisation of abstract concepts. Rule-based teaching and learning do not facilitate conceptual understanding. Prescribed rules supplied by the teacher are often embedded in cognitive structures and are resistant to change (Olivier, 1989; Semenza, et. al., 2006; Levenson, et. al., 2004; Wood, et. al., 1993). Learners discard their own sense-making to satisfy the teacher‟s goals. This is characteristic of the traditional teacher-centred teaching and learning approach. In the past, the belief existed that "learning is the result of teaching", i.e. the learner learnt knowledge that was transmitted by the teacher (Sewell, 2002:24). This approach to teaching and learning is in contrast with the learning theory of

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[3] Clarke (2002:102) described cognitive tools as "the use of symbol systems such as language, mathematics and music notation". Bruner (in Clarke, 2002) maintained that representation of understanding develops gradually from „enactive‟ to „iconic‟ to „symbolic‟ representation.

constructivism, which advocates the construction of knowledge through active participation.

Kilpatrick, et. al. (2001) advocate that learners who develop mathematical competence, create and discover mathematics by themselves and they are dependent on teachers especially for learning mathematics. Effective mathematical learning should thus be both controlled and open-ended. Numeric calculations depend on rules, i.e. procedural steps that should be useful and understood by learners. The development of mathematical proficiency is dependent on, inter alia conceptual understanding and procedural fluency[4]. Learners should understand mathematical ideas, operations and relationships. They should also perform procedures in flexible, accurate, efficient and appropriate ways (Kilpatrick, et. al., 2001). Skemp (1976:2) describes conceptual understanding as "relational understanding", i.e. "knowing what to do and why" while Hiebert, et. al. (1996:3-8) are in agreement with Kilpatrick, et. al. (2001). Hiebert, et. al. (1996) and Kilpatrick, et. al. (2001) promote individual and social cognitive development through internal reflection and communication in the social environment as contexts for learning mathematics. Conscious reflection on their own conceptions and communicating these conceptions with others allow learners to develop "new relationships and connections" in and across mathematical topics, i.e. conceptual understanding. Learners who are allowed to develop, demonstrate, explain and justify their own problem solving procedures develop procedural as well as conceptual understanding, "the primary goal of mathematics instruction" (Hiebert, et. al., 1996:3-8).

Kilpatrick, et. al. (2001) characterise conceptual understanding as a cost-effective manner of dealing with mathematical concepts. For example, a learner who knows that $5 + 5 + 5 = 15$ could be able to apply this knowledge in various situations, i.e. $3 \times 5 = 15$, $5 \times 3 = 15$, $15 - 5 - 5 - 5 = 15 \div 3 = 5$, $15 \div 5 = 3$, $(2 \times 5) + (1 \times 5) = 3 \times 5$, and so on. This understanding could assist learners in avoiding key mistakes and allow them to assess

[4] Kilpatrick, et. al. (2001:116) promote five interrelated *strands* as requirements for the development of mathematical *proficiency*, i.e. *conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition*.

solutions for correctness immediately. It permits learners to perceive underlying correspondences between situations that are superficially unrelated. Hiebert, et. al. (1996) define conceptual understanding as the relationship and connection of existing and new knowledge. Kilpatrick, et. al. (2001) suggest that learners who are capable of developing effective conceptual understanding possess knowledge that is packaged into condensed collections of interwoven mathematical information and beliefs. The knowledge can be recapped into a short phrase and used in various mathematical conditions. When the learner needs to explain an opinion, replicate an idea or learn a new concept, the collection of information and beliefs can easily be unpacked. For example, $3 \times 5 = 15$ and $30 \times 50 = (3 \times 5) \times 10 \times 10 = 15 \text{ x } 100 = 1\ 500$. Hiebert, et. al. (1996) argue that learners who have developed conceptual understanding are familiar with more coherent, holistic knowledge than knowledge broken up into separate facts and techniques. They know why certain concepts are significant and where they can best be applied. Kilpatrick, et. al. (2001) further claim that, the formation of learners‟ conceptions is layered with uncomplicated concepts encapsulated in concepts that are more intricate. The relations among the layers of knowledge allow learners to create understanding of new concepts rather than depend on the recall of concepts learnt by rote. Hiebert, et. al. (1996) maintain that the memorization, application and adjustment of skills to learn to new concepts require conceptual understanding. Constructions are easily remembered and applied and can be restructured or reanimated even when they have been forgotten. If learners understand a fact or strategy, the information can be recollected accurately. They would be able to examine the concepts for sense-making through reflection, self-explanation and even rectify the concepts if needed (Kilpatrick, et. al., 2001; Sewell, 2002).

Learners often internalise or conceptualise meaning of constructions before they are able to articulate their understanding (Sewell, 2002). Evidence of conceptual understanding in the expression of associations among constructions and accounts thereof might thus not be clear. Indications of conceptual understanding might entail different representations of concepts used in different ways for different functions. What is important is the connection between the different representations, their similarities and differences. The quality of conceptual understanding is based on the fluency, depth and scope of the associations the learner is able to make between the connected concepts and strategies in solving problems (Kilpatrick, et. al., 2001). Easy-to-remember facts related to memorized methods and ideas can make it easy to perform mathematical calculations but may not elicit understanding. Proficiency in mathematical abilities is not acquired by mnemonic or memorized techniques (Semenza, et. al., 2006). Knowledge construction that occurred with understanding enables the learner to construct new ideas and to apply existing and new ideas to new and unfamiliar problem situations (as envisaged in the learning theories of Piaget and Vygotsky. Refer to pp. 38-42). The development of conceptual understanding assists learners in making connections between concepts and procedures. They develop their own procedures and confidently express arguments and reasons for the consequences in terms of the facts (Wood, et. al., 1993).

Skemp (1976:2) defines procedural understanding as “instrumental understanding”, i.e. “rules without reasons”. The author claims that learners and teachers who recall and use rules to explain mathematical concepts actually believe that they demonstrate understanding. Kilpatrick, et. al. (2001) maintain that learning procedures without understanding results in the application of only the acquired procedures, without modification or adaptation of the procedures to calculate in easier and smarter ways. Procedural understanding supports conceptual understanding and the investigation of likenesses and dissimilarities between written and mental calculation strategies for the four basic operations. Hiebert, et. al. (1996) maintain that procedural and conceptual understanding develop in support of each other. Learning with understanding allows learners to create procedures that they can recollect, adapt and apply accommodatingly when necessary. Mathematical knowledge and procedures should not be conveyed and demonstrated by teachers and then practised by learners until they develop proficiency in dealing with procedures that they have not constructed themselves.

Kilpatrick, et. al. (2001) further suggest that various classroom and out of school mathematical tasks require the performance of mental or written calculations involving algorithms. While some algorithms are important in their own right, learners could experience difficulties when they apply algorithms that they do not make sense of in isolated cases. Learning procedures without understanding requires consistent practice of the steps so that they are not forgotten. Learners who understand procedures may forget steps but are nonetheless able to reproduce them because the procedures are ingrained in their deep-level structures (Olivier, 1989). Incorrect procedures that have been instilled for years are difficult to remedy by instruction that does elicit understanding (refer to p. 49). Incorrect and inefficient procedures take time to be reconstructed. Conceptual understanding of algorithms allows learners to apply the knowledge to various problems that are related in various contexts. Conceptual understanding supported by effective procedures enables learners to learn more easily and effectively, with less likelihood of forgetting and making mistakes. Effective construction of understanding is thus based on the interplay between procedural fluency and conceptual understanding (Kilpatrick, et. al., 2001). In this study, the interplay between conceptual and procedural understanding implies that knowing the procedure $3 \times 0 = 0$ should be connected to the conceptual understanding, $3 \times 0 = 0 + 0 + 0 = 0$. Star (2005) emphasizes that the construction of procedural fluency and conceptual understanding is equally fundamental in the development of proficiency in learning mathematics.

In this study, I will analyse learners‟ and teachers‟ conception of abstract number concepts in terms of conceptual and procedural understanding, observing the assumed misconceptions they develop in demonstrating understanding. Researchers claim that learners create misconceptions, not because they do not understand, but rather "because they understand something else" (Davis, 1983; Wilcox, 2008:205). In section 2.9, I discuss further views on the development of conceptions that lead to misconceptions.

## 2.9. MISCONCEPTIONS AND ERRORS

Researchers (Olivier, 1989; Booth, 1984; Ginsburg, 1977) connect the terms misconceptions, errors and mistakes to misunderstandings. It appears that they suggest that misconceptions, embedded in cognitive structures, elicit errors or mistakes. Errors or mistakes can of course also occur because of absent-mindedness, haste and carelessness. Olivier (1989) asserts that misconceptions are embedded in the structures that learners conceptualise and thus connect with the acquisition of new concepts, which tends to influence new learning negatively because of resultant errors. Olivier (1989) maintains that misconceptions are fundamental convictions and opinions based in cognitive structures, which cause regular conceptual errors. Booth (1984) describes errors as the result of specific types of misunderstandings, which often occur through carelessness. For Olivier (1989), errors are incorrect solutions arising meticulously in recurrent applications in similar situations and, as such, are indicators of existing conceptual structures. Ginsburg (1977) argues that the mistakes that learners make originate from earlier teaching and make sense to learners. Davis (1983) points out that, misconceptions are often caused by the over-generalisation of previously correct knowledge – knowledge inappropriately extended to new learning, where it becomes invalid.

Sewell (2002) states that learners begin school with an existing reservoir of knowledge. This comprises their own elaborations of how they experience and make sense of the world (learning and teaching experiences in the classroom, in this study), as acquired from personal experiences, social events, the media, people and places. Sewell (2002:24) declares that Giambatista Vico, cited in her research article, could be regarded as the first "constructivist philosopher" because of the assertion he made in 1710 that "one only knows something if one can explain it". Vico probably suggested that language (spoken and/or written) plays a pivotal role in meaning construction, as maintained by social cognitive theory of Vygotsky (in Clarke, 2002). It is imperative that learners communicate the real world conceptions they have of some mathematical concepts, which could often be in disagreement with scientific views of the subject matter. These conceptions could be referred to as misconceptions or inaccurate beliefs. Preconceptions are the foundation for the construction of new conceptions. Thomas Cardinal Wolsey (in Sewell, 2002:24) provided an account of the persistent nature of misconceptions when he cautioned almost five centuries ago that one should "[b]e very, very careful what you put into that head, because you will never, ever get it out".

Learners are often content with what they know (Sewell, 2002). Simply telling them that their existing knowledge is deficient will not motivate them to adapt their incorrect conceptions. Misconceptions are only modified if they become inadequate or distorted and contradict personal convictions. Otherwise, they remain embedded in the cognitive structure with the existing knowledge, creating a flawed basis for future learning (Sewell, 2002). Learners are capable of overcoming misconceptions during traditional instruction and in demonstrating knowledge during a test, for example. But they often revert to the same incorrect preconceptions afterwards because of the misconceptions" persistent nature (Sewell, 2002). This used to happen often in the parrot-style studying and memorisation of dates and facts in subjects such as Mathematics and History, for example. Learners often imitated and (often incorrectly) reproduced rules transmitted by teachers in both primary and high schools (refer to pp. 2-3, Chapter 1). Copying and studying lists of multiplication tables until one knew it by heart and chanted the lists successfully did not guarantee understanding of the concept of multiplication and division (refer to p. 33).

Matz (in Olivier, 1989) has proposed that cognitive performance is guided by two levels of procedures. Surface level procedures are common arithmetical and algebraic algorithms or rules, while deep level procedures generate, adjust, organise and generally direct surface level procedures. The generalisation of number concepts is characteristic of the deep level, which disregards the particular concept and applies a different concept over the original concept. Surface level procedures might otherwise function effectively, but misconceptions are caused by deep level procedures. Surface level structures involve superficial knowledge connected to algorithms learnt by rote (De Jong & Ferguson-Hessler, in Star, 2005). Matz (in Olivier, 1989) claims that misunderstandings are persistent and resistant to change, despite teachers" attempts to remedy the misconceptions or learners" efforts to develop effective understanding. Misconceptions cannot just be wiped out of memory. Attempts to resolve them should involve situations where cognitive conflict occurs, leading to the realisation that something is wrong or about to go wrong. Learners execute the over generalisation of concepts intuitively. Teachers should be capable of making predictions regarding the misconceptions that learners might form in the development of essential concepts. They should assist learners to distinguish between such cases and emphasise the circumstances under which they are relevant.

Ginsburg (1977) suggested that teachers should have insight into learners‟ errors and misconceptions. If there were an understanding of the common principles of cognitive performances, teachers would understand that learners make mistakes, not through stupidity, but rather because the mistakes are sensible attempts to deal with concepts. The constructivist learning theory promotes learning from mistakes. Understanding arises through experiences and actions in the world. Effective understanding could develop from constructive reflections on experience and actions that often involve mistakes (WCED, 2006).

Sewell (2002) suggests that misconceptions could best be remedied when learners are presented with information that is in conflict with their inaccurate cognitive structures. In this way, learners get the opportunity to compare existing knowledge with the information presented. They either adjust or reconstruct the existing knowledge if the new knowledge is relevant, or reject the contrasting knowledge if it is irrelevant. New knowledge presented visually, in meaningful practical demonstrations, in constructive whole class or small group discussions (where teachers or learners play the role of devil‟s advocate) or in thought-provoking written tasks, could assist in eliminating misconceptions Sewell (2002). Attempts to address learners‟ misconceptions can be tedious, but teachers cannot allow learners to build repeatedly on inaccurate conceptions. Teachers who experienced inadequate training and lack significant content and pedagogical knowledge as a result could contribute to learners‟ misconceptions, which might cause further barriers to learning. If teachers have to struggle with unfamiliar content and uncertainties, they might unknowingly transfer their own incorrect conceptions to their learners (Shulman, 1986; Sewell, 2002; Ball, 2003; Kahan, et. al., 2003).

Olivier (1989) reported that misconceptions are opposed to change. Learners do not simply accommodate new concepts as and when required (Davis, 1983). They tend to incorporate new concepts into existing knowledge where the new concept could be misrepresented in relation to the existing knowledge (refer to p. 28). Constructive learning is necessarily dependent on previous constructive learning (Davis, 1983). For example, if learners are allowed to construct their own meaning of division with natural numbers as equal sharing they would possibly be able to relate and apply this understanding to new learning constructively, i.e. equal sharing with remainders that have to be shared (fractions) and even division by zero. Davis (1983) asserts that inaccurate new learning can often be the result of previous inaccurate learning (refer to p. 3, Chapter 1), and inaccurate learning is generally the consequence of previous constructive learning. This implies that incorrect construction of new knowledge could be the result of previous correct understanding. For example, knowing the rule for multiplication by zero and providing the solution $3 \times 0 = 0$ is correct. The inability to explain and illustrate the solution however implies that there was no previous constructive sense-making of the rule. Davis (1983) further reasons that, when learners offer an incorrect solution, they have probably attempted to answer a different question. It is then the teacher‟s responsibility to determine what question it is that they have actually attempted to answer (refer to p. 28). It is this notion of conceptualisation leading to misconceptions that I draw on in the analysis of learner and teacher data in this study.

The discussion in section 2.9 revealed that learners often make sense of the misconceptions they create due to previous correct or incorrect learning. The misconceptions are often deep-rooted and unyielding. Misconceptions are not evident of irrational thinking and reasoning; they are often the result of misinterpretations of the problems they have to solve.

The misconceptions are based on the intuitive structures that learners impose on new concepts to be learnt. The natural structures that learners impose on problems could be constructive but they could also be in conflict with the construction of new understanding. The intuitive structure imposed on problems should be adjusted to accommodate the new concept (Mulligan & Mitchelmore, in Ell, 2001). In section 2.10, I focus on the construction of unproductive cognitive structures that hinder the development of multiplicative thinking – an imperative basis for understanding calculations with zero.

## 2.10. COGNITIVE STRUCTURES

Mulligan & Wright (2000:18) suggest that the development of multiplication and division procedures should depend on the acquiring of an "equal-grouping (composite) structure". Ineffectual development of multiplicative structure could obscure learners" algebraic thinking in higher grades. The use of structure is imperative in organising and interpreting multiplicative conditions displayed in models, diagrams, tables and graphs (Mulligan, 2002). A structure can be regarded as multiplicative if two composite units are manipulated so that one of the composite units is distributed over elements of the other (Mulligan & Wright, 2000). Learners develop initial knowledge of multiplication and division by cognitively restructuring counting, addition and subtraction knowledge, and building on number word sequences, combining and partitioning. Addition and subtraction operations involve the application of equivalent sets as abstract composite units. This implies that the learner "focuses on the unit structure of a numerical composite, for example one ten, rather than on the unit items ten ones". The inverse procedures involved in multiplication and division are the basis of a "developmental model of composite structure" (Lutovac, 2008:32).

Teaching based on rules to develop understanding of multiplication and division by zero could prevent learners from constructing effective structures derived from previous conceptualisations of multiplication and division by natural numbers (Lutovac, 2008). Many learners in higher grades who do attempt to illustrate conceptual understanding of the concept of zero impose incorrect structures on problems. They impose the concept of subtraction as a kind of survival kit in the absence of effective cognitive tools (Polly & Ruble, 2009). If learners insist on using structures of repeated addition, sharing or grouping without composite units, they are not able to demonstrate effective conceptualisation of the concept of zero. The structure of composite units could be applied effectively in the display of conceptual understanding of calculations involving zero (refer to pp. 42-43).

Mulligan & Mitchelmore (in Ell 2001:12) report that "the intuitive model employed to solve a particular problem . . . does not reflect any specific problem feature, but rather the mathematical structure that the student is able to impose on it". A structure is based on the choice and use of a strategy. It establishes learners" views and understanding of numbers and operations with numbers. Learners" intuitive structures could account for strategy choice in problem solving as well as for the common occurrence of misconceptions. These intuitive structures could be useful and accommodate more sophisticated, abstract reasoning, or counteract the achievement of novel insights (such as multiplication and division by zero).

According to Mulligan & Wright (2000), the development and consolidation of the original structures based on repeated addition or subtraction reproductions and sharing representations are followed by the extension of understanding to include recognition of the concept of commutativity and the application of the inverse relationship between multiplication and division. The model of composite structure is dependent on the inverse relation between multiplication and division. The acquirement of knowledge of multiplication and division as inverse operations depends on the learner"s ability to develop both composite structure and commutativity. It also depends on recognition of the connection $p \times q$ where $p$ is the composite unit to be operated $q$ times (Mulligan & Wright, 2000). In our mathematics curriculum (South Africa. DoE, 2002:41; South Africa. DBE, 2010:13), Grade 5 and 6 learners should develop an understanding of zero as the additive inverse, for example $3 + 0 = 3$ and $3 - 0 = 3$. Some learners would probably not be able to apply the inverse relationship between multiplication and division to assert that $0 \times 3 = 0$ and therefore $0 \div 3 = 0$ if they were not explicitly exposed to learning experiences of this concept.

Ell (2001) asserts that the concept of structure is characterised by the difference in development between advanced and slow learners. High achievers have the ability to construct mathematical thoughts by making connections between prior knowledge and new knowledge. Less competent learners do not have the ability to collate detached chunks of knowledge in a meaningful way or to overcome misconceptions. Mulligan & Mitchelmore and Gray & Tall (in Ell, 2001) claim that high achievers construct effective structures built on existing knowledge to make sense. Low achievers construct inappropriate structures that prevent them from constructing new meaning making.

Mulligan & Wright (2000:18) reason that it is possible for learners to employ the same strategies for both multiplication and division, but to create and count "composite units from a known quantity" for division (for example, 12 † 3 = 4 means there are 3 groups of 4 and therefore 0 ÷ 3 = 0 means there are 3 groups of 0). The authors assert that evidence exists that young learners are able to effectively develop multiplication and division strategies. Teaching of multiplication and division concepts generally only takes place from Grade 2 or Grade 3. The study of Wilcox (2008) provides evidence that Grade 1 learners are able to make sense of the concept of zero and negative numbers. Teachers should unravel learners‟ intuitive knowledge and allow them to construct knowledge and understanding of new concepts by building on what they already know. Learners‟ intuitive structures should not be replaced by rules and procedures that separate the problem-solving process from real meaning construction (Wilcox, 2008). Skemp (1976) and von Glaserfeld (2011) emphasize that rules without motivation and the presentation of a series of symbols do not demonstrate conceptual understanding.

The discussion in section 2.10 served primarily as an indication that, regarding zero as *nothing*, together with rule-based learning and the application of ineffective cognitive structures in the absence of conceptual and procedural understanding, can lead to misconceptions concerning the concept of zero. Various instances of learners‟, teachers‟ and ancient mathematicians‟ confusion, uncertainty and inability to justify and explain their understanding of the concept of zero were emphasised. The literature reviewed predominantly accentuated the development of misconceptions resulting from a lack of conceptual understanding. Learners" inaccurate conceptions could be attributed in the main to the application of isolated, unfamiliar rules and procedures, and to their limited knowledge of zero as a number and its behaviour in calculations. These serve to hamper effective meaning construction. Effective practical models and constructive discussion and debate could result in positive reasoning, and the construction of conceptual and procedural understanding of the concept of zero. Allowing learners to engage in healthy, constructive discussions creates opportunities for involvement in a high level of mathematical discourse, thinking, reasoning, debate and sharing, in a non-threatening and relaxed environment. Learners should, however, have a sound understanding of the fundamentals, i.e. of basic operations and number properties, so as to use them in their thinking and reasoning about complex and advanced concepts. This practice requires teachers who have knowledge about teaching and for teaching effectively. In section 2.11, I focus on literature that emphasizes the knowledge that teachers need for effective teaching.

## 2.11. TEACHERS' KNOWLEDGE

Gelman & Gallistel and Hefendehl-Hebeker (in Wheeler & Feghali, 1983) maintain that a constructive teaching approach is required to assist learners in developing an understanding of zero as a number in its own right, i.e. appropriate teaching of the concept requires resourceful, innovative teachers. I consider the possibility, however, that most teachers have not had the opportunities to develop effective conceptualisation of the concept during their own previous learning and training experiences. I also remain mindful of the fact that the current mathematics curriculum and textbooks do not provide significant guidelines for teaching the concept of zero explicitly. In this section, I discuss research literature that draws attention to the content and pedagogical knowledge that teachers require to teach effectively. The intention is to connect the insights I gained from the literature to the data analysis concerning knowledge for effective teaching of the concept of zero in Chapter 5.

### 2.11.1. Teachers’ conceptions of the concept of zero

Research concerning teachers‟ abilities to solve division by zero problems is plentiful. For example, Tsamir, et. al. and Reys & Henry (in Quinn, et. al., 2008) have reported on primary and early high school teachers‟ difficulties with the concept of zero, especially with understanding zero as a number and division by zero. Quinn, et. al. (2008) describe an investigation involving Grade 4 to 8 teachers‟ understanding of division by zero. While some teachers had a good conceptual understanding of division by zero, the majority of teachers‟ understanding of the concept was limited or non-existent (Quinn, et. al., 2008). The study of Wheeler & Feghali (1983) revealed that most pre-service teachers were not able to solve division calculations with zero as a dividend successfully, and erred in calculations with zero as a divisor. Studies by Ball, Blake & Verhille; Even & Tirosh; Reys & Gouws and Tsamir et. al. (in Levenson, et. al., 2007) have all shown that pre- and in-service teachers are often uncertain about the solution in problems involving division by zero. Teachers who know that division by zero is undefined are often not able to explain the concept effectively. According to Quinn, et. al. (2008), research has shown that teachers lack substantial knowledge of the concept of zero. Teachers‟ misconceptions about zero should be identified so that their learning can be developed and supported through professional development. Wheeler & Feghali (1983) share this view, arguing that the development of knowledge regarding the concept of zero should happen at teachers‟ levels of understanding so that they can address the problem at the levels of their learners‟ development. Knowledge should be developed on the cognitive level (accept zero as an attribute for classification); conceptually (identify zero as a number), and computationally (insight into multiplication and division by zero).

Wheeler & Feghali (1983:154) surveyed primary school teachers‟ knowledge of the concept of zero, and claimed that the teachers concerned did not demonstrate sufficient knowledge of the concept of zero and would therefore not be able to teach the concept effectively. Teachers‟ knowledge of the significance and application of the concept of zero will not develop independent from training. Courses in mathematics education should focus on the development of the concept as well as the development of supporting learning and teaching materials, which the authors regarded as “almost nonexistent”.

Kahan, et. al. (2003) maintain that pre-and in-service teachers‟ understanding of key mathematical ideas or facts is inadequate and causes a barrier to learners‟ learning. Interestingly, student teachers who were mathematics majors did not outperform non-experts in mathematics in justifying why division by zero is undefined. Pre- and in-service teachers are often uncertain about the solutions in problems involving division by zero. Shulman (1986) complains that, while teaching matters such as classroom organisation, time allocation, lesson planning, etc. are addressed in research, studies considering the content taught in lessons, the questions posed and the explanations offered to assist learners in learning effectively, were rare. In his study, Shulman (1986:8) was especially concerned with addressing uncertainties about teachers‟ knowledge for teaching effectively, asking “What are the sources of teacher knowledge? What does a teacher know and when did he or she come to know it? How is new knowledge acquired, old knowledge retrieved, and both combined to form a new knowledge base?” The last question relates particularly to constructivist theories of knowledge acquisition and construction as portrayed in the cognitive and social constructivist theories of Piaget and Vygotsky (refer pp. 23-27 in this chapter).

Studies by Wheeler & Feghali (1983), Kahan, et. al. (2003), Quinn, et. al. (2008) and Levenson, et. al. (2007), for example mostly focussed on limitations in teachers‟ knowledge of zero as a number and division by zero. The authors proclaim that pre-service and in-service teachers in both primary and high school experience problems with the concept of zero. Researchers suggest that professional development programmes should empower teachers with the essential content and pedagogical knowledge and tools to teach the concept of zero effectively. The literature portrayed gaps in investigations of teachers‟ conceptualisation of the zero concept. In this study, although I also investigate teachers‟ conception of multiplication by zero, their conception of zero as an even number, and their knowledge of the history of zero, I could not locate any research reports concerning these aspects of teacher knowledge. As far as the strategies used by teachers to teach the concept of zero are concerned, Quinn, et. al. (2008:102) asked Grade 4 to 8 teachers to respond to the question, "Suppose you have a student who ask you what 7 divided by 0 is. How would you respond?" Some of the teachers suggested highly effective strategies to teach the concept while others offered meagre recommendations.

### 2.11.2. Content and pedagogical content knowledge

The studies of various researchers concern the knowledge that teachers need for teaching effectively (Shulman, 1986; Ball, 2003; Kahan, et. al., 2003; Ball & McDiarmid; Ball & Wilson; Ma; Hiebert & Lefevre in Kahan, et. al., 2003; Ball, et. al., 2008). Shulman (1986) was especially concerned with the manner in which student teachers adapt the knowledge they obtain in professional development courses to satisfy the content and pedagogical needs of learners they teach in their classrooms. The researchers mentioned above based their studies on Shulman‟s (1986) work concerning the content and pedagogical content knowledge needed to teach effectively.

Shulman (1986) raised concerns regarding the professional development of teachers. He asked about the sources of teachers‟ explanations, the decisions they make about the subject matter and their presentation thereof. He also identified as matters of concern teachers‟ questioning skills and their dealing with learners‟ misconceptions concerning the subject. Addressing these involved ascertaining teachers‟ existing knowledge, the origin and stages of this knowledge acquirement, the recovering of the knowledge and the assimilation of previous knowledge to accommodate and equilibrate new knowledge. Kahan, et. al. (2003) assert that mathematically strong teachers should be flexible enough to ask impromptu questions and to address unexpected statements or conjectures that arise in the classroom. Breen (in Kahan, et. al., 2003) maintains that mathematically skilled teachers are competent in realising the diverse potential embedded in learner responses. Kahan, et. al. (2003) suggest that the ability of seizing unforeseen opportunities to relate the content to bigger ideas is an element of pedagogical content knowledge.

Shulman (1986) focused especially on the transition from non-specialist student to beginner teacher, and on the transformation of subject matter expertise into structures understandable by learners. According to Ball, et. al. (2008), pedagogical content knowledge is characterised as bridging the gap between mathematical knowledge obtained through professional development and knowledge acquired in actual teaching practice. Pedagogical content knowledge is a combination of knowledge of content, learners and pedagogy, which demands specialised skills, reasoning, a positive attitude and insight into teaching and learning mathematics. Teachers should have the ability to sequence specific content for progressive teaching and learning. They should decide on constructive introductions, effective examples for content development, assessment, remediation and enrichment, and reflect on the teaching and learning process to improve teaching. Shulman (1986) argued that competent teachers should know how to deal with inaccurate content information in textbooks or among puzzled learners. They should know how to apply content proficiency to construct new explications, representations and clarifications. Teachers should also be versed in the use of correlations, figures of speech, examples, re-articulation and demonstration. Shulman (1986:8) was concerned with the "pedagogical prices" that were to be paid if teachers‟ competence in teaching was limited because of ineffective prior education or incompetence.

Pedagogical content knowledge goes beyond the mere knowledge of subject matter to the dimension of subject matter *for teaching*. This particular form of content knowledge embodies the aspects of content most appropriate to how it is taught. Knowledge *for teaching* includes the most useful forms of representation in the most regularly taught topics in syllabus. It entails the best ways of representing and formulating the subject to make it comprehensible to others. There is no single most powerful form of representation, so the teacher requires awareness of a range of alternative forms of representation. These forms can be derived from research or simply originate in the wisdom acquired through practice. Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult – the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons. If the preconceptions are misconceptions, teachers need knowledge of the strategies most likely to help reorganise learners" understanding. They also need an awareness of learners" existing knowledge structures and the relationship between the structures. Dealing with learners" misconceptions requires effective mediation in the Vygotskian sense (Hiebert, et. al., 1996; Steele, 2001; Wink & Putney, 2002; Clarke, 2002; WCED, 2006) by teachers with pedagogical content knowledge, i.e. knowledge *for teaching* (Shulman, 1986; Ball, 2003; Kahan, et. al., 2003; Ball, et. al., 2008).

Ball (2003), Kahan, et. al. (2003) and Ball, et. al. (2008) have built on Shulman"s ideas concerning the knowledge teachers need *for teaching*. Ball (2003) connects these ideas specifically to the teaching of mathematics with reference to the concept of zero, maintaining that mathematics teaching requires quality mathematical knowledge. Teaching mathematics entails a respect for the integrity of the discipline (Shulman, 1986; Ball, 2003) and learning situations where procedures are investigated and reasoned. The efficiency and meaningfulness of those procedures are deeply intertwined. Caring whether a method or idea could be generalised is a core mathematical value. Knowledge of mathematics entails more than knowing it oneself. Knowing mathematics sufficiently *for teaching* requires the ability to unpack ideas and make them accessible as learners first encounter them, and not only in their finished form (for example, a rule stating that, *If you multiply a number by zero, the answer is always zero* is in its finished form). Ball, et. al. (2008:389) define pedagogical knowledge as "knowledge of content and students and knowledge of content and teaching". This type of knowledge embraces an essential sub-field of *pure content knowledge unique to the work of teaching*, i.e. *specialised content knowledge*. This is different from *common content knowledge*, which is acquired by teachers as well as non-teachers. Pedagogical content knowledge is exclusive to teaching. It connects content knowledge and teaching practice. Pedagogical content knowledge implies the knowledge that teachers are expected to be acquainted with to teach mathematics effectively. In short, a combination of knowledge of content and knowledge of pedagogy is imperative for effective teaching.

Kahan, et. al. (2003) explicitly echo the social constructivist theory of Vygotsky by regarding the quality of discourse as a component of knowledge needed for teaching effectively. The discourse of a classroom includes all the ways of representing, thinking, talking, agreeing and disagreeing that occur within it, and is central to what students learn about mathematics as a domain of human inquiry with characteristic ways of knowing. Discourse involves both the way in which ideas are exchanged and what the ideas entail, i.e. who does the talking, what the talking is about, how the talking occurs, what is recorded and how ideas are appreciated. According to the NCTM (in Kahan, et. al. 2003), the discourse, i.e. the occurrence of mathematical communication, is shaped by the tasks in which students engage and the nature of the learning environment. In this context, Kahan, et. al. (2003) related knowledge *for teaching* to the development of language, one of the core principles of constructivism (refer to p. 41). Shulman (1986) asserted that reasoning, insight, understanding and skill to enhance effective learning should occur in everyday classroom situations. This notion of knowledge *for teaching* could be related to mediation, i.e. intentional and well-planned instruction aimed at assisting learners to make sense of new knowledge in the ZPD (refer to p. 42).

According to Ball (2003), because teaching involves cultivating learners" interest in mathematics, teachers have to be curious about and interested in mathematics. Learners" mathematical curiosity should interest teachers. Teachers require insight into learners" fascination with zero and the relation between this and the controversy around zero in the history of mathematics. Thinking properly about content knowledge requires going beyond knowledge of the facts or concepts germane to a subject field (Shulman, 1986). Teachers need not only understand *that* something is so. They should also understand *why* it is so: on what grounds its warrant can be asserted, and under what circumstances the belief in its justification can be weakened and even denied (Shulman, 1986; Kahan, et. al., 2003; Ball, et. al., 2008). Kahan, et. al. (2003:225) described the elements of mathematical content knowledge as (a) "a deep foundation of factual knowledge"; (b) "understanding of the facts and ideas in the context of a conceptual framework"; and (c) "the organisation of the knowledge in ways that facilitate retrieval and application". Kahan, et. al. (2003) exploited

Hiebert & Lefevre‟s idea (in Kahan, et. al., 2003) that knowledge has both procedural (corresponding to facts and ideas, which may include mathematical methods and algorithms) and conceptual (understanding and organising facts and ideas) components. Kahan, et. al. (2003) thus maintain that teachers should have an effective procedural and conceptual understanding of mathematical concepts (refer to pp. 43-46).

Kahan, et. al. (2003) view content knowledge as an important aspect of teachers‟ knowledge, but argue that teachers with extensive content knowledge are not necessarily the best teachers. To apply content knowledge effectively, teachers must find ways to relate the teaching material to learners‟ experience. Teachers with ineffective mathematical content knowledge will probably teach the way they have been taught in tertiary education programmes without connecting the knowledge to learners‟ interests and existing knowledge. This leads to unsuccessful learning. Ball, et. al. (2008) agree with Kahan, et. al. (2003); Ball (2003) and Shulman (1986), noting that a teacher with good content knowledge who does not consider learners‟ prior knowledge, misconceptions and how they acquire new knowledge, might not be an effective teacher. Interviews conducted by Ball, et. al. (2008) revealed the limitations of pre- and in-service teachers‟ knowledge of mathematics essential for effective teaching. Highly educated and knowledgeable teachers may not necessarily be able to explain or demonstrate meaningfully to learners why, for example you multiply by a fraction‟s reciprocal when dividing by that fraction. Dealing effectively with learners‟ unforeseen (correct or incorrect) generalisations often requires teaching for conceptual insight, which should go beyond knowledge of content.

Ball (in Ball, et. al., 2008) incorporated *horizon knowledge* into Shulman‟s (1986) content and pedagogical content knowledge. Teachers should have knowledge of the relationship among the various mathematical subject matters in the curriculum. Primary school teachers should be aware of how the mathematics they teach connects with and impacts on further learning. High school teachers should know how the mathematics they teach relates to mathematics taught in lower grades. This type of knowledge could prevent distortion of the mathematical content required for learning advanced concepts. It could also allow teachers in higher grades to address gaps in learners" thinking by applying basic concepts to develop understanding. In the present context, these capabilities could contribute usefully to the development of the concept of zero. This should include a focus on the behaviour of zero and properties related to zero, in order to build a sound foundation for the development of algebraic thinking. Teachers have to start preparing learners as early as possible to develop a concept of zero informally before moving on to formal meaning and content development (Levenson, et. al., 2007).

Teacher education programs should be research-based and draw upon growing research on the pedagogical structure of learner conceptions (that possible lead to misconceptions), on those features that make particular topics easy or difficult to learn. They should provide teachers with a rich body of prototypes, precedents and parables from which to reason. The ultimate test of understanding rests on the ability to transform one"s knowledge into teaching because “Those who can, do. Those who understand, teach” (Shulman, 1986:14). Teachers essentially need sound knowledge of the topics prescribed for their grade level in the curriculum. The question is however whether teachers" knowledge of mathematical topics is adequate for responding to mathematical requirements. Most importantly, teachers should be able to anticipate and address learners" emergent misconceptions (Shulman, 1986). Ball, et. al. (2008) conclude that the mathematics that teachers learn in college or university is not necessarily the mathematics that they teach in the classroom. For effective mathematics classroom practice, teachers need to make meaning of learners" mathematical thinking and functioning. They should apply appropriate teaching approaches to represent mathematics in influential, meaningful and constructive ways. Professional development courses should be relevant to authentic classroom practice and not merely linked to content knowledge. Mathematical content development courses should be based on learner performance and achievement, i.e. evidence of what learners can and cannot do in the classroom should inform course content in order to improve teachers" content *and* pedagogical content knowledge. Proposed solutions normally suggest that teachers should learn more mathematics, do extra coursework and become subject experts. Yet, an increase in the quantity of mathematics coursework might not necessarily improve the quality of learners" mathematical learning. Professional development programs should empower teachers with more than the curriculum content that they are supposed to teach. Teachers should be equipped to deal with the mathematical demands they are faced with in the classroom (Ball, 2003).

The knowledge that teachers require *for teaching* has been widely reported on. Various researchers have extended the original ideas of Shulman (1986) to characterise this type of knowledge. Shulman (1986) describes different types of teacher knowledge. His description of knowledge for effective teaching, i.e. content and pedagogical content knowledge, remains relevant to the discussion of findings in this study. Content knowledge (knowledge of the subject and its discipline) and pedagogical content knowledge (knowledge of the objectives of the subject and teaching approaches to improve learning) complement each other. Teachers require an understanding of how learners learn and an awareness of the barriers to effective learning. Ball (2003) has extended Shulman"s (1986) proposal of knowledge needed *for teaching* by specifically promoting *knowledge about mathematics* and knowledge *for teaching* mathematics effectively. She incorporates *horizon knowledge*, which requires knowledge of the interrelationships among mathematical concepts at different levels of mathematical development. Kahan, et. al."s (2003) depiction of knowledge *for teaching* emphasises the importance of the acquisition of both mathematical content knowledge and pedagogical content knowledge. Sound mathematical content development does not guarantee sound mathematical teaching strategies: teachers need effective communication skills to promote learner communication and interaction. Ball, et. al. (2008) maintain that pedagogical knowledge involves *knowledge of content and students and knowledge of content and teaching*. They depict knowledge *for teaching* as *specialised content knowledge* which differs from *common content knowledge* and is exclusive to teaching.

These researchers are all in agreement that professional development programmes should be related to the classroom environment and assist teachers in developing knowledge *for teaching* – content and pedagogical content knowledge for the improvement of teaching and learning. Progressive approaches to teaching and learning are advocated in the work of all the researchers canvassed in this section, with a strong emphasis on the constructivist learning theory. Knowledge of subject content, effective teaching approaches, conceptual and procedural understanding, learners‟ conceptions and constructive communication in the learning environment are all relevant to the framework proposed for discussing and generalising the results of this study.

**2.12. CONCLUSION**

The fundamental feature of this study is a commitment to constructivism as a means to develop knowledge construction and conceptual understanding and to address ideas leading to misconceptions.

The data analysis process in this study considers the essential influence of the processes of assimilation, accommodation, equilibrium and mediation on these activities. Misconceptions are bound to occur during the process of knowledge construction that entails the assimilation of unfamiliar concepts which do not fit into learners‟ existing cognitive structures. Teachers who developed misconceptions and experienced a lack of opportunities to conceptualise abstract concepts during their own learning experiences, might not be equipped with effective cognitive structures and mediation skills to deal with the cognitive conflict that learners might experience in dealing with abstract concepts.

Teaching and learning of the abstract concept of zero, can be complicated (Wheeler & Feghali, 1983; Ball, 2003; Levenson, et. al., 2004; Semenza, et. al., 2006; Levenson, et. al., 2007; Ball, et. al., 2008; Quinn, et. al., 2008). Without effective concept development, the difficulties that teachers experience with the concept of zero could become a barrier to learners‟ development of the concept. Efficient teaching requires confident, knowledgeable, enthusiastic, imaginative, creative, interrogative, inquiring, reflective and communicative teachers who are competent enough to predict possible mistakes and misconceptions in learners‟ knowledge construction.

The assumptions made about learners‟ and teachers‟ knowledge construction, the discussion of teacher and learner data production and analysis (Chapters 4 and 5) and the methodology presented in Chapter 3 are receptive to teachers‟ and learners‟ previous learning and teaching experiences concerning the development of the concept of zero, an abstract concept. Development of the concept is not prescribed in the mathematics curriculum but is imperative for laying a foundation for future effective algebraic thinking and reasoning.

Teachers who have the expertise, competence and skills to teach effectively are those who are equipped with powerful meta-cognitive, cognitive and pedagogical tools (Shulman, 1986; Ball, 2003; Kahan, et. al., 2003). These tools assist them in understanding learners‟ interests, experiences and conceptions. Teachers should assist learners in effectively constructing new knowledge in relation to their existing knowledge. Such teachers are concerned with the validity, depth and scope of the discipline and the most comprehensible and flexible ways in which they can make the subject matter available to their learners. Due to the limited duration of professional development courses teachers often do not develop knowledge of every aspect they are confronted with in their actual classrooms. Much of the knowledge that teachers should know *for teaching* effectively depends on experience, wisdom, and on-going interest in research and involvement in communities of practice regarding effective content knowledge and pedagogical content knowledge for teaching and learning.

# CHAPTER 3
# RESEARCH METHODOLOGY

## 3.1. INTRODUCTION

In this chapter, I map out the methodology used in the study and provide a justification for the various research methods employed in the production and analysis of data. I have adopted Gough‟s (2001) term „production‟ who maintained that data is not just sitting and waiting out there to be discovered and accumulated; the researcher produces and creates it through the activities implemented during the research process. Van Maanen (in Gough, 2001:5) suggested that doing research involves "fieldwork, headwork and textwork". „Headwork‟ involves issues concerning methodology, i.e. theories and aspects about the research process; „fieldwork‟ implies the methods applied in the data production and construction process and „textwork‟ entails validation of the „headwork‟ and „fieldwork‟. In the following sections, I discuss the research design in terms of the methodology appropriate to this study (headwork), the sites and sample selection, different research methods and data collection techniques and my role in the management of the research (textwork) and the data production techniques and data analysis strategies (fieldwork).

## 3.2. THEORETICAL CONTEXT

The study was performed in an interpretive orientation with the view that each individual has his/her own reality (Von Glasersfeld, in Brown, 2009) and constructs his/her own meaning of phenomena in the social world. The research was performed in the qualitative orientation with some quantitative elements in the analysis and discussion of data to reflect the effective interpretation of the research problem.

In this study, the quantitative analysis of data is embedded in the data collection process to gain understanding of the different levels of data analysis and to use statistical data to make inferences which inform the qualitative analysis and discussion of data (Creswell, 2003). The qualitative approach – a process of inquiry for performing investigations, making assumptions and generalisations and constructing a multifaceted, intensive depiction of teachers‟ and learners‟ understanding and meaning construction. The main concern was sense-making of the personal world of the experiences of human beings (Cohen & Manion, 2000) by observing learners‟ (and teachers‟) interactions, perceptions and expressions in constructing meaning in their own, authentic learning environments in order to provide detailed descriptions of their actions. Assumptions that are characteristic of the qualitative research paradigm include the researcher‟s involvement as a participant observer of events as they occur. The researcher fuses in with the social environment of the subjects and the incidents that he or she investigates with limited classroom intervention (Babbie, Mouton, Vorster & Prozesky, 2001). Data was collected since 2007 and the formal study was conducted over a period of three years – from 2009 to 2011. In one of the sites, I became more than a participant observer. I performed the role of mediator during one of the classroom support visits (as negotiated with the teacher and learners and because of my role as field worker) to assist learners in constructing meaning of the phenomenon under scrutiny.

The production of data in a qualitative study often involves more than one specific technique (Gough, 2001). Open-ended questions are posed to obtain views of subjects in order to construct broad and general categories and themes as evidence for solutions and generalisations abstracted and organized to derive theoretical conceptions in response to the research question (Gilham, 2000; Creswell, 2003). The data production techniques involved mostly interviews and participant observation. The case studies employed allowed me to explore events and activities in depth over a constant period of time (Creswell, 2003). The case studies entailed single units or multiple individual units in which variables were rigorously examined by interacting with the context/s of the case/s to understand events, actions and processes (Babbie, et al., 2001). The design principles (which also apply to other forms of qualitative research) in case studies focused on conceptualisation, contextual features, multiple data sources and analytical strategies (Yin & Stake in Babbie, et al., 2001).

Qualitative methodology generally involves inductive reasoning in interpreting and categorising data that emerge from various observations of informants‟ accounts or actions. The rich context-bound information that becomes known allows the researcher to realise differences and similarities and directs pattern recognition and the construction of themes or theories, which assist in describing and explaining the phenomena being investigated. The diverse nature of learner and teacher responses in questionnaires required "insight into different levels or units of analysis (Cresswell, 2003:16). I described events, actions and accounts precisely as they have transpired in order to inductively develop and construct new interpretations and theories. The intention was to identify emergent patterns from which I was able to produce generalisations (Connole, 1998; Babbie, et al., 2001; Danermark, Ekstrom, Jakobsen & Karlson, 2002).

Bosk (in Maxwell, 1992) reported that data production processes involving single field workers are subject to questions of validity and reliability. Qualitative studies should generate consistently valid findings so that policies, programs or predictions based on these studies are reliable. Qualitative research studies depend on various understandings and subsequent kinds of validity in descriptions, interpretations and explanations of their research encounters. In a qualitative approach various strategies are employed to guarantee validity and reliability of findings, i.e. triangulation, detailed field notes, member checks, peer review, reasoned consensus, etc. and to prevent misrepresentation of accounts (Babbie, et al., 2001). In this study, I applied triangulation and member checks to validate oral and written responses of the research informants. I also included learner and teacher interview data as far as possible in the reporting in order to invite the reader to review the validity of my arguments drawn from the data.

## 3.3. RESEARCH METHODS

This section entails descriptions of the different research methods employed to respond adequately to the research question in this study. I start with a description of the sampling process and the criteria I regarded as sufficient for the production of data. This research study was performed in three stages involving Grade 3 to 6 learners in Stage 1, in-service teachers on the Bachelor of Education Degree (BEd) and an Advanced Certificate in Education (ACE) course in Stage 2 and Grade 5 learners in Stage 3.

### 3.3.1 SAMPLING

According to Gilham (2000), case studies could involve individual or multiple cases. Individual cases consist of a specific group in a social setting while multiple cases involve two or more groups in the same context. A multiple case study could involve data collection in the same or different grades in different classrooms in the same or in different schools to obtain diverse types of evidence nested in the case context to answer the research question in the best possible way. This study consists of six cases involving learners in three schools and teachers as learners in two in-service professional development courses in the context of teaching and learning mathematics.

As mentioned in Chapter 1, the idea of investing learners" problems concerning the concept of zero occurred to me during my work as a fieldworker in a school-based mathematics development project in the Western Cape since 2004. My engagement with schools provided me with opportunities to access primary school classrooms. During 2006, I established good relationships of trust with learners, teachers and principals in a different three-year school-based project. My proposal for conducting research in some of the classrooms was accepted. In 2007, I started collecting data in a grade 6 classroom because I was mostly engaged with Grade 3 and 6 learners and teachers in the project. I also assumed that, if Grade 7 learners lacked knowledge of the concept of zero the grade 6 learners would probably display the same limited knowledge. In 2008, I decided to conduct investigations in two schools involving Grade 3 to 6 learners. The two schools I targeted have multi-grade classes with manageable numbers of learners and highly dedicated teachers. The sample therefore consisted of four grades but only two classrooms which was beneficial in terms of time spent on data collection. In one of the schools I selected the grade 3 and 4 combined class and in the second school the grade 5 and 6 combined class. It was in one of the selected schools that I originally observed the grade 7 learners" inability to solve mental multiplication and division by zero problems.

I performed a pilot study (Stage 1) during 2007 and 2008 in the two rural Afrikaans medium multi-grade schools. The aim was to trial, improve and supplement the original instrument. The schools were two of thirty-eight schools engaged in a three-year project conducted from 2006 to 2008. The class teachers in the two cases were both enrolled on an ACE course. The schools comprised two of twelve schools that I supported as a field worker and mathematics education specialist. In one of the schools, eleven Grade 3 learners and fourteen Grade 4 learners in a multi-grade class were involved in the study in 2008. In the second school, twenty-three Grade 6 learners in a mono-grade (single grade) class were involved in the study during 2007. Twenty-six Grade 5 learners and eleven Grade 6 learners in a multi-grade class at this school participated in the study in 2008. The class teacher in this school taught these learners in 2007 and 2008.

In 2009, I was employed in a mathematics project at a university in the Eastern Cape and enrolled for the Masters in Education course. I decided to continue with the research in two further stages involving teachers and learners. The idea of including teachers in the study was prompted by the responses of the grade 4 learners and the comment from a teacher I encountered in 2004. The two learners that I interviewed during 2004 both blamed their teachers in previous grades for the misconception they developed related to subtraction involving zero as a digit. My decision to involve Grade 5 learners in the Eastern Cape in Stage 3 of the data production process was based on validity and reliability as well as geographical representation, i.e. two cases of Grade 5 learners in two different provinces represented in the sample to make a comparison between the learner performance in the Western and Eastern Cape if necessary.

Stage 2 of the study involved eight teachers on a BEd Mathematics (in-service) and thirty-nine second year in-service teachers on an ACE course. These teachers were teaching in mostly isiXhosa medium town, rural schools and extreme rural schools across the Eastern Cape. Rural schools are situated in the countryside outside or in small towns. Extreme rural schools in the Eastern Cape are situated many kilometres away from towns (in the Western Cape, schools situated in the countryside more than about 50 km from Cape Town, were regarded as rural for the purpose of geographical location in the project). The teachers attended the accredited Mathematics courses during three contact sessions per year that occurred during the school holidays. The BEd teachers had completed a two-year ACE course in 2008 and qualified to convert the ACE course into a BEd qualification during 2009. The course consisted of isiXhosa-, English- and Afrikaans-speaking teachers teaching in different grades across the Foundation Phase (FP), Intermediate Phase (IP), Senior Phase (IP) and Further Education and Training (FET) Phase. The teachers had an average of fifteen years teaching experience with the minimum being ten and the maximum twenty-two years. I considered involving a local teacher in the Stage 3 data production process, but this was not feasible as the medium of instruction was isiXhosa. The research required learners to use verbal and written language to describe, explain, justify and debate ideas. I do not understand isiXhosa and the data collection techniques required that learners express themselves in their mother tongue. An advantage of conducting the research in an English medium classroom was that it saved time and labour with translation. The majority of ACE teachers were isiXhosa-speaking – only one teacher was Afrikaans speaking. The average years of teaching experience of the ACE teachers was eleven years with the minimum three and the maximum forty-four years. Statistics of BEd and ACE teachers‟ qualifications obtained in one of the written questionnaires are shown in Table 3.1 and Table 3.2 below.

| **Subject** | **Teaching grades** | **Teaching experience** | **Highest Qualifications** |
|---|---|---|---|
| **A** | 4 & 5 | 17 | ACE |
| **B** | 7 | 12 | ACE |
| **C** | 7, 8 & 9 | 11 | ACE |
| **D** | 7, 8 & 9 | 10 | ACE |
| **E** | 7, 8 & 9 | 17 | ACE |
| **F** | 8 & 9 | 17 | ACE |
| **G** | 8 & 9 | 22 | ACE |
| **H** | 9, 10 & 11 | 13 | B. Comm. |

Table 3.1: BEd teachers‟teaching experience and qualifications

| Subject | Teaching Grades | Teaching Experience | Highest Qualification | Subject | Teaching Grades | Teaching Experience | Highest Qualification |
|---|---|---|---|---|---|---|---|
| **1** | 1 | 7 | JPTD | **21** | 4, 5 & 6 | 3 | M+3 |
| **2** | 10 & 11 | 11 | BEdHons | **22** | 5 & 6 | 15 | SPTD |
| **3** | 4 | 21 | M+3 | **23** | 7, 8 & 9 | 3 | SPTD |
| **4** | 7 | 6 | M+3 | **24** | 10 | 3 | SPTD |
| **5** | 5 & 6 | 15 | BEd | **25** | 6 | 21 | M+3 |
| **6** | 5 & 6 | 7 | SPTD | **26** | 5 & 6 | 10 | Std. 10 |
| **7** | 3 | 6 | M+3 | **27** | 6 | 17 | BTech |
| **8** | 7 | 4 | SPTD | **28** | 7, 8 & 9 | 7 | xxx |
| **9** | 7, 8 & 9 | 13 | SPTD | **29** | 4, 5 & 6 | 10 | SPTD |
| **10** | 4, 5 & 6 | 10 | PTD | **30** | 6 & 7 | 8 | SPTD |
| **11** | 3 | 6 | xxx | **31** | 5 | 13 | M+3 |
| **12** | 7 & 9 | 6 | SPTD | **32** | 4, 5 & 6 | 15 | FDE |
| **13** | 4 & 5 | 18 | SPTD | **33** | 4 – 9 | 21 | BEd |
| **14** | 8 | 44 | M+3 | **34** | 8 & 9 | 5 | SPTD |
| **15** | 7, 8 & 9 | 14 | SPTD | **35** | 4, 5 & 6 | 7 | ACE |
| **16** | 4, 5 & 6 | 12 | M+3 | **36** | 3 | 13 | FDE |
| **17** | 4 | 16 | M+3 | **37** | 7, 8 & 9 | 18 | SPTD |
| **18** | 7, 8 & 9 | 5 | SPTD | **38** | xxx | xxx | xxx |
| **19** | 6 | 7 | FDE | **39** | xxx | xxx | xxx |
| **20** | 3 | 7 | SPTD | | | | |

**KEY:**

JPDT: Junior Primary Diploma in Teaching
PTD: Primary Teachers' Diploma
SPTD: Senior Primary Teachers' Diploma
FDE: Further Diploma in Education
Std. 10: Matric

**KEY:**

ACE: Advanced Certificate in Education
M+3: Matric + 3 years teacher training
BEd: Bachelor of Education Degree
BEdHons: BEd Honours' Degree
B.Comm: Bachelor of Commerce Degree
xxx: No information provided

Table 3.2: ACE teachers‟teaching experience and qualifications

The director of the mathematics project I was employed in accompanied me on my first visit to the Eastern Cape school to investigate and negotiate the possibility of conducting research in a grade 5 class. I was granted permission by the principal, class teacher and parents. I pursued with Stage 3 of the study in a grade 5 English medium class involving thirty-four learners. The grade 5 class teacher co-facilitated the initial data production process, which involved the mental calculation tasks.

This study involved six case studies with five opportunity samples that arose in my field of work. The Stage 1 learners (consisting of three case studies) were in project schools in which I was assigned as mathematics facilitator in the Western Cape and the Stage 2 teachers (consisting of two case studies) were involved in the teacher development courses I was engaged in at a university in the Eastern Cape. My involvement in the school project and teacher development courses provided uncomplicated access (opportunities) to the sites and subjects in the study. The fourth sample (Grade 5 learners in Stage 3) was purposefully selected based on the location of the school and the language of teaching and learning. The eight BEd teachers in Stage 2 participated in a focus group unstructured interview while three Grade 5 learners in Stage 3 were randomly selected to partake in a learner focus group semi-structured interview. Two teachers at the school suggested three learners other than the ones I selected. The teachers did not regard the learners as „mathematically competent". I accepted their suggestion in part by replacing the first two learners but included the third learner that I originally selected.

### 3.3.2. DATA PRODUCTION PROCEDURE

Creswell (2003) maintains that data production processes nowadays are not so much about quantitative versus qualitative approaches but rather about a balance between the approaches and how they are used to supplement each other to answer the research question effectively. The participants in this study engaged in responses provided in multiple instruments that required solutions to closed-ended questions as well as written elaborations, illustrations and demonstrations for explaining responses to the closed-ended questions. The closed-ended questionnaires were implemented to obtain evidence for the assumption that learners struggle with the concept of zero while the open-ended tasks were used to establish why and what the difficulties are that learners experience with the concept.

In 2007, I spent one period with a school psychologist in the grade 7 class of one of the schools in Stage 1 of the data collection period. After the psychologist granted me permission to use the mental calculation instrument, I discussed and negotiated my intentions to conduct classroom research at the school with the principal, the grade 6-class teacher and learners. I conducted the mental calculation speed test in this class during my last school visit in 2007 tests (refer to p. 11 for a discussion on the data collection tools used in this study). I assessed the results off-site and realised that I would need data concerning learners" thinking and reasoning about the concept I intended to investigate. I developed an additional questionnaire requiring written elaborations of learners" conceptualisation based on the study of Van den Heuvel-Panhuizen (2001). In 2008, I implemented the mental calculation speed tests in the same teacher"s classroom who then taught a grade 5 and 6 multi-grade class. Feedback was provided to learners during a follow-up visit and they completed the written questionnaire in groups after I explained why I needed their written accounts. The learners were asked to discuss and solve the problems and to explain their solutions as if they were addressing someone who had no knowledge of mathematics at all.

At the beginning of 2008, I negotiated the research plan with the second school. The principal asked the parents for permission to take photographs in the grade 3 and 4 class. The class teacher co-facilitated the implementation of the mental calculation speed tests and offered to assess the learners" solutions. During my next visit to this school, the teacher and I provided the learners with feedback. Their incorrect solutions to problems involving zero were emphasized. We decided to engage in a lesson developing understanding of multiplication by zero because most of the learners solved the problem $4 \times 0$ incorrectly. The learners were requested to work in groups and to make drawings of their understanding of problems involving multiplication by natural numbers and then by zero. When I realised that they were not able to present an understanding of multiplication by zero problems, I intervened and assisted them in developing the concept through a mediated teaching and learning experience. The understanding developed in this process was consolidated by allowing learners" to physically model multiplication problems to demonstrate understanding. I photographed this learning and teaching process, which took about one and a half hours.

The principal of the school in the Eastern Cape mentioned that the school had lost their mathematics teacher in 2008 and the current Grade 5 class teacher was not a mathematics expert. The principal appealed to the project for assistance and I committed myself to assist the grade 5 learners in developing basic number sense during 2009. During the first lesson in this classroom, I engaged the learners in „fun" number activities. The learners enjoyed the activities, which assisted in developing a relationship with them. They completed the mental calculation speed tests during this lesson. I observed that the classroom culture was not well developed so during the following lesson, I addressed co-operative learning principles. The written elaboration questionnaire required learners to discuss and work collaboratively while completing the tasks in groups. This questionnaire was implemented during the third visit to the school. The focus group semi-structured interview was conducted during the next visit. The principal and class teacher agreed that I conduct the interview at the project offices because the learners at the school were rather noisy and the school did not have an appropriate venue.

The research process with the teachers was embedded in the course program. Teacher data was collected during teaching and learning periods ascribed for teacher development of the whole number concept. I included the development of the concept of zero in the learning and teaching periods. The data collection and the focus group unstructured interview were conducted during two ninety-minute teaching periods and the teaching of the concept of zero occurred in a third period. For the ACE teachers data collection and teaching of the concept of zero took place during two ninety-minute periods. Both groups of teachers had to complete an assignment involving implementation of the written questionnaires requiring written solutions and elaborations for subtraction, multiplication and division with zero in their own classrooms. Their learners had to complete the tasks without prior teaching of the concept of zero and teachers had to provide written reflections on the process. Responses to this assignment included accounts of learners" understanding up to the Further Education and Training Phase (FET) Phase and serves as additional data.

The qualitative research design in this study involved the implementation of various data production instruments. As mentioned before, I was not involved in a formal research course during the initiation of the research project. I relied on my own insights concerning the data tools required to investigate learners" conceptions regarding the concept of zero effectively. I realised that various subjects from distinct yet similar contexts and multiple data collection techniques and instruments were imperative in gaining evidence to establish common factors from which I could draw valid inferences and conclusions to construct generalisations.

The methods I employed involved participant observation, questionnaires, practically-based learner tasks, an intervention session, interviews and photographs. Participant observation occurred mainly in the grade 3 to 6 classrooms. Teachers completed the questionnaires individually while learners were involved in individual and group work. The illustration below (Figure 3.3) shows the multi-stage data production process with an indication of the data production period, the types of instruments (numbered 1-3), the nature of the instruments and the number of participants.

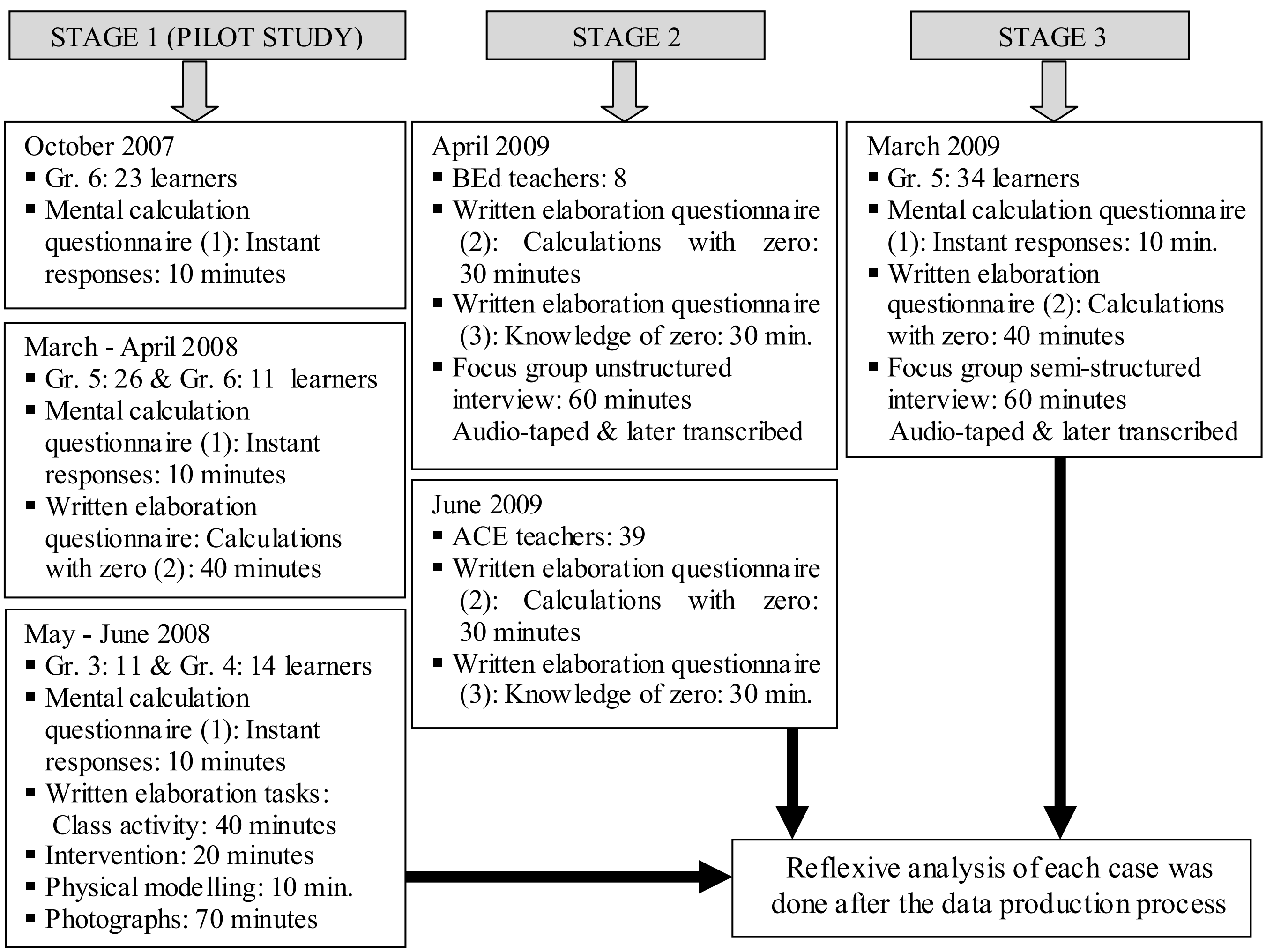



Figure 3.3: Illustration of the data production process

### 3.3.3. DATA SOURCES

Stake (in Creswell, 2003) asserts that data production in case study research entails the collection of detailed information through the use of multiple data sources "over a sustained period of time" (Creswell, 2003:15). The mixed method approach involves both quantitative and qualitative methods to produce data so that the results of one method inform the results of the other to "provide insight into different levels or units of analysis" (Creswell, 2003:16). Data production procedures are coexisting and assembled to offer broad analysis and interpretation of the problem investigated. In this study, I employed closed- and open-ended questionnaires, a practically-based learner task and demonstration, participant observation, photographs and focus group interviews as discussed below.

**a) Questionnaires 1-3**

*(1) The mental calculation questionnaire: Instant responses*

I was originally unsure about labelling the mental calculation speed test that required instant recall of basic calculation facts as a questionnaire because of my understanding that questionnaires are normally associated with surveys involving the broader public on data production of social issues, for example opinions on municipal planning of roads or housing development (Davies, 2007:82). However, according to Irwin (2011), a questionnaire entails a sequence of questions on paper that require written and not oral responses. Questionnaires could be used for qualitative and quantitative data production and the design of the instruments could be done flexibly, i.e. questions could be changed according to assumptions made concerning the research problem and how they could be measured. The kind and quality of responses to closed-ended questions are often controlled by the researcher with limited options for participants (Irwin, 2011:5). In this study, the closed-ended problems requiring numeric solutions on calculations had a specific focus and responses were recorded in spaces provided. The instrument was implemented in Grade 3 to 6 classrooms and individual learners supplied their names, grade and age on the questionnaires. The results obtained from the mental calculation questionnaires were integrated with those of the written elaboration questionnaires performed by Grade 3 to 6 learners to inform the assumption that learners struggle with understanding of multiplication and division by zero.

The original mental calculation speed test template employed by the school psychologist at one of schools involved in this research study consists of four speed tests. Each section consists of thirty close-ended basic number calculations involving 1- and 2-digit numbers to be solved mentally. Basic addition and subtraction questions were printed on one side of the page with multiplication and division calculations printed overleaf. Each section was timed – learners had to complete as many of the thirty questions in each section in one minute. After one minute, they put down their pens and started with the next section on instruction by the facilitator. After completion of the four sections, the psychologist asked for volunteers who wanted to have their responses assessed. Feedback was given to individual learners and areas for improvement were highlighted. The same tests were implemented in different classrooms on a regular basis and learners would attempt to improve their results each time they performed the tests. No formal intervention was conducted. At that stage the school intervention programme in the Western Cape focused on the development of learners‟ basic calculation skills after Grade 3 and 6 results in annual numeracy and mathematics systemic tests revealed that learners lacked basic mental and written calculation skills. I adapted the template to include only the multiplication and division tasks (see Appendix 1) but conducted the exercise in the same way as the school psychologist has done. The grade 3 to 6 learners in this study were timed while attempting to complete to complete as many multiplication and division questions as possible in one minute per section. Responses to two questions, $4 \times 0$ and $0 \div 7$ were used to provide evidence of learners‟ abilities to solve multiplication and division by zero problems.

*(2) The written elaboration questionnaire: Calculations with zero*

This instrument involved the provision of responses to closed- and open-ended questions. The participants had control over what and how they wanted to respond on the open-ended questions while responses to the closed-ended problems were controlled (Irwin, 2011) by the concepts that I regarded as measurable for the purpose of this study. Open-ended questions are generally employed in the production of qualitative data and elicit diverse responses. The responses often provide interesting variations which could be categorised in different themes in the analysis, integration and interpretation of the data (Irwin, 2011).

After assessment and reflective analysis of the grade 6 learners‟ responses in 2007, I realised that I required data in which learners illustrate their thinking and reasoning processes concerning multiplication and division by zero. While I was preparing for a workshop presentation on the intuitive structures that learners apply in calculations, I consulted the book of Van den Heuvel-Panhuizen (2001). I was delighted when I encountered the research study conducted by the Tal Team of the Freudenthal Institute (p. 18, Chapter 2). In their study, Grade 6 learners illustrated procedural and conceptual understanding of division by zero through constructive classroom discussion. I decided to develop a questionnaire involving the tasks that the grade 6 learners (Van den Heuvel-Panhuizen, 2001) had to solve and discuss (see Appendix 2). I included the tasks $1 - 0 = \square$, $0 - 1 = \square$, $1 \times 0 = \square$, $0 \times 1 = \square$, $0 \div 1 = \square$ and $1 \div 0 = \square$. I inserted a space following each closed-ended calculation to provide open-ended written elaborations of the conceptualisation of the concepts. Grade 5 and 6 learners in this study were requested to discuss and solve the problems in their groups. They had to provide explanations of their thinking and reasoning processes in such a way that somebody (an outsider) who did not know mathematics would be able to understand the problems. I assumed that co-operative learning groups would supply richer and more detailed explanations than individual learners. Individuals in the groups recorded the solutions and explanations on behalf of each group. The learners‟ included the grade, group number and the group members‟ names on the questionnaires. The same questionnaire was completed by individual BEd and ACE teachers. The teachers completed the questionnaire individually and anonymously but provided information on the grades they teach. They were requested to explain their conceptual understanding of the problems in a manner that learners would understand. Teachers also had to implement the questionnaire in their own classrooms as an assignment.

*(3) The written elaboration questionnaire: Knowledge and teaching of the concept of zero*

In 2009, I engaged with various research reports concerning learners" and teachers" understanding of the concept of zero. In a study by Wheeler & Feghali (1983), teachers were expected to display knowledge of zero as a number in its own right. The tasks they completed involved counting back and forwards to determine whether they included zero in the counting sequences and a partitioning task, i.e. a problem about two fishermen who caught five fish between the two of them. Teachers were required to record the number combinations that represent the possible number of fish that could have been caught between the two fishermen. They were also requested to elaborate on the question "What is zero?" Levenson, et al. (2007) investigated learners" knowledge of the parity of zero, i.e. zero"s quality as an even number. I decided to develop an open-ended teacher questionnaire (Appendix 3) based on the concepts investigated in the studies of these researchers. I included the questions and instructions:

1. *What is zero?*
2. *Is zero even or odd? Why?*
3. *Count backwards and forwards:*
    *(a) count back in 1s from 10*
    *(b) count back in 2s from 20 and*
    *(c) count forwards in 6s up to 30*
4. *If two cricket players scored 5 runs, what is the number of runs they could possibly have scored between the two of them?*
5. *What is the origin of zero in the history of numbers?*
6. *How do you teach the concept of zero?*

I considered the counting tasks as relevant to the understanding of the concept of zero. Zero is a counting number and should be included in counting sequences involving multiples of whole numbers, for example 0; 6; 12; 18; . . . Counting in sixes, for example on a number line starts from 0; not from 6. But is zero a multiple of 6? If we argue for example, that 12 is a multiple of 6 because $2 \times 6 = 12$ and $12 \div 6 = 2$, without a remainder, then 0 is a multiple of 6 because $0 \times 6 = 0$ and $0 \div 6 = 0$, without a remainder.

I decided to include Question 5 based on Ball‟s (2003) claim that knowledge about mathematics entails knowledge of the character of mathematics, i.e. its origins, development and its exactness (refer to Chapter 2, p. 41). Based on my encounter with the grade 4 learners in 2004, who blamed their previous teachers for the misconception they developed concerning subtraction involving zero, I assumed that teachers mostly teach the concept of zero by the transmission of uninformed rules. I therefore included Question 6 in the questionnaire to find out if and how teachers teach the concept. The BEd and ACE teachers completed the questionnaire individually and anonymously. The questionnaire required of teachers to state their teaching grades, number of teaching years and their highest qualification (see p. 6 in this chapter).

**b) The written elaboration task: Class activity**

The grade 3 and 4 learners in this study could be regarded as an instrumental case study (Stake, 1995). The case was used to provide insight into the cognitive structures that younger learners apply to multiplication by zero. I expected that the learners would use their own intuitive strategies to demonstrate understanding and was not disappointed. Not one of the learners used a rule to describe multiplication by zero. The intervention in this school was more intense. I spent more time at this school than in the second school in Stage 1 of the data production process. The school is situated about 70 km from. I visited this school twice per term while the second school, which is about 400 km from Cape Town, received one visit per term.

After the completion of and feedback on the mental calculation questionnaire (1) the learners were requested to demonstrate understanding of multiplication by zero with drawings. I recorded tasks such as $4 \times 1 = \square$; $4 \times 4 = \square$; $4 \times 3 = \square$ and $4 \times 2 = \square$ on the writing board. The number sentences were recorded randomly to prevent learners from solving the problems through pattern recognition. The expectation was that they would illustrate their thinking and reasoning authentically to display the cognitive structures they imposed on the problems. The problem $4 \times 0 = \square$ was included after the learners solved the problems involving only natural numbers. The grade 3 learners started with problems resulting in multiples of 2; the grade 4 learners with problems involving multiples of 3 and the whole class engaged in problems resulting in multiples of 4. This decision was based on the nature of teaching in multi-grade classes, i.e. differentiated learning and teaching to accommodate the different levels of learner development. The learners worked in co-operative learning groups in their respective grades. One learner per group recorded the group"s accounts on A3-sized paper with a marker pen. Thereafter, the learners" work was displayed on the walls of the classroom (refer to Figure 4.2, p. 4, Chapter 5). This practice gives learners the chance to develop a sense of pride in their work and a sense of ownership of the classroom environment.

**c) The classroom intervention tasks**

Mouton (1998) claimed that educational programs often involve continuous and sustained interventions, which constitute a different category or unit of analysis because they are more structured and patterned. The learners and teachers in the project schools I have supported in the Western Cape were comfortable with my presence and regular interventions in their classrooms. Co-teaching and demonstrations were negotiated with teachers. Interventions were conducted according to the content and pedagogical content needs arising during lessons and not according to a strictly structured plan. There was however an overall year and project phase plan for school-based support. These events included workshops, cluster meetings, whole school development sessions and conference presentations.

When the grade 3 and 4 learners were stuck and not able to demonstrate understanding of multiplication by zero, I negotiated with the teacher that I conduct an intervention to assist the learners in developing understanding of the concept. I used the calculations recorded on the writing board, i.e. $3 \times 1 = \square$; $3 \times 4 = \square$; $3 \times 3 = \square$; $3 \times 2 = \square$ and $3 \times 0 = \square$. I drew frames empty frames next to each calculation to represent composite groups of objects added repeatedly so that the groups of objects represented countable units as suggested by

Steffe (in Mulligan & Wright, 2000). Individual learners were requested to fill in the number of objects in each frame (refer to p. 5, Chapter 4).

**d) Participant observation**

Simpson & Tuson (in Gillham, 2000:45) claimed that observation entails three aspects, i.e. "watching what people do; listening to what they say and sometimes asking them clarifying questions". The authors further asserted that observation is the most straightforward manner of data production because it captures people‟s actual actions, which could reflect in data evidence. Observation allows direct entry to happenings or relations concerning the phenomenon that is investigated. Observation accounts have the potential to present rich, comprehensive depictions of subjects‟ actions in their natural settings. It is however vulnerable to prejudice. Observation sometimes entails the observer‟s opinions and not the actual actions of subjects. Participant observation means being involved in the actions of the subjects in their own socio-cultural contexts. The participant observer performs a different role from that of the detached or structured observer in the observation process, i.e. the role of a co-participant within the natural context.

This study was initiated by an observation of learners‟ incorrect responses to $4 \times 0$ and $0 \div 7$. This observation serves as evidence of the powerful nature of observations – an entire study is based on the observation of two incorrect responses. My role as participant observer involved observation of Grade 3 and 4 learners‟ practically-based written elaboration of multiplication by natural numbers and zero and the group work written elaboration tasks performed by Grade 5 and 6 learners during Stages 1 and 3 of the data production process. The observations that I conducted by photographing Grade 3 and 4 learners‟ actions provided me with rich, authentic data that I could use to answer various aspects of my research question. I seldom had time to make extensive field notes other than the drawing I created of the grade 5 classroom setting in Stage 3 (see Appendix 4). I often became too involved in the actions of participants because of the dual role I fulfilled – that of researcher and co-facilitator (Babbie, et al., 2001). The original observation that I made in the Stage 3 Grade 5 classroom concerning learners‟ behaviour, influenced my decisions to spend time in the classroom to develop a culture of learning by addressing social behaviour and aspects of developing a classroom atmosphere conducive to learning before I could start with the data production process.

The techniques of photographing and physical modelling were both part of the observation process. I discuss each of these data production techniques below.

**e) Photographs**

Du Toit & Sguazzin (in Togo, 2009) declared that photographs are generally used to highlight the experiences of people. Visual materials such as photographs allow opportunities for interpretation that could provide the researcher with understanding of how people construct meaning of phenomena. The reader is however at liberty to interpret the photographs in his or her own manner. Photographs could be perceived as intense text in which the principle for meaning construction is recorded (Walker & Clarke in Togo, 2009). The photographs used in this study illustrated the grade 3 and 4 learners" understanding of multiplication by zero.

The availability of digital cameras has enabled the common practice of taking photographs during school visits and classroom support. Learners" behaviour in classrooms was not influenced by the camera as they were used to this practice. The visual material was included in project reports and presentations to inform various stakeholders about the activities in project schools. I have also used photographs because of the economic manner in which data could be produced. Photographs provide authentic evidence of occurrences and behaviour. Instead of recording lengthy field notes, I was able to capture data in a reliable, complete and exact way in the grade 3 and 4 classroom, which provided insight into the existing cognitive structures and the meaning they constructed of new knowledge in their representations. The use of this form of technology also reduced time-consuming tasks such as the collection, copying, scanning and storing of large sheets of paper. It is not a laborious task to download and insert pictures into documents. Because of the evidence I have captured of learners" work, I did not have to remove their original work from the classroom with the possible risk of not returning it.

**f) Physical modelling**

Babbie, et al. (2001) suggested that qualitative research involves creativity in studying the world in order to answer the research question adequately. After I established that the grade 3 and 4 learners constructed meaning of multiplication by zero during the mediation process involving composite wholes, I decided to consolidate and assess their understanding. The learners were requested to demonstrate their understanding of the multiplication problems physically using groups of learners. The teacher and I were keen to observe how they would deal with multiplication by zero since the physical modelling would imply a diversion from the use of composite wholes used in the mediation process. The groups of learners first demonstrated understanding of multiplication with natural numbers and then demonstrated understanding of multiplication by zero (refer to p. 6, Chapter 4). The learners" physical modelling of the problems served as consolidation of their understanding.

**g) The focus group interviews**

Van der Mescht (2008) maintained that the data produced from interviews is normally the primary source of data. Even if researchers employ various data production instruments, written data would be used as enrichment and complementary to the questionnaire data. The two focus groups in this study involved the eight BEd teachers in Stage 2 and three Grade 5 learners in Stage 3 of the data production process. I chose to use the BEd teachers because they were a smaller group than the ACE teachers. I also assumed that the BEd teachers would probably have more knowledge about the concept of zero because they have completed the ACE course in 2008.

I regarded three learners as representative of the six groups in the grade 5 (Stage 3) case. I originally selected three learners from the three groups that I thought were more actively involved in discussions during the completion of the written questionnaire. The grade 5 teacher and a colleague however pointed out that the three learners I selected were not some of the most advanced learners in the class. They suggested three different learners of which I chose two and kept one of the learners that were in my original selection. During the data analysis and discussion process, I realised that this learner provided a highly positive and constructive response, which was in contrast to his teachers‟ judgement of his abilities (see p. 16, Chapter 4).

Riessman (in Graven, 2002:110) defined interviews as verbal exchanges in which interviewees collaboratively construct common understanding. According to Babbie, et al. (2001), qualitative focus group interviews could be conducted in two ways. The first way is to select about eight to twelve participants and to manage the interview by moving from person to person to ensure that each individual gets an opportunity to raise opinions of the phenomenon to be investigated. This way of conducting focus group interviews however jeopardises the quality of the data. The interviewer ends up with individual views and not the common opinions of the group. The technique applied to interviewing in this study is related to the researchers‟ second suggestion. I interrogated individuals‟ understanding by allowing them to interact through debate to gain insight of the group‟s common and dissimilar conceptions.

The objective for the two focus group interviews was to investigate both individual and group common conceptions based on responses supplied in the questionnaires. The objective was to allow participants to construct meaning of the concept of zero among them. Valuable aspects that I did not consider previously surfaced during the discussions. Participants often changed their opinions to agree or disagree with the assertions of others. I placed a container with counters on the table at the start of the grade 5 interview but did not draw attention to it. The learners ignored the counters during discussion about multiplication but they spontaneously reached for the counters when we discussed the concept of division. My role in the interviews was to set the scene, to initiate and guide the discussions and to probe. I envisaged that I would gain the opinions and knowledge that I thought sufficient to complement and enhance the data provided in the questionnaires to answer my research question effectively. I had to ensure that the questions I posed were well structured and open-ended to elicit effective descriptive responses and to avoid „yes‟ and „no‟ answers.

The researcher who performs unstructured interviews normally has no previously prepared questions or poses one question at the beginning of the interview, which sparks discussion. The questions that follow are based on the responses of the interviewees (Van der Mescht, 2008). Questions are aimed at probing for the construction of meaning and understanding of the interviewees" beliefs, experiences and their way of thinking and reasoning. The grade 5 learner interview could be regarded as a mix between a semi-structured and an unstructured interview. Although I did not have a pre-constructed list of questions, I used the number statements and the learner groups" solutions to the subtraction, multiplication and division tasks in a systematic way to gain insight in the interviewees" meaning making and understanding. The learners often disagreed with one another and even with their class groups" solutions. They often changed their opinions during the interview process. The teachers" interview was unstructured. It was more open and flexible than the learners" interview but both interviews could be regarded as purposeful conversations (Hitchcock & Hughes in Graven, 2000).

In both interviews conducted in this study, I started by introducing myself and stating the purpose of the interviews. I asked the grade 5 learners to introduce themselves and to state their ages. I informed them that we were going to discuss the solutions they provided in the written elaboration questionnaire (2) concerning calculations with zero. I encouraged them to speak freely and to agree or disagree with the opinions of others. In the teacher interview, I related the introduction to the research study, mentioned some of the ethical issues and explained the process of the discussions. Although I started with a question related to the written elaboration questionnaire (2) involving calculations with zero, the discussions developed to address various questions in the written elaboration questionnaire (3) concerning knowledge and teaching of the concept of zero. The conversations were quite free flowing without uncomfortable silent pauses.

## 3.4. ETHICS

In this section, I discuss the ethical issues that surfaced in the study. I have already raised some of the problems in the data production process and extend the discussion below. According to Bassey (1999), conducting research in a democratic culture allows the researcher certain autonomies: the liberty to investigate and question; to provide and obtain information; to articulate opinions; to critique others‟ opinions and to make research findings public. Researchers are however, held accountable by the morality of respecting truth and people. They should be truthful in the process of data production and analysis and the reporting of results. The primary ownership of subjects‟ data should be recognised. Subjects should be regarded as dignified human beings with equal rights whose privacy deserves respect (Bassey, 1999). I developed a creditable relationship of trust and mutual respect with most of the subjects before I embarked on the research process. My relationship with learners, teachers and principals in the first three cases (the pilot study) was more established than with learners in Stage 3. I was involved with the schools in the Western Cape for about two years before I started with the data production process.

The schools in the pilot study were project schools in the in-service mathematics development project I was involved in from 2006 to 2008 as a facilitator providing mathematics development and classroom support. The two teachers were participants on an accredited ACE course on which I lectured. A memorandum of agreement was signed between the project and the Western Cape Education Department (WCED). The teaching staff at each project school signed contracts of commitment during the orientation sessions in 2006. I received the verbal consent of principals and teachers to conduct the research and parents (through the principals) gave their consent for learners‟ photographs to be used in research reports.

Although mathematics content development is an integral part of the BEd course I informed teachers of the research project and asked their permission to use some of the data collected during the course in the research report. The director of the mathematics project I was employed in and I paid a personal visit to the school involved in Stage 3 of the data production process. I negotiate my research intentions with the principal and the grade 5 teacher. The principal and teacher were assured that the school and participants‟ identity and privacy would be honoured. All participants were ensured of anonymity. Formal letters of consent were supplied to the principal, class teacher and the learners‟ parents who signed and returned the letters (see Appendices 5.1 & 5.2).

During the personal visit to the Stage 3 school, the principal informed us that the school‟s mathematics teacher had left the school the previous year. The grade 5 class teacher, who was in a temporary position, did not have extensive mathematics training or Intermediate Phase mathematics teaching experience. The principal asked for assistance in mathematics teaching and learning in the grade 5 classroom. I offered to assist and we agreed that I would visit the school twice per month during the first two terms of the year. My original observations in this classroom showed that, although the desks were arranged in groups, the learners‟ behaviour was not evident of productive social interaction in a constructivist classroom atmosphere. I involved them in activities that they really enjoyed to let them experience that the learning of mathematics could be fun. I managed to capture their attention and interest. I addressed social issues to ensure effective co-operative group learning, which was a requirement for the completion of the written elaboration questionnaire (2). This experience emphasised the effect that disruptions in schools have on the teaching and learning process.

I transcribed the audio-recorded teacher interviews and gave them to the teachers to check for accuracy. The data gathered from various sources (written calculation and explanation questionnaires, interviews and observations) were used to check the validity of my original understanding (that learners and teachers experience problems with the concept of zero). The interview data was used to supplement data in order to obtain comprehensive interpretation and representation of the data (Bergman, 2008) and to ensure validity of findings. I translated the Afrikaans learner responses provided in the Stage 1 written elaboration tasks into English.

Pseudonyms were used in the transcripts of learner and teacher interviews and teachers completed the questionnaires anonymously to protect their identities. In presenting and reporting the findings, I had to be respectful of issues related to teacher identities and socio-cultural realities. Although teachers" responses reflected a lack of knowledge of various characteristics of zero, reporting these phenomena had to be done in such a way that the reader or audience would not find the statements concerning teachers" knowledge offensive.

## 3.5. DATA ANALYSIS

When I initiated this study, I had no particular approaches to data production and analysis in mind. The knowledge and skills that I applied during these processes were originally intuitive until I enrolled in the Master"s program and became more familiar with formal research methods. Looking back at the processes, I realised that my intuitive approach to the data collection, recording and analysis processes was in accordance with case study research methods. I captured the raw data and systematically analysed and sorted it into categories that I thought significant for answering the research question. Bassey (1999) mentioned that case study research is unique in the sense that it does not require specific methods of producing or analysing data but rather methods that researchers find suitable to and convenient for their studies. For Bogdan & Biklen (in Cantrell, 1993) the analysis of data is about the engagement, organization, reduction and synthesis of the data. It involves pattern recognition, determination of the important issues, possible learning and the decision of what would be communicated to readers. Bassey (1999) suggested that the data recording process should be systematic and analysis should occur as the data is produced. Ideas are constructed about possibilities for methods of analysis, about the reliability of data and additional aspects that should be investigated. Analytical statements are constructed based on summaries of participant responses to different aspects of the research question and the focus they have on the initial hypotheses.

Wolcott (1994:9) asserts that researchers who apply the qualitative approach to data production for the first time do not have a problem with producing data but rather with the decision of "what to do" with the data they produce. The challenge of converting data that could be overwhelming into convincing written interpretations remains a daunting task. I immediately captured data using tables in Word and Excel sheets after each production process in each stage. I carefully and systematically looked for key features and patterns (Wolcott, 1994) in responses within and then across the three stages which allowed me to reduce the data in all three stages into manageable categories using different colour codes. The next step was to organise the learner and teacher data separately into various themes and link the themes in the different data sources. Emerging patterns were categorised, represented in tables and described to reflect different levels of learners" and teachers" conceptualisation and perceptions about the concept of zero. By the time that the focus group interviews were transcribed, I had a clear picture of the different categories that I wished to report on but the transcripts reflected additional categories worthy to report on. The analysis and description of data involved a mix between qualitative and quantitative methods. I looked for connections between common patterns (in accurate conceptions, errors, misconceptions, thinking processes, discussions and explanations) occurring in this study and those in existing literature to make generalisations. The emerging themes allowed me to construct analytical statements regarding learners" and teachers" conceptions concerning multiplication and division by zero. This process was followed by a search for similarities in the different sets of themes displayed in data tables, summaries and bar graphs. This allowed me to compare learners" to learners" and learners" to teachers" understanding, which lead to some of the final generalisations and conclusions. As suggested by Wolcott (1994), during the entire analysis process I consistently had to keep a critical view on the categories to be included and remind myself of their purpose and relevance to addressing the problem.

The next step was to make connections between the findings and cognitive learning theories as well as theories concerning the knowledge that teachers should acquire to teach so that learners understand the content. This was a challenging task. The themes regarding the teacher responses were overwhelming. I engaged in various research literature to gain insight in existing findings in studies and established theories to support and confirm generalizations in this study. According to Connole (1998), the connection between theory and evidence is pivotal because of the observation (data) process and in the ways that evidence substantiates theory. The process of inductive reasoning "proceeds from specific observations (data) to general principles (laws)". Generalizable relationships or patterns in data that could be tested and confirmed repeatedly result in finding legitimate relationships. Inductive generalizations could be considered as authentic if they are constructed from a big pool of data and diverse circumstances. Generalizations made by inductive reasoning could however not be regarded as cast in stone – there is no guarantee that evidence of the past could not be inverted by future events or occurrences that prove the past evidence invalid. "The best theories are those which can be subjected to decisive attempts to refute them, but which continue to withstand such efforts" (Connole (1998:4-6). I decided to omit teacher responses to two of the questions in the third questionnaire, i.e. the written elaboration questionnaire: *Knowledge and teaching of the concept of zero* (3). I reasoned that the study of Wheeler & Feghali (1983) already reported on teachers‟ considerations of zero as a number in the counting and partitioning questions. I could possibly use the responses in this study as additional data. I also reasoned that the remaining themes provided me with sufficient data to construct analytical statements and generalisations about the conceptions that teachers experienced with the concept of zero.

The analysis process provided me with comprehensive understanding of the research participants‟ personal thoughts, beliefs and experiences (Davies, 2007). One of the challenges that I experienced was the consistent consideration of the focus of the study. I had to remind myself constantly of the disadvantages that teachers experienced during their own learning years, the limited training that most teachers had and the disregard for development of the concept of zero in curriculum documents.

### 3.6. VALIDITY

To err is human but mistakes, misinterpretations and untruths could be unfavourable and even have legal implications if you have to describe, interpret, theorise, generalise, evaluate and report on the actions and thoughts of other people (Maxwell, 1992). Threats of validity in a research study should be addressed so that the researcher‟s credibility is unquestioned. When I started with the study and analysed the responses in the first questionnaire, I soon realised that the findings were not sufficient to make claims about learners‟ competence. The data did not provide evidence of learners‟ thoughts and reasoning. I therefore had to develop additional instruments and consider creative techniques to validate the original findings in order to make effective inferences. I have done this by developing additional questionnaires and included interviews and observations as techniques in order to make claims about my research question. The qualitative multi-source data, i.e. the mental calculation tasks, the written elaboration tasks, the written responses in the classroom activity, the observations involving visual material and the focus group interviews assisted me to cross-check the responses in order to, describe, interpret, theorise, generalise and make evaluative judgement of the results.

I had to ensure that the transcripts of the audio-recorded interviews represented the interviewees‟ actual responses. The teachers‟ responses were sent to them for member checking, but I did not do this for the learners. I did double- check the transcripts, which were done by a student. I listened to the audio-recordings and crosschecked the exactness of the transcripts.

To guarantee consistency and accuracy in discussing the findings in the various data instruments and techniques I applied the process of triangulation. Precision and soundness in communicating results are among researchers‟ main concerns. They generally attempt to establish coherence between the findings and the reporting of those findings through triangulation. Triangulation entails the employment of various insights to shed light on understanding, i.e. ". . . to clarify meaning by identifying different ways the phenomena is being seen" (Flick and Silverman in Stake 1995). Cross-referencing among the different data production tools and techniques allowed me to make inferences about learners‟ and teachers‟ conception of the concept of zero.

## 3.7. LIMITATIONS

One of my main regrets is that I was not able to conduct interviews with the learners and teachers in Stage 1 of the study due to time limitations and practicalities. I also regret not being involved with the grade 5 and 6 learners in Stage 1 in a similar intervention as in the grade 3 and 4 class. Some of the grade 5 and 6 learners in the Western Cape displayed highly effective multiplicative thinking in written explanations of their understanding of multiplication and division by zero. It would have been worthwhile to have interrogated their cognitive construction of these concepts during an interview. The class teacher of the grade 5 class in Stage 3 was not as involved in the study as the Stage 1 teachers. She was often not in the classroom during my visits to this class because of the practice of subject teaching in different classes in the Intermediate Phase. Although I assisted the grade 5 learners in the Eastern Cape to develop an understanding of zero as a number during my support visits, I did not have the time to follow up with the development of understanding of multiplication and division by zero. It would have been a benefit to this study to compare the learners‟ conceptualisation of division by zero to the grade 6 learners‟ understanding in the study of Van den Heuvel-Panhuizen (2001).

## 3.8. CONCLUSION

In this chapter, I have sketched my personal view on the research process – the fieldwork, headwork and textwork as proposed by Van Maanen (in Gough, 2001). I have provided insight into the theory behind the research methodology and design of the study. The research methodology is located in the interpretive paradigm with the belief that individuals make their own sense of events and experiences in their social environment and the world. I have applied a mixed method approach to the data production and analysis processes but the study could be characterized as primarily qualitative. Applying the qualitative method allowed me to make sense of learners‟ and teachers‟ perceptions, interpretations and comprehension in their authentic learning environments while I was actively and dynamically involved in the data production process. The quantitative data allowed me to make sense and inferences of the various levels of understanding of individuals and groups of learners. The mixed method approach assisted in the identification of patterns, differences and commonalities in responses to construct generalisations which I was able to report through the use of statistical and narrative accounts. I applied a multi-tool approach involving both closed- and open-ended questions to provide results which would inform and answer the research question in the best possible ways. The process of triangulation assisted in ensuring validity of findings and conclusions. The research could be described as a multi-case study involving subjects in different settings with a similar teaching and learning context. I consistently ensured that the analysis and reporting of data are reliable and trustworthy.

# CHAPTER 4
# LEARNER RESULTS

## 4.1. INTRODUCTION

This research study focuses on the problems that primary school learners (Grades 3 to 6) experience with the concept of zero and teachers" contributions to learners" misconceptions especially with regard to multiplication and division by zero. The results of the study imply that both learners and teachers have limited knowledge and understanding of the concept of zero. Effective teaching and learning of the concept of zero are either insufficient or non-existent.

This chapter provides a discussion of the learner results obtained in Stage 1 (Grades 3, 4, 5 and 6) and Stage 3 (Grade 5) of the data production process. The discussion is based on findings obtained in the mental calculation questionnaire (1), written elaboration questionnaire (2), observations, photographs and a focus group semi-structured interview involving three Grade 5 learners in Stage 3 of the study. First, I highlight the performance of Grade 3 and 4 learners regarding multiplication by zero. Next, I focus the discussion on Grade 5 and 6 learner results concerning multiplication and division by zero. I further provide a summary in which I highlight key features of the findings reflecting Grade 3 to 6 learners" understanding of multiplication and division by zero.

## 4.2. GRADE 3 AND 4 LEARNERS' UNDERSTANDING OF MULTIPLICATION BY ZERO

The development of mental calculation strategies is fundamental in mathematics – in mental and written calculations. In this study, Grade 3 to 6 learners were required to provide instant responses to 1- and 2-digit basic multiplication and division problems. In the next section, the focus is on learners" ability to instantly recall solutions to the problems $4 \times 0 = \square$ and $0 \div 7 = \square$ to investigate understanding of multiplication and division by zero.

#### 4.2.1. Responses to the mental calculation tasks

The objective of this exercise was to compare learner responses in mental calculations to written elaborations involving multiplication and division by zero. For the mental calculation problem $4 \times 0 = \square$, 18% of Grade 3 and 50% of Grade 4 learners supplied correct solutions while 27% of Grade 3 and 36% of Grade 4 learners solved $0 \div 7 = \square$ correctly (refer to Figure 4.6, p. 102). Incorrect responses were stated as $4 \times 0 = 4$ by 82% of Grade 3‟s and 50% of Grade 4‟s, while 63% of Grade 3‟s and 50% of Grade 4‟s reckoned that $0 \div 7 = 7$. It was surprising to observe that Grade 3 learners performed better in the division than the multiplication task. It is generally accepted that learners perform better in multiplication than division tasks (Reid, 1956; Lutovac, 2008; Quinn, et al., 2008).

Only 34% of the learners in the multi-grade class knew that $4 \times 0 = 4$ and 32% knew that $0 \div 7 = 7$. The low scores is probably an indication that the concept of zero was not taught. It was questionable whether the correct responses were evident of conceptual understanding of the concept of zero. Instant mental responses could be based on the recall of memorised facts (refer to pp. 4-5, Chapter 1; 11-12, 26-27 & 46, Chapter 2) or even guessing, which does not necessarily imply conceptual understanding of the facts. It is not clear whether teaching of multiplication and division by zero occurred previously when these leaners were in the Grade 1 and 2 multi-grade class. Correct responses of the Grade 3 and 4 learners could not have been a result of teaching multiplication and division by zero. The class teacher did not focus on the teaching of zero as a number in counting or calculation activities before the project intervention (E. de Bruin, personal communication, 2007). My involvement with teachers in classrooms and lectures made me realise that teachers do not teach the concept of zero explicitly. The current mathematics curriculum (South Africa. DoE, 2002) does not require teaching and learning of the concept of zero. The lack of teaching the concept is supported by the report of an ACE teacher in this study, *I never taught zero in my class at al.*

The grade 3 and 4 class teacher offered to assess the mental calculation tasks. After checking and analysing the results, I realised that she marked the response $4 \times 0 = 4$ correct in eight of the learners‟ scripts. I questioned why a teacher who had proved to be competent would make such a mistake. The teacher reported at the project conference held towards the end of September 2008 that it appears as if learners of all ages have problems with the concept of zero. She was referring to her incorrect assessment of 4 × 0 in the learners‟ responses as reflected in Figure 4.1. below.

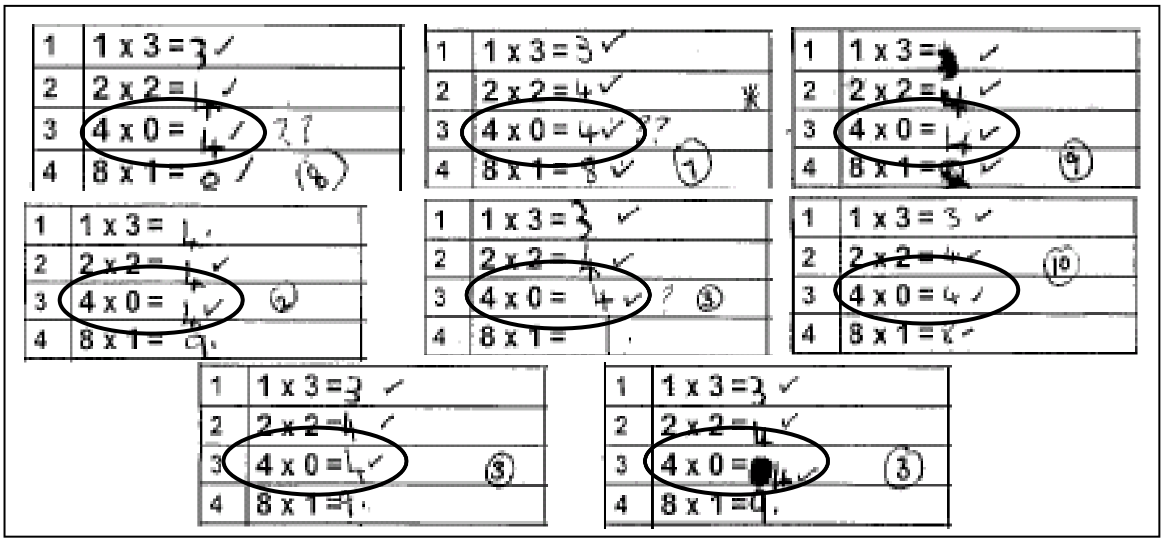
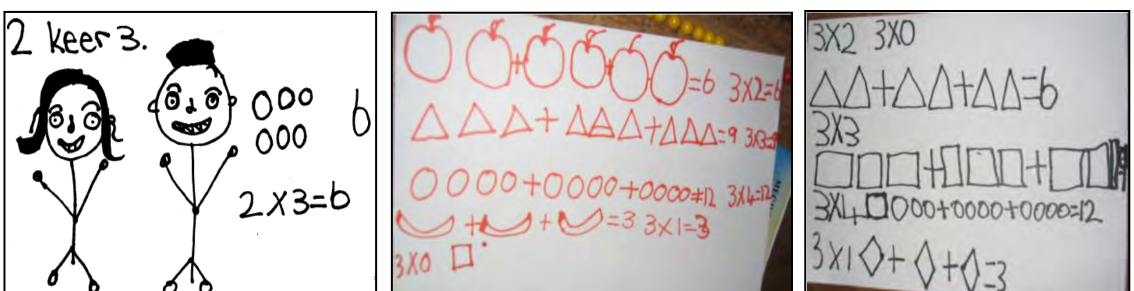


Figure 4.1: Extracts of eight Grade 3 and 4 learners‟ scripts indicating teacher assessment of 4 × 0

### 4.2.2. Understanding of multiplication by zero

During a follow-up classroom support visit, the teacher and I provided learners with feedback on their mental calculation tasks. The teacher and learners agreed that we engage in a lesson focussing on concept development of multiplication by zero. Grade 3 learners performed weaker in the problem 4 × 0 = ☐ in the mental calculation tasks. The learners worked in cooperative learning groups in their respective grades. They were asked to make drawings to display understanding of multiplication calculations written on the writing board. Calculations such as 1 × 4 = ☐; 3 × 4 = ☐; 2 × 4 = ☐ and 4 × 4 = ☐ were randomly recorded on the board. The purpose with the random presentation of the problems was to prevent learners from solving the problems through pattern recognition. I added the problem 4 × 0 after they have solved the problems involving only natural numbers. After learner feedback on these problems, I used the same calculations connected to models of equal grouping with composite sets of single objects to mediate understanding of multiplication by zero (refer to pp. 43 & 50-51, Chapter 2). The learners engaged in physical modelling of the problems after the intervention. This classroom intervention procedure involved firstly, systematic illustration of learners‟ practically based and mathematically based understanding, followed by practical teacher modelling with learner interaction and lastly a demonstration to consolidate concept construction of multiplication by zero.

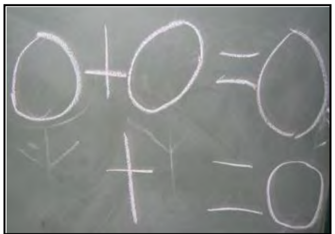
Although Grade 3 and 4 learners in this study displayed good understanding of multiplication with single-digit natural numbers, they were unable to make sense of multiplication by zero (refer to pp. 26-28, Chapter 2). The learners intuitively assimilated counting strategies into repeated addition to represent multiplication by illustrating pictorially, for example that 3 × 4 = □□□□ + □□□□ + □□□□ = 12 or 3 × 1 = □ + □ + □ = 3 thinking about multiplication as, for example *three times four*, i.e. a group of four individual objects repeated three times. They displayed competence in linking practically based illustrations to symbolic representations, for example 2 × 3 = 6; 3 × 4 = 12, etc. as displayed in Figure 4.2 below.

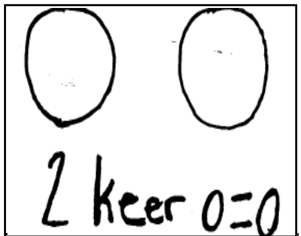



**2 times 3**

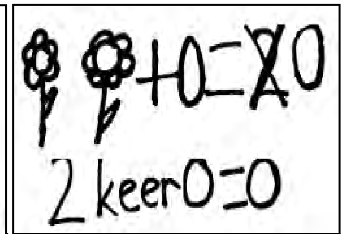
Figure 4.2: Grade 3 and 4 learners‟ representations of multiplication with single-digit natural numbers.

The learners‟ effective practical models became problematic in illustrating understanding of 3 × 0 as seen in pictures two and three in Figure 4.2 above. The learners did not know how to deal with this problem. The pictorial structure used in the multiplication with natural numbers tasks did not assist the operation with zero. The learners could not accommodate multiplication by zero into their existing cognitive structure as suggested by Davis (1983).








| | | | |
|---|---|---|---|
| | **2 times 0 = 0** | **I can't draw 0** | **2 times 0 = 0** |

Figure 4.3: Grade 3 and 4 learners‟struggle to represent multiplication by zero pictorially and symbolically

In their attempts to solve 2 × 0 (Figure 4.3 above), learners recorded the solution correctly as *2 times 0 = 0*. They could however not accommodate and find equilibrium between the problem (unknown knowledge) and the pictorial and symbolic representations (known knowledge) they attempted to impose. In the first picture above, the first group correctly applied repeated addition to 2 × 0 by asserting that *0 + 0 = 0*. They could not relate the structure to multiplication and left two empty spaces on both sides of a plus sign followed by the equal sign and zero. The second group sketched two empty sets, omitted the operation sign but asserted mathematically that *2 times 0 = 0*. This representation could be interpreted as two empty sets resulting effectively in *2 × 0 = 0*. This was however not the learners‟ intention because they attempted to relate the problem to repeated addition and omitted the plus sign. The third group drew two objects, wrote down *2 × 0 = 0* but asserted that, *I can't draw 0*. Group 4 represented the problem as ☐☐ + 0 = 0 (they originally wrote 2 as the solution) and recorded *2 times 0 = 0*.

The 34% of Grade 3 and 4 learners, who solved 4 × 0 correctly in the mental calculation task, were not able to present effective models or representations to demonstrate conceptual understanding of 4 × 0, 3 × 0 and 2 × 0 in the written elaboration tasks. Some groups could not structure their drawings correctly. Others were unsure or confused about the application of the addition and multiplication signs to show repeated addition. The learners intuitively knew that 2 times 0 is equal to 0, but could not represent the problem practically as with multiplication involving natural numbers. Representing multiplication by zero through repeated addition based on the structure for multiplication with natural numbers caused cognitive conflict in the conceptualisation of multiplication by zero as suggested by Davis (1983); Mulligan & Wright (2000); Levenson, et. al. (2004).

After realising that the learners‟ existing cognitive structure hampered the demonstration of understanding multiplication by zero, I represented the problems on the writing board (Figure 4.4 below). I applied a structure of composite units, i.e. equal grouping in which a collection of single objects is regarded as one set so that the unit is countable. The learners were able to experience for example, 3 × 2 as *three two's* as suggested by Steffe (in Mulligan, 2002).

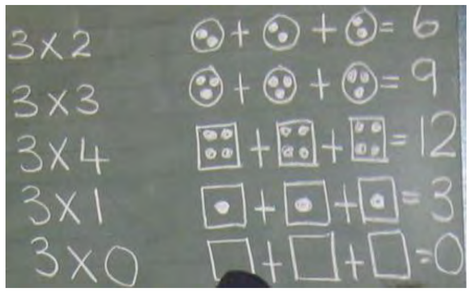



Figure 4.4: Restructuring the representation of multiplication using composite units to assist Grade 3 and 4 learners in understanding multiplication by zero.

Mediation (refer to pp. 37 & 42, Chapter 2) of multiplication by zero using composite units, assisted learners in meaning construction of 3 × 0 = ☐ + ☐ + ☐ = 0. They understood the multiplicative structure as *three zero's* indicated by three empty sets. When requested to model multiplication problems physically, they were able to organise themselves into equal groups to demonstrate understanding (Figure 4.5 below) while I took photographs of their demonstrations. One of the learners in a grade 3 group who had to demonstrate understanding of 3 × 0, told two learners to position themselves at the door. He then took a seat at the teacher‟s table. When asked to explain what they were doing, the learner suggested that I take a picture of the empty space in front of the classroom because 3 × 0 is equal to zero!

| 3 × 2 or 3 groups of 2 | 3 × 0 or 3 groups of zero = the empty space! |
|---|---|

Figure 4.5: Grade 3 and 4 learners‟ physical representation of multiplication problems.

The grade 3 and 4 learners applied both practically based diagrams (drawings of apples, triangles, squares and circles) and mathematically based elaborations. For example, they recorded *2 times 3* and symbolically stated next to the drawings $3 \times 2 = 6$ for three groups with two items each. None of the learners supplied rule-based explanations like learners in higher grades who often provide rules to explain multiplication by zero. For example, a grade 5 learner replied in the study of Levenson, et al. (2004) that, ". . . I know zero times one is zero. . ." (refer to p. 33, Chapter 2). This indicated that the Grade 3 and 4 teacher did not teach multiplication by zero nor supplied rules to explain the concept.

The findings show that Grade 3 and 4 learners were able to construct meaning of multiplication by zero. The process of mediation – guiding and directing understanding without telling – in a constructive social learning environment involved adaptation of their intuitive cognitive structure and elicited conceptual and procedural understanding (refer to pp. 41-43, Chapter 2). The application of only one particular structure could obscure learners" algebraic thinking and cause problems in higher grades (refer to pp. 26-29, Chapter 2). The use of the area model, for example could be useful for developing understanding of multiplication. Employing effective structures is imperative in organising and interpreting multiplicative conditions in higher grades (Mulligan, 2002). It is not clear at this stage whether teachers expect FP learners to distinguish between, for example the expressions $2 \times 3$ and $3 \times 2$, which is important in the development of understanding of the commutative property (refer to p. 52, Chapter 2). What became clear in this study is that the grade 3 and 4 learners" existing knowledge of multiplication with natural numbers was well developed. The correct responses supplied for multiplication by zero in the mental calculation task was not evident of conceptual understanding. It became apparent that they did not have previous learning experiences of the concept of multiplication by zero. The learners demonstrated that knowledge of multiplication with natural numbers does not imply conceptualisation of multiplication by zero. The learners did not have the intuitive cognitive tools to represent multiplication by zero problems effectively. Their existing (accurate) conception prevented them from demonstrating conceptual understanding although they procedurally illustrated, for example that *2 times 0 = 0.* They only made sense of these problems when their cognitive conflict was addressed in the zone of proximal development with constructive support (refer to p. 41-42, Chapter 2) as reflected in Figures 4.4 and 4.5 above.

On leaving the classroom, I turned to a learner sitting close to the door and posed the question, „Six times zero?" The learner replied, „Six . . . u-huh . . . zero!" The learner was, however quick in rectifying his thinking. This was unfortunately my last visit to this school and, as a result, I was not able to observe and co-facilitate the learners" understanding of division. I could also not arrange for opportunities to interview the learners or the teacher to investigate their thinking and reasoning concerning the concept of zero as a number.

## 4.3. GRADE 5 AND 6 LEARNERS' UNDERSTANDING OF MULTIPLICATION AND DIVISION BY ZERO

In this section, I discuss Grade 5 and 6 learner results obtained in the mental calculation questionnaire (1) that required instant recall of solutions to $4 \times 0 = \square$ and $0 \div 7 = \square$. I discuss the responses provided in the written elaboration questionnaire (2) for the problems $1 \times 0 = \square$; $0 \times 1 = \square$; $1 \div 0 = \square$ and $0 \div 1 = \square$, which required written explanations and justifications for the solutions. I compare the numerical solutions to the mental calculations to those provided in written elaboration task.

### 4.3.1. Responses in the mental calculation tasks

Grade 5 learners in Stage 1 scored 88%, Grade 6 learners (2007) 83%, Grade 6 learners (2008) 64% and Grade 5 learners in Stage 3 scored 41% in solving $4 \times 0$ correctly. In the division task, Grade 5 learners in Stage 1 scored 46%, Grade 6 learners (2007) 70%, Grade 6 learners (2008) 91% and Grade 5 learners in Stage 3 scored 41% in solving $0 \div 7$ accurately. The difference in scores in both tasks of the 2007 and 2008 grade 6 learners is interesting. The same teacher taught these learners across the two years. The same observation could be made for the difference in scores of the 2008 grade 5 and 6 learners. These learners were in a multi-grade class taught by the same teacher. This phenomenon could be an indication that multiplication and division by zero were not taught formally or explicitly. I am not certain about this because I did not have such a discussion with the teacher. Grade 5 learners (Stage 1) in the South Cape outperformed Grade 5 learners (Stage 3) in the Eastern Cape in both tasks (Figure 4.6) by 47% and 5% for the multiplication and division tasks respectively. Grade 4 learners in Stage 1 also outperformed Grade 5 learners in Stage 3 by 9%. Both groups of Grade 5 learners (Stage 1 and Stage 3) achieved scores below 50% in the division task while Grade 5 learners in Stage 3 achieved the lowest score in the multiplication task. Incorrect responses to 4 × 0 included answers such as 2, 4 and 40 while some learners reckoned that 0 ÷ 7 results in solutions such as 2, 7, 9 and 20.

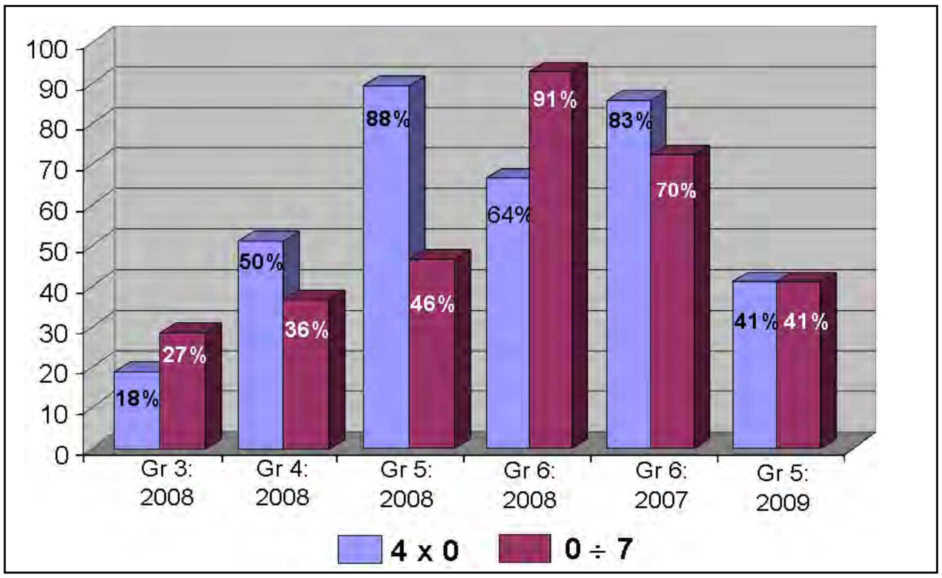



Figure 4.6: Grade 3 to 6 learner performance in the mental calculation tasks.

Grade 5 and 6 learners in Stage 1 scored 78% on average in the multiplication task and 81% of Grade 6 learners in Stage 1 solved the division task correctly. My assumption is that these high achievements and even achievements below 50% in Grade 5 and 6 are probably the result of memorized rules transferred by the teacher. Learning of the concept of zero is generally based on rule-based teaching which results in the development of procedural understanding; not conceptual understanding. Learners in higher grades are often not allowed to construct conceptual understanding of the concept of zero. The retrieval of rules for multiplication and division by zero from memory often reflect procedural knowledge disconnected from conceptual understanding (refer to pp. 43-46, Chapter 2). I further assumed that most of these learners would probably provide rules to explain their understanding as learners in higher grades often do (Levenson, et. al., 2004; Ball, et. al., 2008; Semenza, et. al., 2006). Recorded representations showing procedural and conceptual understanding should be a requirement for mental calculation solutions. Such tasks should be employed to understand learners" thinking processes and the structures that they create and utilise in their thinking and perception of numbers (refer to p. 3, Chapter 1; p. 29 & 44, Chapter 2).

The teaching of rules and algorithms that learners are not able to explain in the primary school is not a means to concept development. Providing a numerical solution or a series of symbols to problems (Skemp, 1976; von Glaserfeld, 2011) does not always display conceptual understanding of the numbers and their relationships (refer to pp. 35 & 46, Chapter 2). Knowing that any number multiplied or divided by zero results in zero does not serve as evidence that a learner is able to express and demonstrate understanding of this rule, as indicated by one of the groups of Grade 5 and 6 learners" in Stage 1 written account in this study.

> *The reason why I cannot solve these problems so well is because I do not understand why the answer is zero and I am not good at explaining mathematical terms. I know how I got the answer but to explain it is a bit of a problem. It is difficult for me to put the explanation of the sum in words. If you divide zero by any number, it stays zero. For example: 0 ÷ 9= 0, 0 ÷ 900 =0, 0 ÷ 9000 = 0.*

The discussion of learners" written responses regarding multiplication and division by zero in the next two sections emphasises the importance of the establishment of connections between mental calculations and written explanations of learners" procedural and conceptual understanding. The discussion is supplemented with extracts obtained in the semi-structured focus group interview conducted with Grade 5 learners in Stage 3. I need to repeat that I placed a container with counters in the centre of the table during the interview. The learners ignored the counters during knowledge illustration of multiplication by zero but reached for them when I interrogated their conceptualisation of division by zero.

### 4.3.2. Responses in the written elaborations for multiplication by zero

The data in the table in Figure 4.7 below indicates the written responses for the problems 0 × 1 = ☐; 1 × 0 = ☐; 0 ÷ 1 = ☐ and 1 ÷ 0 = ☐ provided by Grade 5 and 6 learners in Stages 1 and 3. Although these problems could be solved by the instant recall of responses, this might not have been the case in answering the questions during the data production process because the learners had enough time to discuss and reflect on the solutions in their groups.

| Problems | Grade 5 (2008) | Grade 5 (2009) | Grade 6 (2007) | Grade 6 (2008) |
|---|---|---|---|---|
| 0 × 1 = 0 | 100% | 75% | 100% | 100% |
| 0 × 1 = 1 | | 25% | | |
| 1 × 0 = 0 | 100% | 50% | 100% | 100% |
| 1 × 0 = 1 | | 50% | | |
| 0 ÷ 1 = 0 | 100% | 88% | 100% | 100% |
| 0 ÷ 1 = blank | | 12% | | |
| 1 ÷ 0 = 0 | 100% | 37% | 100% | 100% |
| 1 ÷ 0 = 1 | | 63% | | |

Figure 4.7.: Grade 5 and 6 learners‟responses to the written multiplication and division problems

All of the grade 5 and 6 learners involved in Stage 1 knew that 0 × 1 = 0 and 1 × 0 = 0 while 75% of the Grade 5 learners in Stage 3 reasoned that 0 × 1 = 0 and 50% of them reasoned that 1 × 0 = 0. Only 25% of the Stage 1 learners displayed multiplicative thinking through the concept of grouping and composite wholes although they referred to zero as *nothing*, for example "*If you multiply nothing once, it stays nothing*", "*I have no groups of one then I have nothing*" or "*I have one group of zero. I have nothing*". As I have assumed, the majority of these learners (75%) were not able to justify solutions sensibly. None of the Grade 5 learners in Stage 3 was able to provide sensible justifications for multiplication by zero. I expected that the grade 5 and 6 learners would apply rules to explain multiplication by zero. I was therefore surprised when I realised that they rather over generalised or exploited the concept of subtraction in elaborations. Matz (in Olivier, 1989) argues that the over generalization of concepts occurs intuitively (refer to p. 49, Chapter 2). The learners used terminology such as „*lost it*‟, „*ate it*‟ and „*give away*‟ and thus provided flawed explanations to explain the action of multiplication by zero. Learners responded, for example, "*I have nothing. I got one sweet and lost it*" and "*I have no strawberries I ask for* *one strawberry and I eat it all up and then I have no strawberries left*" (implying that 0 + 1 – 1 = 0); "*I have one orange and I give my friend that orange and I will go home with nothing* (implying that 1 – 1 = 0) or "*I have one apple and 1 don't eat that apple and still have that one leftover*" (implying that 1 – 0 = 1). The mathematical decoding of the learners" reasoning indicates accurate calculations but the learners did not explain the problems they were supposed to explain, i.e. 0 × 1 = 0 and 1 × 0 = 0 (refer to p. 50, Chapter 2). Some of the learners found it difficult to represent their thinking processes relating to multiplication by zero. One group in the grade 5 and 6 multi-grade class (Stage 1) asserted that multiplication by zero was easier said than done, did not make sense and was difficult to explain (refer to p. 33, Chapter 2):

> *I took the sum zero multiplied by one. It looked very easy to do but it was not as easy as I thought. I feel it is a bit difficult. I never understood it. It is also difficult to explain. But I tried to explain. At the beginning it was easy to write down the answer. But I struggled to write down the explanation.*

Some Grade 5 and 6 learners (25%) in Stage 1 supplied rule-based explanations displaying procedural but not conceptual understanding, for example "*If you take 1 × 0 the answer will be 0*". The grade 5 learners (25%) in Stage 3 who incorrectly responded that 0 × 1 = 1, provided rule-based explanations, for example "*If you have 0 × 1 the answer will be 1*" (refer to pp. 33, Chapter 2). Some learners (50%) who claimed that 1 × 0 = 1 asserted that multiplying by zero is impossible because ". . . *you can't times/×* [sic] *zero by a big number*" probably alluding to the misconception that you cannot subtract a big number from zero as in the grade 4 learner"s reasoning (refer to p. 3, Chapter 1). Some learners used the term *times* for explaining multiplication, for example "*If I have 1 sweet and times by 0 I'll still have 1*". They related multiplication to addition and subtraction, for example „. . . *my friend give me more I go home with nothing*" and „*I don't eat . . . and still have that one leftover*" (implying that 0 + 1 = 0 and 0 + 1 = 1). As suggested by Davis (1983), these learners obviously answered a different question for 1 × 0 and 0 × 1 (refer to p. 50, Chapter 2).

During the focus group semi-structured interview, Grade 5 learners in Stage 3 demonstrated some understanding of multiplication as repeated addition. They were, however consistent in their misunderstanding that 1 × 0 = 1 although Aziz‟s group maintained that 1 × 0 = 0 in the written task in the classroom. I used multiplication of natural numbers in an attempt to help them understand that 1 × 0 is *one group of zero.* Aziz demonstrated understanding of commutativity when he recognised that 3 × 1 = 1 × 3 because the numbers are *reversed*. He however claimed that both expressions mean 1 + 1 + 1 while 1 × 3 means 1 group of 3. Aziz claimed that 1 × 0 is impossible because zero is *nothing*. The learners could not make a connection between 3 × 1 as *three one's* and 1 × 0 as *one zero* (refer to pp. 23-24, Chapter 2).

*Z:* *Aziz, in your group you said one times zero is zero.*

*Aziz:* *I disagree with that, Ma'am. Because if you got one and you have zero here you can't times it with nothing, then it still remains one. You can't times it with anything.*

*Z:* *If I have one and I multiply it by three, what will the answer be?*

*Zinzi:* *I say it's three.*

*Z:* *It's three. Why three?*

*Adam:* *Because you times one by three.*

*Aziz:* *Because you add one three times.*

*Z:* *You add one three times? There we said three times one is equal to one plus one plus one. And here, one times three.*

*Aziz:* *It's still the same thing. You just reversed. (Others confirm). You can still say one plus one plus one.*

Grade 5 learners in Stage 3 over generalised the concept of doubling in relation to multiplication by zero (0 × 1) by asserting that "*You double zero by one and you still get zero*" instead of *you multiply zero by one*. During the grade 5 focus group semi-structured interview, I attempted to create a situation of cognitive conflict to help learners realise that multiplication is connected to repeated addition (refer to pp. 28-29; 48-49, Chapter 2).

*Z:* *What does this mean? Two multiplied by three?*

*Aziz:* *You times that two by three and it gives you six. It's like you double the three by two. (The other two confirmed.)*

*Z:* *Ok, you double the three. That's a good answer…*

I regarded the use of the concept of doubling for multiplication by two as good thinking until I discovered that the grade 5 learners used *double* to refer to multiplication by any number, not only by two.

*Z:* *So, now we say zero multiplied by one.*

*Aziz:* *…(Giggling) You double zero by one and you still get zero. . .*

*Z:* *Is it double or . . . ? Let's say you have three multiplied by one.*

*Aziz:* *Ok. I think you double that one by three and then it still gives you that three.*

*Z:* *Is it double? Here you said you double it. So, it's two times three. Ok? And then you said you have two three's which is actually three plus three, which is six . . . Let's do this one, two times one. What is that?*

*Aziz:* *You double that one, it's two.*

*Adam:* *No, two times one is two.*

Although $2 \times 1 = 1 + 1$ (double 1), Adam disagreed with Aziz when he realised that the situation was not about the concept of doubling but rather multiplication so that *two times one is two*. At this stage, I thought the confusion between *double* and *times* had been resolved. I pursued the use of repeated addition to help them make sense of multiplication. The learners however insisted that "*You can't double nothing*", once again disregarded zero as a number by claiming that $0 \times 1$ is impossible – an indication of the persistent nature of misconceptions (Davis, 1983; Ell, 2001).

*Z:* *So, it's equal to one plus one is equal to two.* (Learners chorus). *Now here you have three times one. What is that equal to?*

*Adam:* *Three.*

*Z:* *Three. And if you have to write a plus sum for that, what will it be?... Here you say two times three is three plus three. Here you say two times one is one plus one. So what is three times one?*

*Chorus: One plus one plus one.*

*Z:* *Ok, right. Now let's come to this one. Zero times one and you also said it is zero. How does that work?*

*Aziz:* *You have nothing here. And then, if you say zero double one it gives you nothing because you can't double zero.*

*Aziz:* *. . . You can't double nothing.*

*Adam: You can't double nothing.*

*Z:* *But are we still busy with doubling? You need to multiply. It's times. Zero times one. Two times one; three times one…is one plus one plus one. Now zero times one.*

The most assertive learner in the group, Aziz realised that he was in a spot. He then blamed his Grade 3 teacher for his misunderstanding (refer to p. 3 Chapter 1; p. 30, Chapter 2) and offered a rule to explain multiplication by zero (refer to pp. 27, Chapter 2).

> *I think in Grade three then the teacher taught us about zero times but she did not explain why. So we don't really, really know why. But I know zero times one is zero but we don't know the explanation . . . She's just told us zero times one is zero or if you can say zero times one hundred it still remain zero, but she did not explain clearly that. Like when I said, teacher can you please explain that sum to me but she did not explain it.*

Most of the grade 5 and 6 learners in Stages 1 and 3 did not display previous knowledge of multiplication to impose on multiplication by zero. It appeared that they mostly provided explanations for different problems from the ones they were supposed to solve and explain (Davis, 1983). Very few learners applied multiplicative thinking to conceptualise the two concepts $0 \times 1$ and $1 \times 0$. Those who displayed multiplicative thinking explained the problems as *nothing once*, *no groups of one* or *one group of zero*. The rest of the learners rather relied on their personal interpretation of the problems, which resulted in the actual mathematics and the rooted practical objects having a different meaning from that implied by the problems. Learners" elaborations on multiplication tasks involving zero were faulty because of their persistent referral to the concepts of subtraction, addition and doubling. These misconceptions hampered the construction of effective mathematical structures. The learners knew that $1 \times 0 = 0$ and $0 \times 1 = 0$, which could be the result of rule-based teaching without opportunities to construct meaning of multiplication by zero or multiplication with natural numbers for that matter. The grade 5 and 6 learners did not apply existing knowledge of multiplication as repeated addition as illustrated by the grade 3 and 4 learners in this study. If Grade 5 and 6 learners realised, for example that $4 + 4 + 4 = 12$ means *3 times 4*, *three groups of four* or *three fours*, they should have been able to understand that *1 times 0* is *one group of zero* and *0 times 1* is *zero groups of one*. Learner responses such as, "*I have no pens and I have to give one to my friend. My friend gets nothing.*" (implying that $0 - 1 = 0$) reflected no connection to the conceptualization of multiplication whatsoever (refer to pp. 28 & 50, Chapter 2).

Grade 5 learners in the focus group interview claimed that $1 \times 0 = 1$. They reasoned that zero cannot be multiplied by any number – thus implying that the concept $1 \times 0$ does not exist but in contrast mentioned that the answer is 1. One of the learners in the focus group interview recognised that $3 \times 1 = 1 \times 3$. He however claimed that, $3 \times 1 = 1 + 1 + 1$ and $1 \times 3 = 1 + 1 + 1$ because the numbers have just been *reversed*; implying that there is no difference in the structure of the two statements. This learner however asserted that $1 \times 0$ "*remains one*", i.e. $1 \times 0 = 1$, but in the same breath mentioned that multiplication by zero is impossible. The three learners also maintained that you cannot "*double*" zero so that $0 \times 1 = 0$. The learner, who applied the concept of commutativity to assert that $3 \times 1 = 1 \times 3$ was not capable of recognising that $1 \times 0 = 0 \times 1$. This is an indication of learners" tendency to separate zero from the natural numbers (refer to pp. 20-21, Chapter 2).

#### 4.3.3. Responses in the written elaborations for division by zero

The discussion in this section focuses on the written accounts of learners working cooperatively in groups to solve calculations with zero as a dividend and divisor and the reasons they provided to justify their solutions. The discussion draws on responses obtained in the grade 5 focus group semi-structured interview conducted in Stage 3. I present the discussion in two sections because of the diverse nature of the learner responses.

#### 4.3.3.1. Division with zero as the dividend: 0 ÷ 1 = □

All the grade 5 and 6 learners in Stage 1 of the study solved the problem 0 ÷ 1 = 0 successfully although an average of 69% of them solved 0 ÷ 7 = 0 accurately in the mental calculation task. Forty-three percent of these learners were able to justify solutions effectively by referring to *sharing* and *grouping*. They used everyday discourse and multiplicative structures to justify solutions (refer to pp. 29 & 52, Chapter 2). They reasoned, for example that, "*I have no apples. I share it with one friend and nothing is left. I have nothing to share. I cannot give anybody anything*" and "*There are no groups of one in zero so the answer stays zero*". Some learners in Stage 1 (29%) responded that they found it difficult to explain or perform division by zero although they knew the answer. They afforded a general rule stating that if you *divide zero by any number the answer would be zero*, for example 0 ÷ 9 = 0; 0 ÷ 900 = 0 and 0 ÷ 9000 = 0, a result of rule-based teaching (refer to pp. 33-34, Chapter 2). They further mentioned that the problem originally appeared easy but that it was more complex than they thought. Some of these learners (28%) applied the concept of subtraction (refer to pp. 48-49, Chapter 2) to illustrate division by asserting that, "*Mother buys me one pencil. I lost it now I have nothing*" and "*I have one rand and I lost it. Now I have nothing*" (implying that 1 – 1 = 0).

Most of the grade 5 learners in Stage 3 (88%) reasoned that, 0 ÷ 1 = 0 although only 41% of them solved the calculation 0 ÷ 7 = 0 correctly in the mental calculation task. Fifty percent of the learners provided reasonable validations for their reasoning using real life simulations. Some groups reasoned that no division would occur. Others implied that no-one would get anything; there would be zero items left or if there is nobody to share with, the answer is zero. They asserted, for example that, "*There is nobody, one ball and no one to play with it so its equal to zero*" and "*If I have no banana's so then I can't devide anything to anyone*". They used mathematical language to explain rules, for example, "*When I have 0 and divide it by 1 I will still have 0 left*". The learners (38%) applied the subtraction concept to describe division by zero by referring to *give away* (refer to pp. 51-52, Chapter 2). For example, "*If you have four chocolet you give away four chocolet to four children then you have 0 left*" (implying that 4 – 4 = 0 or 4 ÷ 4 = 0).

During the grade 5 focus group interview I checked the learners" understanding of the concept of division and realised that they defined the concept quite accurately (refer to p. 51, Chapter 2). They connected the concept to equal sharing while they explained multiplication as "*You times*", which did not display conceptual understanding. The learners used the counters to demonstrate conceptual understanding. They imposed real life situations and mathematically based accounts in elaborations (refer to pp. 36, 44, 49, Chapter 2). They were able to demonstrate effectively that 10 ÷ 5 means 2 + 2 + 2 + 2 + 2 and 17 ÷ 2 = 8 remainder 1 or 8½.

*Z:* *Now for zero divided by one is equal to zero. All of you said it is zero. What does division mean? You said multiplication means you times. What does division mean?*

*Aziz:* *It's to share. You share something with friends and friends. Say you have ten sweets and there are five people. Then it's all equal. Two, two, two, two, two.*

*Zinzi:* *Division is… (takes counters, counts out seventeen and divides into two groups).*

*Aziz:* *Share it amongst two people. Say me and Adam.*

*Zinzi:* *Ok. Adam, he gets eight and then Aziz gets eight. Then it's one left. And there's two of them. So, then I cut that one in half and give each one so that they get eight and a half.*

*Z:* *…So, what does division mean?*

*Aziz:* *You share so that people get equal parts.*

When I made the selection of three learners to participate in the focus group semi-structured interview during the data collection process, two teachers at the school suggested three different learners from the ones I chose. They suggested Adam be excluded because "He talks a lot but doesn"t think much"! I was pleasantly surprised by Adam when he applied multiplicative thinking to explain that 0 ÷ 4 = 0 means you could give four friends each one zero; implying that 0 × 4 = 0, which is 0 + 0 + 0 + 0 = 0. This reasoning involved quite effective and constructive thinking and reasoning! Adam even rectified Aziz during the learners" confusion of *doubling* and *multiplying* (p. 107). The learners managed to construct a generalisation for division by zero (refer to pp. 45-46, Chapter 2).

*Z:* *. . . Now, zero divided by one, you say it's equal to zero.*

*Aziz:* *I have nothing to share with Adam so then we cannot use the counters. He's not going to get anything because I don't have anything, any counters.*

*Z:* *Why do you say you agree?*

*Zinzi:* *If you have no counters and Adam wants counters, you can't give him.*

*Z:* *And if I have zero divided by four, what will the answer be?*

*Zinzi:* *It will be zero.*

*Z:* *Zinzi, you explain that one?*

*Zinzi:* *If you have nothing, you can't give anyone something.*

*Adam:* *Like when you have four friends you can also give each one one zero.*

*Z:* *So even zero divided by a hundred? Why do you shake your head, Aziz?*

*Aziz:* *It will still remain that zero… because you have nothing to give. If you have hundred people and you have nothing to give . . . Let's say they must get their salaries and I don't have any money. I can't give them anything, any money because I have no money.*

*Z:* *So nobody will get money.*

*Aziz:* *Nobody. Not even one of them.*

The grade 5 and 6 learners" in the two stages conceptual understanding of division with zero as a dividend was more effective than multiplication by zero. Only 25% of the learners were able to justify multiplication by zero effectively. They imposed concepts of grouping and composite wholes on multiplication while 43% of them were capable of providing accurate and sensible justifications for division by zero. Adam, in the focus group interview, even regarded zero as a number rather than *nothing*. He displayed multiplicative thinking by claiming that four friends could each get *one zero* if you divide zero by four. He co-ordinated two composite units so that one composite unit is distributed over components of the other as suggested by Steffe (in Mulligan & Wright, 2000).

#### 4.3.3.2. Division with zero as the divisor: 1 ÷ 0 = □

Concerning the concept of division by zero, I realised that it was an unfair question to pose to Grade 5 and 6 learners. I did not expect them to know that division by zero is undefined. The problem however provided insightful understanding of the learners" conceptualisation of this extraordinarily abstract concept.

All of the grade 5 and 6 learners in Stage 1 reasoned that 1 ÷ 0 = 0. Forty per cent of them asserted that, if there is one object but no subject nobody will get anything because nothing can be given away or shared. These learners indirectly implied that no division would or could be performed (refer to pp. 30-32, Chapter 2). They reasoned that, "*If I have one apple. I divide it among no friends so there is nothing. I cannot give to anybody*" and "*If I have one banana and I give nobody anything, then nobody gets anything*". I assumed that the use of the term *give* in the second explanation referred to sharing and not to subtraction as *giving away* per se in the way the learners over generalised subtraction to explain multiplication by zero. In sharing situations, the concept of *giving away* objects until zero objects are left is a constructive strategy to display conceptualization of division through repeated subtraction. A lack of understanding or awareness of zero"s behaviour as dividend pressurised these learners to supply a numerical solution, i.e. 1 ÷ 0 = 0, instead of asserting that you cannot divide by zero or that it is senseless to divide by zero as implied by the responses above.

The majority of the grade 5 learners (63%) in Stage 3 reasoned that 1 ÷ 0 = 1. They claimed that, because there is one object and no one to share it with, the original object remains undivided or is left intact as depicted in the reasoning of the grade 6 learner in the study of Van den Heuvel-Panhuizen (2001).

> *Zinzi:* *Let's say I have one slice of bread and there's no-one to share it with, it will stay one because no-one is going to eat it. That's my answer.*
>
> *Z:* *So what will your answer be?*
>
> *Aziz:* *It's one. Because that slice of bread will remain one because I didn't give to anyone.*

*Z:* *But in your answer you said if you have one and if there is one child and you want to give him something but you have nothing, then it will be nothing. Which answer is correct?*

*Aziz:* *I think it's this one.*

*Z:* *Which one?*

*Aziz:* *One divided by zero is equal to one. Because you have something and there's no-one to share it with it is still one. I have not given it to anyone.*

*Z:* *Zinzi, what do you say?*

*Zinzi:* *I say over here we had nothing and you want to share it with someone (For 0 ÷ 1 = 0). So you still have nothing. And over here you have one divided by zero. You have something but no one to share it with.*

*Z:* *So what will your answer be?*

*Zinzi:* *One.*

These learners used mathematical discourse to display understanding of division, i.e. *divide*, *share* and *give away*. Aziz‟ group reasoned that 1 ÷ 0 = 0 in the written tasks, but the learner refuted this reasoning in the interview because Zinzi‟s reasoning probably made more sense to him. Connecting real life reasoning to division, for example 8 ÷ 4 = 2, means you have 8 sweets and 4 children to share it with so each one gets 2, leads to reasoning that 1 ÷ 0 = 1. But, 1 ÷ 0 implies that you have 1 sweet and zero children to share it with so that should result in no sharing at all, which should be the answer. So, what happens to the sweet? The learners answer, i.e. 1 ÷ 0 = 1 actually implies the result 1 ÷ 0 = „no sharing‟ remainder 1. They however illustrated mathematically that 1 ÷ 0 = 1 because they have no other way of explaining this peculiar mathematical situation. This is a complex and abstract concept, which could even be overwhelming to adults who are not aware that division by zero is meaningless, not allowed, indeterminate or undefined (refer to pp. 18-19; 29-30, Chapter 2).

Grade 5 learners (37%) who asserted that 1 ÷ 0 = 0, linked division to subtraction. Some learners indirectly asserted that division by zero is impossible. They claimed that, "*If I have* *no bananas so then I can't devide anything to anyone*" but still maintained that the solution is zero. This could be the result of how mathematics is taught or understood – that there should be numerical solutions to problems as proposed by Tsamir, et al. (in Quinn, et al., 2008). Primary school learners are not aware that they can assert that division by zero is not allowed or that it is stupid, silly or meaningless to divide by zero. As expected, none of the grade 5 and 6 learners solved the problem 1 ÷ 0 successfully, i.e. 1 ÷ 0 is undefined (or impossible or not allowed). The learners alluded to the fact that no sharing or division would be performed or that it is impossible to divide by zero, which implies that they are able to make sense of the concept. Learners should be motivated to answer that 1 ÷ 0 is impossible or cannot be done. Why should you divide if you have nothing to be divided? Developing conceptualisation of division by zero requires powerful mediation skills, which allow opportunities for interactive inquiry, exploration and discovery (refer to p. 32, Chapter 2).

The findings in the grade 5 and 6 results indicate that no previous constructive learning occurred regarding multiplication and division by zero. Learners had difficulty in assimilating and accommodating these concepts in their existing cognitive structures, which resulted in the development of various misconceptions. This could be ascribed to the fact that the development of the concept of zero is not featured in the mathematics curriculum until Grade 5 when learners are expected to develop understanding of zero as the additive inverse (South Africa. DoE, 2002:41). Zero is not included in counting or calculation concepts in the Foundation or Intermediate Phase. In this study, some learners‟ conceptions concerning multiplication and division by zero was however a revelation. Developing understanding of multiplication and division by zero is within learners‟ zone of proximal development. Researchers claim that learners find division more difficult than multiplication (Reid, 1956; Lutovac, 2008; Quinn, et al., 2008). Evidence in this study contradicts this view because the grade 5 and 6 learners demonstrated that they have a better conceptual understanding of division than multiplication.

In the next section, I present a summary of the key features that surfaced in the findings of Grade 3 to 6 learners‟ mental and written responses to highlight learners‟ conception of multiplication and division by zero.

## 4.4. KEY FEATURES IN LEARNER RESULTS

The search for and identification of patterns that reflect key aspects, commonalities and differences from which I could make generalisations revealed the following common features:

4.4.1. Accurate mental responses does not indicate conceptual understanding of multiplication by zero

4.4.2. Using practically based models could assist in the development of abstract thinking and reasoning

4.4.3. Knowing rules for multiplying and dividing by zero does not indicate conceptual understanding

4.4.4. Learners experience difficulties in expressing mathematical understanding

4.4.5. Disregarding zero as a number causes problems in conceptual understanding of the concept

4.4.6. Ineffective multiplicative structures are barriers to sense-making of the concept of zero

4.4.7. Misconceptions develop during sense-making of multiplication and division by zero

4.4.8. Learners blame teachers for their limited knowledge of calculations with zero

4.4.9. Learners intuitively connect division to sharing and grouping.

4.4.10. Learners do not find division necessarily more difficult than multiplication.

4.4.11. Learners intuitively know that division by zero is impossible.

I discuss each of these features below.

### 4.4.1. *Accurate mental responses does not indicate conceptual understanding of multiplication by zero*

In the mental calculation speed tests, 34% of Grade 3 and 4 supplied correct solutions for $4 \times 0 = 0$ while 70% of Grade 5 and 6 learners correctly responded that $4 \times 0 = 0$. Learners generally provide instantaneous solutions in mental speed test without consideration of the significance of the solutions (Kouba & Franklin, 1995). In the written elaboration tasks entailing multiplication by zero as a multiplicand, 25% of Grade 5 learners in Stage 3 asserted that $0 \times 1 = 1$ and 75% claimed that $0 \times 1 = 0$. For multiplication with zero as a multiplier, 50% maintained that $1 \times 0 = 0$ and 50% declared that $1 \times 0 = 1$. Grade 5 and 6 learners in Stage 1 provided 100% correct responses in solving these problems (I suspect that the teachers transferred rules for calculations with zero during the data production process).

The grade 3 and 4 learners were not able to illustrate understanding of multiplication by zero through pictorial models. They however demonstrated effective cognitive structures for multiplication by natural numbers. They were able to display procedural understanding of multiplication by zero, for example $2 \times 0 = 0$ without conceptual understanding of the concept. These learners assimilated a repeated addition structure into multiplication by reproducing abstractly internalised concrete models. Cognitive conflict occurred, however when they attempted to apply the structure to multiplication by zero. The structure prevented them from demonstrating conceptual understanding of the concept as suggested by Anghileri; Carpenter, Ansell, Franke, Fenema & Weisbeck; Clark & Kamii; Kouba; Mulligan & Mitchelmore; Steffe (in Mulligan & Wright, 2000:17).

The majority of Grade 5 and 6 learners displayed procedural understanding but were unable to illustrate conceptual understanding of $0 \times 1 = 0$ and $1 \times 0 = 0$ (refer to pp. 44-46, Chapter 2). Some of these learners imposed preconceptions, i.e. subtraction, addition and doubling on these problems, which lead to misconceptions. As proposed by Davis (1983), it appears that the learners interpreted the problems differently. The reasons for that interpretation need to be established. Matz (in Olivier, 1989) suggested that the over generalisation of concepts is based on deep level procedures which often direct surface level procedures. This process allows learners to impose a different concept on the concept at hand, which results in a misconception without the realisation that something is wrong. The application of inaccurate structures on concepts could be assigned to a lack of effective conceptual multiplicative tools, which forces the application of the only available (inappropriate) tools without consideration of the mathematical implications imbedded in the problem (Davis, 1983; Olivier, 1989; Hiebert, et. al., 1996; Kilpatrick, et. al., 2001; Sewell, 2002).

The findings provided evidence that accurate responses in mental calculation tasks are not always proof of learners‟ conceptual understanding, for example 2 × 0 is *two zero's* or procedural understanding, for example *2 times 0 = 0 + 0*. Mental calculation tasks should be connected to written or oral explanations to establish the strategies and procedures that learners implement to illustrate understanding. The successful solutions that learners produced in the mental and written tasks were in contrast to the learners‟ explanations and justifications of the responses, which were often inaccurate (Ebbutt & Askew, 1997; Semenza, et. al., 2006). A distorted view of learners‟ abilities could occur if the correct responses in tasks that require instant responses are used as the only benchmark for understanding.

### 4.4.2. *Using practically based models could assist in the development of abstract thinking and reasoning*

It became apparent that the grade 3 and 4 learners had not acquired intuitive cognitive mathematical tools to illustrate understanding of multiplication by zero problems. Although they were competent in solving multiplication problems involving 1-digit natural numbers through repeated addition, the learners were having difficulties when they had to provide evidence for understanding multiplication by zero. They were unsuccessful in applying the intuitive pictorial model to illustrate understanding of 3 × 0 as they have done with, for example 3 × 2. They repeated groups with single objects to show that 3 × 2 = ◆◆+ ◆◆ + ◆◆ = 6, and thus experienced problems with modelling 3 × 0. They struggled to represent three groups containing zero objects. This barrier resulted in a group of learners asserting that they could not draw zero. They were not able to acquire the process of equilibrium between multiplication with natural numbers and multiplication by zero. This led to a situation of cognitive conflict (refer to pp. 21-22, Chapter 2). Some learners correctly recorded that "*2 times 0 is 0*" and *2 × 0 = 0*. Moyer (2000) maintains that learners represent concepts practically or pictorially before they are able to demonstrate symbolic representations. The learners in this study represented multiplication by zero symbolically before they were able to demonstrate the concept practically. In this study, this was an indication that zero‟s behaviour in calculations differs from calculations with natural numbers as suggested by Reid (1956) and Kaplan (1999).

Requesting Grade 3 and 4 learners to illustrate their understanding with drawings, explicit teaching and modelling the multiplication problems physically using groups of learners enforced their conceptual understanding. Bonotto, Freudenthal, Linchevski & Williams, Piaget, Bruner and Skemp (in Levenson, et. al., 2007) promote this practice. The learners intuitively employed practically based strategies to illustrate conceptual and procedural understanding of multiplication with natural numbers. They were able to connect mathematically based accounts to their practically based illustrations. Practically based models involving composite units assisted learners in conceptualising multiplication by zero. The learners were able to demonstrate understanding by physically modelling multiplication by zero after mediation, which entailed constructive discussion and interaction (refer to p. 49, Chapter 2).

Young learners have a tendency to employ concrete materials or everyday life simulations to demonstrate their mathematical reasoning or exploration of mathematical ideas. They might often not be able to provide sophisticated mathematical accounts of their practical representations but they could supply informal though mathematically sound justifications as suggested by Levenson, et al. (2004). The complexity of the concept of zero was clearly illustrated by Grade 3 and 4 learners‟‟ accounts in this study. They were able to provide mathematically based accounts but not practically based models for multiplication by zero (refer to pp. 21-23, Chapter 2).

Unlike the grade 3 and 4 learners, most of the grade 5 and 6 learners did not offer practical models but rather provided rules and real life connections to illustrate understanding of multiplication and division by zero in the written elaboration tasks. The grade 5 learners in the focus group interview provided practical real life and sound mathematically based models for explaining division, for example "*Say you have ten sweets and there are five people. Then it's all equal. Two, two, two, two, two*". These learners ignored the available counters when they had to demonstrate understanding of multiplication. They however spontaneously reached for the counters when they had to illustrate understanding of division. They successfully conceptualised division with zero as a dividend by asserting that, "*I have nothing to share . . . so then we cannot use the counters. He's not going to get anything because I don't have anything, any counters*".

### 4.4.3. *Knowing rules for multiplying and dividing by zero does not indicate conceptual understanding*

An average of 25% of Grade 5 and 6 learners provided rules to explain multiplication with zero, for example *the answer will be 0* in 1 × 0 and for 0 × 1 *the answer will be 1* (incorrect). One of the grade 5 learners in the focus group semi-structured interview reported that ". . . *I know zero times one is zero but I don't know the explanation* . . ." Grade 5 and 6 learners (29%) were not able to explain why 0 ÷ 1 = 0. The learners offered rule-based accounts by reporting that ". . . *If you divide zero by any number, it stays zero* . . ." and provided a list of calculations to justify this rule. Some of the grade 5 learners in Stage 3 replied that, when zero is divided by one, you *will still have 0 left.* The grade 5 learners in the focus group semi-structured interview did not offer rules for 0 ÷ 1 = 0 but rather used counters to demonstrate conceptualization practically.

The grade 3 and 4 learners did not provide rules to explain multiplication by zero. The learners were not exposed to teaching and learning experiences of the concept of zero prior to the conduct of the research study, according to the class teacher. The application of uninformed rules, for example *any number multiplied by zero is equal to zero* was therefore not a barrier to these learners‟ conceptualization of multiplication by zero. Levenson, et. al. (2004) report that the application of rules to explain understanding of the concept of zero often occurs among learners in higher grades but rule-based teaching does not facilitate conceptual understanding. Grade 5 and 6 learners in their study knew that 3 × 0 = 0 and

$0 \times 3 = 0$ but could not explain why the solutions are zero. The authors claim that there is a decline in the use of mathematically and practically based explanations for calculations with zero because of rule-based teaching and learning in the higher grades.

The illustration of understanding using a range of numbers and operations does not reflect conceptual understanding because isolated facts could be easily recalled from memory without conceptual understanding. Effective explanation of rules needs understanding of how and why they work, i.e. conceptual understanding which goes beyond the fluent and accurate use of the rules. Rules for calculations with zero should be constructed and generalized by learners not transferred by teachers. Conceptual understanding of algorithms and procedures allows learners to make connections between concepts, operations and relationships in flexible and appropriate ways (Skemp, 1976; Shulman, 1986; Wood, et. al., 1993; Kilpatrick, et. al., 2001; Ball, 2003; Semenza, et. al., 2006; Ball, et. al., 2008; Von Glaserfeld, 2011). Learners should first construct meaning of, for example $8 \div 4 = 2$. They could interpret the statement as „How many groups of 4 are in 8?‟ then interpret 0 † 3 = 0 and 0 † 5 = 0 as „How many groups of 3 are in 0?‟ and „How many groups of 5 are in 0?‟ before they generate the rule that *zero divided by any number is equal to zero*.

#### 4.4.4. *Learners experience difficulties in expressing mathematical understanding*

Twenty-five per cent of the grade 5 and 6 learners in Stage 1 reported that it was difficult for them to explain why $0 \times 1 = 0$ because they were ". . . *not good at explaining mathematical terms*. . . ". The learners argued that the problem originally looked easy. They were not able to explain the problem because they never understood it and ". . . *It is also difficult to explain*. They ". . . *struggled to write down the explanation*". During the focus group semi-structured interview Grade 5 learners in Stage 3 asserted that they knew $0 \times 1 = 0$ but did not know how to justify the solution and asserted that they ". . . *don't know the explanation* . . .". Grade 5 and 6 learners in Stage 1 claimed that they were not able to explain why $0 \div 1 = 0$ because they did not understand why the answer is zero. They knew how to get the solution but did not have the cognitive mathematical tools and terminology to justify the answer. The theory of constructivism, concerned with cognitive development and progressive development of thinking and reasoning, is about individual and social actions and interactions. The development of language and interaction with language is central to the principles of constructivism. Explanation, negotiation, sharing of and reflection on ideas occur during meaningful learning experiences that allow opportunities for exploring and creating conceptions (Clements, et al., 1990; Yackel, et al., 1990; Kamii & Lewis, 1990; Wood, et. al., 1990; Cobb, 1994; Hiebert, et. al., 1996; Clements, 1997; Yager, 2000; Clarke, 2002; Boghossian, 2006; WCED, 2006; von Glaserfeld, 2011). It appears that the grade 5 and 6 learners in this study were not exposed to learning experiences in which they were allowed to explain, discuss, argue and reflect on their knowledge construction. Teaching and learning in most classes in the IP in our schools are still teacher-centred with limited or no opportunities for learners to engage in constructive language interaction to make sense of concepts. Many learners are not confident in expressing mathematical understanding if they are required to.

While observing mathematics lessons in IP classrooms, I noticed that the teaching approach was mostly based on teacher talk and chalk. In most classrooms, learners are not allowed opportunities for discussion and sharing their understanding of mathematical concepts. They are not expected to construct their own meaning of concepts, express difficulties they experience with concepts or question and challenge teachers" explanations of concepts. If learners are deprived of opportunities to reflect on and communicate their thinking and reasoning or their opinions are not valued and respected, they will not develop enthusiasm, confidence and excitement to express understanding (Hiebert, et. a., 1996). The constructivist teacher creates a classroom environment where mutual respect and trust are developed and where learners are motivated to interpret the understanding of the teacher and their peers to restructure their own understanding.

#### 4.4.5. Disregarding zero as a number causes problems in conceptual understanding of the concept

Although learners in Stage 1 solved and explained solutions to calculations with zero more effectively than Stage 3 learners, 75% of them referred to zero as *nothing*. Only 25% of

Stage 3 learners related zero to *nothing*. Even learners who displayed conceptual understanding of calculations with zero diminished the value of zero by claiming, for example "*If you multiply nothing once, it stays nothing*". Learners in the focus group semi-structured interview implied that multiplication by zero is not possible because "*You can't double nothing*" and "*You can't times it* (zero) *with anything*". It appears that learners refer to zero as *nothing* because of the real life meaning and use of the concept. They use the term *nothing* because of limited teaching and learning experiences of the concept (Reys & Gouws, 1975; Wheeler & Feghali, 1983; Anthony & Walshaw, 2004; Levenson, et. al., 2004; Levenson, et al., 2007). Regarding zero as *nothing,* prevent learners from making sense of the concept of zero.

Grade 3 and 4 learners did not refer to zero as *nothing* at all. The learner in the grade 3 group who physically demonstrated understanding of 3 × 0 declared that there were no groups of zero because *three times zero is zero*. The last learner to whom I posed the question "Six times zero?" responded "...zero", not *nothing*. A positive culture of learning was established in the grade 3 and 4 multi-grade class and these learners focused on the mathematics to be learnt. The grade 5 and 6 learners" reference to zero as *nothing* could be a result of ineffective teaching because teachers often reduce the importance to *nothing*. In the written multiplication and division with zero elaboration tasks, 45% of the teachers related the concept of zero to *nothing* (refer to pp. 23-24, Chapter 5).

### 4.4.6. *Ineffective multiplicative structures are barriers to sense-making of the concept of zero*

The mathematics curriculum (South Africa. DoE, 2002:44-45) requires of IP learners to "recognise, describe and use the reciprocal relationship between multiplication and division" and apply the commutative, associative and distributive properties to reason mathematically. The majority of Grade 5 and 6 learners did not master the concept of multiplicative thinking. One of the grade 5 learners identified that 3 × 1 = 3 and 1 × 3 = 3 during the focus group semi-structured interview. This was an indication that the learner"s thinking was at an advanced level because he connected multiplication to the commutative property. Levenson, et. al. (2004) suggest that learners in higher grades should be expected to use this property to justify, for example that $0 \times 3 = 0$ because $3 \times 0 = 0$. Mulligan & Wright (2000) state that strategies involving counting, repeated addition or subtraction and equal sharing and grouping should lead to understanding and application of commutativity and the inverse relationship between multiplication and division. The grade 5 learner correctly reported that the answer is the same for $3 \times 1$ and $1 \times 3$ because the numbers have just been *reversed*. He however did not recognize a difference in the structure of the two expressions and maintained that both expressions mean $1 + 1 + 1$, which indicates a lack of conceptual understanding concerning composite structure. Learners who apply composite structures effectively are able to recognize that $3 \times 1$ means *3 groups of 1* or *three one's* and $1 \times 3$ means *1 group of 3* or *one three.* The grade 5 learner was not able to apply his intuitive structure of *reversed numbers* to multiplication by zero because of his persistence in referring to multiplication by zero as "*You can't double nothing*". He could not make the connection to realize that $3 \times 1 = 3$ and $1 \times 3 = 3$ and therefore $0 \times 1 = 0$ and $1 \times 0 = 0$. This learner maintained that $1 \times 0 = 1$.

Learners who have developed multiplicative skills are able to demonstrate understanding of multiplication and division by connecting existing knowledge of counting, repeated addition or subtraction, commutativity and inverse operations. The ability to employ composite structures is based on the relationship between multiplication and division. Ell (2001), Mulligan & Mitchelmore and Gray & Tall (in Ell, 2001) maintained that less competent learners are not capable of connecting prior knowledge to new concepts. High achievers have the ability to construct effective structures based on previous learning. It is unlikely that 50% of Grade 5 learners in Stage 3 (who claimed that $1 \times 0 = 1$) could be classified as low achievers (Some of these learners reported in an informal discussion during the focus group interview that they would like to become doctors and lawyers). It is more likely that these learners were not exposed to learning situations where they were explicitly expected to make connections between known and unknown knowledge. It is also possible that they were not allowed the opportunity to construct meaning of the concept of multiplication. Most of these learners displayed effective procedural and conceptual understanding of the division concept.

### 4.4.7. *Misconceptions developed during knowledge construction of multiplication and division by zero*

Grade 5 and 6 learners‟ elaborations for multiplication by zero often reflected the development of misconceptions. Learners‟ persistent references to *subtraction*, *addition* and *doubling* prevented conceptual understanding of the concept. Their justifications generally implied calculations such as $0 + 1 - 1 = 0$; $1 - 1 = 0$ and $1 - 0 = 1$ instead of $0 \times 1 = 0$ and $1 \times 0 = 0$. Grade 5 learners (25%) in Stage 3 claimed that $0 \times 1 = 1$ and $1 \times 0 = 1$ because it is impossible to multiply zero *by a big number*, for example. A grade 5 learner in the focus group semi-structured interview insisted that $3 \times 1$ and $1 \times 3$ both means $1 + 1 + 1$. Although the learner illustrated awareness of the commutative property, he misunderstood the structure of the expressions, which could cause difficulties in the application of composite units to display understanding of multiplication. Grade 5 and 6 learners (38%) applied subtraction to explain $0 \div 1 = 0$. Their accounts implied that $1 - 1 = 0$ and $1 \div 1 = 0$. Some learners maintained that division by zero as a dividend is impossible. The learners generally concluded that $1 \div 0 = 0$ although some of them regarded the operation as impossible. Learners in the focus group interview insisted that $1 \div 0 = 1$ because "*. . . there's no one to share it with, it will stay one . . .*" In a study conducted by Polly & Ruble (2009:599) the researchers required of learners to construct and record their own everyday life, story problems to model understanding of the expression $10 \div 5$. They asserted that more than 50% of the class instinctively constructed subtraction problems to reflect their understanding of the division expression. The learners consistently referred to *take away* items instead of separating them into equal groups (refer to p. 51, Chapter 2).

Grade 5 and 6 learners‟ incorrect reasoning often resulted in answering questions that were not intended by the problems. It often appeared that they answered different questions than those they were supposed to solve, as suggested by Davis (1983) and Wilcox (2008). Their justification of $0 \times 1 = 0$ implied calculations such as $0 + 1 - 1 = 0$; $0 + 1 = 0$ and $0 + 1 = 1$.

For 0 ÷ 1 = 0, they implied that 1 – 1 = 0; 4 – 4 = 0 and 4 ÷ 4 = 0. Learners who reasoned that 1 ÷ 0 = 1 actually made sense if you consider the real life sharing they employed. Their reasoning however implies that no sharing occurs and the original object remains. Mathematically, this reasoning should however be decoded as 1 ÷ 0 = 0 remainder 1. This reasoning is similar to that of the grade 6 learner in the study of Van der Heuvel-Panhuizen (2001) and a BEd teacher in this study (refer p. 143, Chapter 5).

Learners and teachers often develop misconceptions because they undervalue the importance of zero. This tendency prevails from primary to high school learners and teachers (Levenson, et. al., 2007). Grade 3 and 4 learners did not develop misconceptions during knowledge construction of multiplication by zero. These learners did not relate zero to *nothing* or applied rules to display understanding. The intuitive structure they imposed on the concept constrained sense-making. This problem could be the result of teaching and learning one specific structure. The structure complied with multiplication of natural numbers but did satisfy multiplication by zero. The structure requires modification during mediation to facilitate constructive sense-making (Davis, 1983). Conceptualization of multiplication by zero might not develop independently. Learners should make sense of the concept if sense-making is mediated by resourceful teachers (Ginsburg, 1977; Sewell, 2002; Ball, 2003; Kahan, et. al., 2003).

#### 4.4.8. *Learners blame teachers for their limited knowledge of calculations with zero*

During the focus group semi-structured interview, one of the grade 5 learners blamed his Grade 3 teacher for his incompetence in explaining why 1 × 0 = 0. This learner claimed that he knew multiplication by zero results in zero but could not justify the solution because the teacher never provided them with an explanation. Henry and Reys (in Quinn, et. al., 2008) argue that high school learners often blame their teachers for not explaining rules. It appears that primary school learners have the same tendency. Rules supplied by teachers are often ingrained in learners‟ cognitive structures and difficult to change (Davis, 1983; Olivier, 1989; Kilpatrick, et. al., 2001; Star, 2005). Learners abandon their own understanding to please the teachers‟ objectives. Justification and explanations of rules need more than knowledge of the fluent and accurate use of the rules. Explaining and justifying algorithms requires conceptual understanding, i.e. knowledge of why processes or procedures work and why they are valid (Davis, 1983; Shulman, 1986; Wood, et. al., 1993; Sewell, 2002; Ball, 2003). Rule-based teaching often entails a traditional approach to teaching and learning in which teachers perceive one-word answers as evidence for understanding. A constructivist approach to teaching involves initiation, discussion, negotiation and mediation of sense-making depending on the character, quality and degree of the teacher‟s own sense-making, beliefs and attitude towards mathematical teaching and learning (Wood, et. al., 1993).

It appears that, if learners experience cognitive conflict and are unable to explain their understanding of concepts, they find an easy way out by blaming their teachers for supplying rules without explanations. During a classroom support session in a grade 4 classroom in 2004, a learner asserted that 50 – 18 = 48 because a number could not be subtracted from zero. The learner claimed that her Grade 1 teacher supplied the rule stating that, "*You can never subtract a number from zero*" (refer to p. 3, Chapter 1). It was two and a half years since this learner was in Grade 1 and she still recalled this misconception. This serves as proof that this deep level construction resisted change over a long period, and was imposed over the surface level construction (Olivier, 1989; Sewell, 2002; Star, 2005). The grade 1 teacher was actually a barrier to the learner‟s knowledge construction.

### 4.4.9. *Learners intuitively related division to concepts of sharing and grouping*

In solving multiplication by zero problems, only 25% of Grade 5 and 6 learners in Stage 1 were able to apply multiplicative thinking to demonstrate understanding of the concept. Most of the other Grade 5 learners used real life simulations, which did not justify the processes and solutions for $0 \times 1 = 0$ and $1 \times 0 = 0$. The use of real life simulations often reflected misconceptions in the application of inaccurate cognitive structures. Although learners in the focus group semi-structured interview displayed understanding of multiplication with natural numbers, they were not able to show conceptual understanding of multiplication by zero. On average, 43% of Grade 5 and 6 learners in Stage 1 and 3 were able to justify 0 ÷ 1 = 0 sensibly. The learners applied real life simulations and multiplicative thinking effectively. They used terminology such as *sharing*, *grouping* and *divide*. Grade 5 learners in the focus group semi-structured interview used practically and mathematically based explanations to demonstrate understanding of division with natural numbers. They were competent in expressing the meaning of division by asserting that *it means to share . . . so that people get equal parts*. One learner applied equal sharing with a remainder that is shared equally. The learners used the counters to display conceptual and procedural understanding by explaining that he had *nothing to share . . . we cannot use the counters*. Another learner applied multiplicative thinking and explained that 0 ÷ 4 = 0 because four friends will each get *one zero*. The learners were able to generalize that 10 ÷ 5 means 2 + 2 + 2 + 2 + 2; 17 ÷ 2 = 8 remainder 1 or 8½; 0 ÷ 4 means 0 + 0 + 0 + 0 and generalized that zero ÷ 100 = 0. Although Grade 5 and 6 learners were not competent in reasoning that 1 ÷ 0 is undefined or not allowed, they connected the concept to *sharing* and *dividing* (refer to p. 51, Chapter 2).

It appeared that learners are more exposed to everyday discourse involving equal sharing in their socio-cultural environments than to situations involving multiplication. They were naturally inclined to explain understanding of division effectively using real life simulations even though Levenson, et. al. (2007) maintain that learners struggle with division with zero as a dividend as well as zero as a divisor.

#### 4.4.10. ***Learners intuitively know that division by zero is impossible***

Grade 5 and 6 learners were not able to report that division by zero as a divisor is *senseless*, *not allowed* or *undefined*. Thirty-nine per cent of the learners responded that 1 ÷ 0 = 0. Their reasoning however implied that no sharing would occur if there were nobody to share an object with. They reported, for example that they *cannot give anybody*, *nobody gets anything* or *cannot divide anything*. It is highly probable that the learners did not experience teaching and learning opportunities of division by zero as a divisor or division by zero at all as proposed by Quinn, et. al. (2008). The learners" explanations indicate that they are capable of inferring that division by zero as a divisor is impossible. Tsamir, et. al.

(in Quinn, et. al., 2008) conclude that learners believe numerical problems require numerical solutions. This leads to a barrier in understanding division by zero as a divisor. The belief that problems result in numerical solutions holds true for most numerical problem situations and division by zero as a divisor is probably the only exception to the rule. Henry and Reys (in Quinn, et. al., 2008) maintain that division by zero is normally not taught in primary school. The study of Van den Heuvel-Panhuizen (2001) provides evidence that primary school learners are able to make sense of division by zero as a divisor if they are allowed opportunities to engage in constructive, mediated discovery and exploration. The process however requires teachers with effective pedagogical content knowledge, i.e. specialized mathematical reasoning and questioning skills and the ability to deal with incorrect content (Shulman, 1986; Ball, 2003; Kahan, et. al., 2003; Ball, et. al., 2008).

**4.4.11. *Learners do not find division necessarily more difficult than multiplication.***

An average of 57% of the grade 3 to 6 learners solved the multiplication problem $4 \times 0$ successfully while 51% of them solved the division problem $0 \div 7$ correctly. The learners generally performed better in the multiplication than the division task. According to Quinn, et. al. (2008), learners of all ages find division by zero confusing and the teaching of division by zero is often bypassed in primary school. Quinn, et. al.‟s assertion however relates to division with zero as a divisor, which is more complicated and abstract than division by zero as a dividend. It could be argued that $0 \div 7 = 0$ because there are *zero groups of seven in zero*. This understanding could be based on existing knowledge of $7 \div 7 = 1$, i.e. *one group of seven in seven.* This understanding could be assimilated to accommodate conceptualization of division with zero as a dividend. It is noticeable that the grade 3 and the grade 6 (2008) learners performed better in the division task while the grade 5 (Stage 3) learners‟ responses in the two tasks reflect an equal scores.

Grade 5 and 6 learners (25%) illustrated efficient understanding of multiplication by zero while 42% of them explained competently why $0 \div 1 = 0$. Grade 5 learners in the focus group semi-structured interview struggled to make sense of multiplication by zero but effectively conceptualized division with zero as a dividend. The real life connections that learners applied to explain multiplication by zero often lead to the construction of misconceptions. In explaining division by zero, learners used concepts such as *sharing*, *grouping* and *dividing* effectively. Although some replied that 1 ÷ 0 = 0, their reasoning implied that division with zero as a divisor is impossible.

Various researchers claim that learners find division more difficult than addition, subtraction and multiplication in calculations involving zero (Reid, 1956; Levenson, et. al., 2007; Lutovac, 2008; Quinn, et. al., 2008). Others however argue that learners might find division easier than multiplication because of their real life experiences with equal sharing (Davis & Pikethly in Roberts, 2003; Mulligan & Wright, 2000). The learners in this study illustrated more accurate and appropriate conceptual understanding of division than multiplication with zero.

## 4.5. CONCLUSION

The findings in this study illuminated the difficulties that Grade 3 to 6 learners experienced with the concept of zero concerning multiplication and division by zero. The importance of meaning construction of zero as a number denoting the empty set should develop gradually from the FP through constructive learning experiences. Misconceptions concerning the concept of zero that developed at an early stage could be persistent. If they are not identified and addressed, these misconceptions could be damaging to the development of knowledge of number concepts, properties, relationships and future algebraic thinking and reasoning. Teachers should be aware of the damaging effect of teaching by the transfer of rules that learners do not create themselves to develop understanding. Teaching and learning of “rules without reasons” (Skemp, 1976:2) that learners are expected to recover from memory often have a destructive effect on mathematical concept development.

Grade 3 and 4 learners in this study showed more competence in solving multiplication by zero problems than the grade 5 and 6 learners. The older learners offered responses such as 2, 4 and 40 for 4 × 0 while some of these learners provided 2, 7, 9 and 20 as solutions for 0 † 7. Grade 3 and 4 learners‟ incorrect solutions to these problems were predominantly

4 × 0 = 4 and 0 † 7 = 7. The younger learners‟ thinking was not clouded by uninformed rules supplied by the teacher. Most of the grade 5 and 6 learners did not apply effective intuitive structures to demonstrate conceptual understanding of multiplication by zero. Instead, they exploited subtraction, addition and doubling concepts, which created misconceptions. They also applied teacher-supplied rules that made no sense to them. The learners did not display multiplicative thinking involving repeated addition structures, the commutative property and inverse relationships between multiplication and division to justify solutions. The existing repeated addition structure applied by the grade 3 and 4 learners made it easy for them to accommodate understanding of multiplication by zero when the structure was adjusted. This serves as proof that the concept of zero can be mediated in young learners‟ ZPD. I suggest that the development of the concept of zero should not be delayed as advocated by Inhelder & Piaget (1969); Oesterle in Anthony & Walshaw (2004); Wellman & Miller in Semenza, et al. (2006). The concept should be developed gradually and progressively through meaningful learning experiences appropriate to learners‟ levels of understanding (Fischer, 1980).

Grade 5 and 6 learners in this study illustrated effective conceptual understanding more in division than multiplication by zero. It appeared that learners‟ intuitive concepts of sharing and grouping related to division were more naturally accommodated in learners‟ existing knowledge than repeated addition and equal grouping related to multiplication. This occurrence could probably be assigned to equal sharing activities that are more often applied in their social environment. This finding is in agreement with research studies that reported that learners might be more efficient in solving division than multiplication problems. Young learners have the ability to share problems involving fair sharing through the process of partitioning (Davis & Pitkethly, in Roberts, 2003; Mulligan & Wright, 2000).

The learners‟ abilities to explain conceptual understanding of division by zero more effectively than multiplication by zero, refutes some researchers‟ (Reid, 1956; Lutovac, 2008; Quinn, et al., 2008) claims that learners of all ages struggle more with division than the other basic operations. Claims were offered that learners might find division with zero as a dividend or divisor more difficult than addition, subtraction and multiplication with zero calculations. The majority of learners in this study applied mathematical discourse and terminology effectively in illustrating conceptual understanding of division but not for multiplication. Learners should be encouraged to assert that division by zero is not allowed or impracticable through constructive class discussions and debates.

Grade 3 to 6 learners" correct responses in mental calculation speed tests were not proof of their conceptual and procedural understanding of multiplication and division by zero. They struggled to make sense of multiplication by zero in written tasks although they had a sound understanding of multiplication by natural numbers. The intuitive physical model of repeated addition imposed on multiplication by natural numbers, i.e. repeatedly adding groups with single objects, obstructed demonstration of conceptual understanding of multiplication by zero. One can draw two or three individual flowers in a set that is not combined, but how do you draw zero flowers in a set where the objects are not merged to form a countable unit? Mental calculation tasks should be connected to written and oral elaboration tasks to obtain insight in learners" thinking and reasoning processes as suggested by Cooper, Heirdsfield & Irons and Fuson & Smith (in Ell, 2001). Misconceptions, for example relating multiplication by zero to subtraction and doubling and arguing that $1 \div 0 = 1$ should be addressed accordingly. The study of van den Heuvel-Panhuizen (2001) illustrates the importance of discourse, reflection and knowledge of related mathematical concepts to develop conceptual understanding procedural and conceptual understanding.

Learners should be allowed to model calculations with tasks using practically based strategies and intuitive multiplicative thinking, i.e. counting, repeated addition and subtraction, sharing and grouping smaller quantities into equivalent composite sets and using the area model to develop understanding of multiplication and division by zero. Although the application of practical models present in Grade 3 and 4 learners" existing cognitive structure developed cognitive conflict with multiplication by zero, their intuitive structure allowed them to develop conceptual understanding of multiplication by zero when the structure was modified. Grade 5 learners in the focus group interview demonstrated conceptual understanding of division with zero as a dividend when they engaged with physical objects.

The findings in this study highlight the need for IP learners to engage in intuitive practically based representations. Learners in the FP are often required to work with practical materials and models to illustrate understanding. This practice should continue into Grades 4 to 6. Teachers should allow learners to demonstrate their intuitive thinking processes (Ebbutt & Askew, 1997). Teachers should build on learners" intuitive multiplication and division structures and use mistakes and misconceptions as building blocks for learning. The teacher should create cognitive conflict by challenging inappropriate structures and mediating conflicting understanding in the ZPD to achieve equilibrium. Learners should receive guidance and direction in adjusting and improving the incorrect intuitive models and structures they impose on new concepts (Ginsburg, 1977; Davis, 1983; Olivier, 1989; Sewell, 2002). They should listen to the understanding of others, discuss, share and negotiate meaning of multiplication and division by zero. Knowledge construction of counting, repeated addition and area models should ultimately lead to the effective application of knowledge of number properties and inverse relationship to explore and develop conceptual and procedural understanding of division and multiplication with zero. This knowledge should assist learners to make sense of and relate isolated concepts to construct knowledge of new concepts.

# CHAPTER 5

# TEACHER RESULTS

## 5.1. INTRODUCTION

My encounters with the learners in various grades in primary schools made me realise that their problems with the concept of zero could be connected to the way their teachers understood the idea. Learners who provided uninformed rules to explain their understanding of calculations with zero often blamed teachers for their lack of understanding of the concept. Teachers, on the other hand blamed their limited understanding of the concept on their previous learning experiences. The development of the concept of zero is not specified either in the current mathematics curriculum (South Africa. DBE, 2002) or in most textbooks. It is also likely that teacher development and education programs did not focus on the development of conceptual understanding of the concept, its uses as a number and its history. It is for these reasons that I incorporated an investigation of teachers‟ understanding and knowledge of the concept of zero into this study.

I produced the teacher data in 2009, during Stage 2 of the data production process. The process involved eight teachers on an in-service BEd course and thirty-nine teachers on an accredited ACE course. The results revealed that most teachers displayed limited knowledge of the concept of zero. Teachers themselves often held their former teachers responsible for their lack of knowledge of the concept. Most of the teachers had more than ten years teaching experience. It became apparent that teaching the concept of zero is mostly based on the transmission of rules that teachers were not able to conceptualise.

This chapter presents and discusses the BEd and ACE in-service teacher knowledge based on findings obtained in two written elaboration questionnaires. One of the questionnaires is the same as the one completed by Grade 5 and 6 learners in this study. These instruments required of teachers to demonstrate knowledge of the four basic operations with zero and general knowledge of zero as a number in its own right. I do not make a distinction between

BEd an ACE teachers‟ knowledge in the discussion. I am however mindful of the fact that the BEd teachers‟ might have had some knowledge development of the concept during the two-year ACE course training that preceded the BEd course. I present and discuss the findings in in terms of teachers‟ understanding of multiplication and division with zero and responses to the questions, „What is zero?‟, „Is zero even or odd?‟, „What is the origin of zero in the history of numbers?‟ and „How do you teach the concept of zero?‟ The discussion is supplemented by data obtained in a focus group unstructured interview with the eight BEd teachers. The concept of multiplication by zero was not a feature in the interview. I assumed that teachers experienced more difficulties with division by zero.

## 5.2. UNDERSTANDING OF MULTIPLICATION AND DIVISION BY ZERO

In this section, I present and discuss findings concerning teachers‟ explanations of multiplication with zero as a multiplier ($1 \times 0$), zero as a multiplicand ($0 \times 1$), division with zero as a dividend ($0 \div 1$) and zero as a divisor ($1 \div 0$). The graph in Figure 5.1 on the next page reflects the outcomes of the BEd and ACE teachers‟ knowledge concerning calculations with zero. A distinction is made between BEd and ACE teachers‟ responses in the graph. The results reflect that the BEd teachers generally performed better than the ACE teachers. In the discussion of the results I combine the results where both groups are represented and provide average percentages. This is done for the responses to $0 \times 1 = 0$; $1 \times 0 = 0$; $0 \div 1 = 0$; $0 \div 1 =$ undefined; $1 \div 0 = 0$ and $1 \div 0 =$ undefined. The rest of the responses indicated on the graph were supplied by ACE teachers and I indicate this in the discussion. These responses are $0 \times 1 = 1$; $1 \times 0 = 1$; $0 \div 1$ does not exists; $0 \div 1$ (Blank – no solution provided); 1 † 0 = 1 and 1 † 0 = ∞. The key generalizable features of the responses in the two groups are quite similar and I found it more suitable to use average percentages in the responses indicated above to display the general trends in teachers‟ responses.

### 5.2.1. Teachers' understanding of multiplication with zero

Only 15% of the teachers displayed accurate conceptualisation of $0 \times 1$ although 98% of them knew that $0 \times 1 = 0$. Those who displayed conceptual understanding assimilated the concept of repeated addition to accommodate multiplication by zero. They illustrated effective multiplicative thinking by asserting that, for example *This means repeated addition of 0 and therefore there is no repeated addition*; *0 is the same whether you have 10 of them. It still means that you want to add 0s (repeated addition)*; . . . *Multiplication is about adding the same number of things a certain amount of times"*; *You have multiplied nothing so there is no repetition*. . . and *If you multiply 0 by 1 you get zero because you repeat zero 1 time*. One teacher applied the rule for the multiplicative identity of 1 by claiming that, *Anything multiplied by 1 remains the same. If you multiply the number only once you will get that same number*, which implies that 3 × 1 = 3; 2 × 1 = 2; 1 × 1 = 1 and 0 × 1 = 0 (refer to pp. 12, 25, 26, 29, 31, 3-34, Chapter 2).

The majority of teachers (62%) who knew that 0 × 1 = 0, supplied uninformed rules as explanations, which did not reflect conceptual understanding. For example, *Any number multiply by zero the answer is zero because it is a mathematical law*; . . . *because it is a mathematical method* or *Zero multiplied by any number is zero*. Some of these teachers explained the rule by stating that . . . *you multiply nothing and that destroys the value of the number you multiply with*. Although some referred to the commutative property, i.e. *1 × 0 is the same as 0 × 1*, they did not explain why 0 × 1 = 0 (refer to pp. 11, 16, 18, 26, 28-29, Chapter 2).

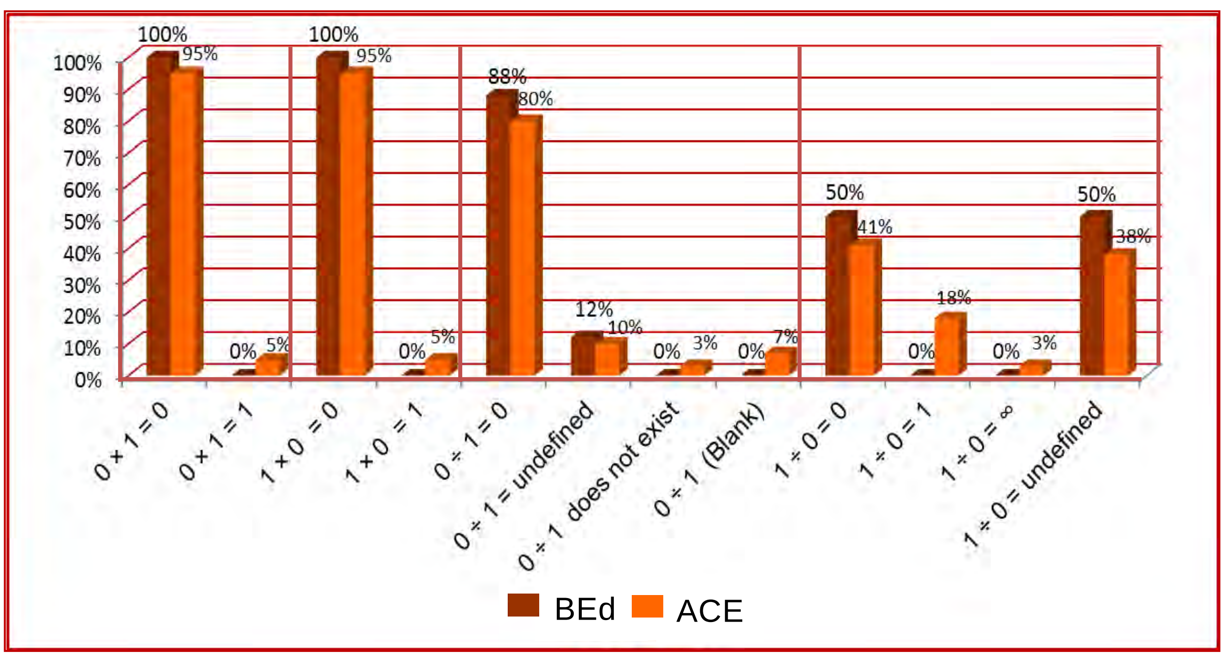



Figure 5.1.: Teachers" knowledge demonstration of multiplication and division by zero

Teachers (10%) who raised uncertainty and confusion about solving the problem $0 \times 1$ claimed, for example that the rule for multiplication by zero is a theory, i.e. *I don't know, it's just the theory that I have by having a zero in your multiplication your answer will be zero*. One of these teachers admitted that she taught this concept in her classroom but found the idea puzzling. She reported that, *I teach this but even I, I get confuse why you multiply the number with zero does not change*. Others simply replied *Confused* or *Not sure why* (refer to pp. 37 & 40, Chapter 2).

Some of the teachers (13%) mentioned that multiplication by zero is impossible although they solved the problem as $0 \times 1 = 0$. They claimed for example, that *Zero cannot be multiplied because there is nothing to be multiplied by 1*; *0 cannot multiply any number and give you an answer*" or *You cannot multiply something that you don't have*. A small number of ACE teachers (5%) maintained that $0 \times 1 = 1$. One of the teachers characterised zero as an identity element by asserting that, *In this case any number multiplied by zero is that number which means now zero here is an identity element of any number*. This statement implies that, $0 \times 2 = 2$ or $0 \times 10 = 10$. Another teacher made a connection to commutativity by referring to her previous solution, i.e. $1 \times 0 = 1$. She claimed it is the same as $0 \times 1 = 1$ because *Same applies here, it's just that you twisted the numbers* (refer to pp. 7-8, Chapter 1; 29-32, Chapter 2)

Only 8% of the teachers provided reasonably accurate explanations to justify the solution to $1 \times 0$ although 98% of them knew that $1 \times 0 = 0$. Teachers who displayed conceptual understanding of the concept related multiplication to repeated addition. They claimed, for example that, *Multiplication is repeated addition. This means that multiplying by 0 is repeated addition by nothing and therefore nothing has been repeated*; . . . *1 is multiplied 0 times* and . . . *In other words you are adding zero 1s (ones)*. The teacher who made the latter comment regarded $1 \times 0$ as difficult to explain. Another teacher commented that multiplying by zero eliminates (*destroys*) the value of the number you multiply it by.

Seventy-five per cent of the teachers who knew that $1 \times 0 = 0$, provided rules to justify solutions for multiplication by zero. This implies that the teachers have procedural knowledge but not conceptual understanding of the concept. Some of these teachers asserted that the answer is zero because it is a mathematical rule, for example *When you multiply any number by 0 you get 0. It's a mathematical law* or *If any number is multiplied by zero the product is zero*. Others maintained that they could not explain the rule because they did not know the reason neither did they have any proof, for example *Any number multiplied by zero becomes zero. I just know this as a rule of mathematics, but does not know the reason behind it*. Some teachers declared that it was only during the task that they realised they did not understand the rule, for example *I've written zero here . . . and this doesn't make sense to me now*. Others stated that this was what they were taught or told when they were at school, i.e. *I've only known it this way because I was taught this at school*. Teachers also mentioned that they (and their learners) were confused and uncertain of the rule, for example *Here even the children is getting confused if I told them that any number multiply by zero the answer is zero.*

Some teachers (17%) provided incorrect explanations. They were confused about the commutative property of numbers and claimed that, $1 \times 0$ *is like* $0 \times 1$ *but it does not give the same answer*, implying that $1 \times 0 \neq 0 \times 1$. Some argued multiplying any number by zero is impossible. For example, *You cannot multiply nothing with nothing; You cannot multiply any number by zero*; *Multiplying anything by nothing has no multiplication at all* and *Multiplication by zero does not give any answer*. It appears that the teachers imposed the concept of division by zero on multiplication by stating that multiplication by zero is impossible or results in no answer. A few teachers over generalised the concept of doubling to describe multiplication. They stated that, *1 multiply by zero means you're not doubling the number at all that's why the answer is zero* and *You can't double nothing*. Two of the ACE teachers (5%) mistakenly claimed that $1 \times 0 = 1$ because zero as a multiplier has *no impact* on the number it is multiplied by. Even if you multiply any number by zero *ten times* you will get that number as the answer. The response implies that, $10 \times 0 = 10$ or

$25 \times 0 = 25$! (refer to pp. 29-32, Chapter 2). The teachers are not the same two teachers who responded that $0 \times 1 = 1$.

A significant number of teachers (34%) reduced the importance of zero to „nothing" and regarded multiplication by zero as non-existent in justifying why $1 \times 0 = 0$. They asserted that, for example *Anything multiplied by 0 is 0 because 0 is less than 1 and we tend to say 0 is nothing so if you multiply a number by nothing you won't get a thing*. Some teachers claimed that, . . . *zero represents no number* and . . . *there is no value for the number zero*. In explaining understanding of $0 \times 1 = 0$, 25% of the teachers diminished the significance of zero to *nothing* by explaining for example that, *You can't multiply nothing with something and get something* or *When 0 (nothing) is multiplied by 1 or any no the solution will always be 0* (refer to p. 7, Chapter 1; pp. 5-6, Chapter 2).

### 5.2.2. Teachers understanding of division with zero

An average of 84% of the teachers solved $0 \div 1$ correctly but only 15% of them were able to provide sensible explanations for why $0 \div 1 = 0$. For example, *There is 0 ones in zero*; *Because there are no one(s) in zero*; *You have nothing to share, but one child to give something. That child gets nothing* and *I have divided nothing by 1 and it gives me nothing*. These teachers used concepts of grouping and sharing to explain division. Two of these explanations reflected effective multiplicative thinking, i.e. *zero ones in zero* as, for example there are *two fours* in eight for $8 \div 4 = 2$. The use of cognitive structures involving composite wholes, demonstrates effective procedural and conceptual understanding (refer pp. 29-33 & 40, Chapter 2). Although the statements connected to real life discourse make sense, the use of „nothing" shows a disregard for zero as a number.

Although the majority of teachers responded correctly that $0 \div 1 = 0$, 48% of these teachers could not supply significant explanations other than uninformed rules to support their thinking and reasoning. For example, *If zero is divided by any number then the answer is zero*. One teacher asserted that it was an *Inherited solution* suggesting that the concept was transferred over the years (refer to p. 30-31, Chapter 2). Teachers (10%) who responded that 0 ÷ 1 = 0, applied rules involving properties of numbers to explain the problem. They maintained for example that, *Any number divided by one gives the same number* and *Anything divided by 1 is equal to that number*. . . implying, for example that 5 ÷ 1 = 5 or 6 ÷ 1 = 6 and therefore 0 ÷ 1 = 0, which makes sense. One teacher confused the identity rule by asserting that, *Zero is the identity element of division*. One teacher claimed that, *One is bigger than 0 therefore you cannot divide* . . . while another asserted that one *Cannot divide o by 1because 1 is a whole number*. . . probably suggesting that zero is not a whole number (refer to pp. 13, 16, 26, 28, 30, 32, 36, Chapter 2).

Although only one ACE teacher mistakenly claimed that 0 ÷ 1 *does not exist*, 21% of the teachers confused 0 ÷ 1 with 1 ÷ 0 by explaining that there would be no solution, for example *There will be no answer if you divide a number by zero*. They also reasoned that dividing zero by a number was impossible in reasoning for example that, *You cannot divide something that does not exist* but replied that 0 ÷ 1 = 0. Teachers (6%) who correctly responded that 0 ÷ 1 = 0 mentioned that they were confused and declared that, for example *Even I too is getting confused about the answer* (refer to p. 5-8, 29-32, Chapter 2).

Teachers (11%) claimed that 0 ÷ 1 is undefined. They asserted that zero divided by a number is undefined because, for example *You cannot share nothing with someone that is impossible*; *You don't divide any number by zero* and *Devision by zero cannot be defined*. These statements displayed confusion with the rule for 1 ÷ 0.

Although 44% of the teachers were certain that 1 ÷ 0 is undefined, not allowed, result in no answer or does not exist, only 12% of them were able to justify explanations accurately although they diminished the importance of zero. These teachers asserted that, *It means that any number cannot be shared by nothing. There is nothing there. So it means you cannot share it. Its like not sharing at all* and *When dividing by zero you divide by nothing and that cannot be done*. These teachers were competent in explaining why division by zero is impossible by asserting that you cannot share at all when you divide by zero (refer to pp. 13-14, 19, 21-22, 36-37, Chapter 2).

Teachers (28%) correctly asserted that division by zero is undefined but 82% of them were not competent in explaining the rule or the term undefined effectively (refer to pp. 36- 37, Chapter 2). These teachers maintained, for example that *If you divide any number by zero you cannot define the process. Anything divided by nothing cannot be defined*; *Division by zero cannot be defined* and *We do not divide by zero in maths, so the answer is undefined*. One teacher made an unsuccessful attempt to explain the term undefined and referred to the statement, which is displayed on a calculator screen when you attempt to divide by zero. She stated that, *You can't divide any no by zero the solution will always be undefined. Undefined means – something that is there i.e. a no can never be divided by nothing (0). There must be an error or a mistake*. A number of teachers (4%) also claimed that division by zero is not allowed and has no answer without significant explanations. One of these teachers referred to the fraction, $\frac{1}{0}$ by maintaining that, *One cannot divide something with nothing, because division by zero is not allowed and a number with a zero denominator is not in the real number system*. One teacher replied *No answer* and referred to the everyday terminology used for zero. This teacher was also adamant in mentioning that zero is a number by claiming that, *Any number cannot be divided by zero. Zero is nil, naught, nothing and zero is a number*.

Some teachers (24%) responded that they did not know why division by zero is undefined. They mentioned that they were told that it is so, that it is a mathematical law and the solution was transferred without sense making. The teachers reasoned that division by zero is *Undifined. I don't know, I was told that if you divide a number by 0 it is undivisible so it is undefined*. Their statements reflected the result of rule-based teaching, for example, *I had been taught that a number divided by 0 is undefined*; *Inherited solutions but with no sense even to me* and *As a mathematical strategy you are not allowed to divide any number with zero* (refer to p. 14, 36, Chapter 2).

A significant number of teachers (46%) reasoned that 1 ÷ 0 = 0 and 50% of them supplied inaccurate rules by over generalising the rules for multiplication by zero and division with zero as a dividend. They claimed, for example that *If you divide any number by 0 the answer will always be 0*. A number of these teachers (29%) were accurate in their assertion that division by zero is senseless and undefined by claiming that no sharing would occur and there would be no answer; they however solved the problem as 1 ÷ 0 = 0. They used mathematical terminology to assert that, for example *Division by zero is indefinite. It does not give any answer. It is impossible to divide by zero* and *Because any number divided by zero is undefined* but responded that 1 ÷ 0 = 0 (refer to pp. 16, 29-32, Chapter 2).

Uncertainty and confusion were evident in 14% of the responses to 1 ÷ 0. The teachers related division by zero to their own learning experiences (refer to p. 3, Chapter 1; pp. 26-29, Chapter 4). They mentioned that, *I do not know how to put this mathematically but what I only know is that 0 means nothing*; *Our teachers always taught us without explanations that 1 ÷ 0 = 0. 1 ÷ 0 = 1 could also be an answer because if you have 1 sweet and don't give it away or eat it, then you still have 1* and *I do not know why if you divide 1 by zero the answer is zero*.

A number of teachers, who asserted that 1 ÷ 0 = 0 in their written responses, actually implied that 1 ÷ 0 = 1 in their explanations. Some of these teachers (75%) offered false generalisations, for example *If you have a number and divide it by 0, it remains the same . . .* ; *One remains because you divide 1 by nothing*; *When a number is divided by zero it is equal to that number* and *If you divide any number by zero you will get the same number* (refer to p. 6-7, Chapter 2). These statements imply that, if you have an object and no one to share it with the object would remain undivided. Reasoning that 1 ÷ 0 = 1 and applying the permanent rule, i.e. the property of inverses means that 1 × 0 = 1 or 1 ÷ 1 = 0, which defies this principle of mathematics. ACE teachers (13%) explained that there is no answer to the problem 1 ÷ 0 and referred to zero as non-existent in reasoning that, *If you divide an existing thing by the thing which is not existing it will be the existing thing* (refer to pp. 29-33, 36-37, Chapter 2).

One ACE teacher (3%) claimed that the solution to $1 \div 0$ is infinity. Another ACE teacher offered both *infinity* and *undefined* as solutions to this problem, i.e. $1 \div 0 = \infty$ (undefined). The teacher provided an unexplained rule and asserted that, *You cannot divide any number by zero*. Some of the teachers implied that the number of zero‟s in any number is infinite because, *If you divide any number by zero the answer is undefined because nobody can decide how many zeros are there in any number, here one* and *You cannot divide by 0 because if you ask how many zeros in 1 you will find that it is undefined* (refer to pp. 13-15, Chapter 2). These teachers displayed effective conceptualisation of the concept but should have asserted that $1 \div 0$ is undefined and not $1 \dagger 0 = \infty$ because infinity is not a number.

Teachers often referred to zero as „nothing‟ or non-existent in their elaborations. For example 34% of them used the term for explaining $0 \times 1$ while 25% referred to zero as „nothing‟ to explain the solution to $1 \times 0$. In explanations for $0 \div 1$, 43% of them labelled zero as „nothing‟ while 46% of the teachers used the term in explaining solutions to $1 \dagger 0$. ACE teachers who claimed that $1 \dagger 0 = \infty$ or related infinity to the fact that division by zero is underfined, did not characterise zero as „nothing‟ (refer to pp. 6-8, Chapter 2).

In explaining division with zero as a divisor, 29% of the teachers imposed real life discourse on the problem. They claimed, for example that, *If you have 1 sweet you share it with no one means that no sharing had happened*; *I divided 1 thing amongst no people then no one is going to get anything*; *You have one orange but have no-one to share it to. So I have no one to give the orange to. No-one gets the orange* and *If I have a cake and there is no-one to divide it into then it means I won't divide the cake and there is no fraction*. Some of the teachers‟ elaborations (25%) implied that division would not be performed and 1 remains because . . . *there is no dividing taking place. If you have one sweet and divide it by 0 you will still have 1 sweet* and *You have not shared it, it remains the same*. They indirectly implied that $1 \div 0 = 0$ (no action) remainder 1 so that $0 \times 0 + 1 = 1$, which is sensible reasoning based on the real life discourse but it is mathematically incorrect (refer to pp. 14-15, 37, Chapter 2).

Extracts from the BEd focus group unstructured interview reflected teachers" diverse conceptualisation of division by zero. The FET teacher started by asserting accurately that division by zero is senseless. Her reasoning became flawed by her reference to fractions and claiming that the value of the numerator, 1 would be unaffected, which resulted in 1 † 0 = 1. This teacher"s reasoning implied that, $\frac{1}{0} = 0$ so that $1 \times 0 = 1$ and $\frac{1}{1} = 0$.

M: *I think there is no reason for answers when we divide by zero, because for us to divide by zero is meaningless, because we are still going to get that number.*

Z: *Which number?*

M: *The numerator because when we divide by zero, zero becomes the denominator so that zero has no effect in the numerator.*

Z: *So you're talking about fractions now. The numerator on top and the denominator at the bottom?*

M: *Yes, yes. So if you divide any number by zero, it becomes that number. So it means that zero has no effect on the number.*

J, a Senior Phase teacher refuted M"s statement. He claimed that 1 † 0 = 0 because zero did have an effect on the numerator according to the rule he was familiar with. K, a Senior Phase teacher replied with a reasonable, logical and mathematically sound account claiming that division means ". . .*you count how many times the number goes into the dividend*". N, an Intermediate Phase teacher responded with her perception of division by zero and in fact echoed K"s reasoning using a real life simulation (refer to p. 18, Chapter 4). She implied that no division would occur. She however asserted that the problem was complicated and that zero has no value.

J: *I am not on the same point as the last speaker when she says zero has no effect on the dividing number whereas, the results according to the information I have that if you divide a number by zero you'll get zero, so that means zero has an effect on that number because that number has changed now from one divided by zero to zero. Zero is a number although zero means nothing . . . So, zero is making, stating the number there, that* *one divided by zero is zero. So, zero has something to do with that one. Thank you.*

*K:* *We say a number divided by zero is undefined. We cannot divide a number by zero. To divide by a number you count how many times the number goes into the dividend. So which means you can't have zero as a divisor.*

*N:* *I'm thinking now at the level of the learners. If you have one piece of cake, and you are asked to divide it, take that one cake as a whole and divide it among no one, then you are not going to divide that cake. You are not going to divide at all. So why divide by zero? It means that you can't divide by zero. Because you must divide it among so many. So, there's no body to divide it amongst. So, if I say divide four by two. So, which means that if there are four people, how many sweets will each person get? Maybe if you have four sweets and you're going to divide it by two, so how much will each person get? So it's going to get two. So, this one, you can't get it. It's difficult.*

*Z:* *So, this one says dividing by zero is not allowed and you say dividing by zero is meaningless?*

*N:* *It's senseless. I can't divide it. Because it says how many times are you going to divide it? Zero is nothing.*

K was accurate in her claim that division by zero is undefined and her reasoning implied, for example that $6 \div 2$ means 2 goes into 6 three times so $6 \div 2 = 3$ and $3 \times 2 = 6$. This cannot be done with zero as a dividend because, for example if $6 \div 0 = 0$ then $0 \times 0 = 6$, which is untrue). J was however persistent in his reasoning that $1 \div 0 = 0$ and eventually influenced N"s real life reasoning that she would not divide a cake if there was no-one to share it with.

X, a Senior Phase teacher confirmed that division by zero is not allowed but he only knew this because it is a rule he heard from his teacher when he was in school. He had no understanding of why division by zero is not allowed.

*X:* *What I was told by teachers is that you are not allowed to divide by zero. You are not given any explanation. Undefined or error, I don't know why you can't divide by zero.*

Much later during the interview, J steered the discussion back to N"s example of not sharing the cake if there was nobody to share it with. It became apparent that J was not satisfied with N"s claim that she was not going to divide the cake at all. N provided a simulation and claimed that 4 ÷ 2 = 2 so, if you share 4 sweets each one will get 2 sweets but this could not be done if you divide by zero. An exchange between J and N followed in which J eventually convinced N that the issue is about the action taken when you divide by zero. He reasoned that if you have done nothing, 1 ÷ 0 = 0.

*J:* *I am still not convinced about the example that was put here as a cake as a whole divided by zero. Then, when the speaker said nothing has been done on that cake, so that one, which means now, you said the result will remain one, hey?*

*N:* *I said I'm not going to divide it.*

*J:* *But at the end we talked about this cake and this cake is to be divided by zero. So nothing has been done to the cake. That's what you said? Cool. So, if nothing has how I took it in my mind now. So, how come that, you divide one by zero, and you get the same original number, the same cake.*

*N:* *If I have divided I would have said I've done that. But I didn't do anything. I was thinking now.*

*J:* *Then now, what if nothing has been done, then the result would be nothing. Not the same cake.*

*N:* *Not one?*

*J:* *Yeh.*

*N:* *No, because the cake I didn't touch. If I have divided, I would have said I've done it. So, I didn't do anything. So, it's nothing. I didn't divide.*

*J:* *So that will be nothing. Ok. You get it?*

*N:* *Yes, man. I thought the same. Because you didn't do anything. There's nothing left. You didn't even touch it.*

| | |
|---|---|
| *J:* | *Sure. Ok.* |
| *N:* | *Because now you must say to the child, division is a fraction or fractions is division. You're not going to do anything. You won't even touch that thing. You're won't even think of dividing. You won't do anything. Because where are the people you are going to divide? How many times? (Laughter)* |
| *J:* | *So the point is what you have done. Not what you have. What have you done to that cake – I have done nothing.* |

Teachers (44%) knew that division by zero is undefined, does not exist, has no answer or is not allowed. Some related the solution to infinity but could not specify clear explanations for their reasoning. Teachers could not explain the term undefined or why division by zero is not allowed. One teacher connected the solution to the display on a calculator („error") implying that a mistake occurred in the calculation. Some teachers admitted that they were uncertain why division by zero is undefined because they were told so by their teachers during their own learning experiences. Others maintained that they knew the concept because it is a mathematical law.

## 5.3. KEY FEATURES OF TEACHERS' KNOWLEDGE OF MULTIPLICATION AND DIVISION BY ZERO

In this section, I offer discussions on the key features reflected in the findings that characterise teachers" conceptualisation of multiplication and division. The search and identification of patterns reflecting key aspects, differences and commonalities allowed me to make the following general inferences:

5.3.1. The application of multiplicative structures facilitate conceptual understanding

5.3.2. Rule-based explanations do not elicit conceptual understanding of multiplication and division by zero

5.3.3. Knowledge construction of multiplication and division by zero developed misconceptions

5.3.4. The value of zero as an important number is being disregarded

5.3.5. Understanding of multiplication and division by zero is related to previous learning experiences

5.3.6. Real life discourse obscure conceptual understanding

5.3.7. Real life simulations are used more in division than multiplication by zero

I present discussions of these features below.

**5.3.1. *The application of multiplicative structures facilitate conceptual understanding***

Applying the concept of repeated addition to conceptualise and justify solutions to $1 \times 0 = 0$ and $0 \times 1 = 0$ allowed 12% of the teachers to demonstrate effective understanding of multiplication by zero. Teachers applied a structure of repeated addition to explain, for example that, *Multiplication is repeated addition. This means that multiplying by 0 is repeated addition by nothing and therefore nothing has been repeated*. One of these teachers applied multiplicative thinking linked to composite wholes to reason accurately that *Multiplication is about adding the same number of things a certain amount of times.* This thinking and reasoning is a requirement for the development of advanced levels of sophisticated thinking and reasoning about numbers and operations.

Most teachers knew that $0 \div 1 = 0$ but only 15% of them were able to provide rational explanations to justify solutions. Some of these teachers imposed real life structures on the problem by relating the problem to sharing and reasoning that if you have nothing to share no one would get anything. They asserted for example that, *You have nothing to share, but one child to give something. That child gets nothing.* Although the real life simulations represented sensible justification, the references to zero as „nothing" did not make the statements mathematically sound. Others applied multiplicative thinking by imposing the concept of grouping or the structure of composite wholes on the problems. For example, *There is 0 ones in zero.* This teacher made reference to the number of one"s or groups of one in zero, which is zero. Only 12% of the teachers displayed conceptual understanding of $1 \div 0$. One of the teachers used real life discourse to explain the concept by asserting that . . . *any number cannot be shared by nothing. There is nothing there. So it means you cannot* *share it.* Another teacher offered a mathematically based explanation by stating that, *When dividing by zero you divide by nothing and that cannot be done* but related zero to *nothing*. Multiplicative reasoning skills develop from the application of counting strategies to repeated addition and ultimately to the application of the properties of number, inverse relationships and knowledge of basic number facts to solve problems (Mulligan & Mitchelmore, in Ell, 2001). Conceptualisation of the concept based on the cognitive structure of composite wholes reflects the ability to assimilate deep level structures to accommodate surface level structures effectively (Matz, in Olivier, 1989). The process results in the achievement of equilibrium between known knowledge (for example $6 \div 2 = 3$ means three groups of two in six) and unknown knowledge ($0 \div 1 = 0$ is zero groups of one in zero). This dynamic organisation of knowledge demonstrated by multiplicative thinkers allows them to construct absolute knowledge across various mathematical concepts at any time in any particular learning situation. According to the cognitive theory of Piaget, learners construct and reconstruct concepts and consciously reflect on them when solving problems through the processes of assimilation, accommodation and equilibrium (Hiebert, et. al., 1996; Harries & Spooner, 2000; Clarke, 2002; Vianna & Stetsenko, 2006).

The low percentages of teachers who were able to display conceptual and procedural understanding of multiplication and division by zero are alarming. Teachers" mathematical content knowledge should reflect a profound basis of number facts, conceptual understanding of the facts and the ability to consolidate the facts so that they are recollected and applied effectively. Effective content knowledge is characterized by procedural knowledge (of mathematical strategies and rules) and conceptual knowledge (understanding and organizing mathematical facts and concepts). Teachers should know why rules work, how they work and when to apply them in flexible and appropriate ways to facilitate the development of new relationships (Shulman, 1986; Hiebert, et. al., 1996; Kilpatrick, et. al., 2001; Kahan, et. al., 2003; Ball, et. al., 2008).

### 5.3.2. *Rule-based explanations do not elicit conceptual understanding*

Teachers generally displayed procedural but not conceptual understanding of multiplication by zero. Rules for explaining $1 \times 0 = 0$ and $0 \times 1 = 0$ were offered by 69% of the teachers while 57% of them provided rules for explaining $0 \div 1$ and $1 \div 0$. The presentation of rules that they learnt during their own schooling years showed limited meaning construction of multiplication and division by zero. Teachers mentioned that they were confused, uncertain and puzzled, were unable to explain and make sense of the rule, did not know the reason for and could not provide proof for the rules they presented.

Orton (2004) connected the transmission of algorithms without sense making to the traditional teaching approach. If learners were able to recall and apply isolated memorized facts the teacher accepted that learners understood the facts. Freire (in Clarke, 2002:95) characterised this teaching and learning practice as the „banking" and „deposit" of knowledge. The mnemonic recollection of the rule for calculations by zero often involves incoherent procedural and conceptual understanding (Semenza, 2006; Von Glasersfeld, 2011). Acquiring knowledge inactively does not enhance understanding. Understanding develops through active, meaningful learning experiences in the learner"s own and social worlds. Understanding should be negotiated, related and contextualised in order to develop new understanding (Clements & Battista, 1990; Hiebert, et. al., 1996; Harries & Spooner, 2000; Clarke, 2002; Carpenter 2003; Vianna & Stetsenko, 2006). The presentation of a rule for multiplying by zero should be supported by understanding why the rule is true and what makes it work because understanding occurs most effectively if it is created and communicated (Wood, et al.; 1993; Ball, 2003; von Glaserfeld, 2011). Most teachers in this study were not able to assimilate existing knowledge of multiplication to accommodate multiplication by zero because this concept is isolated from the concept of natural numbers. If they connected their existing knowledge of counting, multiplication and pattern recognition to the concept of zero, they could have reasoned that, for example $5 \times 5 = \mathbf{25}$, $5 \times 4 = \mathbf{20}$, $5 \times 3 = \mathbf{15}$, $5 \times 2 = \mathbf{10}$ and $5 \times 1 = \mathbf{5}$ so therefore $5 \times 0 = \mathbf{0}$.

### 5.3.3. *Knowledge construction of multiplication and division by zero developed misconceptions*

Teachers‟ understanding of the concept of multiplication by zero was unsound and reflected various misconceptions. It appears that the previously learnt rules for calculations with zero are embedded in their cognitive structures and impacted negatively on sense-making of the concept of zero. If non-productive structures are preserved and recovered without knowledge of the conditions under which they are justified, they prevent the development of productive structures (Ginsburg, 1977; Davis, 1983; Olivier, 1989).

Teachers maintained that multiplication by zero is impossible and that there is no answer when you multiply with zero (over generalising the rule for division by zero). Some reasoned that 0 × 1 = 1 because zero is the identity of any number (refer to 5.4.1.4 in this chapter). Others related the concept to doubling by claiming that you cannot double zero. In cases where reference was made to the commutative property, it was used inaccurately, for example 1 × 0 *is like* 0 × 1 but the answers differ. This teacher also maintained that 1 × 0 = 1 and argued that 0 × 1 = 1 because the numbers have just been *twisted*. Teachers who asserted that 1 × 0 = 1 and therefore 10 × 0 = 10 because zero is the identity of any number or that zero has no influence on the number, suggested indirectly that 1 ÷ 1 = 0 or 1 ÷ 0 = 1, which is untrue. Their reasoning that 1 × 0 = 1 because zero has no influence on the number it is multiplied by, implied indirectly that, for example 25 × 0 = 25 or 100 × 0 = 100.

Teachers inaccurately applied the identity rule to explain understanding of 0 ÷ 1 by claiming that zero is an identity element for division. Zero is the identity element for addition because, when you add zero to any number the sum is that same number, but 0 ÷ 1 ≠ 1. Some teachers related the problem to the properties of 1 by generalizing that a problem with 1 as a divisor results in the dividend as a solution refer to. This explanation could be sensible to learners if, for example the behaviour of 1 is facilitated in a variety of similar division problems. They should be allowed to investigate that 0 ÷ 1 = 0 because 4 ÷ 1 = 4; 3 ÷ 1 = 3; 2 ÷ 1 = 2 and 1 ÷ 1 = 1, provided that the concept of division is explored, discussed and understood through grouping and sharing. Questions such as, „If there are 4 sweets, how many children can each get one sweet?‟; „How many ones are there in 4?‟; „If there are zero sweets and 4 children how many sweets does each child get?‟ and „How many 4‟s are in zero?‟

Misconceptions occurred in some teachers‟ explanations when they mentioned that zero cannot be divided by one because, *one is bigger than 0*. This understanding implies for example that, ¼ ÷ 1 or -6 ÷ 1 is not possible. Stating that zero divided by one is impossible because *1 is a whole number* implies that zero is not a whole number. Teachers also displayed a lack of conceptual understanding when they responded that 0 ÷ 1 results in no answer and that *something that does not exist* cannot be divided although they stated that 0 ÷ 1 = 0. If we apply the concept of inverse operations and argue that 0 ÷ 1 = 0 therefore 0 × 1 = 0, we prove that 0 ÷ 1 = 0 is a valid solution. Some teachers misconceived the rule for 1 ÷ 0 as undefined by asserting that 0 ÷ 1 is undefined. They stated that a solution to the problem is impossible and that 0 ÷ 1 does not exist. If you apply the grouping concept and ask, „How many ones are there in zero?‟ you can claim as one of the teachers did that, *There is 0 ones in zero,* which proves that the problem is not impossible.

A considerable number of teachers incorrectly claimed that 1 ÷ 0 = 0 and supported their reasoning with meaningless rules. Teachers did not consider explanations that involved the law of inverses to justify their reasoning. Some teachers implied that division by zero is meaningless and undefined but provided zero as a solution. These teachers have not developed the confidence to indicate in the solution, that 1 ÷ 0 is meaningless. They rather stated the solution as zero, a numerical figure, possibly because of the belief that calculations need numbers as solutions (Quinn, et. al., 2008).

Teachers incorrectly maintained that 1 ÷ 0 = 1. Some indicated that division by zero results in the number indicated by the dividend while others claimed that there was no sharing or division so that the dividend remains the same. Teachers also referred to zero as non-existent and the *existing thing* (number) would remain the same if divided by *the thing that does not exist* (zero). One teacher related the expression to fractions but asserted that zero as a denominator does not influence 1 as a numerator, which implied that 1 ÷ 0 = 1, as asserted by the Indian mathematicians, Mahãvira and Brahmagupta (refer to pp. 3-4, Chapter 2).

Teachers who claimed that 1 † 0 = ∞, maintained that the number of zero‟s in 1 or any number cannot be defined therefore division by zero is undefined. These teachers displayed effective multiplicative thinking by applying the concept of composite units. Their reasoning could be proven as valid if you apply the problem in a long division process as in the study of Van den Heuvel-Panhuizen (2001). Providing infinity as a solution implies, however that if 1 † 0 = ∞, then ∞ x 0 = 1 or 1 † ∞ = 0, which is impossible. Infinity is not regarded as a number in mathematics (refer to pp. 14-15, Chapter 2).

The teachers obviously did not consider the mathematical implications implied by their misconceptions. Misconceptions could be caused by over generalising previously accurate knowledge acquirement which is inaccurately connected to new concepts and therefore becomes erroneous (Ginsburg, 1977; Davis, 1983). Knowledge of the properties of numbers is one of the cornerstones of mathematics on which advanced algebraic thinking and reasoning is based. The teachers‟ misconceptions reflect the pedagogical price that they are paying for their prior ineffective learning. Teachers who struggle with unfamiliar content and confusion with concepts could transmit inaccurate conceptions to their learners. The structures they imposed on calculations with zero are deep-rooted and became invalid and misrepresented when they connected it to the concept of zero (Davis, 1983; Shulman, 1986; Olivier, 1989; Sewell, 2002; Kahan, et. al., 2003; Star, 2005).

### 5.3.4. *The value of zero as an important number is being disregarded*

A considerable number of teacher responses confirmed that they did not regard zero as a number. They referred to zero as „nothing‟ or conceived multiplication with zero as non-existent because *You won't get a thing*. More teachers however used the term *nothing* in elaborations for the expression 1 × 0 (34%) than for 0 × 1 (25%) indicating inconsistency in the tendency to diminish the value of zero. Even those teachers who displayed multiplicative thinking by relating the problem to repeated addition asserted, for example that repeatedly adding *nothing* results in *nothing*.

An average of 45% of teachers related the value of zero to *nothing* regarding division by zero while 30% referred to zero as *nothing* in the multiplication by zero tasks. The teachers who responded that 1 † 0 = ∞ did not diminish the value of zero to nothing but rather regarded it as a valid number by asking the question, *How many zero's in . . .* ?"

Although a zero amount of objects represents the empty set, zero is a number in its own right and an element of the set of whole numbers (refer to pp. 8- 9, Chapter 2). The set of integers includes zero and if extended to include the negative numbers zero is not referred to as *nothing*. Real life (mathematical) discourse involving zero seems to acknowledge zero as a number or digit, for example in reading temperature. On the other hand, it appears that the characterisation of zero as *nothing* mainly relates to the empty set, which occurs in number representation and calculation as suggested by Reid (1956) and Kaplan (1999).

### 5.3.5. *Understanding of multiplication and division by zero is related to previous learning experiences*

Teachers who were not able to demonstrate conceptual understanding of multiplication by zero mentioned that their teachers told them that multiplication by zero results in zero. One teacher asserted that *I've only known it this way because I was taught this at school*. They realised that they did not make sense of the rule, had no proof to justify it and were confused. One teacher reported that *I've written zero here . . . and this doesn't make sense to me now*. Another teacher admitted that she even confused her own learners when she transmitted the rule to them. It became apparent that the provision of rules was the only way they could explain the solutions.

Teachers stated that they were confused with the rule for division by zero that they *inherited*. They did not know the reason and could not justify the rule because their teachers taught the rule without explaining it. A teacher in the BEd focus group interview mentioned that, *What I was told by teachers is that you are not allowed to divide by zero. You are not given any explanation. Undefined or error, I don't know why you can't divide by zero*. Teachers who are regarded as mathematical experts could often not conceptualize and explain why division by zero is undefined or not allowed. Teacher educators should become aware of teachers" limited knowledge in order to assist them in developing cognitive, conceptual and computational understanding of the concept of zero (Wheeler & Feghali, 1983; Kahan, et al., 2003; Quinn, et al., 2008)

### 5.3.6. *Real life discourse obscure conceptual understanding*

It became apparent during the BEd teacher interview that teachers in the FET, Senior Phase and IP experienced misconceptions and confusion regarding division by zero. Some teachers offered reasonable justifications to support solutions for division by zero but allowed their reasoning to be influenced by inaccurate statements offered by one of the Senior Phase teachers. The teacher who referred to the cake that would not be divided if there was no one in fact provided three different conditions for division by zero: $1 \div 0$ has no answer; $1 \div 0$ is meaningless and $1 \div 0 = 1$ as the grade 6 learner in the study of Van den Heuvel-Panhuizen, 2001) and a grade 5 learner in this study (refer to p. 18, Chapter 4). Interview exchanges concerning division by zero in this study are evident of the prevailing confusion regarding division by zero. Everyday discourse generated a trivial configuration of zero and caused confusion, which successfully prevented effective conceptualisation of the concept of zero.

### 5.3.7. *Real life simulations are used more in division than multiplication by zero*

Teachers used real life simulations to explain their thinking more in the division than in multiplication problems. The use of mathematical terminology was also more prevalent in the division situations, i.e. the use of sharing, grouping and dividing. An average of 12% of teachers was able to explain multiplication by zero effectively while 14% of them explained division by zero accurately. It appears that the concept of division is more intuitively embedded in cognitive structures and a more natural process than multiplication. Imposing a real life structure on division by zero ($1 \div 0$), however caused a misconception when teachers argued that division would not be performed and the original object would therefore remain untouched. This reasoning thus suggests that $1 \div 0 = 0$ remainder 1, which is false.

The results in this section reflect that the majority of teachers are not equipped to teach the concept of zero effectively. Some teachers honestly declared that they have limited knowledge of division by zero, they could not express their understanding in mathematically based expansions and blamed their own teachers for not explaining rules for calculating with zero. They thus implied that their own teachers had limited knowledge of the concept because the solutions provided were often incorrect. The teachers‟ indecisiveness and puzzlement about division by zero are proof of their own ineffective learning but also an indication that their misconceptions have not been addressed during their teaching career. Teachers who do not make sense of algorithms would not assist learners in conceptualising the concept (Ball, 2003). Learners should be allowed to construct meaning of the rules *Any number multiplied by zero is zero*; *Zero divided by a number is always zero* and *Division by zero is not allowed* Quinn, et al. (2008) reported on teachers‟ limited knowledge concerning division by zero. Teachers‟ uncertainties and confusion lead to learners‟ misconception of the concept.

Shulman (1986) suggested that competent teachers transform the content matter to make it more understandable to learners. They should deal with learners‟ difficulties and uncertainties of the content matter and apply proficient knowledge of content resulting in effective construction of new knowledge. The misconceptions reflected in the teachers‟ thinking and reasoning about multiplication by zero is proof of the pedagogical price they have to pay because of previously ineffective learning. Their reasoning reflects inconsideration for the properties of number, which occurs because of the influence of the deep level directing theory as promoted by Matz (in Olivier, 1989).

Teacher educators should involve teachers in learning experiences in which they reflect on their own understanding and negotiate shared meaning with others so that they connect and adjust their own sense making to more sophisticated constructions (Wood, et. al,. 1993; Hiebert, et. al., 1996). Cognitive development of the concept of zero should be directed by social mediation in the ZPD (Clarke, 2002; Wink & Putney, 2002). Effective questioning skills and constructive communication of symbolic representations and cognitive structures should be employed to develop advanced stages of conceptualization (Hiebert, et. al., 1996; von Glaserfeld, 2011). Teachers should be allowed to communicate their existing unproductive cognitive structures at a social level in order to create generalisations that demonstrate conceptual and procedural understanding. They should develop knowledge about mathematics and for teaching mathematics. (Shulman, 1986; Ball, 2003; Kahan, et. al., 2003; Ball, et. al., 2008).

## 5.4. TEACHERS' KNOWLEDGE OF THE CHARACTERISTICS OF ZERO

In this section, I present and discuss responses to four aspects concerning the character of the concept of zero as supplied by BEd and ACE teachers in a questionnaire that required individual written reflections of teachers‟ general knowledge of the concept. Responses to the questions „What is zero?‟, „Is zero even or odd?‟ and „What is the origin of zero in the history of numbers?‟ are discussed. I further discuss teachers‟ responses to the question „How do you teach the concept of zero?‟ as reflected in the written elaborations. I draw on data obtained in the BEd teacher focus group interview to supplement the discussion. I do not make a distinction between responses provided by the two groups of teachers. The quantitative data is displayed as average percentages to reflect teachers‟ general understanding of the characteristics of zero. Some responses to the question, „What is zero?‟ were only provided by ACE teachers. In addressing these responses I refer to the ACE teachers only. Where only ACE teachers In some cases I had to supplement the discussion with literature pertaining to specific mathematical concepts that are not included in Chapter 2. This inclusion I regarded as necessary to clarify and develop understanding of certain attributes that teachers ascribed to zero.

### 5.4.1. Responses to 'What is zero?'

This question elicited various disconnected descriptions from teachers. A discussion of the attributes that teachers assigned to zero, follows in this section. Some categories might have common characteristics. Teachers often ascribed more than one quality to zero.

#### 5.4.1.1. *Zero is nothing or a representation of the empty set*

The majority of teachers (69%) referred to zero as a number but 41% of them claimed that zero is has no value, does not represent anything or has no meaning. Teachers who related zero to the empty set did not elaborate and justify this characteristic of zero. The use of zero in real life is often connected to *nothing* and the *empty* or *null set* is often perceived as *none* or *nothing left*. People often tend to disregard zero as a number. The empty set, a collection with zero elements is often experienced as an abstract concept because zero is generally not included in early counting exercises (Anthony & Walshaw, 2004; Semenza, et. al., 2006) and ordering activities although learners use number lines and measuring instruments where zero is presented as a valid number. Disregarding zero as a number and regarding it as valueless in real life discourse lead to difficulties in knowledge construction of calculations with zero and understanding negative numbers. Zero has a prominent place in the set of whole numbers.

Although zero is commonly not classified as a natural number, it satisfies the same conditions as natural numbers because it responds to the question „How many?" (Reid, 1956). The history of the development of the number system reflects ancient mathematicians" reluctance to accept zero as a number. Indian mathematicians invented zero as a number in the sixth century. Zero as a number was introduced in Europe during the twelfth century when its importance was still not valued (Reid, 1956; O"Connor & Robertson, 2000; Quinn, et. al., 2008). Today, during the twenty-first century, many learners and teachers still disregard zero as an important number although it was proven years ago that we can operate with zero in the same way as with the natural numbers, except for its peculiar behaviour in division by zero as a divisor. Reid (1956) maintains that zero is the only number that can be divided by all other numbers ($0 \div 7 = 0$; $0 \div 70 = 0$, etc.) but also the only number that cannot divide any other number ($7 \div 0$ is not allowed).

During the unstructured focus group interview, the BEd teachers in this study afforded reasons why they regarded zero as a number in its own right and an important number in the number system.

*Z:* *You said zero is a number. You were very adamant that zero is a number. Why?*

*J:* *Because when you count you start counting from nothing . . . And there are numbers beyond zero. So you can't just let zero to be not be a number. From nothing you went to something.*

*A:* *I think J is right when he says zero is a number. Because when you do measurement, to be accurate you need zero. So, zero is a number. You can't do without zero.*

*K:* *Can I just add something. If you look at the number line, you find zero on the number line . . . you find numbers on a number line and zero is one of those numbers on the number line. So, because of that I will say zero is a number.*

*N:* *Zero is a counting number. So I think, when you count numbers because zero is also a counting number, you start from zero when you count.*

*Z:* *And what about the set of integers?*

*N:* *Yes, it's part of the integers.*

*M:* *I just want to support the fact that zero is a number. Because when you extend the natural numbers and you add zero to the set of natural numbers, zero becomes the first number to make the set of whole numbers.*

These teachers supported their reasoning by referring to the use of zero in counting, the extension of numbers beyond zero, accurate measurement and the number line. The vignettes provide evidence that discussions about the concept of zero could elicit effective meaning construction of the concept. The discussion assisted 38% of the BEd who characterised zero as *nothing* and a non-integer in their written elaborations, to develop an understanding of zero as an important number.

#### 5.4.1.2. *Zero is a whole number. . .*

Some teachers (69%) conceived zero as a whole, counting or natural number. They claimed, for example that *Whole numbers start from zero*; *Zero is a number that we start with while counting. And it is a natural number* and *Zero is not an integer*. It is not clear what one teacher implied when she stated that, *It is used with other numbers to give a positive answer*. Teachers further mentioned that zero is a neutral number between natural numbers and integers. The teacher who maintained that zero is not an integer also asserted that zero is a whole number.

Gullberg (1997) claimed that whole numbers, which include negative numbers, zero and positive numbers, are called integers. Positive numbers are also called natural numbers. In the graphical representation of whole numbers below (Figure 5.3) zero is classified as an integer, a whole number and a counting number but not as a natural number. In my opinion zero is an integer, a whole number and a counting number.

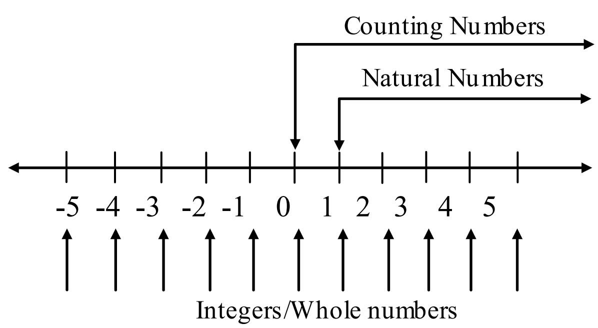



Figure 5.3: A graphical representation of the set of whole numbers or integers.

I consulted different mathematical dictionaries to find out how zero is classified in resources that we often consider trustworthy. The Mathematical Dictionary for Schools (Bolt & Hobbs, 1998) describes the set of counting numbers as the whole numbers 1; 2; 3; . . ., which are used for counting. The authors claimed that counting numbers are sometimes called natural numbers. The natural numbers (or counting numbers) are the numbers 1; 2; 3;

. . . used for counting. Zero is sometimes included as a natural number. The set of whole numbers is the set of counting numbers 1; 2; 3; . . . and the set if integers is the set -3; -2; -1; 0; 1; 2; 3; . . . They can be thought of as positions on a number line or as changes of positions. The Illustrated Maths Dictionary (de Klerk, 1999) depicts a counting number as a member of the set of numbers used in counting, i.e. 1; 2; 3; . . . and states that zero is not a counting number. A natural number is one of the counting numbers 1; 2; 3; . . . and the set of whole numbers is described as zero together with all the counting numbers, i.e. 0; 1; 2; 3; . . . while integers are portrayed as positive and whole numbers including zero. The Mathematics Study Dictionary (Tapson, 1996) portrays counting numbers as congruent to natural numbers. Natural numbers are the set of numbers 1; 2; 3; . . . as used in counting. The author claims that it is a matter of choice whether 0 is included or not. He stated that a whole number is a term used rather loosely to mean either the natural numbers or the integers. It depends on the context. Integers are numbers made up from the natural numbers (including 0) by putting a positive or a negative sign in front. The positive sign is often omitted.

Reid (1956:11) questioned whether natural numbers are more natural than other numbers and whether they are natural because they are used for counting. We can however also count in rational and negative numbers. Zero is not regarded as one of the natural numbers although it logically and naturally fits in with other numbers and replies in a similar way as the counting numbers to the question „How many?" (refer to p. 6, Chapter 2).

#### 5.4.1.3. *Zero is the identity element of any number in an operation . . .*

A significant number of teachers (47%) classified zero as an identity element. These teachers claimed that zero is the identity element for any number, for addition and subtraction and for multiplication. Some of the teachers justified their reasoning by mentioning that numbers do not change when operated with zero or zero does not make any changes in operations. Gullberg (1997) defines an identity element as a number that ensures validity of variables in common sets to create the equality of expressions even when the law of commutativity is applied. For example, zero is the identity element for addition because $0 + a = a$; and $a + 0 = a$ for all $a$ so that $0 + a = a - 0$. Zero is not an identity element for subtraction because $0 – a = -a$; and $a – 0 = a$ so that $0 – a \neq a – 0$. In multiplication, $0 \times a = 0$ and $a \times 0 = 0$ resulting in $0 \times a = a \times 0$ and for division $0 \div a = 0$ and $a \div 0$ = undefined so that $0 \div a \neq a \div 0$. It appears though that zero is the identity element for addition and multiplication. Gullberg (1997) however argues that zero is not an identity element for multiplication although it has a unique and general application in multiplication because whenever you multiply a number by zero the result is always zero.

#### 5.4.1.4. *Zero is the additive inverse of addition and subtraction*

Some of the ACE teachers (8%) defined zero as the additive inverse of addition and subtraction but failed to provide any elaborations to justify the quality they attributed to zero, for example *Zero is an additive inverse of addition and subtraction*. The acceptance of the identity elements gives meaning to the idea of an inverse. The number -2 is the additive inverse of 2 because their sum is 0. Two numbers resulting in a sum of zero are additive inverses of each other, i.e. $-2 + 2 = 0$. Two numbers resulting in a product of one are multiplicative inverses of each other. Since $4 \times ¼ = 1$, ¼ is the multiplicative inverse of 4, vice versa. Referring to addition and subtraction in their statements the teachers could have alluded to positive and negative values of the same number being added or subtracted to give a sum or difference of zero, i.e. $-4 + 4 = 0$ but $-4 – 4 = -8$. Zero is therefore the additive inverse for addition but not for subtraction. Zero presents a problem in the operation of division. Since zero multiplied by any number is zero, i.e. $n \times 0 = 0$, zero divided by zero, i.e. $0 \div 0$, may be any number. Thus, $0 \div 0$ is called an indeterminate symbol because it may be the name for any number. In fact, $n \div 0 =$ is impossible for all numbers, including $n = 0$ (Gullberg, 1997:45).

#### 5.4.1.5. *Any base to the exponent zero is equal to 1*

One of the BEd teachers referred to zero‟s attribute as an exponent, i.e. . . . *Any base to the exponent zero is = 1*, i.e. $10^0 = 1$ or $a^0 = 1$ in the questionnaire. Significant discussion about this quality surfaced in the BEd focus group interview. The FET Phase teacher could not provide a justification for why any base to the power zero is equal to one. The IP teacher made an attempt to justify why, for example $10^0 = 1$. She rationalized the concept by stating that the sequence of powers of ten, i.e. $10^1$; $10^2$; $10^3$; . . would be incomplete without $10^0$, which represents 1. It would therefore make sense to have $10^0$; $10^1$; $10^2$; $10^3$; . . . , which represents the sequence 1; 10; 100; 1 000.

*M:* *. . . in algebra whether it is the letters of the alphabet or a number and zero is the exponent, the whole term becomes one. So, zero is a number, it has an effect. For example, the base is ten to the power zero, then the whole thing becomes one. If you have a letter of the alphabet, a to the power three, I mean to the power zero, the whole term becomes one.*

*Z:* *Can you explain to us why it becomes one?*

*M:* *It's a rule. I cannot explain it further (laughter). It's a rule that any number, any base, any base to the power zero is one. I don't have a further explanation.*

*Z:* *Is it an axiom? Something that they say in mathematics is true without any arguments? It's just like that. It's a rule. And we…*

*N:* *Accept it.*

*Z:* *Accept it without any arguments.*

*N:* *Maybe it is because there's no term there. Because when you do this calculation it says times ten to the power one, times ten to the power two. So, that first one must be to the power zero. So it's ten to the power zero, ten to the power two, ten to the power. See, it's ten to the power zero, ten to the power one, ten to the power two, ten to the power three. So that must be ten to the power zero. That's why…*

The concept of exponents relates to more sophisticated interactions in mathematics. Multiplication is regarded as stylish addition. Raising numbers to powers could be seen as well-groomed multiplication, which represents more advanced thinking about multiplication (Kaplan, 1999). It is clear that the BEd teachers across the different phases did not have the necessary conceptual understanding of the concept of zero raised to a power although some of them probably teach the concept in their classrooms.

### 5.4.2. Responses to 'Is zero even or odd?'

I present these findings in the different categories suggested by responses. The question extracted various responses, which often reflected limited conceptualisation, confusion and uncertainty regarding the parity of zero. Some teachers (7%) honestly admitted that they did not know or could not say whether zero is an even or odd number, for example *I don't know* and *You can't say because it got no value. It's nothing*.

#### 5.4.2.1. *Zero is neither an odd nor an even number*

The majority of teachers (41%) argued that zero is neither an odd nor an even number. They asserted, for example that . . . *all whole numbers can start with zero*; *It is not in the list of even or odd numbers*; . . . *it is something that is not there*; . . . *it has no value*; *Any number below zero is negative and above zero is positive* and . . . *odd numbers start with 1 and even numbers start with 2 and also zero is not a natural number while odd and even numbers are natural numbers*. It is not clear what the teacher meant by reasoning that whole numbers can start with zero. I know that some teachers (on the ACE course) had a tendency to use a numbering method, 01, 02, 03, . . . when they numbered pages. This practice is also applied in coding items in real life but it has no relation to even and odd numbers. Zero is included in the set of even numbers in the sequence of even integers, i.e. -4; -2; 0; 2; 4; . . ., which contests one teacher"s argument that odd and even numbers are natural numbers. One of the teachers in this category argued that zero is neither odd nor even but she also thought . . . *it is even because all even numbers have a zero* reflecting the confusing nature of this quality of zero, i.e. the parity of zero.

#### 5.4.2.2. *Zero is either an odd or an even number*

Some of the ACE teachers (10%) asserted that zero is either an even or an odd number. They reported that zero is *Both – we start from zero when counting* and it is odd or even . . . *depending on the position of the place it is holding in numbers especially when it is a unit*. These teachers possibly implied that the counting sequences 0; 2; 4; 6; . . . and 0; 3; 6; 9; . .

. implicated zero as an even and an odd number and that numbers ending in zero, for example 10, 350, etc. are even numbers and numbers such as 101 or 3 053 are odd numbers!

#### 5.4.2.3. *Zero is an odd number*

Teachers (10%) claimed that zero is an odd number because, for example . . . *any number when multiply by the number has a remainder* and . . . *because it is not divisible by 2*. These teachers reasoning implied, for example that 3 ÷ 2 = 1 remainder 1 or 5 ÷ 2 = 2 remainder 1 so that 3 and 5 are odd numbers and 6 ÷ 2 = 3 or 8 ÷ 2 = 4 so that 6 and 8 are even numbers. However, 0 ÷ 2 = 0 leaves no remainder and zero is divisible by 2.

#### 5.4.2.4. *Zero is an even number*

About one third of the teachers (32%) correctly classified zero is an even number by applying common sense in their reasoning. Some teachers claimed that they have never given the parity of zero any thought. These teachers provided logical explanations based on counting forward and backward, for example:

> *I never thought about it but I think zero is an even number because if you count the first number it is 1 or -1 which are both odd.*
>
> *I think zero is even because when I count in even numbers I start at 2 and skip 3 to 4. So if I can reverse I can count back 2 skip 1 and land on 0.*
>
> *Even, because between odd numbers are even numbers e.g. 0; 1; 2; 3; 4, therefore zero, being the first number will be followed by an odd number.*
>
> *Even number because odd numbers starts at one.*

Other teachers referred to place value, which did not clarify the parity of zero as a number but rather as a symbol, a digit in numbers. They argued, for example that . . . *when you look at the tens they end with zero and any number ended with zero is even*, . . . *zero is a place holder to any number when it is placed at the end of a number* and . . . *zero is a place holder of even numbers like 10, 20, 30*.

One teacher justified the concept in relation to multiples, for example . . . *because even numbers always ends in multiples of two including 0 e.g. 2, 4, 6, 8, 10 . . . 20 . . . 30 etc.* I was originally under the impression that the teacher referred to counting in multiples of two, i.e. 0; 2; 4; 6; 8; . . . to justify that zero is even but then realised that the argument was related to the symbol because the multiples of 10 were emphasised.

The BEd teachers could not reach agreement about the parity of zero during the focus group unstructured interview.

*X:* *This issue about is zero an even or odd number.*

*Z:* *Oh yes. Is zero an even or an odd number?*

*X:* *Even. Any number ending with zero can be divided by two.*

*N:* *(Laughter) It means, zero is nothing. But when you move from nothing, you got something. When you move from zero, the next step is one, which is odd. Then follows even and odd. It means you can't move odd to odd. So you move from even to odd, even, odd, even, odd.*

*K:* *I don't agree fully with her. What if you going to count in twos. You know, multiples of two. Then you're gonna have zero, two, four, six and then? If you're gonna count in multiples of uneven numbers, naught, one, three, five. So, you can't say naught is now an even number. Because we're counting now in uneven numbers.*

*W:* *Yes, we count in uneven numbers. You say one, three. You can't say zero, three.*

*Z:* *Oh, I see what you say. When you say zero, two, then you skip one.*

*K:* *Ok, but with multiples of three.*

*Z & W: Zero, three, six, nine*

*W:* *When you talk of multiples of three, zero is not a multiple. And zero is not a multiple of two.*

*Z:* *Is it a multiple of any number?*

*N:* *No, I don't think so. I think zero is neutral. Zero is not an even or an odd number.*

*A:* *I want to support that zero is not an even number. Because, because, no, it is not even or odd. I'm not sure in that area. I think zero is not an even number because all even numbers are divisible by two. They're all divisible by two. So we can't say zero is an even number.*

*W:* *If you go back to patterns, we've got a rule for even numbers, that is 2n. So, you can't get zero, zero n. If you use the rule for even numbers, you can't get zero. No, zero is an odd number.*

*Z:* *So, zero is odd?*

*W:* *It's neither. Neither. Even nor odd.*

*J:* *So, you can just say zero is neutral, neutral.*

Misconceptions occurred in teachers‟ reasoning concerning the parity of zero during the focus group unstructured interview. X originally argued that zero is an even number because numbers ending in zero are divisible by two, thus referring to multiples of 10 and including zero in this set but later agreed that zero is not a multiple of any number. N confirmed that zero is even by using the concept of skip counting to illustrate that zero is even, one is odd, two is even, etc. K did not „fully‟ agree with N because when you count in *multiples* of uneven numbers you get 0; 1; 3; 5 so zero is not an even number. W resolved this mistake by clarifying that zero should be excluded in the sequence – when you count in uneven numbers you start with one, three. . . W then claimed that zero is not a multiple of two or three and N echoed that zero is not a multiple of any number because it is *neutral* and zero is not an even or odd number. A declared that zero is not an even number and then continued that it is not even or odd because even numbers are divisible by two. W related the problem to the algebraic generalisation of even numbers, which is $2n$ and asserted that $0 \times n$ is impossible so that zero is not contained in the set of even numbers if the general rule for creating even numbers is considered. The teacher thus reasoned that zero is an odd number but then disputed this classification by claiming that it is neither odd nor even. J concluded that zero is neutral.

Doubt, confusion, inconsistence and contradiction were reflected in these teachers‟ reasoning. N was one of the first teachers to maintain that zero is an even number but afterwards claimed that zero is neither odd nor even because it is *neutral* and zero is not a multiple of any number. If multiples of a number are divisible by that number and 6, for example is a multiple of 2 and $6 \div 2 = 3$ then 0 is a multiple of 2 because $0 \div 2 = 2$. W (an FET Phase teacher) misconceived the generalisation for even numbers by claiming that zero is not an even number because „zero *n*‟ is unattainable but if the variable $n$ is 0 than $2n = 2 \times 0 = 0$.

Teachers are not expected to develop learners‟ understanding of the parity of zero, which explains their ignorance of this concept. They do not include zero in the set of even numbers. Investigating the parity of zero could lead to discussion and understanding of various related concepts. The study of Levenson (2007) provides insight into learners‟ difficulties with the conceptualisation of zero as an even number (refer to pp. 9-10, Chapter 2).

### 5.4.3. Responses to 'What is the origin of zero in the history of numbers?'

The forty-seven teachers in this study displayed limited knowledge of the origin and discovery of the idea of zero denoting the empty set or the invention of zero as a number in its own right (see Appendix 6). A few teachers (6%) offered insignificant and inaccurate explications of what they thought the history of zero is in relation to ancient times. They claimed that,

> *When people were using natural numbers they found out that when you lost all your possessions, for example sheep, they could not represent the empty space with a natural number they put zero.*
>
> *The Arabs discovered the use of 0.*
>
> *Zero comes from Greek word which means that there is nothing in the set.*
>
> *They found out that there were numbers beyond 1, . . . you start having nothing and then move up. Numbers beyond the ground have to start from something that was not there, then the number zero was discovered.*

Some teachers (31%) admitted that they were not knowledgeable about the history of zero by stating, for example *Ashamed to say I don't know*; *No idea*; *I don't know about such history. Sorry* and *I never hear about where it comes from*. One teacher suggested that she would investigate while another claimed that he had forgotten what the origin of zero was.

Most teachers (60%) provided descriptions related to counting, whole numbers or operations that are totally irrelevant to the history of zero in asserting, for example *It originated when one natural number was subtracted from the same natural number i.e. 2 – 2 = 0*, *0 is the first number in whole numbers* and *Zero is the starting point of all scales*.

Knowledge development of the inclusion of zero in the number system would not only open up the rich field of the history of number, but also understanding of different ancient number systems and the mathematical concepts that were developed many centuries ago (refer to pp. 2-4, Chapter 2). Involvement in research concerning the history of zero could especially assist in developing knowledge of place value and calculations with zero. Research on the history of number is plentiful and learners would be fascinated by stories about the history of zero. Teachers have a responsibility to motivate and stimulate learners" interest in mathematics (Ball, 2003) to develop curiosity and appreciation for the marvels and beauty of the subject. Learners are naturally fascinated by the behaviour of numbers if they are allowed the opportunity to develop a love for the subject. Teachers need to be empowered with knowledge about mathematics so that they are aware of the origins, development and exactness, i.e. the character, development and integrity of the subject (Ball, 2003). Ball, et al. (2008) advocated the development of specialised content knowledge – a combination of content and pedagogical content knowledge (refer to pp. 38-43, Chapter 2).

### 5.4.4. Responses to 'How do you teach the concept of zero?'

This discussion is based on teacher responses reflected in the written questionnaire that required individual teachers to record their general knowledge concerning the concept of

zero and indications of how they teach the concept of zero in their classrooms. The discussion is supplemented with data obtained in the BEd focus group unstructured interview.
Some of the ACE teachers (28%) reported that they had never taught or did not explicitly teach the concept of zero. Some of them reported that they had never considered teaching the concept. They claimed that, for example that *I never taught zero in my class at all*; *I don't get into details* and *I have never taught it in detail because I have never given the value of zero much thought. I assume they* (high school learners) *know.*

The majority of teachers (69%) mentioned that they provided learners with uninformed rules by telling or explaining facts to transmit knowledge of the concept of zero (as they were taught previously). They related these rules or facts to nothingness and concepts such as counting, place value, basic operations and number properties without allowing learners to construct knowledge of the concept. Teachers" accounts revealed that the teaching and learning of the concept of zero is based on the traditional approach. They report that, *I tell my learners that zero represents nothing*; *Telling learners that when counting we start from zero and zero is less than any number*; *I will tell them that zero is a place holder of unit numbers in tens, hundreds and thousands*; *I give them calculations in addition and subtraction and end up explaining . . .* or *I can say zero means nothing because you can't add zero to the number then you get a something. The numbers does not change if you add zero.*

About one third of the teachers (31%) asserted that they allowed their learners to develop the concept of zero through discovery and exploration using real life situations and/or concrete learning materials. One of the teachers claimed that:

> *I let my learners count forward and backwards until they come to zero and beyond. I also let them identify the different sets of numbers, patterns they find. I specifically ask them to find the results in 4 basic operations using 0 and 1. The results will tell that there is no change in basic operations to the original number.*

This teacher probably referred to zero‟s behaviour in addition ($1 + 0 = 1$; $0 + 1 = 1$) and subtraction with zero as the subtractor ($1 - 0 = 1$). The *original number* however, does change in subtraction with zero as the subtrahend ($0 - 1 = -1$), in multiplication ($1 \times 0 = 0$) and in division ($1 \div 0$ = undefined). A second teacher claimed that:

> *I demonstrate by putting concrete objects. Then I ask them to take nothing. Each time they will see that nothing happened. By counting with the real object and all of them to see that after 1 there's nothing.*

This strategy could assist in developing understanding of subtraction with zero as a subtractor, for example $3 - 0 = 3$, $30 - 0 = 30$, etc. to generalize that *a number remains unchanged if zero is subtracted from it.* The teacher also declared that learners would be able to *see that after 1 there's nothing* when learners count the *real objects.* This learning experience could assist learners in developing understanding of the empty set, i.e. a collection with zero objects described by Kaplan (1991:37) as an abstract concept because "... names belong to things but zero belongs to nothing". This activity could be extended to the use of a number line and calculator so that learners discover that *zero comes after 1* when counting back. Another teacher mentioned that:

> *I may give two learners 1 sweet each ask one learner to give his/her sweet to another and ask how many sweets is he/she left with.*

This teacher‟s explanation implied that her learners would be able to understand that $1 - 1 = 0$ after one of two learners gave his/her only sweet to the other. The activity could be used effectively to generate the rule that *any counting number subtracted from itself results in zero*, for example $3 - 3 = 0$, $30 - 30 = 0$, etc. Understanding of zero as the additive inverse could be constructed in this way. One of the teachers also claimed that she would:

> *Give learners exercises that include zero and check their meaning of zero using concrete objects.*

It is not clear how the teacher intended doing this but the strategy could assist learners to conceptualize that $3 \times 2 = 6$, $3 \times 1 = 3$ and therefore $3 \times 0 = 0$ as illustrated in Figure 5.4 below.

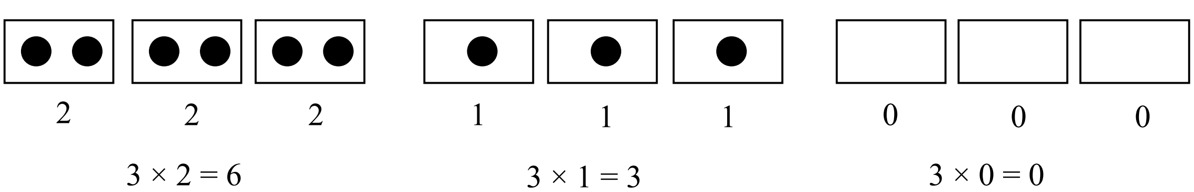



Fig. 5.4.: A practical demonstration to conceptualize multiplication by zero

Some of the teachers claimed that they assisted learners to make sense of the concept by allowing them to engage in practical learning situations such as counting, pattern recognition and basic operations. These teachers‟ responses implied that they allowed active learner participation to assist constructive understanding of the concept of zero.

Learners have to construct their own knowledge of rules such as *multiplication by zero always results in zero* in an inquiry-based learning environment. Wood, et al. (1993) suggest that learners should be allowed to make sense of abstract concepts by adjusting their cognitive structures in learning situations that mediate meaning. The development of unproductive cognitive structures without conceptual understanding is a product of the traditional teaching approach. The lack of conceptual understanding is often not observed because teachers assume that learners understand if they reproduce algorithms that they are not expected to explain. Learners who develop conceptual understanding are able to validate concepts and apply them accurately when necessary in different but related mathematical concepts (Kilpatrick, et. al., 2001). If they realise that they apply concepts inaccurately, they are able to examine, reflect and rectify their thinking and reasoning.

Teachers‟ knowledge of mathematical concepts should be well developed so that they can confidently address learners‟ confusion, uncertainties and misconceptions. Teachers should be knowledgeable about the concepts they teach, why and how they teach them so that they teach in the most enjoyable and constructive way to ensure conceptual understanding. Professional development and training programs should support teachers in developing knowledge of how learners learn best and how to deal with incorrect learning (refer to pp. 38-43, Chapter 2).

## 5.5. CONCLUSION

The findings in this study revealed that some teachers have developed effective conceptualisation of the concept of zero. More teachers however have limited knowledge of the concept. Their thinking and reasoning reflect misconceptions, which were mostly caused by traditional, teacher-centred teaching and learning. The findings allow me to generalise confidently that most teachers are not equipped to teach the concept of zero. They did not develop knowledge and understanding of multiplication and division by zero, of zero as a number and the pedagogical content knowledge needed for teaching the concept of zero. Most teachers in this study will therefore not be able to predict and deal with the misconceptions that learners experience with the concept. The findings concur with research studies reporting on the limited knowledge displayed by pre-service and in-service teachers in this regard (Wheeler & Feghali, 1983; Ball & McDiarmid, 1989; Tsamir, et al., 2000; Kahan, et al., 2003; Levenson, et al., 2007; Quinn, et al., 2008). Studies on teachers‟ understanding of multiplication by zero, the parity of zero, the history of zero and how the concept of zero is taught are limited or non-existent, however. Teacher responses reflected a lack of knowledge of the history of zero although the topic has been widely researched and documented (Reid, 1956; Gullberg, 1997; Kaplan, 1999; O‟Connor & Robertson, 2000; Anthony & Walshaw, 2004; Ball, 2005; Quinn, et al., 2008).

Teachers are products of the education system that policymakers and educationists attempt to improve (Ball, 2003). Most teachers experienced inadequate and erratic teaching during their school years, which was based on the traditional teaching approach. Teachers lack conceptual understanding. Explaining their knowledge of the concept of zero mainly involved rule-based accounts. It is not surprising that many teachers did not display effective understanding of the concept of zero. Problems in the teaching and learning of the concept of zero have not been highlighted although an acknowledgement that teachers experience difficulty with the concept have been reported for many years, for example Wheeler & Feghali (1983). The problem gets passed along from generation to generation.

Teachers‟ misconceptions could have pedagogical implications if they are expected to teach the concept of zero. Grade 5 and 6 learners are expected to develop, for example knowledge of zero as the additive inverse (Grades 5 and 6) but only 8% of (ACE) teachers in this study connected zero to this concept – some of these teachers even regarded zero is the subtractive inverse. Very few teachers displayed effective multiplicative thinking. Concepts such as commutativity and inverse relationships, a basis for advance mathematical reasoning, are based on the development of multiplicative thinking. Grade 7 learners are expected to develop the concept of exponents (South Africa. DoE, 2002) but teachers in the focus group interview could not explain why a number to the power zero is one. An FET teacher could not relate zero to the general rule for even numbers, i.e. $2n$.

There is currently no school textbook support for the development of the concept of zero. Authors of mathematics dictionaries should develop a common understanding of the classification of zero as a number (de Klerk, 1990; Tapson, 1996; Bolt & Hobbs, 1998). Citations in the three dictionaries revealed contrasting views of zero as an element of the set of whole numbers, which could contribute to teachers" confusion and uncertainty about the concept.

The teaching and learning of the concept of zero require teachers with effective mediation skills, which promote interaction, communication, reflection to develop sense making in a social environment where meaning is negotiated (Wood, et. al., 1993; Hiebert, et. al., 1996; Harries & Spooner, 2000, Kahan, et. al., 2003; Steele, 2001; Clarke, 2002; Wink & Putney, 2002). I am in agreement with researchers (Quinn, et. al., 2008), for example who assert that teaching the concept of zero requires competent teachers because of the abstract nature of the concept (which is probably the reason why zero has been side-lined in curriculum documents). Professional development courses and teacher development programs should focus on both content and pedagogical content knowledge development concerning the concept of zero. Textbook authors, curriculum developers and professional development educators should become aware of the difficulties that teachers and learners experience with the conceptualisation of zero"s qualities reported by various researchers (Wheeler & Feghali, 1983; Reys & Gouws in Wheeler & Feghali, 1983; Shulman, 1986; Ball, 1988; Cockburn, 1999; Ball, 2003; Kahan, et al., 2003; Ball, et al., 2008; Quinn, et al., 2008; Wilcox, 2008). Training and development programmes should raise awareness of the qualities and uses of the concept of zero because what the teacher believes, knows, thinks and decides has a profound effect on the way they teach as well as on students" learning in their classrooms (Fenema, Carpenter & Loef, 1989).

# CHAPTER 6
# CONCLUSION

## REFLECTION

This multi-case study was conducted within an interpretive, qualitative framework. The multi-tool data production method allowed me to make extensive inferences about the conceptualization of abstract mathematical concepts. I focused on learners" and teachers" understanding of the concept of zero, which has implications for learning and teaching mathematics in the classroom, for curriculum development, materials development and for the training and professional development of teachers.

The empirical field of the study involved learners in different grades, different schools in three different geographical locations in South Africa. It also involved teachers with varied experience in teaching. The first and second cases entailed data production in a grade 6 class and a grade 5 and 6 multi-grade class in a rural school in the Southern Cape. The third case involved Grade 3 and 4 learners in a multi-grade class in the Western Cape. These cases served as a pilot study in Stage 1 of the data production process. Stages 2 and 3 involved learners and teachers in the Eastern Cape which formed cases four and five. The teachers, based in rural and extreme rural areas across the Eastern Cape, attended professional development university courses during the period of the data production process. The grade 5 learners were from a local school in Grahamstown.

The diverse nature of the sample contributed to the validity of the findings, which provided confirmation that the research problem is widespread and prevalent. The data production methods required learners to demonstrate their knowledge of multiplication and division by zero in mental calculation and written elaboration tasks. Teachers had to demonstrate an understanding of multiplication and division by zero, the attributes of zero, the history of zero, and they had to explain how they teach the concept. The majority of learners and teachers in the sample did not display conceptual prior knowledge of the concept of zero from which to construct their own meaning of the concepts of multiplication and division by zero. Most of the teachers did not display conceptual understanding of zero as a number, the parity of zero and the history of the inclusion of zero in the number system. Teachers‟ accounts of how they teach the concept of zero mostly reflected rule-based teaching while a significant number of teachers indicated that they did not teach the concept at all. The study served to develop awareness of the concept of zero and the difficulties involved in teaching and learning the concept.

**KEY ARGUMENTS ADVANCED IN THE STUDY**

**Learners' and teachers' understanding of multiplication and division with zero**

1. *Knowing that zero multiplied or divided by any number results in zero is not evident of conceptual understanding.*

Grade 3 and 4 learners were not able to demonstrate conceptual understanding of multiplication by zero although some of these learners provided correct solutions for 4 × 0 and 0 ÷ 7 in the mental calculation tasks. Every Grade 5 and 6 learner in Stage 1 knew that zero multiplied and divided by any number results in zero, for example 0 × 1 = 0; 1 × 0 = 0 and 0 ÷ 1 = 0. The majority of Grade 5 and 6 learners were however not competent in illustrating conceptual understanding of the calculations in written elaboration tasks. Most teachers could not illustrate conceptual understanding of multiplication and division by zero although they knew that multiplication and division by zero as a dividend (0 ÷ 1) result in zero and division by zero as a divisor (1 ÷ 0) is undefined.

2. *Rule-based teaching does not facilitate sense-making.*

Grade 3 and 4 learners did not apply prescribed rules to illustrate understanding of multiplication by zero. Grade 5 and 6 learners were not able to construct and communicate meaning of the rules they have previously learnt for multiplication and division by zero. Teachers did not make sense of the rules they offered to explain understanding of multiplication and division by zero. The application of rules transmitted by teachers during learners‟ and teachers‟ previous learning experiences caused barriers in knowledge construction of calculations with zero.

3. *Limited knowledge of calculations with zero is often related to earlier learning experiences.*

A grade 5 learner attributed his inability to make sense of multiplication by zero on his learning experience in Grade 3. He claimed that the class teacher did not explain why a number multiplied by zero always results in zero. Teachers claimed that they *inherited* the rules for calculations with zero and asserted that they could not make sense of the rules.

4. *Intuitive structures hamper understanding of calculations with zero.*

The practically based structure of repeated addition using single objects applied by the grade 3 and 4 learners was effectual for multiplication with natural numbers but did not suffice the operation with zero. Grade 5 and 6 learners developed misconceptions when they imposed intuitive structures of subtraction, addition and doubling on multiplication and division with zero. Teachers over generalized the concept of doubling and number properties and applied real life contexts which became invalid in relation to knowledge demonstration of calculations with zero.

5. *Difficulties are experienced in expressing mathematical understanding.*

Grade 3 and 4 learners struggled to illustrate multiplication by zero pictorially and concluded that they could not draw zero. Grade 5 and 6 learners asserted that they could not explain rules for multiplying and dividing by zero *mathematically*. Teachers were not able to express conceptual understanding of rules for calculations with zero. Reports often reflected confusion, puzzlement and inability to express understanding of multiplication and division by zero.

6. *Multiplicative thinking that develops advance concepts is limited.*

Learners in this study did not use knowledge of the inverse relationships and the commutative property to explain multiplication and division by zero. A grade 5 learner illustrated conceptual understanding of commutativity applied to natural numbers but could not connect this understanding to multiplication by zero. Teachers‟ attempts to connect the commutative property to multiplication by zero were inaccurate.

*7. Misconceptions develop during sense making of multiplication and division by zero.*

Grade 3 and 4 learners did not develop misconceptions during meaning construction of multiplication by zero. Grade 5 and 6 learners‟ over generalization of existing knowledge of subtraction, addition and doubling developed misconceptions. Teachers imposed existing knowledge of rules (often inaccurate) for calculations with zero, doubling and number properties on calculations with zero which facilitated the development of misconceptions.

*8. Disregarding zero as a number causes problems in conceptual understanding.*

Grade 3 and 4 learners did not relate zero to *nothing*. Grade 5 and 6 learners often referred to zero as *nothing* which contributed to their inability to make sense of calculations with zero. Teachers often diminished the value of zero to *nothing* and disregarded zero as a number in its own right by claiming that *zero does not exist* or *has no value*. Learners and teachers who constructed effective understanding of calculations with zero even related zero to *nothing*.

*9. Division with zero as a divisor is challenging.*

Grade 5 and 6 learners were unable to provide the solutions *undefined* or *not allowed* for the problem 1 ÷ 0 although they indirectly claimed that division with zero as a divisor is impossible. The learners reported that 1 ÷ 0 = 0. Teachers who knew that division by zero is undefined were not competent in developing conceptual understanding of division by zero as a divisor.

*10. Division is intuitively connected to sharing and grouping.*

The grade 5 and 6 learners used real life and mathematical discourse more effectively in explaining division than multiplication. They used terminology and concepts of *dividing, sharing* and *grouping* and real life simulations to illustrate understanding of division by zero. Teachers applied concepts of *sharing, grouping, dividing and composite wholes* to explain division by zero more than in explanations for multiplication by zero. For division with zero as a divisor, the teachers employed mathematical terminology such as *undefined, cannot be defined, indivisible, indefinite, divide* and *not existing*.

*11. Understanding that division by zero is impossible is an intuitive notion.*

Grade 5 and 6 learners expressed conceptual understanding regarding the impossibility of division by zero as a divisor although they concluded procedurally that $1 \div 0 = 0$. Teachers who responded that $1 \div 0 = 0$ provided explanations that reflected conceptual understanding that division by zero as a divisor is impossible.

*12. Division by zero is not necessarily more difficult than multiplication by zero.*

Grade 5 and 6 learners made more sense of division than multiplication by zero. Teachers were more competent in conceptualizing division by zero than multiplication by zero.

*13. Understanding of the concept of zero developed with assistance during constructive learning experiences.*

Grade 3 and 4 learners developed conceptual and procedural understanding of multiplication by zero during constructive learning experiences. Grade 5 learners demonstrated understanding of division with zero as a dividend during constructive discussion. Teachers constructed meaning of zero as a number during productive discussion and debate.

*14. Practically based models could assist in the development of abstract thinking and reasoning.*

The intuitive practical models of repeated addition that Grade 3 and 4 learners originally applied in demonstrating understanding of multiplication with natural numbers caused a barrier to their sense-making of multiplication by zero. The learners developed conceptual and procedural understanding of multiplication by zero when the structure was modified and adapted, however. Grade 5 learners were able to conceptualize division with zero as a dividend when they used physical objects to demonstrate understanding. Teachers did not apply practically based models to illustrate understanding of calculations with zero.

*15. Learners and teachers are able to make sense of calculations with zero.*

Grade 3 and 4 learners made sense of multiplication by zero when the existing structure of repeated addition was adapted during interactive and shared knowledge construction. Some of the grade 5 and 6 learners applied effective existing multiplicative structures to assimilate and accommodate multiplication and division by zero. Some of the teachers established equilibrium between existing cognitive structures and multiplication and division by zero.

**Teachers' understanding of zero as a number**

Teachers offered diverse characteristics of zero as a number. They did not have a common understanding of zero as a whole number or integer, zero as a counting number and zero as an even number. A significant number of teachers related the value of zero to *nothing*, had no clear understanding of zero representing the empty set and did even not regard zero as a number. Teachers who classified zero as an identity element, an additive inverse and an exponent resulting in a base of 1 provided inaccurate explanations, over generalizations and rules. The teachers were often not competent in explaining and justifying the characteristics they ascribed to zero as a number. Justifications often reflected misconceptions. It became apparent that the majority of teachers had limited content knowledge of zero as a number in its own right. The characteristics of zero that developed from the teachers" responses are in themselves justification for the necessity of concept development and explicit teaching of the concept of zero. Knowledge of zero as a whole number or an integer, a counting number, an even number, a representation of the empty set, a place holder, an identity element, an additive inverse and an exponent is an essential foundation for learning mathematics.

**Teachers' knowledge of the origin of zero in the history of numbers**

Teachers displayed unawareness of the importance of zero in the development of the number system.

**Teaching the concept of zero**

1. *Explicit teaching of the concept of zero is limited.*

ACE teachers disclosed that they did not include the concept of zero in the teaching and learning of mathematics or they did not teach the concept noticeably.

2. *Teaching the concept of zero is based on uninformed rules.*

Most of the teachers reported that they teach the concept by the transmission of rules. They either tell or explain facts or procedures concerning positional calculations and arithmetical operations with zero. The rules or explanations offered by teachers in illustrating how they teach the concept of zero often displayed misconceptions developed by the teachers.

3. *Teaching the concept of zero, when it occurs, happens through guided discovery.*

A significant number of teachers reported that they involved learners in counting, pattern recognition and basic calculation activities to explore and discover the use of zero. Some of these teachers reported that they used demonstrations and concrete material to develop understanding of the concept of zero.

**LIMITATIONS**

- The school-based project in which I was involved in the Western Cape was in its third year of implementation, and classroom support visits ended in the middle of the third year. The two schools involved in the study are situated many kilometres outside Cape Town. I was not able to conduct interviews with the learners to enhance my understanding of the misconceptions reflected in their written explanations. I also had no opportunity to interview the two class teachers in the case to ascertain their understanding of the concept of zero. It would have been ideal to conduct an intervention in the grade 5 and 6 multi-grade classroom. This would have allowed me to obtain knowledge of the intuitive, practically based structures that they could possibly apply to the understanding of multiplication and division by zero. This would have allowed me to draw comparisons between Foundation and Intermediate Phase learners" application of intuitive cognitive structures.

- I observed that learners in Stage 3 of the data production process were not familiar with constructive cooperative learning, which was a requirement for the written elaboration task concerning calculations with zero. The temporary class teacher was not a mathematics teacher – the expert mathematics teacher had left the school the previous year. My relationship with the learners in Stage 1 was well established and the classroom atmosphere in the two classrooms was conducive to learning. In the Stage 3 classroom, I had to develop a relationship of trust with the learners. I had to address learners‟ behaviour consistently, which required direct intervention. My first visit to the school occurred at the end of January 2009. I only completed the data collection process at the end of March that year. I started teaching in this class twice a month to develop learners‟ number sense, which included the understanding of zero as a number. I could however not continue with these lessons for any protracted length of time because of my work commitments.

- After the data production process in Stage 2, I requested the ACE and BEd teachers to implement the written elaboration task involving calculations with zero in their own classrooms (without pre-teaching of the concept), and asked them to reflect on the learners‟ interactions. This task served as an assignment. The resultant data contains learner responses and teacher reflections from FP to the FET Phase. This data is not included in this study but serves as useful supplemental data. Including this data in the study would have provided me with the opportunity to investigate FP learners‟ conceptual understanding of division by zero, a current gap in this study.

- During Stage 1, I was not involved in a formal research program. I had to rely on my intuition for the data capturing and analysis process. I had limited knowledge of research design and methodology and was not familiar with literature concerning the topic. In 2009, I registered for the part-time MEd course but did not attend the research design session during that year. Uncertainty about this aspect of the thesis was a barrier in the writing-up process.

**RECOMMENDATIONS**

The study is concerned with the problems that learners and teachers experience with the concept of zero, in particular multiplication and division by zero. I also focused on teachers" knowledge regarding the characteristics of zero, the history of zero and the teaching of the concept of zero. I offer the following recommendations based on the findings.

- Development of the concept of zero should be specified explicitly in the mathematics curriculum. Learners should gradually develop conceptual understanding from an informal to a sophisticated level. Learners should be allowed to make sense of zero as representing the empty set and discouraged from classifying zero as *nothing* from the reception year of schooling. Zero should be included in counting exercises and acknowledged as an important number and an element of the set of even numbers. Learners should become aware that numbers extend beyond 1. They should regard the number zero with the same respect as the natural numbers. FP learners should be able to make sense of calculations with zero through practically based real life situations based on understanding of calculations with natural numbers. They should be granted opportunities to make sense of division by zero and encouraged to assert, for example that $1 \div 0$ is senseless. Development of the concept of zero in the FP should lay the basis for developing conceptual and procedural understanding of counting and calculating on a number line, the set of negative integers, decimal numbers, number properties, identity elements, inverse relationships, exponents, coordinates, accurate measurement, the construction of basic and trigonometric graphs, etc.

- Mental calculations should be connected to written or verbal knowledge demonstration to establish learners" understanding of memorized, easy-to-recall facts. Knowing and reporting that multiplication and division by zero results in zero and division by zero as a divisor is undefined is not evident of conceptual understanding. Assessment tasks that require the recall of facts only might not reflect understanding of concepts. Learners should be allowed to communicate and reflect on their thinking and reasoning processes because "One only knows something if one can explain it" (Vico, in Sewell, 2002:24).

- Learners in the IP should be motivated to display understanding through practical models. In the FP, learners often engage in practically based illustrations intuitively. Practical models should however develop into sound mathematically based accounts leading to more advanced and sophisticated thinking and reasoning. Intuitive counting and repeated addition structures should be adjusted to include alternative structures that accommodate understanding of calculations with zero, for example the area model for multiplication and division. I suggest the application of a structure of composite wholes because it assisted Grade 3 and 4 learners in this study to effectively construct meaning of multiplication by zero.

- The application of number properties should be developed so that learners are able to apply them flexibly and accurately in justifying calculations with zero. Multiplicative thinkers are able to manage composite units, to easily recollect or obtain existing multiplication and division facts, to connect them to related facts and to use properties of numbers and inverse relationships to construct new understanding. Using the inverse relationship to justify why $3 \times 0 = 0$ and $0 \div 3 = 0$ for example, might however be problematic; $3 \div 0$ does not have an inverse relationship. The commutative principle of multiplication and the inverse relationship between multiplication and division is within learners" zone of proximal development.

- Learners and teachers should be able to construct and explain the rules for calculating with zero because "The human mind can only know what the human has made" (Vico, in von Glaserfeld, 2011:4). Previous rule-based learning, embedded at a deep cognitive level, could cloud demonstration of knowledge of the concept of zero because the rules are not related to learners" existing knowledge of calculations with natural numbers. To this extent, zero is separated from the natural numbers because learners are not engaging in making sense of zero as a number.

- Teachers should develop a classroom atmosphere conducive to constructive communication and reflection so that learners express mathematical thinking and reasoning confidently. They should listen to learners‟ verbal accounts and arguments. Learners should present written reflections of their thinking. The use of mathematical language, in the form of verbal communication and the representation of symbolic models, is a basic requirement of sense making. Learners develop problem-solving skills through practically and mathematically based demonstrations and illustrations, the spoken language and written symbolic equations, in constructive whole-class or small-group discussions. These practices allow teachers to make sense of the learners‟ levels of meaning construction.

- The accurate expression of thinking should be used as a set of building blocks for new, advanced knowledge construction. Teachers should thereby become aware of learners‟ existing misconceptions and predict the misconceptions that learners might develop because ". . . one should be very, very careful what you put into that head, because you will never, ever get it out" (Vico, in Sewell, 2002:24). Teachers should mediate cognitive conflict by using the most understandable comparisons, images, demonstrations and debates to address misunderstandings (Shulman, 1986).

- The teaching of division should not be delayed until the third year of schooling. Division is an everyday human activity, and actions like sharing and grouping often arise in learners‟ social environment. Teachers should build on this intuitive knowledge. Learners should investigate the relationship between multiplication and division. The concept of multiplication by zero requires more intensive knowledge construction. Learners intuitively know that division by zero as a divisor is impossible, but they feel obliged to provide some numeric solution.

- Teachers need effective content and pedagogical content knowledge to teach the concept of multiplication and division by zero. Professional development educators and trainers should become aware of teachers‟ limited content knowledge concerning the concept of zero. Teachers in this study were confused by zero‟s characteristics as a whole number and integer, zero as an identity element and additive inverse, zero as an even number and zero raised to a power. Knowing mathematics is knowing how it develops as it is learned, how ideas can be structured and connected in the field, and how they might be unfolded and connected across time as students grow in terms of mathematical skills and maturity (Shulman, 1986; Ball, 2003).

- Teachers should engage in research assignments to become aware of the history of zero‟s inclusion in the number system. Developing short stories about the history of number could raise interest, appreciation for and fascination with mathematics. The mathematics curriculum (South Africa. DoE, 2002) included the development of the history of number to develop understanding and appreciation of ancient cultures‟ contributions to mathematics. This aspect has now been omitted in the latest mathematics curriculum (South Africa. DBE, 2010) for the IP and teachers are expected to develop learners‟ knowledge of the history of measurement only.

**IDEAS FOR FURTHER RESEARCH**

This research study could be of significant value to the teaching and learning of mathematics in the South African context. The findings could facilitate awareness and enhance understanding on the part of mathematics educators, curriculum developers, textbook authors and teachers of learners‟ and teachers‟ limited knowledge of the concept of zero. The study could raise awareness of the importance of the development, characteristics and use of the concept of zero in order to assist learners and teachers to develop effective conceptual and procedural understanding of the concept. This study revealed findings not previously reported, i.e. teachers‟ inability to conceptualize multiplication by zero and the parity of zero and their limited knowledge of the history of the inclusion of zero in the number system.

Further research should be conducted concerning the content and pedagogical content knowledge that teachers require in order to teach the concept of zero so that it makes sense to learners without the transfer of uninformed rules. South African learners‟ conception of zero as a number and its relation to problems with zero as a placeholder in number representation and calculations could be additional topics for investigation. Development of the concept of zero is included in the new CAPS curriculum (South Africa, DBE, 2011) from Grade R. This opens a field for further studies concerning learners‟ sense-making.

**A CONCLUDING NOTE**

This study advocates the importance of developing conceptual understanding of the concept of zero to address the difficulties that learners and teachers experience with the concept. It is acknowledged that teachers were not expected to develop learners‟ understanding of the concept until the fifth grade, which implies that zero was isolated in various ways in the lower grades. Teachers with effective content and pedagogical content knowledge probably teach the concept effectively. But those who teach concepts stipulated by the curriculum without connecting within and across the mathematical curriculum content are unlikely to teach the concept at all. Teachers and learners experience problems with the concept of zero because it is separated from teaching and learning the natural numbers. Because the concept is not assimilated within existing knowledge, development of the unknown concept often leads to cognitive conflict and the construction of misconceptions. Teachers who struggle with unknown content, uncertainties and confusion might unwittingly transmit ineffective conceptions to their learners.

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# APPENDICES

**APPENDIX 1**

Parent consent letter

30 January 2009

THE PARENTS OF GRADE 5 LEARNERS

…………. PRIMARY

Dear Parent

I, Zonia Jooste, an ex-mathematics teacher, deputy-principal, textbook author and teacher developer and an ex-Rhodian has recently started in a mathematics lecturer post at Rhodes University. I am also enrolled on the Master"s in Education program. After collecting research data in the Western Cape, I am currently in the process of collecting data involving Grade 5 learners as well as teachers on our university courses.

I will be involved in your Grade 5 learner"s mathematics classroom about twice per month assisting the learners with the development of effective mental calculation strategies – a major problem among our learners.

I wish to request your consent for the involvement of your child in my research study, which will also benefit him/her in learning mathematics. I can assure you that your child will by no means be identified in the research study. Strict anonymity will be performed.

I"m looking forward to be involved in your child"s mathematics teaching and learning.

Yours in Education

…………………………..

Ms Zonia Jooste

Lecturer/Facilitator/Researcher

**APPENDIX 2**

School consent letter

30 January 2009

THE PRINCIPAL

…………. PRIMARY

……………………….

Dear ……………..

In response to our verbal conversation on Friday 23 January 2009, I hereby wish to express my sincere gratitude to you for granting me the opportunity to collect data in your Grade 5 class for my Master‟s Research Study in mathematics.

It will also be my greatest pleasure to assist with your school‟s dilemma regarding the lack of a permanent specialist mathematics teacher. I therefore wish to make myself available to teach mathematics in the Grade 5 class twice per month. As per our discussion, I will assist your learners in developing/improving their basic mental calculation skills and strategies.

I wish to assure you that learners‟ or any other possible participants will not be identified in the research study. I also attach a letter addressed to the parents of the Grade 5 learners to request their permission for learners to be involved in the study. I will bring copies of the letter on my first visit to the classroom on Tuesday 3 February 2009.

Yours in Education

……………………………….

Ms Zonia Jooste

Lecturer/Facilitator/Researcher

**APPENDIX 3**

Mental calculation questionnaire (1)

| Multiplication & Division Mental Calculations | | | |
|---|---|---|---|
| Name: | Surname: | Age: | Grade: |
| ONE MINUTE MULTIPLICATION | | | ONE MINUTE DIVISION |
| 1 | 1 × 3 = | 1 | 4 ÷ 1 = |
| 2 | 2 × 2 = | 2 | 0 ÷ 7 = |
| 3 | 4 × 0 = | 3 | 4 ÷ 2 = |
| 4 | 8 × 1 = | 4 | 3 ÷ 3 = |
| 5 | 5 × 2 = | 5 | 10 ÷ 5 = |
| 6 | 3 × 3 = | 6 | 5 ÷ 5 = |
| 7 | 2 × 4 = | 7 | 12 ÷ 4 = |
| 8 | 4 × 3 = | 8 | 50 ÷ 5 = |
| 9 | 3 × 5 = | 9 | 6 ÷ 3 = |
| 10 | 7 × 2 = | 10 | 70 ÷ 10 = |
| 11 | 4 × 4 = | 11 | 15 ÷ 3 = |
| 12 | 1 × 5 = | 12 | 25 ÷ 5 = |
| 13 | 6 × 4 = | 13 | 9 ÷ 3 = |
| 14 | 9 × 3 = | 14 | 16 ÷ 4 = |
| 15 | 2 × 8 = | 15 | 20 ÷ 5 = |
| 16 | 3 × 10 = | 16 | 18 ÷ 9 = |
| 17 | 4 × 5 = | 17 | 12 ÷ 2 = |
| 18 | 8 × 3 = | 18 | 14 ÷ 7 = |
| 19 | 9 × 2 = | 19 | 27 ÷ 9 = |
| 20 | 5 × 5 = | 20 | 21 ÷ 7 = |
| 21 | 7 × 4 = | 21 | 18 ÷ 3 = |
| 22 | 6 × 3 = | 22 | 30 ÷ 6 = |
| 23 | 7 × 5 = | 23 | 42 ÷ 7 = |
| 24 | 6 × 6 = | 24 | 45 ÷ 5 = |
| 25 | 5 × 8 = | 25 | 28 ÷ 7 = |
| 26 | 7 × 6 = | 26 | 24 ÷ 4 = |
| 27 | 4 × 9 = | 27 | 54 ÷ 6 = |
| 28 | 9 × 5 = | 28 | 49 ÷ 7 = |
| 29 | 6 × 5 = | 29 | 63 ÷ 9 = |
| 30 | 5 × 8 = | 30 | 35 ÷ 5 = |

**APPENDIX 4**

Written Calculations Elaboration Questionnaire (2)

Teaching Grade/s: ……………. Date: ………………..

- Solve the following calculations.
- Give a written explanation for each solution.

1 – 0 = ………

..........................................................................................................................

..........................................................................................................................

0 – 1 = ………

..........................................................................................................................

..........................................................................................................................

0 × 1 = ………

..........................................................................................................................

..........................................................................................................................

1 × 0 = ………

..........................................................................................................................

..........................................................................................................................

0 ÷ 1 = ………

..........................................................................................................................

..........................................................................................................................

1 ÷ 0 = ………

..........................................................................................................................

..........................................................................................................................

**APPENDIX 5**

Written elaborations: knowledge and teaching of the concept of zero (3)

Dear BEd/ACE Student,

Please complete the following questionnaire. Your responses will be used as data in a research project. Please be ensured that your name will not be used in the research. Your co-operation is highly appreciated.

Teaching Grade/s: ………………   No of years in teaching: …………….

Highest qualification:……………..

1. What is zero?

…………………………………………………………………………………………

…………………………………………………………………………………………

3. Is zero even or odd? Why?

…………………………………………………………………………………………

…………………………………………………………………………………………

2. (a) Count back in 1s from 10

…………………………………………………………………………………………

(b) Count back in 2s from 20:

…………………………………………………………………………………………

(c) Count forwards in 6s up to 30:

…………………………………………………………………………………………

3. Five runs are scored during the first ten minutes of a cricket match. What are the combinations that the two batsmen could have scored?

…………………………………………………………………………………………

4. How do you go about teaching the concept of zero in your classroom?

…………………………………………………………………………………………

…………………………………………………………………………………………

5. What is the origin of zero in the history of numbers?

…………………………………………………………………………………………

…………………………………………………………………………………………

Thank you for your co-operation

**APPENDIX 6**

Presentation of teachers" knowledge of the concept of zero

| What is zero? | BEd | ACE |
|---|---|---|
| • Identity element for +, -, x | 75 | 18% |
| • Counting/whole/neutral no | 87,5 | 51% |
| • Nothing/no value/empty set | 37,5 | 44% |
| • Any base to exponent zero = 1 | 12,5 | ---- |
| • Additive inverse of + and – | ---- | 8% |
| • Placeholder | ---- | 15% |
| • A number less than 1 | ---- | 5% |
| • Next to no: increase value | | 5% |

| Is zero odd or even? | BEd | ACE |
|---|---|---|
| • Even | 37,5 | 39 |
| • Odd | 12,5 | 8 |
| • Odd & even | ---- | 10 |
| • Not odd or even | 37,5 | 44 |
| • Don't know | ---- | 2 |
| • Can't say. Nothing | 12,5 | ---- |

| How do you teach the concept of zero? | BEd | ACE |
|---|---|---|
| • Never taught the concept | ---- | 28% |
| • Represents nothing | ---- | 23% |
| • Relate to concepts such as place value, basic operations, number properties, etc. | 62,5 | 57% |
| • Rule-based | 62,5 | 75% |
| • Learner discovery/practical exploration | 37,5 | 25% |

| What is the origin of Zero in the history of numbers? | BEd | ACE |
|---|---|---|
| • Don't know/Forgot | 25% | 36% |
| • First/Start of counting/whole numbers | 62,5% | 23% |
| • Started counting from nothing | 12,5% | 8% |
| • Origin is multiplication | | 2% |
| • Greek word for nothing | | 2% |
| • Arabs discovered zero | | 2% |
| • Empty set/placeholder/nothing | | 2% |

**APPENDIX 7**

Layout of a grade 5 classroom