TANGENTIALLY SYMPLECTIC FOLIATIONS

by

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Abstract

This thesis is concerned principally with tangential geometry and the applications of these concepts to tangentially symplectic foliations. The subject of tangential geometry is still at an elementary stage. The author here systematises current concepts and results and extends them, leading to the definition of vertical connections and vertical G—structures. Tangentially symplectic foliations are then characterised in terms of vertical symplectic forms. Some significant particular cases are discussed.

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Introduction

The subject of this thesis is tangentially symplectic foliations. This is a novel concept. The notion is natural : the leaves of the foliation carry a symplectic structure in such a way that together they yield an almost symplectic structure of a special kind on the underlying manifold. To investigate tangentially symplectic foliations I have preferred to use the concepts of tangential geometry. These seem better suited to the subject, since they carry more naturally the properties of this kind of foliation.

Transversal geometry is well studied [see below for references]. Tangential geometry on the other hand has received relatively little attention and has not yet been systematised. In this thesis therefore I have found it necessary to develop in detail the subject of tangential geometry before applying it to tangentially symplectic foliations. While some aspects of this study have involved compilation and reworking of known concepts and results, the major portion of it is, as far as I am aware, novel in its content.

To set the scene for my exposition I will now briefly review some aspects of the background of tangentially symplectic foliations.

Poisson structures

Poisson structures were first introduced by *Sophus Lie* [Lie] in connection with his study of Lie groups and their realisation by canonical transformations.

Their importance was rediscovered in the 1970's in connection with linear representation theory and as a setting for generalisations of Hamiltonian mechanics. A Poisson manifold as introduced by *André Lichnerowicz* [Li1] is essentially a union of symplectic manifolds which fit together in a smooth way.

Foliation theory

Foliation theory has its origins in the study of solutions of ordinary differential equations.

A first approximation to the concept of a foliation is a dynamical system and the resulting decomposition of a domain by its trajectories.

A second approximation is the idea of a foliation as a decomposition of a manifold into [immersed] submanifolds, all of the same dimension.

The global study of foliations on manifolds was begun in the 1940's by *Charles Ehresmann* and *Georges Reeb* [ER]. Since then the study of foliated manifolds, particularly from the point of view of global topology and global differential geometry, has produced an extraordinarily rich and varied collection of works. At the moment, the subject is the focus of a great deal of research activity.

Tangential and transversal geometry

A foliation on a manifold gives rise to two types of geometries, namely tangential and transversal.

The tangential geometry [or leaf geometry] of a foliation is the geometry infinitesimally modeled by the tangent bundle of the foliation. This concept is, in a sense, dual to that of transversal geometry of a foliation, which is the geometry infinitesimally modeled by the transverse bundle of the foliation.

A geometric structure is said to be transverse [for a given foliation] if it is "locally projectable along the leaves". For example, the condition for a foliation to be Riemannian is a "transverse property", being given by the existence, on the local quotient manifolds, of a Riemannian structure that is invariant by sliding along the leaves.

The study of such structures constitutes the transversal geometry of a foliation, which can be thought of as the differential geometry of the [singular] space of leaves. These ideas have their origin in the work of *C. Ehresmann* [E] and *A. Haefliger* [H2].

Since the beginning of the theory of foliations on manifolds it has been realised that the transverse properties are the most important. So, the most part of the theorems on foliated manifolds are of the following type : a certain transverse property implies a property of the foliation.

Transversal geometry has received a good deal of attention in the literature (see, e.g., [H3], [He1], [He3], [Mo1], [Mo3], [Mo7], [Wo]).

In contrast, there are relatively few works on tangential geometry. However, some good results have been obtained on totally geodesic foliations (see [BH1], [C1], [C2], [C3], [C4], [CG], [G1], [JW1], [JW2]), tangentially affine foliations (see [I], [IM]), tangentially projective foliations (see [IM]), Lagrangian foliations (see [I], [W1]), and Legendre foliations (see [I], [J], [Lb2], [P]).

If one prefers, the tangential geometry is precisely the differential geometry of the leafwise manifold associated to the foliation [which can be thought of as a "tangential model" for the foliation].

So, for the study of tangential geometry of a foliation there are two essentially equivalent options : either working on the vertical bundle [i.e., the tangent bundle of the foliation] or investigating the geometry of a single leaf. In other words, instead of imposing "geometric constraints" in the transverse direction that are invariant along leaves [i.e., in the tangent direction], we consider geometric structure "along leaves" and require to be, additionally, invariant in the direction of some given "horizontal bundle" [i.e., a complement of the vertical bundle].

Note that the tangential geometry depends upon the choice of a horizontal bundle, which is not the case in transversal geometry. In tangential geometry we "act" on the vertical bundle of the foliation [which is obviously integrable] but need also to consider at the same time a "horizontal bundle". The latter is, in general, not integrable, so the tangential geometry is, in this sense, "weaker" than the transversal one. There is more "freedom", and so less "order".

A particular type of foliation, Riemannian foliations, was introduced by *Bruce L. Reinhart* [Re1] in 1959. This notion is quite intuitive : one imposes the existence of a "bundle–like" Riemannian metric on the underlying manifold [i.e., a metric for which the leaves of the foliation remain locally at constant distance from each other]. The condition is so natural that Riemannian foliations appears as a good candidate for modeling situations drawn from mechanics or physics.

B.L. Reinhart established some important properties of these foliations (cf. [Re1], [Re2], [Re3]).

In the late 1970's and early 1980's *Pierre Molino* developed a comprehensive structure theory for Riemannian foliations (cf. [Mo3], [Mo5], [Mo6] ; see also [Mo7]).

Tangentially symplectic foliations

A tangentially symplectic foliation is a foliation such that the ambient space [i.e., the underlying manifold] is covered by a collection of distinguished charts for which the coordinate transformations are symplectic in the direction tangent to the leaves and such that the symplectic structure "along the leaves" is invariant in the direction of the given "horizontal bundle" for the foliation [i.e., a complement of the vertical bundle].

This notion is, in a sense, dual to that of [transversely] symplectic foliation ([Kt2], [Mc], [Mo8]).

Such a structure [i.e., a tangential symplectic structure] induces a [regular] Poisson structure on the underlying manifold.

So, the study of tangentially symplectic foliations combines, in a sense, the theory of Poisson manifolds with foliation theory.

Tangentially symplectic foliations appear in several branches of mathematics :

(a) The characteristic foliation of a regular Poisson manifold inherits a natural tangential symplectic structure (see [HMS]).

(b) A canonical manifold, as defined by *André Lichnerowicz* (cf. [FLS], [Li1]), is naturally equipped with a codimension 1 tangentially symplectic foliation (see also [M]).

(c) A symplectic fibre bundle, as defined in [GLSW], determines a tangentially symplectic foliation.

The concept of Poisson–Riemann structure has been introduced by *Pierre Molino* [Mo9] in 1990. A [regular] Poisson–Riemann structure on a given [connected] Riemannian manifold (M,g) is a Riemannian tangentially symplectic foliation with the following property : the complete [global] vector fields tangent to the foliation, that preserve the induced Poisson structure and the horizontal bundle of the foliation, act transitively on the leaves.

[In this case the horizontal bundle is the orthogonal complement (with respect to the "bundle–like" metric g) of the tangent bundle of the foliation].

Some results have been obtained by *P. Molino* and *C.M. Diop* [DM], [D], but the theory is far from complete.

An important way to realise Poisson manifolds [and consequently tangentially symplectic foliations] is as quotients of symplectic manifolds by group actions.

Let G be a Lie group acting freely on a symplectic manifold M. Assume that the quotient space M/G is a manifold [this is the case when G is compact ; moreover, in this case, M has the structure of a principal fibre bundle over M/G with structure group G]. Then the manifold M/G has a Poisson structure for which the projection $p : M \longrightarrow M/G$ is a Poisson morphism.

Outline of thesis

The material in this thesis is distributed as follows. The first chapter is a compilation of some basic material [concepts, notations and results] concerning structures on manifolds, in a form suitable for later use.

I review the following standard objects and results in sections 1.1 to 1.3 : manifold, tangent bundle, tangent mapping, vector field, Lie bracket, cotangent bundle, differential form, Liouville form, exterior differentiation, interior product, 1–parameter group of transformations, [immersed] submanifold, tensor field, Lie differentiation, Riemannian metric, fibre bundle, principal fibre bundle, reduced bundle, vector bundle, symplectic manifold, standard symplectic form, canonical symplectic form, symplectic transformation, Hamiltonian vector field, locally Hamiltonian vector field, Poisson bracket, Poisson manifold, characteristic foliation of a regular Poisson manifold.

In section 1.4, I review basic notions that underlie the theory of foliations : leaf, distinguished (or foliated) chart, foliated atlas, simple foliation, local automorphism, plaque, leaf topology, leafwise manifold, distinguished map, basic function, tangent and transverse bundle, foliate and transverse field. I also present some standard general methods of constructing foliations, and some examples of foliations.

Finally, in section 1.5, I indicate how a Poisson manifold is naturally associated with a foliation.

In chapter two, the tangential geometry of a foliation is considered. Basic concepts such as : tangential functions, horizontal, vertical and tangential vector fields, vertical and tangential vector fields, vertical and tangential forms, sheets and tangential holonomy, and vertical G–structures are introduced. I then investigate several aspects of tangential geometry, and give examples of vertical G–structures. Finally the notion of tangentially symplectic foliation is defined.

In chapter three, I investigate tangentially symplectic foliations, and establish some basic properties of this structure.

In chapter four, I present three interesting candidates for applications and identify relevant features of these that are worthy of investigation. These are : canonical manifolds, symplectic fibre bundles and [regular] Poisson–Riemann structures.

Contents

Introduction

Chapter 1. PRELIMINARIES 1

Chapter 1

PRELIMINARIES

In this chapter I briefly review basic material concerning structures on manifolds : I introduce the "standard" objects of this dissertation [e.g., manifold, submanifold, tensor field, fibre bundle and foliation] and recall some standard results. I also use this chapter as an occassion to establish notation and terminology.

Throughout, the *Einstein convention* applies, that is, a repeated index, one upper and one lower, denotes summation over its range.

All manifolds, mappings, tensor fields, foliations, curves, etc. are assumed to be smooth [i.e., of class C^∞] unless otherwise stated.

1.1 MANIFOLDS AND TENSOR FIELDS ([Bt], [KN], [YK])

By a *manifold* of dimension m is meant a Hausdorff space with a countable basis of open sets together with a differentiable structure [or a manifold structure] of class C^∞ and dimension m. An arbitrary *chart* in a given manifold M is denoted by (U,φ). We say that U is the *domain* of the chart [or a *coordinate neighbourhood*]. The system of functions

$$x^1 = pr^1 \circ \varphi, \cdots, x^m = pr^m \circ \varphi$$

defined on U is called a *local coordinate system* in U.

By a *mapping* [or a *map*] of a manifold into another is meant a differentiable mapping of class C^∞. A *function* on M is a differentiable mapping of class C^∞ of M into $\mathbb{R}$.

The set $\Omega^0(M)$ of all functions on a given manifold M has the structure of an algebra over $\mathbb{R}$ [in particular, the structure of a ring].

A diffeomorphism of M onto iself is said to be a *transformation* of M. The set of all transformations of M, denoted by Diff(M), forms a group.

Let M be a manifold.

The set T_xM of all *vectors* tangent to M at the point $x \in M$ has the structure of a real linear space, called the *tangent space* to M at x.

Consider the union

$$TM := \bigcup_{x \in M} T_xM$$

of the tangent spaces to M at all points $x \in M$. Then the set TM has the natural structure of a manifold.

(1.1.1) **DEFINITION.** The manifold TM is called the *tangent bundle* of the manifold M. Let

$$p\colon TM \longrightarrow M$$

be the *natural projection.* The preimages T_xM of the points $x \in M$ under the mapping p are called *fibres* of the bundle TM.

Let $f\colon M \longrightarrow N$ be a mapping of a manifold into another and let

$$f_{*,x} : T_xM \longrightarrow T_{f(x)}N$$

denote the induced linear mapping of the corresponding tangent spaces.

This mapping is called the *tangent mapping* of f [at x].

The mapping

$$f_* : TM \longrightarrow TN,\ f_*\big|_{T_xM} = f_{*,x}$$

maps the fibres of TM linearly into the fibres of TN.

Let (U,φ) be an arbitrary chart on M with local coordinates $(x^1,\cdots,x^m)$.
Notice that

$$p^{-1}(U) = TU \approx U \times \mathbb{R}^m.$$

Let $(e_1,\cdots,e_m)$ be the standard basis of the linear space $\mathbb{R}^m$.

We write

$$T\varphi(v) = (x^1,\cdots,x^m,\dot{x}^1,\cdots,\dot{x}^m),$$

where $v = (x,e) \in T_xM$, $x \in U$ and $e = \dot{x}^1e_1 + \cdots + \dot{x}^me_m$.

The pair

$$(TU,\ T\varphi)$$

is a chart on TM, called the *natural chart* induced by (U,φ). The corresponding local coordinates will be called *adapted local coordinates* [on TM].

Let M be a manifold with tangent bundle TM.

(1.1.2) DEFINITION. A *vector field* X on M is a mapping $X : M \longrightarrow TM$ such that the mapping $p \circ X : M \longrightarrow M$ is the identity mapping 1_M or equivalent such that the diagram

$$\begin{tikzcd}
 & TM \arrow[dr, "p"] & \\
M \arrow[ur, "X"] \arrow[rr, "1_M"'] & & M
\end{tikzcd}$$

is commutative [i.e., $p(X(x)) = x$ for all $x \in M$].

If (U,φ) is a chart with local coordinates $(x^1,\cdots,x^m)$ then the tangent mapping of φ at x

$$\varphi_{*,x} : T_xM \longrightarrow T_{\varphi(x)}\mathbb{R}^m \approx \{\varphi(x)\} \times \mathbb{R}^m$$

is an isomorphism and therefore the vectors

$$\left.\frac{\partial}{\partial x^i}\right|_x := \varphi_{*,x}^{-1}(\varphi(x),e_i) \quad (i = 1,\cdots,m)$$

form a basis for T_xM.

Thus, every vector field X on M can be written uniquely as

$$X\Big|_U = X^1\frac{\partial}{\partial x^1} + \cdots + X^m\frac{\partial}{\partial x^m}.$$

We denote by $\mathfrak{X}(M)$ the set of all vector fields on a given manifold M. $\mathfrak{X}(M)$ is a $\Omega^0(M)$–module and its elements act as derivations on this ring.

If $X \in \mathfrak{X}(M)$ and $f \in \Omega^0(M)$, then :

$$(Xf)(x) := \left.\frac{d}{dt}\right|_{t=0}(f \circ c_x)(t),$$

where $c_x : [0,1] \longrightarrow M$ is a curve passing through x and representing the vector $X(x) \in T_xM$.

If X and Y are in $\mathfrak{X}(M)$, the *Lie bracket* $[X,Y]$ is the vector field on M defined by :

$$[X,Y]f := X(Yf) - Y(Xf).$$

With respect to this bracket operation, $\mathfrak{X}(M)$ is a Lie algebra over $\mathbb{R}$.

In particular, we have *Jacobi's identity* :

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0,$$

for all $X,Y,Z \in \mathfrak{X}(M)$.

The following relation holds :

$$[fX,gY] = fg[X,Y] + f(Xg)Y - g(Yf)X,$$

for $f,g, \in \Omega^0(M)$ and $X,Y \in \mathfrak{X}(M)$.

A transformation f of M induces an automorphism f_* of the Lie algebra $\mathfrak{X}(M)$, defined by :

$$(f_*X)(x) := f_{*,y}X(y), \text{ where } f(y) = x.$$

Let M be a manifold.

For a point $x \in M$, the dual linear space T_x^*M of the tangent space to M at x is called the *space of covectors* at x [or the *cotangent space* at x].

Consider the union

$$T^*M = \bigcup_{x \in M} T_x^*M$$

of all cotangent spaces to M at all points $x \in M$. Then the set T^*M has a natural structure of a manifold.

(1.1.3) DEFINITION. The manifold T^*M is called the *cotangent bundle* of the manifold M. Let

$$q : T^*M \longrightarrow M$$

be the *natural projection.* The preimages T^*M of the points $x \in M$ under the mapping q are called *fibres* of the bundle T^*M.

Let (U,φ) be an arbitrary chart on M with local coordinates $(x^1,\cdots,x^m)$.
Notice that :

$$q^{-1}(U) = T^*U \approx U \times (\mathbb{R}^m)^*.$$

Let $(\epsilon^1,\cdots,\epsilon^m)$ be the standard cobasis of the linear space $(\mathbb{R}^m)^*$.
We write

$$T^*\varphi(\xi) = (x^1,\cdots,x^m,p_1,\cdots,p_m),$$

where $\xi = (x,\epsilon) \in T_x^*M$, $x \in U$ and $\epsilon = p_1\epsilon^1 + \cdots + p_m\epsilon^m$.

The pair

$$(T^*U, T^*\varphi)$$

is a chart on T^*M, called the *natural chart* induced by (U,φ).

The corresponding local coordinates will be called *adapted local coordinates* [on T^*M].

Let T^*M be the cotangent bundle of a manifold M.

(1.1.4) DEFINITION. The *Liouville form* [or the *canonical 1–form*] on T^*M is the form

$$\lambda \equiv \lambda^{(M)} : T^*M \longrightarrow T^*(T^*M),$$
$$a \longmapsto \lambda_a = (q^*)_a(a)$$

where $q : T^*M \longrightarrow M$ is the natural projection and

$$(q^*)_a : T^*_{q(a)}M \longrightarrow T^*_a(T^*M)$$

is the transpose of the tangent mapping $q_{*,\alpha} : T_\alpha(T^*M) \longrightarrow T^*_{q(\alpha)}M$ at $\alpha \in T^*M$.

In adapted local coordinates $(T^*U, T^*\varphi)$ on T^*M, the Liouville form λ may be expressed as

$$\lambda\Big|_{T^*U} = p_1dx^1 + \cdots + p_mdx^m.$$

Let M be a manifold with cotangent bundle T^*M. Let $\wedge\, T^*_xM$ be the exterior algebra over T^*_xM.

(1.1.5) DEFINITION. An r–*form* ω on M is an assignment of an element of degree r in $\wedge\, T^*_xM$ to each point x of M, i.e., a mapping

$$\omega : M \longrightarrow \bigcup_{x \in M} \overset{r}{\wedge}\, T^*_xM, \quad x \longmapsto \omega_x.$$

If (U,φ) is a chart with local coordinates $(x^1,\cdots,x^m)$, then the $dx^i(x)$ $(i = 1,\cdots,m)$ are a basis for T^*_xM, so the

$$dx^{i_1}(x) \wedge \cdots \wedge dx^{i_r}(x) \quad (1 \leq i_1 < \cdots < i_r \leq m)$$

are a basis for $\overset{r}{\wedge}\, T^*_xM$.

Thus every r–form ω can be written uniquely as

$$\omega\Big|_U = \sum_{i_1 < \cdots < i_r} \omega_{i_1 \cdots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r}$$
$$= \frac{1}{r!} \omega_{i_1 \cdots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r} .$$

An r–form ω can be defined also as a skew–symmetric r–linear mapping over $\Omega^o(M)$ of $\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)$ [r–times] into $\Omega^o(M)$. We denote by $\Omega^r(M)$ the set of all r–forms on a given manifold M. $\Omega^r(M)$ has the structure of a $\Omega^o(M)$–module. We set

$$\Omega^\bullet(M) := \bigoplus_{r=0}^{m} \Omega^r(M).$$

With respect to the exterior product, $\Omega^\bullet(M)$ has the structure of an algebra over $\mathbb{R}$, called the *exterior algebra* of M.

Let $f: M \longrightarrow N$ be a mapping of a manifold into another.

(a) If $\omega \in \Omega^o(N)$, then we denote by $f^*\omega$ the mapping $\omega \circ f$.

(b) If $\omega \in \Omega^r(N)$ $(r \geq 1)$, then we consider the *pull–back* $f^*\omega$ of ω by f, defined as

$$(f^*\omega)(x)(X_1, \cdots, X_r) := \omega_x(f_{*,x}X_1, \cdots, f_{*,x}X_r),$$

where $x \in M$ and $X_1, \cdots, X_r \in T_xM$.

The mapping

$$f^*: \Omega^\bullet(N) \longrightarrow \Omega^\bullet(M), \ \omega \longmapsto f^*\omega$$

is a homomorphism of algebras.

(1.1.6) PROPOSITION. *The Liouville form* λ *on the cotangent bundle* T^*M *is the unique 1–form on* T^*M *which satisfies*

$$\omega^*\lambda = \omega$$

for every 1–form ω *on* M.

Let M be a manifold with exterior algebra $\Omega^\bullet(M)$. *Exterior differentiation* $d \equiv d_M$ can be characterised as follows :

(i) d is an $\mathbb{R}$–linear mapping of $\Omega^\bullet(M)$ into itself such that

$$d(\Omega^r(M)) \subset \Omega^{r+1}(M) ;$$

(ii) if $f \in \Omega^0(M)$, then

$$df(X) = Xf \text{ for all } X \in \mathfrak{X}(M)$$

[i.e., df is the total differential] ;

(iii) if $\omega \in \Omega^r(M)$ and $\eta \in \Omega^s(M)$, then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^r \omega \wedge d\eta ;$$

(iv) $d^2 = 0$.

Let ω be an r–form. Then we have :

$$(d\omega)(X_1,\cdots,X_{r+1}) = \frac{1}{r+1}\sum_{i=1}^{r+1} (-1)^{i+1} X_i(\omega(X_1,\cdots,\hat{X}_i,\cdots,X_{r+1}))$$

$$+ \frac{1}{r+1} \sum_{1 \leq i < j \leq r+1} (-1)^{i+j} \omega([X_i,X_j], X_1,\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots, X_{r+1}),$$

where the symbol ^ means that the term is omitted.

In terms of local coordinates, if

$$\omega = \sum_{i_1 < \cdots < i_r} \omega_{i_1 \cdots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r},$$

then

$$d\omega = \sum_{i_1 < \cdots < i_r} d\omega_{i_1 \cdots i_r} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r}.$$

The exterior differentiation d commutes with f^*:

$$d(f^*\omega) = f^*(d\omega),$$

where $f\colon M \longrightarrow N$ is a mapping and $\omega \in \Omega^r(N)$.

Let M be a manifold with exterior algebra $\Omega^\bullet(M)$ and let X be a vector field on M. The *interior product* ι_X with respect to X of $\Omega^\bullet(M)$ can be characterised as follows:

(i) ι_X is an $\mathbb{R}$–linear mapping of $\Omega^\bullet(M)$ into itself such that

$$\iota_X(\Omega^r(M) \subset \Omega^{r-1}(M) ;$$

(ii) if $f \in \Omega^0(M)$, then

$$\iota_X f = 0 ;$$

(iii) if $\omega \in \Omega^1(M)$, then

$$\iota_X \omega = \omega(X) ;$$

(iv) if $\omega \in \Omega^r(M)$ and $\eta \in \Omega^s(M)$, then

$$\iota_X(\omega \wedge \eta) = \iota_X\omega \wedge \eta + (-1)^r \omega \wedge \iota_X\eta.$$

If ω is an r–form, then :

$$(\iota_X\omega)(X_2,\cdots, X_r) = \omega(X,X_2,\cdots,X_r).$$

In terms of local coordinates, if

$$\omega = \frac{1}{r!}\,\omega_{ii_2\ldots i_r}\,dx^i \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_r},$$

then

$$\iota_X\omega = \frac{1}{(r-1)!}\,\omega_{ii_2\ldots i_r}\,X^i dx^{i_2} \wedge \cdots \wedge dx^{i_r}.$$

REMARKS.

(i) $\iota_X^2 = 0$.

(ii) If $f\colon M \longrightarrow N$ is a diffeomorphism, then :

$$\iota_X(f^*\omega) = f^*(\iota_{f_*X}\,\omega)$$

Let X be a vector field on a manifold M. A curve $c\colon J \subset \mathbb{R} \longrightarrow M$ in M is called an *integrel curve* of X if, for every $t \in J$, the vector $X_{c(t)}$ is tangent to the curve c at $x = c(t)$. For any point $x \in M$, there is a unique integral curve c of X, defined for $|t| < \epsilon$ for some $\epsilon > 0$, such that $c(0) = x$.
A 1–*parameter group of transformations* of M is a mapping

$$\mathbb{R} \times M \longrightarrow M,\ (t,x) \longmapsto \varphi_t(x)$$

which satisfies the following conditions :

(i) For each $t \in \mathbb{R}$,

$$\varphi_t\colon M \longrightarrow M,\ x \longmapsto \varphi_t(x)$$

is a transformation of M.

(ii) $\varphi_{t+s} = \varphi_t \circ \varphi_s$ for all $t,s, \in \mathbb{R}$.

Each 1–parameter group of transformations (φ_t) induces a vector field.

A *local* 1–*parameter group of local transformations* can be defined in the same way, except that $\varphi_t(x)$ is defined only for t in a neighbourhood of 0 and x in an open set of M.

(1.1.7) PROPOSITION. *Let* X *be a vector field on a manifold* M. *For each point* $x \in M$, *there exist a neighbourhood* U *of* x, *a positive number* δ *and a local 1–parameter group of local transformations* $(\varphi_t)_{|t| < \delta}$, *which induces the given* X.

We shall say that X generates a local 1–parameter group of local transformations $(\varphi_t)_{|t| < \delta}$ in a neighbourhood of x.

If there exists a [global] 1–parameter group of transformations of M which induces X, then we say that X is *complete.*

On a compact manifold M, every vector field X is complete.

Let $f: M \longrightarrow M$ be a transformation of M. If a vector field X generates a local 1–parameter group of local transformations (φ_t), then the vector field $f_* X$ generates $(f \circ \varphi_t \circ f^{-1})$. As a corollary, a vector field X is invariant by f [i.e., $f_* X = X$] if and only if f commutes with (φ_t).

Let $f: M \longrightarrow N$ be a mapping of a manifold M into another. We say that f is an *immersion* [resp. a *submersion*], if its tangent mapping [at x]

$$f_{*,x}: T_x M \longrightarrow T_{f(x)} N$$

is injective [resp. surjective] for every point $x \in M$. When an immersion $f: M \longrightarrow N$ is such that f is a homeomorphism of M onto $f(M) \subset N$, then we say that f is an *imbedding* [of M into N].

(1.1.8) DEFINITION. A *submanifold* of a manifold M is a subset L of M possessing a manifold structure such that the canonical injection $i_L: L \longrightarrow M$ is an imbedding [of L into M].

When the canonical injection i_L is just an immersion [but not necessarily an imbedding], we say that L is an *immersed submanifold* of M.

Let M be a manifold.

Let $\mathscr{T}^r_s(M)$ denote the $\Omega^o(M)$–module of all $\Omega^o(M)$–multilinear mappings

$$\Omega^1(M) \times \cdots \times \Omega^1(M) \times \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \longrightarrow \Omega^o(M).$$

$$[\Omega^1(m) \text{ r times}, \mathfrak{X}(M) \text{ s times}]$$

We put

$$\mathscr{T}^r_o(M) = \mathscr{T}^r(M)$$

$$\mathscr{T}^o_s(M) = \mathscr{T}_s(M)$$

and

$$\mathscr{T}^o_o(M) = \Omega^o(M).$$

(1.1.9) DEFINITION. A *tensor field* K or M of type (r,s) is an element of $\mathscr{T}^r_s(M)$. Such a tensor field K is said to be *contravariant* of degree r and *covariant* of degree s. In particular, the tensor fields of type (0,0), (1,0) and (0,1) on M are just the functions on M, the vector fields and the 1–forms on M, respectively.

If $x \in M$, we consider the tensor product

$$T^r_s(x) := \overset{r}{\otimes} T_xM \otimes \overset{s}{\otimes} T^*_xM$$

and we put

$$T^o_o(x) = \mathbb{R}.$$

The tensor field K gives rise to a mapping [denoted by the same letter]

$$K\colon M \longrightarrow \bigcup_{x \in M} T^r_s(x), \; x \longmapsto K_x.$$

Conversely, a mapping as above defines a tensor field K of type (r,s) such that

$$K_x = K(x) \quad \text{for all } x \in M.$$

If (U,φ) is a chart with local cooordinates $(x^1,\cdots,x^m)$, then

$$\left.\frac{\partial}{\partial x^{i_1}}\right|_x \otimes \cdots \otimes \left.\frac{\partial}{\partial x^{i_r}}\right|_x \otimes dx^{j_1}(x) \otimes \cdots \otimes dx^{j_s}(x)$$

are a basis for $T^r_s(x)$.

Thus every tensor field K can be written uniquely as

$$K\Big|_U = K^{i_1\cdots i_r}_{j_1\cdots j_s}\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_r}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_s},$$

where $K^{i_1\cdots i_r}_{j_1\cdots j_s}$ are called the *components* of K with respect to the above basis.

We set

$$\mathcal{T}(M) := \bigoplus_{r,s=0}^{\infty} \mathcal{T}^r_s(M).$$

With respect to the tensor product, $\mathcal{T}(M)$ has the structure of an algebra over $\Omega^0(M)$ called the *mixed tensor algebra* of M.

Every transformation f of M induces an algebra automorphism $\tilde{f}$ of $\mathcal{T}(M)$, which preserves type.

Let X be a vector field on M and let (φ_t) be a global 1–parameter group of transformations of M generated by X.

For each t, $\tilde{\varphi}_t$ is an automorphism of the mixed tensor algebra $\mathcal{T}(M)$.

For any tensor field K on M, we set

$$(\mathcal{L}_X K)_x = \lim_{t\to 0}\,[K_x - (\tilde{\varphi}_t K)_x]$$

(1.1.10) DEFINITION. The mapping

$$\mathcal{L}_X:\ \mathcal{T}(M) \longrightarrow \mathcal{T}(M),\ K \longmapsto \mathcal{L}_X K$$

is called the *Lie differentiation* with respect to X.

REMARK. There is no difficulty in modifying the definition of Lie differentiation when X is not complete.

Lie differentiation $\mathcal{L}_X$ with respect to a vector field X satisfies the following conditions:

(i) $\mathscr{L}_X$ is a derivation of $\mathscr{T}(M)$

[i.e., it is linear and satisfies

$$\mathscr{L}_X(K \otimes K') = (\mathscr{L}_X K) \otimes K' + K \otimes (\mathscr{L}_X K')$$

for all $K, K' \in \mathscr{T}(M)$].

(ii) $\mathscr{L}_X$ is type–preserving :

$$\mathscr{L}_X(\mathscr{T}^r_s(M) \subset \mathscr{T}^r_s(M).$$

(iii) $\mathscr{L}_X f = Xf$ for every $f \in \Omega^0(M)$.

(iv) $\mathscr{L}_X Y = [X,Y]$ for every $Y \in \mathfrak{X}(M)$.

Let K be a tensor field of type (r,s). Then we have :

$$(\mathscr{L}_X K)(\eta^1,\cdots,\eta^r; Y_1,\cdots,Y_s) = \mathscr{L}_X(K(\eta^1,\cdots,\eta^r; Y_1,\cdots,Y_s) + + \sum_{i=1}^{r} K(\eta^1,\cdots,L_X\, \eta^i,\cdots,\eta^r; Y_1,\cdots,Y_s) - \sum_{j=1}^{s} K(\eta^1,\cdots,\eta^r; Y_1,\cdots,\mathscr{L}_X Y_j,\cdots,Y_s).$$

In terms of local coordinates, if

$$K = K^{i_1 \dots i_r}_{j_1 \dots j_s}\, E^{j_1 \dots j_s}_{i_1 \dots i_r} ,$$

then

$$\mathscr{L}_X K = \left(X^l \frac{\partial K^{i_1 \cdots i_r}_{j_1 \cdots j_s}}{\partial X^l} - \sum_{h=1}^{r} K^{i_r \dots i_{h-1} l i_{h+1} \dots i_r}_{j_1 \dots j_s} . \frac{\partial X^{i_h}}{\partial X^l} + \sum_{K=1}^{s} K^{i_1 \dots i_r}_{j_1 \dots j_{k-1} l j_{k+1} \dots j_s} \cdot \frac{\partial X^l}{\partial X^{j_k}} \right) E^{j_1 \dots j_s}_{i_1 \dots i_r},$$

where $E^{j_1 \dots j_s}_{i_1 \dots i_r} = \frac{\partial}{\partial X^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial X^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}$.

As relations among d, $\mathscr{L}_X$ and ι_X, we have :

(v) $\mathscr{L}_X = d \circ \iota_X + \iota_X \circ d$ for every $X \in \mathfrak{X}(M)$.

(vi) $\iota_{[X,Y]} = \mathscr{L}_X \circ \iota_Y - \iota_Y \circ \mathscr{L}_X$ for every $X,Y \in \mathfrak{X}(M)$.

(1.1.11) DEFINITION. *A Riemannian metric* on M is a tensor field $g \equiv g_M$ of type (0,2), which satisfies the following conditions :

(i) It is symmetric :

$$g(X,Y) = g(Y,X) \text{ for any } X,Y \in \mathfrak{X}(M).$$

(ii) It is positive–definitive :

$$g(X,X) \geq 0 \text{ for every } X \in \mathfrak{X}(M)$$

and

$$g(X,X) = 0 \text{ if and only if } X = 0.$$

A manifold M equipped with a Riemannian metric g is called a *Riemannian manifold.* A Riemannian metric gives rise to an inner product on each tangent space T_xM to M at x.

In terms of local coordinates, the components of g are given by

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \; i,j = 1,2,\cdots,m.$$

We call g_{ij} the *covariant components* of g. The *contravariant components* g^{ij} of g are defined by

$$g^{ij} = g(dx^i, dx^j), \; i,j = 1,2,\cdots,m.$$

We have then

$$g_{ij}g^{jk} = \delta^k_i = \begin{cases} 1, & k = i \\ 0, & k \neq i. \end{cases}$$

Let M and F be manifolds.

(1.1.12) DEFINITION. A *fibre bundle over* M *with typical fibre* F consists of a manifold E and a surjective submersion $\pi : E \longrightarrow M$ such that the following condition of *local triviality* is satisfied :

(LT) Every point x of M has a neighbourhood U such that $\pi^{-1}(U)$ is isomorphic with $U \times F$ in the sense that there is a diffeomorphism

$$\varphi_U : \pi^{-1}(U) \longrightarrow U \times F$$

[called a U–isomorphism] such that the following diagram commutes :

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\varphi_U"] \arrow[dr, "\pi"'] & & U\times F \arrow[dl, "pr^1"] \\
& U &
\end{tikzcd}$$

[i.e., $pr^1 \circ \varphi_U = \pi$ or, equivalently,

$$\pi(\varphi_U^{-1}(x,t)) = x \text{ for all } x \in U \text{ and } t \in F \text{].}$$

- A fibre bundle will be denoted by (E,π,M), $E \xrightarrow{\pi} M$ or simple E.

We call E the *total space*, M the *base space* and π the *projection.*

For each $x \in M$, the inverse image $E_x = \pi^{-1}(x)$ is a closed submanifold of E, called the *fibre over* X. It is diffeomorphic to F by the restriction $\varphi_{U,x}$ of φ_U to E_x, where $x \in U$.

For each $e \in E$, the submanifold $E_{\pi(e)}$ is called the *fibre through* e.

A pair (U, φ_U) is referred to as a *local trivialization* of the fibre bundle.

Let (U,φ_U) and (V,φ_V) be two local trivializations of a fibre bundle $E \xrightarrow{\pi} M$ with typical fibre F. If $U \cap V \neq \emptyset$ we get the following commutative diagram :

$$\begin{tikzcd}
(U\cap V)\times F \arrow[dr] & \pi^{-1}(U\cap V) \arrow[l, "\varphi_U"'] \arrow[r, "\varphi_V"] \arrow[d] & (U\cap V)\times F \arrow[dl] \\
& U\cap V &
\end{tikzcd}$$

and the mapping

$$\varphi_{UV}\colon U\cap V \longrightarrow \mathrm{Diff}(F),$$
$$x \longmapsto \varphi_{U,x} \circ \varphi_{V,x}^{-1}$$

satisfies the following two conditions :

(i) $\varphi_{UU}(x) = 1_F$ for any $x \in U$;

(ii) $\varphi_{UW}(x) = \varphi_{UV}(x) \circ \varphi_{VW}(x)$ for any $x \in U \cap V \cap W$.

The mapping φ_{UV} is called the *coordinate transformation* over $U\cap V$.

Let $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi} M$, two fibre bundles over the manifold M. By a *fibre bundle morphism* of E into E′ we mean a mapping

$$f\colon E \longrightarrow E'$$

such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

[i.e., f preserves the fibres].

For many fibre bundles there is an additional structure which may be described by the following data:

(FB–1) A Lie group G and an effective action $\mathcal{A}\colon G \times F \longrightarrow F$ of G on the typical fibre F.

(FB–2) For every two local trivializations (U,φ_U) and (V,φ_V) such that $U\cap V \neq \emptyset$, a mapping

$$a_{UV}\colon U\cap V \longrightarrow G$$

such that :

$$[\varphi_{UV}(x)](t) = \mathcal{A}(a_{UV}(x),t) \text{ for any } x \in U\cap V \text{ and } t \in F.$$

Note that :

(i) $a_{UU}(x) = 1_G$ for any $x \in U$;

(ii) $a_{UW}(x) = a_{UV}(x) \circ a_{VW}$ for any $x \in U \cap V \cap W$.

(1.1.13) DEFINITION.

(i) A set of local trivializations of a fibre bundle $E \xrightarrow{\pi} M$ satisfying conditions (FB–1) and (FB–2) is called a G–*atlas.*

(ii) Two G–atlases are said to be *equivalent* if their union is again a G–atlas.

(iii) A fibre bundle together with an equivalence class of G–atlases is called a *fibre bundle with structure group* G [or a G–*bundle*].

If, in particular, F = G and the action of G on itself is left translation, then fibre bundles structure group G and typical fibre G are called *principal fibre bundles with structure group* G [or *principal* G–*bundles*].

An equivalent definition is the following.

Let M be a manifold and G a Lie group.

(1.1.14) DEFINITION. *A principal fibre bundle over* M *with structure group* G consists of a manifold P and an action of G on P satisfying the following conditions :

(PB–1) G acts freely [i.e., without fixed points] on P on the right :

$$P \times G \longrightarrow P, \ (u,a) \longmapsto R_a(u) = ua.$$

(PB–2) M is the quotient space of P by the equivalence relation induced by G: $M = P/G$ and the canonical projection π: $P \longrightarrow M$ is differentiable.

(PB–3) P is locally trivial, that is, every point $x \in M$ has a neighbourhood U such that $\pi^{-1}(U)$ is isomorphic with $U \times G$ in the sense that there is a diffeomorphism

$$\varphi : \pi^{-1}(U) \longrightarrow U \times G$$

such that

$$\varphi(u) = (\pi(u), \tilde{\varphi}(u)),$$

where $\tilde{\varphi}$ is a mapping of $\pi^{-1}(U)$ into G satisfying :

$$\tilde{\varphi}(ua) = \tilde{\varphi}(u)a \text{ for all } u \in \pi^{-1}(U) \text{ and } a \in G.$$

A principal fibre bundle will be denoted by P(M,G), $G \longrightarrow P \xrightarrow{\pi} M$ or simple P.

A *homomorphism* f of a principal bundle $P'(M',G')$ into another principal bundle P(M,G) consists of a mapping :

$$f' : P' \longrightarrow P$$

and a homomorphism

$$f'' : G' \longrightarrow G$$

such that

$$f'(u'a') = f'(u')f''(a') \text{ for all } u' \in P \text{ and } a' \in G'.$$

For the sake of simplicity, we shall denote f' and f'' by the same letter f.

A homomorphism

$$f: P'(M',G') \longrightarrow P(M,G)$$

is called an *imbedding* [or *injection*], if $f: P' \longrightarrow P$ is an imbedding, and $f: G' \longrightarrow G$ is an injective homomorphism. If $f: P' \longrightarrow P$ is an imbedding, then the induced mapping $f: M' \longrightarrow M$ is also an imbedding.

By identifying P' with $f(P') \subset P$, G' with $f(G') \subset G$ and M' with $f(M') \subset M$, we say that $P'(M',G')$ is a *subbundle* of P(M,G).

If, moreover, $M' = M$ and the induced mapping $f\colon M' \longrightarrow M$ is the identity transformation of M,

$$f\colon P'(M',G') \longrightarrow P(M,G)$$

is called a G–*reduction* of P(M,G). The subbundle P′(M′,G′) is called a *reduced bundle* [of P(M,G)].

Let M be a manifold.

(1.1.15) DEFINITION. An ℓ – *dimensional* [real] *vector bundle* over M [or a *vector bundle over* M *with typical fibre* $\mathbb{R}^\ell$] is a fibre bundle E over M with typical fibre $\mathbb{R}^\ell$ together with the structure of an ℓ–dimensional [real] vector space on each fibre E_x such that the following condition of *local triviality* is satisfied :

(LT)′ Each point x of M has an open neighbourhood U and an U–isomorphism

$$\varphi_U : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^\ell$$

such that the restriction

$$\varphi_{U,x} : E_x \longrightarrow \mathbb{R}^\ell$$

is a vector space isomorphism for each $x \in U$.

Let $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M$ be two vector bundles over the manifold M. A *vector bundle morphism* [over M] of E into E′ is a fibre bundle morphism

$$f\colon E \longrightarrow E'$$

such that the restriction

$$f_x : E_x \longrightarrow E'_x$$

is $\mathbb{R}$–linear for each $x \in M$.

A vector bundle morphism f is said to be of *constant rank* k if the restriction

$$f_x : E_x \longrightarrow E'_x$$

is of rank k for each $x \in M$.

(1.1.16) PROPOSITION. *Let* f: $E \longrightarrow E'$ *be a vector bundle morphism* [*over* M] *of constant rank* k. *Then* ker f, im f and coker f *are vector bundles* [*over* M].

The usual terminology of exact sequences carries over to vector bundles and vector bundle morphisms of constant rank. For example, if

$$0 \longrightarrow E' \xrightarrow{f} E \xrightarrow{g} E'' \longrightarrow 0$$

is an exact sequence [i.e., f is a monomorphism, im f = ker g and g is an epimorphism], E and ker g are isomorphic, and E" and coker f are isomorphic. This is called a *short exact sequence.*

1.2 SYMPLECTIC MANIFOLDS ([AG], [AM], [LM], [W2])

Let M be a manifold of dimension $m = 2k$.

(1.2.1) DEFINITION. A *symplectic structure* on M is given by a 2–form σ on M, called a *symplectic form*, satisfying the following properties :

(S–1) It is non–degenerate :
$\sigma^k \neq 0$ at all points of M.

(S–2) It is closed: $d\sigma = 0$.

A manifold M equipped with such a structure is called a *symplectic manifold.*

Let σ be a 2–form on M. We write

$$\sigma^\flat(v) = -\iota_v\sigma,$$

where $v \in T_xM$.

In other words, $\sigma^\flat(v)$ denotes the unique element of T_x^*M such that, for every $w \in T_xM$:

$$\sigma^\flat(v)(w) = -\sigma(v,w).$$

The associate map

$$\sigma^\flat : TM \longrightarrow T^*M,\ v \longmapsto \sigma^\flat(v)$$

is a vector bundle morphism.

(1.2.2) PROPOSITION. *The 2–form σ is non–degenerate if and only if the associate map $\sigma^\flat$ is an isomorphism.*

Let (M,σ) be a symplectic manifold and let $\sigma^\flat$ be the corresponding isomorphism. Denote the inverse isomorphism by $\Lambda^\#$.
For every $x \in M$, $\xi \in T_x^*M$ and $v \in T_xM$, one has:

(i) $i_{\Lambda^\#(\xi)}\sigma = -\xi$;

(ii) $\sigma(\Lambda^\#\xi),v) = -\xi(v)$.

REMARKS.

(i) In order to simplify the notation, if $x \in M$, $v \in T_xM$ and $\xi \in T_x^*M$, we shall write

$$^\flat v = \sigma^\flat(v)\ ;\ ^\#\xi = \Lambda^\#(\xi).$$

(ii) The same notation will also be used for vector fields and 1–forms. For every vector field X and every 1–form η, we shall denote by $^\flat X$ and $^\#\eta$, respectively, the 1–form and the vector field which satisfy :

$$^\flat X = \iota_X\sigma\ ;\ \iota_{^\#\eta}\sigma = -\eta$$

(1.2.3) DEFINITION. Let (M,σ) be a symplectic manifold. A vector field X on M is said to be *Hamiltonian* if the 1–form $^\flat X = -\iota_X\sigma$ associated with it is exact.
Then every function $f : M \longrightarrow \mathbb{R}$ satisfying

$$^\flat X = df$$

[or, equivalently,

$$X = {}^{\#}(df)\ ]$$

is called a *Hamiltonian* associated with X.

The set $H(M,\sigma)$ of all Hamiltonian vector fields has the structure of a Lie algebra.

(1.2.4) DEFINITION. Let (M_1,σ_1) and (M_2,σ_2) be symplectic manifolds of the same dimension. A diffeomorphism

$$\varphi : M_1 \longrightarrow M_2$$

is called a *symplectic transformation* [or a *symplectomorphism*] if

$$\varphi^* \sigma_2 = \sigma_1.$$

The set Aut (M,σ) of all symplectic transformations of a symplectic manifold (M,σ) onto itself is a group, which is not a finite–dimensional Lie group.

(1.2.5) DEFINITION. Let (M,σ) be a symplectic manifold. A vector field X on M is called a *symplectic infinitesimal automorphism* [or a *locally Hamiltonian vector field*] if the local 1–parameter group $(\varphi_t)_{|t| < \delta}$ associated to X has the following property : for all t, the [local] diffeomorphism φ_t is a symplectic transformation from the open subset D_t of M onto its image $\varphi_t(D_t) = D_{-t}$.

1.2.6 PROPOSITION. *Let* X *be a vector field on the symplectic manifold* (M,σ). *The following statements are equivalent* :

(i) X *is a symplectic infinitesimal automorphism.*

(ii) *The Lie derivative of* σ *along* X *is zero :*

$$\mathscr{L}_X \sigma = 0.$$

(iii) *The 1–form* ${}^{\flat}X = -\,\iota_X \sigma$ *associated with* X *is closed :*

$$d({}^{\flat}X) = 0.$$

The set $\mathrm{aut}(M,\sigma) = L(M,\sigma)$ of all symplectic infinitesimal automorphisms of (M,σ) has the structure of a Lie algebra.

(1.2.7) REMARKS.

(i) The Lie algebra $H(M,\sigma)$ of all Hamiltonian vector fields is not of finite dimension.

(ii) The set $H(M,\sigma)$ is an ideal of the Lie algebra $\mathrm{aut}(M,\sigma) = L(M,\sigma)$, which contains the derived ideal:

$$[L(M,\sigma), L(M,\sigma)] \subset H(M,\sigma).$$

Let (M,σ) be a symplectic manifold and $\Omega^0(M)$ the ring of functions on M. We define a bilinear map, called the *Poisson bracket*, as follows :

$$\{\ ,\ \} : \Omega^0(M) \times \Omega^0(M) \longrightarrow \Omega^0(M)$$
$$(f,g) \longmapsto \{f,g\} := \sigma({}^{\#}(df), {}^{\#}(dg)).$$

When equipped with this composition law, the set $\Omega^0(M)$ is a Lie algebra and the map

$$\Omega^0(M) \longrightarrow H(M,\sigma),\ f \longmapsto {}^{\#}(df)$$

is a surjective Lie algebra homomorphism.

(1.2.8) EXAMPLES.

(i) The space $\mathbb{R}^{2k}$, equipped with the 2–form

$$\sigma_0 = dx^1 \wedge dp_1 + \cdots + dx^k \wedge dp_k,$$

called the *standard symplectic form*, is a symplectic manifold. [Here $\mathbb{R}^{2k}$ is identified with

$$T^*\mathbb{R}^k = \mathbb{R}^k \times (\mathbb{R}^k)^* \ ;$$

suitable coordinates in $\mathbb{R}^{2k}$ are therefore written as

$$(x^1,\cdots,x^k,p_1,\cdots,p_k)\ ].$$

(ii) The cotangent bundle T^*M of a manifold M has a natural symplectic structure, which is called the *canonical symplectic structure.*

The *canonical symplectic form* $\sigma_c \equiv \sigma_c^{(M)}$ giving this structure is the exterior derivative $d\lambda$ of the Liouville form λ on T^*M.

Furthermore, for every closed 2–form η on M, the form

$$\sigma = d\lambda + q^*\eta,$$

where λ is the Liouville form on T^*M and $q^*\eta$ is the pull–back of η by the canonical projection

$$q : T^*M \longrightarrow M,$$

defines a symplectic structure on T^*M.

1.3 POISSON MANIFOLDS ([Li1], [LM], [M], [W3], [W4])

Let M be a manifold.

(1.3.1) DEFINITION. A *Poisson structure* on M is a map

$$\{\cdot,\cdot\}\colon \Omega^0(M) \times \Omega^0(M) \longrightarrow \Omega^0(M),$$

called a *Poisson bracket*, satisfying the following properties :

(P–1) It is $\mathbb{R}$–bilinear.

(P–2) It is skew–symmetric : $\{g,f\} = -\{f,g\}$.

(P–3) It is a derivation in each of its arguments:

$$\{f_1f_2,g\} = \{f_1,g\}f_2 + f_1\{f_2,g\}$$
$$\{f,g_1g_2\} = \{f,g_1\}g_2 + g_1\{f,g_2\}.$$

(P–4) It satisfies the *Jacobi identity* :

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0.$$

A manifold M equipped with such a structure is called a *Poisson manifold.*

On every Poisson manifold M, there exists a unique 2–times contravariant skew–symmetric tensor field Λ, such that for all f,g $\in \Omega^{0}(M)$,

$$\{f,g\} = \Lambda\ (df,dg).$$

Λ is called the *Poisson tensor* [or the *structure tensor*] of the Poisson manifold.

REMARKS.

(i) Let M be a manifold and Λ a 2–times contravariant skew–symmetric tensor field on M. We may associate with Λ a Poisson bracket on M by seetting

$$\{f,g\} = \Lambda\ (df,dg),$$

for all f,g $\in \Omega^{0}(M)$.

Then conditions (P–1), (P–2) and (P–3) of Definition (1.3.1) are satisfied, but, in general, condition (P–4) is not satisfied. Lichnerowicz ([Lil]) has shown that condition (P–4) is satisfied if and only if

$$[\Lambda\ \Lambda] = 0,$$

where $[\cdot\,,\cdot\,]$ is the Schouten–Nijenhuis bracket for skew–symmetric contravariant tensor fields (see also [Kz]).

Thus, a Poisson structure on a manifold M is defined by a 2–times contravariant skew–symmetric tensor field Λ on M, such that the Schouten–Nijenhuis bracket of Λ with itself, $[\Lambda\ \Lambda]$, vanishes.

In what follows we shall refer to a manifold M equipped with a Poisson structure, with structure tensor Λ, as the Poisson manifold (M,Λ).

(ii) The set $\Omega^{0}(M)$ of all real functions on the Poisson manifold (M,Λ) has a natural structure of Poisson algebra [i.e., a real vector space equipped with an associative algebra structure and a Lie algebra structure coupled by property (P–3) of (1.3.1)].

(1.3.2) DEFINITION. Let (M,Λ) be a Poisson manifold and $f \in \Omega^o(M)$. The *Hamiltonian vector field* on M associated with f [or admitting f as Hamiltonian] is the unique vector field X_f on M such that, for any $g \in \Omega^o(M)$,

$$X_f \cdot g = \{f,g\}.$$

The map

$$f \longmapsto X_f$$

is a Lie algebra homomorphism from $\Omega^o(M)$ into $\mathfrak{X}(M)$.

Let (M,Λ) be a Poisson manifold. For every x in M and every ϵ in T_x^*M, we denote by $\Lambda_x^{\#}(\epsilon)$ [or simple by $\Lambda^{\#}(\epsilon)$] the unique element in T_xM such that

$$\eta(\Lambda^{\#}(\epsilon)) = \Lambda(x)(\epsilon,\eta),$$

for every $\eta \in T_x^*M$.

The map

$$\Lambda^{\#} : T^*M \longrightarrow TM$$

so defined is a vector bundle morphism. In particular,

$$X_f = \Lambda^{\#}(df)$$

for $f \in \Omega^o(M)$.

(1.3.3) DEFINITION. Let (M,Λ) be a Poisson manifold.

(i) The *characteristic field* $C \equiv C_\Lambda$ of its structure is the image of the morphism $\Lambda^{\#}$:

$$C_\Lambda := \Lambda^{\#}(T^*M).$$

(ii) The *rank* at a point, x of M of its structure is the dimension of the characteristic space C_x.

REMARKS.

(i) The annihilator of the characteristic space C_x at x is the kernel of the linear map

$$\Lambda^{\#}_x : T^*_x M \longrightarrow T_x M.$$

Also, note that

$$C_x = \{X_f(x) \mid f \in \Omega^o(M)\},$$

where X_f is the Hamiltonian vector field on M associated with f.

(ii) The rank of Λ at a point x is an even integer; in general this integer depends upon the point x under consideration. Therefore, the characteristic field C_Λ is not, in general, a vector subbundle of TM.

(1.3.4) DEFINITION. Let (M,Λ) be a Poisson manifold. The structure tensor is called *regular* [resp. *nondegenerate*] if it has constant rank [resp. maximal rank that equals dim M].

(1.3.5) PROPOSITION. *Let* (M,Λ) *be a regular Poisson manifold* [*i.e., a Poisson manifold with regular structure tensor*]. *Then the characterastic field* C_Λ *is an integrable subbundle of* TM.

The corresponding foliation $\mathcal{F}_\Lambda$ on M is called the *characteristic foliation.* The leaves of $\mathcal{F}_\Lambda$ are immersed symplectic submanifolds, called the *symplectic leaves* of (M,Λ).

The dimension of $\mathcal{F}_\Lambda$ is said to be the *symplectic dimension* of (M,Λ).

REMARK. The notion of a nondegenerate Poisson structure on an even–dimensional manifold is equivalent to that of a symplectic structure.

(1.3.6) DEFINITION. Let (M_1,Λ_1) and (M_2,Λ_2) be two Poisson manifolds. A map $\varphi : M_1 \longrightarrow M_2$ is called a *Poisson morphism* if it satisfies one [and therefore all] of the following equivalent properties :

(i) The induced map

$$\varphi^* : \Omega^0(M_2) \longrightarrow \Omega^0(M_1),\ f \longmapsto \varphi^* f = f \circ \varphi$$

is a Lie algebra morphism, [i.e.,

$$\varphi^*\{f,g\} = \{\varphi^* f, \varphi^* g\}$$

for every $f,g \in \Omega^0(M_2)$].

(ii) The structure tensors Λ_1 and Λ_2 are compatible with the map φ [i.e.,

$$\Lambda_1(x)({}^t(\varphi_{*,x})(\epsilon),\ {}^t(\varphi_{*,x})(\eta)) = \Lambda_2(\varphi(x)(\epsilon,\eta)),$$

for every $x \in M_1$ and $\epsilon,\eta \in T^*_{\varphi(x)}M_2$].

(iii) The Hamiltonian vector fields $X_{f\circ\varphi} = \Lambda_1^\#(d(f\circ\varphi))$ and $X_f = \Lambda_2^\#(df)$ for every $f \in \Omega^0(M_2)$ are φ–related, [i.e.,

$$\varphi_{*,x}(X_{f\circ\varphi}(x)) = X_f(\varphi(x))$$

for every $x \in M_1$ and $f \in \Omega^0(M_2)$].

(iv) The morphisms $\Lambda_1^\# : T^*M_1 \longrightarrow TM_1$ and $\Lambda_2^\# : T^*M_2 \longrightarrow TM_2$ satisfy :

$$\Lambda_2^\#(\varphi(x)) = \varphi_{*,x} \circ \Lambda_1^\#(x) \cdot {}^t(\varphi_{*,x})$$

for every $x \in M_1$.

If φ is a Poisson morphism and a diffeomorphism [resp. a transformation], then we say that φ is a *Poisson diffeomorphism* [resp. a *Poisson automorphism*].

The set $\mathrm{Aut}(M,\Lambda)$ of all Poisson automorphisms of (M,Λ) is a group.

(1.3.7) DEFINITION. Let (M,Λ) be a Poisson manifold. A vector field X on M is called a *Poisson infinitesimal automorphism* if the local 1–parameter group $(\varphi_t)_{|t| < \delta}$ associated to X has the following property : for all t, the local diffeomorphism φ_t is a Poisson morphism from the open subset D_t of M on which it is defined into its image $\varphi_t(D_t) = D_{-t}$.

(1.3.8) PROPOSITION. *Let* X *be a vector field on the Poisson manifold* (M,Λ). *The following statements are equivalent* :

(i) X *is a Poisson infinitesimal automorphism.*

(ii) X *is a derivation of the Poisson algebra* $\Omega^0(M)$ [*i.e.*,

$$X\cdot\{f,g\} = \{X\cdot f,g\} + \{f,X\cdot g\},$$

for every $f,g \in \Omega^0(M)$].

(iii) *The Lie derivative of the structure tensor* Λ *with respect to* X *vanishes* :

$$\mathscr{L}_X\Lambda = 0$$

[i.e., $X\cdot\Lambda(\epsilon,\eta) + \Lambda(\mathscr{L}_X\epsilon,\eta) + \Lambda(\epsilon, \mathscr{L}_X\eta) = 0$,

for every $\epsilon,\eta \in \Omega^1(M)$].

The set $\mathrm{aut}(M,\Lambda)$ of all Poisson infinitesimal automorphisms of (M,Λ) is a Lie algebra.

(1.3.9) PROPOSITION. *Let* ϵ *be a 1–form on the Poisson manifold* (M,Λ). *The vector field* $\Lambda^{\#}(\epsilon)$ *is a Poisson infinitesimal automorphism if and only if* ϵ *satisfies* :

$$d\epsilon(X_f,X_g) = 0$$

for all f,g $\in \Omega^o(M)$.

In this case we say that the 1–form ϵ is C–*closed*, and that the vector field $\Lambda^{\#}(\epsilon)$ is a *locally Hamiltonian vector field* associated with ϵ.

REMARKS.

(i) Every locally Hamiltonian vector field on a Poisson manifold (M,Λ) is a Poisson infinitesimal automorphism but the converse, in general, is not true.

(ii) The set $LH(M,\Lambda)$ of all locally Hamiltonian vector fields of (M,Λ) is an ideal of the Lie algebra $aut(M,\Lambda)$, [i.e.,

$$[aut(M,\Lambda),\ LH(M,\Lambda)] \subset LH(M,\Lambda)\ ].$$

(iii) The set $H(M,\Lambda)$ of all Hamiltonian vector fields of (M,Λ) is an ideal of $LH(M,\Lambda)$ which contains the derived ideal, [i.e.,

$$[LH(M,\Lambda),\ LH(M,\Lambda)] \subset H(M,\Lambda)\ ].$$

(1.3.10) EXAMPLES.

(i) A 2–times contravariant tensor field Λ on $\mathbb{R}^m$,

$$\Lambda = \Lambda^{ij}\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^j} \quad (i,j = 1,\cdots,m)$$

defines a Poisson structure on $\mathbb{R}^m$ if the functions Λ^{ij} satisfy:

$$\Lambda^{ji} = -\Lambda^{ij}$$

and

$$\Lambda^{\ell i}\frac{\partial\Lambda^{jk}}{\partial x^\ell} + \Lambda^{\ell j}\frac{\partial\Lambda^{ki}}{\partial x^\ell} + \Lambda^{\ell k}\frac{\partial\Lambda^{ij}}{\partial x^\ell} = 0 \quad (i,j,k,\ell = 1,\cdots,m).$$

(ii) A symplectic manifold (M,σ), with the usual Poisson bracket defined by its symplectic structure [i.e.,

$$\{f,g\} = \sigma(X_f,X_g) \quad \text{for } f,g, \in \Omega^o(M)\ ],$$

is a Poisson manifold.

(iii) Let $\mathfrak{g}$ be a real, finite–dimensional Lie algebra. Its dual $\mathfrak{g}^*$ has a canonical Poisson structure, called the *Lie–Poisson structure* :

$$\{f,g\}(\xi) = \xi([df(\xi), dg(\xi)]),$$

for $f,g, \in \Omega^o(\mathfrak{g}^*)$, $\xi \in \mathfrak{g}^*$ and $df(\xi), dg(\xi) \in \mathfrak{g}$ [identified with its bidual].

Let *ad* be the adjoint representation of $\mathfrak{g}$ [i.e., $A \longmapsto ad_A(B) = [A,B]$ for $A,B \in \mathfrak{g}$].

The Poisson tensor Λ on $\mathfrak{g}^*$ is defined by :

$$\Lambda(\xi)(A,B) = \xi([A,B]) = \xi(ad_A(B)),$$

for $\xi \in \mathfrak{g}^*$ and $A,B \in \mathfrak{g}$.

The morphism

$$\Lambda^{\#} : T^*(\mathfrak{g}^*) \cong \mathfrak{g}^* \times \mathfrak{g} \longrightarrow T(\mathfrak{g}^*) \cong \mathfrak{g}^* \times \mathfrak{g}^*,$$

associated with Λ, is defined by :

$$\Lambda^{\#}(\xi,A) = (\xi, \xi \circ ad_A),$$

for $\xi \in \mathfrak{g}^*$ and $A \in \mathfrak{g}$.

1.4 FOLIATIONS ([CN], [H1],[H2], [HH], [KT1], [KT4], [L], [Mo7] ,[R], [Re4] ,[Rem] ,[To] ,[V])

Let M be a manifold of dimension m. Roughly speaking, a foliation on M corresponds to a decomposition of M into a union of disjoint connected submanifolds of the same dimension, which are described, locally, by a family of parallel planes in $\mathbb{R}^m$. More precisely, one can formulate the following

(1.4.1) **DEFINITION** (The "pictorial" approach). A *foliation* $\mathcal{F}$ on M of dimension ℓ and codimension n [$\ell + n = m$] is a partition

$$\left\{ L_\lambda \right\}_{\lambda \in \mathcal{L}}$$

of M into connected subsets with the following property :

(F) For every point of x of M there exists a local chart (U,φ) with coordinates (x^i,x^α), whose domain U contains x, such that the connected components of $U \cap L_\lambda$ are described by the equations :

$$x^\alpha = \text{const.} \quad (\alpha = \ell + 1,\cdots,m)$$

[Throughout, we use the following convention : the indices $i,j,\cdots$ run over the range $1,2,\cdots,\dim \mathcal{F}$ and $\alpha,\beta,\cdots$ run over the range $\dim \mathcal{F} + 1,\cdots,\dim M$].

Such a chart is called a *distinguished chart* for $\mathcal{F}$ [or a *foliated chart* on M]. The domain of the chart is said to be a *distinguished open set* and the corresponding local coordinates (x^i,x^α) are called *distinguished coordinates* in U.

REMARKS.

(i) Without loss of generality one can assume that $\varphi(U) = U^1 \times U^2$, where U^1 is an open disk in $\mathbb{R}^\ell$, and U^2 is an open disk in $\mathbb{R}^n$.

(ii) The sets L_λ are called the *leaves* of the foliation $\mathcal{F}$. Every leaf L_λ is an ℓ–dimensional immersed submanifold of M.

(iii) The presence of the foliation $\mathcal{F}$ on the manifold M may be expressed by the symbol $(M, \mathcal{F})$. One uses to say that $(M, \mathcal{F})$ is a *foliated manifold.*

(1.4.2) EXAMPLES.

(i) The family

$$\mathcal{F}_0 = \{\mathbb{R}^\ell \times \{\lambda\}\}_{\lambda \in \mathbb{R}^n}$$

is a foliation of dimension ℓ on $\mathbb{R}^\ell \times \mathbb{R}^n$, called the *canonical* [or *model*] *foliation* on $\mathbb{R}^\ell \times \mathbb{R}^n$

(ii) On every manifold M there exist two trivial foliations : the *discrete foliation* $\mathcal{F}_M^\bullet$, whose leaves are the points of M and the *indiscrete foliations* $\mathcal{F}_M^0$, whose leaves are the connected components of M.

Obviously, the foliation $\mathcal{F}^{\bullet}_{M}$ has dimension 0, while the foliation $\mathcal{F}^{o}_{M}$ has dimension m = dim M.

(iii) Let $\mathcal{F}$ be a foliation of dimension ℓ on the manifold M and let U be an open subset of M. Then the connected components of the traces of the leaves of $\mathcal{F}$ on U constitute a foliation $\mathcal{F}|_{U}$ of dimension ℓ, called the *foliation induced* by $\mathcal{F}$ on U [or the *restriction* of $\mathcal{F}$ to U].

(iv) Let $\mathcal{F}$ [resp. $\mathcal{F}'$] be a foliation of dimension ℓ [resp. ℓ'] on the manifold M [resp. M′]. One can construct, in a natural manner, a foliation $\mathcal{F} \times \mathcal{F}'$ of dimension $\ell + \ell'$ on the product manifold $M \times M'$, called the *product foliation* of $\mathcal{F}$ and $\mathcal{F}'$. The leaves of $\mathcal{F} \times \mathcal{F}'$ are products of leaves of $\mathcal{F}$ with leaves of $\mathcal{F}'$.

(v) Let $f: M' \longrightarrow M$ be a map transverse to a foliation $\mathcal{F}$ on M, [i.e.

$$T_{f(x)}M = f_{*,x}(T_x M') + T_{f(x)}(L_{f(x)})$$

for each $x \in M$, where $L_{f(x)}$ is the leaf of $\mathcal{F}$ passing through the point $f(x) \in M$].

Consider the connected components of the inverse images by f of the leaves of $\mathcal{F}$. Thus we obtain a foliation $f^*\mathcal{F}$ on M′, called the *pull–back* of $\mathcal{F}$ by f.
Note that

$$\operatorname{codim} f^*\mathcal{F} = \operatorname{codim} \mathcal{F},$$

that is

$$\dim f^*\mathcal{F} = \dim M' - \dim M + \dim \mathcal{F}.$$

In particular, if the transverse map f is the canonical inclusion of a submanifold M′ of M into M, then the foliation $f^*\mathcal{F} = \mathcal{F}|_{M'}$ is the foliation induced by $\mathcal{F}$ on the transverse submanifold M′.

> Another particular case is a submersion and the discrete foliation on M. In this situation the foliation $\mathcal{F}' = f^* \mathcal{F}^{\bullet}_{M}$ on M′ is said to be defined by the submersion f: M′ $\longrightarrow$ M and is called a *simple foliation.* The leaves are then closed imbedded submanifolds.
>
> If U is a distinguished open set, then the foliation $\mathcal{F}|_{U}$ is simple. In other words, every foliation is locally [but no necessarily globally] simple.
>
> Note that every fibre bundle E over M is a manifold equipped with a natural simple foliation.

Let $\mathcal{F}$ be an ℓ–dimensional foliation on the manifold M, and let $(U_\mu,\varphi_\mu),(U_\nu,\varphi_\nu)$ be two distinguished charts for $\mathcal{F}$ which overlap [i.e., $U_\mu \cap U_\nu \neq \emptyset$]. Then the coordinate transformation

$$\varphi_{\nu\mu} = \varphi_\nu \circ \varphi_\mu^{-1} : \varphi_\mu(U_\mu \cap U_\nu) \longrightarrow \varphi_\nu(U_\mu \cap U_\nu)$$

has the form :

$$\varphi_{\nu\mu}(x^i,x^\alpha) = (\varphi^1_{\nu\mu}(x^i,x^\alpha), (\varphi^2_{\nu\mu}(x^\alpha)) \in \mathbb{R}^\ell \times \mathbb{R}^n.$$

In other words, the map $\varphi_{\nu\mu}$ and the second projection pr^2: $\mathbb{R}^\ell \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ commute.

We are conducted towards the following

(1.4.3) DEFINITION. An atlas $\mathcal{A} = \{(U_\mu,\varphi_\mu)\}_{\mu \in \mathcal{M}}$ on the manifold M is called a *foliated atlas* of dimension ℓ and codimension n if it satisfies the following condition : For any two charts (U_μ,φ_μ) and (U_ν,φ_ν) belonging to $\mathcal{A}$, which overlap, the coordinate transition map $\varphi_{\nu\mu}$ has the form above.

Let $\mathcal{A}^{(\ell)}(M)$ be the family of all foliated atlases of dimension ℓ on M. We define an equivalence relation $\sim$ on $\mathcal{A}^{(\ell)}(M)$ by

$$\mathcal{A}_1 \sim \mathcal{A}_2 \text{ iff } \mathcal{A}_1 \cup \mathcal{A}_2 \in \mathcal{A}^{(\ell)}(M).$$

Notice that, given a foliated atlas $\mathcal{A} \in \mathcal{A}^{(\ell)}(M)$, the union $\mathcal{A}^*$ of all foliated atlases $\mathcal{A}' \in \mathcal{A}^{(\ell)}(M)$ such that $\mathcal{A}' \sim \mathcal{A}$ is a foliated atlas of dimension ℓ on M. We say that the foliated atlas is *maximal* [or *complete*] if it is a maximal element in the set of $\mathcal{A}^{(\ell)}(M)$, ordered by the inclusion relation. A foliated atlas $\mathcal{A}$ is maximal if and only if $\mathcal{A} = \mathcal{A}^*$.

If $\mathcal{A}, \mathcal{A}' \in \mathcal{A}^{(\ell)}(M)$ and $\mathcal{A} \subset \mathcal{A}'$, then we will say that the foliated atlas $\mathcal{A}'$ prolongs [or completes] $\mathcal{A}$. Note that given a foliated atlas $\mathcal{A} \in \mathcal{A}^{(\ell)}(M)$, $\mathcal{A}^*$ is the unique complete foliated atlas that prolongs $\mathcal{A}$. We say that $\mathcal{A}^*$ is the *completion* of $\mathcal{A}$.

(1.4.4) DEFINITION. A *foliated structure* on M of dimension ℓ and codimension n is an equivalence class of foliated atlases of dimension ℓ on M.

REMARKS.

(i) A foliated structure on M can be given by choosing a foliated atlas $\mathcal{A}$ on M. Without loss of generality one can assume that $\mathcal{A}$ is complete. The corresponding foliated structure is denoted by $\hat{\mathcal{A}}$.

(ii) Let $\varphi : \Omega \longrightarrow \Omega'$ be a local diffeomorphism of $\mathbb{R}^m$. We say that φ is a *local automorphism* of the model foliation $\mathcal{F}_0$ on $\mathbb{R}^m = \mathbb{R}^\ell \times \mathbb{R}^n$ if it preserves the foliation $\mathcal{F}_0|_\Omega$ [i.e., sends leaves onto leaves]. This means that, in a neighbourhood of a point of Ω, φ projects to a diffeomorphism of $\mathbb{R}^n$.

The family $\Gamma_{m,n}$ of all these local automorphisms is a pseudogroup of transformations of $\mathbb{R}^m$.

> Note that a foliated atlas is a $\Gamma_{m,n}$–atlas on M. So, a foliated structure on M of dimension ℓ and codimension n is a $\Gamma_{m,n}$–structure on M.

One has already seen that to any foliation $\mathcal{F}$ on M there corresponds a unique foliated structure $\hat{\mathcal{A}}$. Conversely, let $\hat{\mathcal{A}}$ be a foliated structure of dimension ℓ on M. Let $(U,\varphi) \in \mathcal{A}^*$ be a foliated chart. [Recall that

$$\varphi = (\varphi^1,\varphi^2) : U \longrightarrow U^1 \times U^2 \subset \mathbb{R}^{\ell} \times \mathbb{R}^{n}].$$

The sets

$$\varphi^{-1}(U^1 \times \{\lambda\}),$$

where $\lambda \in U^2$ are called *plaques* in U. The plaques are connected submanifolds of dimension ℓ of M. Moreover, if P and P′ are two plaques in U, then either $P \cap P' = \emptyset$ or $P = P'$. Note that all the plaques in U are diffeomorphic to U_1.

A *path of plaques* is a finite sequence $(P_1,\cdots,P_k)$ of plaques such that $P_i \cap P_{i+1} \neq \emptyset$ for all $i = 1,\cdots,k-1$.

Define the equivalence relation
"$x \sim y$ iff there exists a path of plaques $(P_1,\cdots,P_k)$ such that $x \in P_1$ and $y \in P_k$",
and set

$$\mathcal{F} = M/_{\sim}.$$

The family $\mathcal{F}$ is a foliation of dimension ℓ on the manifold M. Consequently, the foliations on M are, up to conjugacy [see Annexe A], in a natural 1–1 correspondence with the foliated structures on M. So, one has :

(1.4.5) DEFINITION (The "microscopic" approach). A foliation $\mathcal{F}$ on M of dimension ℓ and codimension n is [in fact, can be identified with] a foliated structure on M of dimension ℓ and codimension n.

Let

$$\mathbb{R}^{\ell}_{m} = \mathbb{R}^{\ell} \times (\mathbb{R}^{n})^{\delta}$$

be the product space $\mathbb{R}^{\ell} \times \mathbb{R}^{n}$, where the space $\mathbb{R}^{n}$ is equipped with the discrete topology.

$\mathbb{R}^{\ell}_{m}$ is an ℓ–dimensional manifold and the identification of $\mathbb{R}^{\ell} \times \mathbb{R}^{m}$ $(m = \ell + n)$ is a bijective immersion of $\mathbb{R}^{\ell}_{m}$ into $\mathbb{R}^{m}$.

Let Ω be an open subset of $\mathbb{R}^{m}$, it follows that Ω is also an open subset of $\mathbb{R}^{\ell}_{m}$. The structure of open submanifold of dimension ℓ of $\Omega \subset \mathbb{R}^{\ell}_{m}$ will be denoted by $\Omega_{(\ell)}$. Clearly, $\Omega_{(\ell)}$ is an ℓ–dimension manifold immersed in Ω.

Let $\mathcal{F}$ be an ℓ–dimensional foliation on the m–manifold M. There exists an unique manifold structure on M, denoted by M^{δ}, satisfying the following condition : for each distinguished chart (U,φ) for $\mathcal{F}$, U is an open subset of M^{δ} and

$$\varphi : U \subset M^{\delta} \longrightarrow \varphi(U)_{(\ell)} \subset \mathbb{R}^{\ell}_{m}$$

is a diffeomorphism.

The manifold M^{δ} [also denoted by $M_{\mathcal{F}}$] is called the *leafwise manifold* associated to the foliation $\mathcal{F}$.

REMARKS

(i) M^{δ} is an ℓ–dimensional manifold immersed in M. The connected components of M^{δ} are just the leaves of $\mathcal{F}$.

(ii) The topology of M^{δ} is called the *leaf topology* on $(M, \mathcal{F})$. The leaf topology is stronger than the manifold topology of M.

Let $\mathcal{F}$ be an ℓ–dimensional foliation on M. Let (U,φ) be a distinguished chart for $\mathcal{F}$. The map

$$\varphi^2 = pr^2 \circ \varphi : U \longrightarrow U^2 \subset \mathbb{R}^n$$

is a submersion, called an *elementary distinguished map* of $(M, \mathcal{F})$.

(1.4.6) DEFINITION. A map

$$f : U \longrightarrow \mathbb{R}^n,$$

where U is an open set of M, is called a *distinguished map* [or a *local first integral*] of (M,f) if, in a neighbourhood of a point of U, f is an elementary distinguished map.

REMARKS

(i) A map $f : U \longrightarrow \mathbb{R}^n$ is a distinguished map of $(M, \mathcal{F})$ if and only if f is constant on each leaf of $\mathcal{F}|_U$.

The components $f^1,\cdots,f^n$ of a distinguished map

$$f = (f^1,\cdots,f^n) : U \longrightarrow \mathbb{R}^n$$

are called [local] *basic functions.*

The set $\Omega^o_B(M, \mathcal{F})$ of all [global] basic functions is a subring of the ring $\Omega^o(M)$ of functions on M.

(ii) If $f: U \longrightarrow \mathbb{R}^n$ is a distinguished map of $(M, \mathcal{F})$, then for each leaf L of $\mathcal{F}$ and each point $x \in L \cap U$, one has :

$$T_xL = \ker(d_xf: T_xM \longrightarrow \mathbb{R}^n).$$

We are interested now in the infinitesimal object associated to a foliation, [i.e., the subbundle of the tangent bundle TM of M consisting of all vectors tangent to the leaves of the foliation].

Let $\mathcal{F}$ be an ℓ–dimensional foliation on M.

A vector $v \in T_xM$ is called *tangent to the leaf* L_x passing through the point $x \in M$ if there exists a distinguished map $f: U \longrightarrow \mathbb{R}^n$ of $(M, \mathcal{F})$ such that :

$$(d_xf)(v) = 0.$$

The set $T\mathcal{F}$ [also denoted by $\mathcal{V}(M,\mathcal{F}]$ of all vectors tangent to the leaves of $\mathcal{F}$ is a vector subbundle of TM, called the *tangent bundle* [or the *vertical bundle*] of the foliation $\mathcal{F}$. This is a vector subbundle over M with typical fibre $\mathbb{R}^\ell$. It is clear that the tangent bundle of $\mathcal{F}$ is involutive [i.e.,

$$X,Y \in \Gamma\,(U,T\mathcal{F}) \Longrightarrow [X,Y] \in \Gamma\,(U,T\mathcal{F}),$$

where $\Gamma\,(U,T\mathcal{F})$ denotes the sections of $T\mathcal{F}$ over an open subset U].

Conversely, let $E \subset TM$ be a vector subbundle of dimension ℓ.

(1.4.7) DEFINITION.

(i) We denote by E_x the fibre of E over $x \in M$. A submanifold P of M is called a *integral manifold* [or a *plaque*] of E if the inclusion map $i_P\colon P \longrightarrow M$ satisfies the following conditions :

$$(i_P)_{*,x}(T_xP) = E_x \quad \text{for all } x \in P.$$

(ii) The vector subbundle E is called *integrable* if for every point $x \in M$ there exists a local chart (U,φ), $\varphi = (\varphi^1,\varphi^2) : U \longrightarrow \mathbb{R}^\ell \times \mathbb{R}^n$, whose domain U contains x, such that for every $\lambda \in \mathbb{R}^n$ [in fact, for every $\lambda \in \varphi^2(U) \subset \mathbb{R}^n$] the submanifold

$$P = \varphi^{-1}(\mathbb{R}^\ell \times \{\lambda\})$$

is an integral manifold of E.

A vector subbundle E of TM is integrable if and only if it is the tangent bundle of a foliation on M. It is the content of the classical *theorem of Frobenius* that every involutive vector subbundle E of TM is integrable [i.e., it is the tangent bundle of a foliation $\mathcal{F}$ on M]. The leaves of $\mathcal{F}$ are constructed as the maximal connected integral manifolds of E.

Consequently, the foliations on M are in a natural 1–1 correspondence with the involutive vector subbundles of TM. So one has :

(1.4.8) DEFINITION. (The infinitesimal approach). A foliation $\mathcal{F}$ of dimension ℓ on M is [in fact, can be identified with] an involutive vector subbundle of TM.

Let $\mathcal{F}$ be an ℓ–dimensional foliation on M. The *transverse* [or *normal*] *bundle* $\nu\mathcal{F}$ of the foliation $\mathcal{F}$ is the quotient bundle

$$\nu\,\mathcal{F} = \mathrm{TM}/_{\mathrm{T}\,\mathcal{F}}\,.$$

This is a vector bundle over M with typical fibre $\mathbb{R}^n$.

REMARK. By suitable choice of a Riemannian metric g_M on M, we may view $\nu\mathcal{F}$ as the subbundle of TM orthogonal to T $\mathcal{F}$:

$$\nu\,\mathcal{F} = (\mathrm{T}\,\mathcal{F})^{\perp}.$$

We have the exact sequence

$$0 \longrightarrow \mathrm{T}\,\mathcal{F} \longrightarrow \mathrm{TM} \longrightarrow \nu\mathcal{F} \longrightarrow 0$$

of vector bundles over M.
The exactness of this sequence is equivalent to the exactness of the sequence of dual vector bundles

$$0 \longrightarrow \nu^*\,\mathcal{F} \longrightarrow \mathrm{T}^*\mathrm{M} \longrightarrow \mathrm{T}^*\,\mathcal{F} \longrightarrow 0.$$

The vector bundle

$$\mathrm{T}^*\,\mathcal{F} = \mathrm{T}^*\mathrm{M}/_{\nu^*\,\mathcal{F}}$$

is called the *cotangent bundle* of the foliation $\mathcal{F}$. Note that the vector subbundle $\nu^*\,\mathcal{F} \subset \mathrm{T}^*\mathrm{M}$ can be identified with the annihilator of the tangent bundle of $\mathcal{F}$:

$$\nu^*\,\mathcal{F} = (\mathrm{T}\,\mathcal{F})^{\mathrm{o}}.$$

(1.4.9) DEFINITION. Let $\mathcal{F}$ be a foliation on M. A diffeomorphism φ of M is called an *automorphism* of $\mathcal{F}$ if it satisfies one [and therefore all] of the following equivalent properties :

(i) φ preserves the foliation $\mathcal{F}$ [i.e., sends leaves onto leaves].

(ii) $\varphi^*\,\mathcal{F} = \mathcal{F}$
[$\varphi^*\mathcal{F}$ is the pull–back of $\mathcal{F}$ by φ].

(iii) φ is compatible with the [maximal] foliated atlas $\mathcal{A}^*$ associated to $\mathcal{F}$:

$$(U_\mu, \varphi_\mu) \in \mathcal{A}^* \Longrightarrow (\varphi(U_\mu),\ \varphi_\mu \circ \varphi^{-1}) \in \mathcal{A}^*.$$

(iv) The local expression of φ in every distinguished charts (U_μ, φ_μ) and (U_ν, φ_ν)

$$\varphi_\nu \circ \varphi \circ \varphi_\mu^{-1} : \varphi_\mu(U_\mu) \longrightarrow \varphi_\nu(U_\nu)$$

belongs to $\Gamma_{m,n}$ [i.e., it is a local automorphism of the model foliation $\mathcal{F}_0$ on $\mathbb{R}^\ell \times \mathbb{R}^n$].

(v) The tangent map φ_* induced by φ preserves the tangent bundle of $\mathcal{F}$:

$$\varphi_*(T\mathcal{F}) = T\mathcal{F}.$$

REMARK. Let $G_{\ell\,;\,m}$ denote the group of all linear transformations of $\mathbb{R}^m = \mathbb{R}^\ell \times \mathbb{R}^n$ which map the subspace $\mathbb{R}^\ell \times \{0\}$ into itself. The matrix with respect to the coordinate basis of any such transformation has the form

$$\begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix},$$

where $a_1 \in GL(\ell\,;\mathbb{R})$ and $a_3 \in GL(n\,;\mathbb{R})$. a_1 is a matrix for the induced linear transformation on the subspace $\mathbb{R}^\ell \times \{0\}$.

The set $\mathrm{Aut}(M, \mathcal{F})$ of all such automorphisms is a group.

(1.4.10) DEFINITION. Let $\mathcal{F}$ be a foliation on M. A vector field X on M is called an *infinitesimal automorphism* [or a *foliate vector field*] of $\mathcal{F}$ if the local 1–parameter group $(\varphi_t)_{|t| < \delta}$ associated to X has the following property : For all t, the local diffeomorphism φ_t preserves the foliation.

(1.4.11) PROPOSITION. *Let* X *be a vector field on the foliated manifold* $(\mathrm{M}, \mathcal{F})$. *The following statements are equivalent :*

(i) X *is an infinitesimal automorphism of* $\mathcal{F}$.

(ii) $[\mathrm{X},\mathrm{Y}] \in \Gamma(\mathrm{T}\mathcal{F})$ *for all* $\mathrm{Y} \in (\mathrm{T}\mathcal{F})$.

(iii) *In every distinguished chart* (U,φ) *with distinguished coordinates* (x^i,x^α) X *is of the form*

$$\mathrm{X}\big|_{\mathrm{U}} = \mathrm{X}^i(x^1,\cdots,x^m)\frac{\partial}{\partial x^i} + \mathrm{X}^\alpha(x^{\ell+1},\cdots,x^m)\frac{\partial}{\partial x^\alpha}.$$

The set $\mathrm{aut}(\mathrm{M}, \mathcal{F})$ [also denoted by $\mathscr{L}(\mathrm{M}, \mathcal{F})$] of all infinitesimal automorphisms of $\mathcal{F}$ is the normalizer in $\mathfrak{X}(\mathrm{M})$ of the Lie subalgebra $\Gamma(\mathrm{T}\mathcal{F})$. In particular, $\mathrm{aut}(\mathrm{M},\mathcal{F})$ is a Lie subalgebra of $\mathfrak{X}(\mathrm{M})$.

REMARK. Every vector field $\mathrm{X} \in \mathfrak{X}(\mathrm{M})$ defines a section $\bar{\mathrm{X}}$ of $\mathcal{F}$. In particular, if X is foliate, we will call $\bar{\mathrm{X}}$ *the transverse field* associated to Y. The set $\ell(\mathrm{M}, \mathcal{F})$ of all transverse fields has a natural structure of Lie algebra. We have the exact sequence

$$0 \longrightarrow \Gamma(\mathrm{T}\mathcal{F}) \longrightarrow \mathrm{aut}(\mathrm{M},\mathcal{F}) \longrightarrow \ell(\mathrm{M},\mathcal{F}) \longrightarrow 0$$

of Lie algebra.

(1.4.12) EXAMPLES.

(i) Let X be a non–vanishing vector field on M. The orbits of the local 1–parameter group associated to X fit together to form a 1–dimensional foliation on M.

More precisely, let $\mathrm{E} \subset \mathrm{TM}$ be the line bundle with fibre E_x spanned by $\mathrm{X}(x)$ for $x \in \mathrm{M}$. It is involutive [in fact, any line bundle $\mathrm{E} \subset \mathrm{TM}$ is involutive] and so it defines a 1–dimensional foliation on M.

(ii) Let η be a nowhere zero 1–form on M. Let $E^* \subset T^*M$ be the line bundle with fibre E^*_x spanned by η_x for $x \in M$. The vector subbundle $E \subset TM$ defined by

$$E_x = \ker \eta_x \subset T^*_xM$$

is involutive if and only if $\eta([X,Y]) = 0$ for local vector fields X and Y such that $\eta(X) = 0$ and $\eta(Y) = 0$.
This is equivalent to the local representability of $d\eta$ as

$$d\eta = \epsilon \wedge \eta$$

for a local 1–form ϵ.

[More generally, any involutive vector subbundle $E \subset TM$ of codimension 1, is locally given by a 1–form η as described above].
So, the involutive vector subbundle E defines a codimension 1 foliation on M.

Note that on a compact manifold M a codimension 1 foliation exists if and only if the Euler characteristic $\chi(M)$ equals zero.

(iii) Let $M \xrightarrow{\pi} N$ be a fibre bundle with typical fibre F and discrete structural group. Under this assumption, the codimension n [$n = \dim F$] foliations on $\pi^{-1}(U)$ given by the submersions $\pi^{-1}(U) \longrightarrow U \times F \longrightarrow F$ [$\pi^{-1}(U) \longrightarrow U \times F$ is a local trivialization] fit together to give a foliation on M.

(iv) Let G be a Lie group acting locally freely on a manifold M [i.e., for each $x \in M$ the isotropy subgroup

$$G_x = \{g \in G \mid gx = x\}$$

is discrete]. Then the orbits of G form a foliation on M of dimension dim G.

1.5 POISSON MANIFOLD ASSOCIATED WITH A FOLIATION ([Li3])

Let $\mathcal{F}$ be an ℓ–dimensional foliation on a manifold M [dim M = m = ℓ+n].

Let

$$W = T^*\mathcal{F}$$

be the cotangent bundle of the foliation $\mathcal{F}$, and let

$$q^0 : W \longrightarrow M$$

be its natural projection.

Note that :

(i) $W = T^*M/_{(T\mathcal{F})^o}$, where $(T\mathcal{F})^o \subset T^*M$ is the annihilator of $T\mathcal{F}$.

(ii) W is a vector bundle over M with typical fibre $(\mathbb{R}^\ell)^*$.

(iii) dim W = 2ℓ+n.

Let

$$\mathcal{F}_W := (q^0)^*\mathcal{F}$$

be the pull–back of the foliation $\mathcal{F}$ by q^0. It is a foliation of dimension 2ℓ and of codimension n on the manifold W.

Let

$$\sigma_c \equiv \sigma_c^{(M)}$$

be the canonical symplectic form on T^*M.

In adapted local coordinates

$$(x^1,\cdots,x^m, p_1,\cdots,p_m)$$

on T^*M, σ_c has the form :

$$\sigma_c\big|_{T^*U} = dp_1 \wedge dx^1 + \cdots + dp_m \wedge dx^m.$$

The corresponding structure tensor

$$\Lambda_c \equiv \Lambda_c^{(M)}$$

is defined by :

$$\Lambda_c(df,dg) = \sigma_c(X_f,X_g).$$

[Recall that X_f denotes the Hamiltonian vector field associated with f :

$$\iota_{X_f}\sigma_c + df = 0 \ ].$$

In adapted local coordinates $(x^1,\cdots,x^m, p_1,\cdots,p_m)$ on T^*M, Λ_c has the form :

$$\Lambda_c\big|_{T^*U} = \frac{\partial}{\partial p_1}\wedge\frac{\partial}{\partial x^1} + \cdots + \frac{\partial}{\partial p_m}\wedge\frac{\partial}{\partial x^m}.$$

Now, let (U,φ) be a distinguished chart for the foliation $\mathcal{F}$, with the distinguished local coordinates (x^i,x^α).

Let

$$(T^*U = q^{-1}(U),\ T^*\varphi)$$

be the corresponding natural chart on T^*M,

$$T^*\varphi : T^*U \longrightarrow \mathbb{R}^m \times (\mathbb{R}^m)^*,$$

with adapted local coordinates

$$(x^i,x^\alpha,p_i,p_\alpha).$$

Define on T^*U the [local] vector fields

$$P_\alpha : = \Lambda_c^\#(dx^\alpha).$$

One has

$$P_\alpha = P_\alpha^i\frac{\partial}{\partial x^i} + P_\alpha^\beta\frac{\partial}{\partial x^\beta} + P_\alpha^{\bar{i}}\frac{\partial}{\partial x^{\bar{i}}} + P_\alpha^{\bar{\beta}}\frac{\partial}{\partial x^{\bar{\beta}}},$$

where $\bar{i} = m+i$, $\bar{\beta} = m+\beta$, $p_i = x^{\bar{i}}$ and $p_\beta = x^{\bar{\beta}}$.

According to the definition of P_α, it follows that :

(i) $P_\alpha^i = 0$;

(ii) $P_\alpha^\beta = 0$;

(iii) $P_\alpha^{\bar{i}} = 0$;

(iv) $P_{\alpha}^{\bar{\beta}} = \delta_{\alpha}^{\beta}$.

So,

$$P_{\alpha} = \frac{\partial}{\partial x^{\bar{\alpha}}} = \frac{\partial}{\partial p_{\alpha}}.$$

Notice that P_{α} are [local] sections of $(T\mathcal{F})^{o}$.

As

$$\mathscr{L}_{P_{\alpha}} \Lambda_c = 0,$$

it follows that Λ_c projects well onto $\Lambda_{(\mathcal{F})}$ on $W = T^{*}\mathcal{F}$.

Let

$$(U^{o} = (q^{o})^{-1}(U), \varphi^{o})$$

be the corresponding natural chart on $W = T^{*}\mathcal{F}$

$$\varphi^{o} : U^{o} \longrightarrow \mathbb{R}^{m} \times (\mathbb{R}^{\ell})^{*},$$

with adapted local coordinates :

$$(x^{i}, x^{\alpha}, p_{i}).$$

In adapted local coordinates on W, $\Lambda_{(\mathcal{F})}$ has the form :

$$\Lambda_{(\mathcal{F})}\Big|_{U^{o}} = \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial x^1} + \cdots + \frac{\partial}{\partial p_{\ell}} \wedge \frac{\partial}{\partial x^{\ell}}.$$

Clearly, $\Lambda_{(\mathcal{F})}$ is a [regular] Poisson tensor on W.

REMARK. The characteristic foliation of $(W, \Lambda_{(\mathcal{F})})$ is $\mathcal{F}_W$:

$$\mathcal{F}_{\Lambda_{(\mathcal{F})}} = \mathcal{F}_W.$$

The leaves of $\mathcal{F}_W$ will be called the *cotangent fibres* of the leaves of $\mathcal{F}$. The regular Poisson manifold

$$(W = T^{*}\mathcal{F}, \Lambda_{(\mathcal{F})})$$

is said to be the *canonical Poisson manifold* associated to the foliation $\mathcal{F}$.

Chapter 2

TANGENTIAL GEOMETRY

The *tangential geometry* [or *leaf geometry*] of a foliation is the geometry infinitesimally modelled by the tangent bundle of the foliation. So, for the study of the tangential geometry of a foliation there are two essentially equivalent options : either working on the vertical bundle [i.e., the tangent bundle of the foliation] or investigating the geometric structure of a single leaf. There is no extant systematic treatment of tangential geometry in the literature. The purpose of this chapter is to rectify this deficiency. Further, most of the contents of this chapter are, to my knowledge, original, [except section 2.2, 2.5 partially, and 2.3 entirely]. The concepts and definitions presented here have been developed specifically for use in this thesis. My aim here is to create a general framework for the study of foliations equipped with an [integrable] vertical G–structure. I also define and describe in detail, the fundamental concept of a vertical G–structure on a foliated manifold.

Throughout this chapter, M is an m–dimensional connected manifold and $\mathcal{F}$ is an ℓ–dimensional foliation on M ; $m = \ell + n$.

2.1 HORIZONTAL BUNDLES

Let TM be the tangent bundle of M and let $\mathcal{V} \equiv \mathcal{V}(M, \mathcal{F})$ be the vertical bundle of the foliated manifold $(M, \mathcal{F})$.

(2.1.1) DEFINITION. A vector subbundle $\mathcal{H}$ of TM is said to be a *horizontal bundle* for $\mathcal{F}$ if it is complementary to the vertical bundle $\mathcal{V}$ [i.e., if it satisfies

$$\mathrm{TM} = \mathcal{V} \oplus \mathcal{H} \text{]}.$$

Any general vector X decomposes as

$$X = X^v + X^h \in \mathcal{V} \oplus \mathcal{H} .$$

We call X^v [resp. X^h] the *vertical* [resp. *horizontal*] *component* of X.

Let (U,φ) be a distinguished chart for $\mathcal{F}$ with distinguished local coordinates (x^i,x^α). Now

$$\frac{\partial}{\partial x^\alpha} \in TU ; \quad \alpha = \ell + 1, \cdots, m.$$

We set

$$N_\alpha := \left(\frac{\partial}{\partial x^\alpha}\right)^h .$$

We have

$$\left(\frac{\partial}{\partial x^\alpha}\right)^v = t^i_\alpha \frac{\partial}{\partial x^i},$$

and therefore

$$N_\alpha = \frac{\partial}{\partial x^\alpha} - t^i_\alpha \frac{\partial}{\partial x^i}; \quad i = 1,\cdots,\ell \text{ and } \alpha = \ell+1,\cdots,m.$$

REMARK. The *horizontal subspace* $\mathcal{H}_x$ at x is generated by $N_\alpha\big|_x$, $x \in U$.

Notice that if in distinguished local coordinates X has the form

$$X = X^i \frac{\partial}{\partial x^i} + X^\alpha \frac{\partial}{\partial x^\alpha},$$

then

$$X^v = (X^i + t^i_\alpha X^\alpha) \frac{\partial}{\partial x^i};$$

$$X^h = X^\alpha N_\alpha = - t^i_\alpha X^\alpha \frac{\partial}{\partial x^i} + X^\alpha \frac{\partial}{\partial x^\alpha}.$$

The [local] basis $\left(\frac{\partial}{\partial x^{\alpha}}, N_{\alpha} \right)$ will be called the $\mathcal{H}$ – *adapted basis* associated to the distinguished chart (U,φ). The dual basis $(\theta^{i},\theta^{\alpha})$ will be called the $\mathcal{H}$ – *adapted cobasis* associated to (U,φ).

The following conditions are satisfied :

(1) $\theta^{i}\left(\frac{\partial}{\partial x^{j}} \right) = \delta^{i}_{j}$;

(2) $\theta^{i}(N_{\alpha}) = 0$;

(3) $\theta^{\alpha}\left(\frac{\partial}{\partial x^{i}} \right) = 0$;

(4) $\theta^{\alpha}(N_{\beta}) = \delta^{\alpha}_{\beta}$.

Set

$$\theta^{i} = A^{i}_{j}\, dx^{j} + B^{i}_{\alpha}\, dx^{\alpha}$$

and

$$\theta^{\alpha} = A^{\alpha}_{i}\, dx^{i} + B^{\alpha}_{\beta}\, dx^{\beta}$$

Then (1) implies

$$A^{i}_{j} = \delta^{i}_{j}$$

and (2) implies

$$\theta^{i}_{\alpha} = t^{i}_{\alpha}\,.$$

So

$$\theta^{i} = dx^{i} + t^{i}_{\alpha}\, dx^{\alpha}\,.$$

Also (3) implies

$$A^{\alpha}_{i} = 0$$

and (4) implies

$$B^{\alpha}_{\beta} = \delta^{\alpha}_{\beta}\,.$$

So

$$\theta^{\alpha} = dx^{\alpha}.$$

REMARK. A horizontal bundle $\mathcal{H}$ for the foliation $\mathcal{F}$ is described, locally, by a set of 1–forms θ^{i}, $i = 1,\cdots,\ell$, as above [i.e., the 1–forms θ^{i} constitute a local basis for the annihilator $\mathcal{H}^{o}$ of $\mathcal{H}$].

Calculating the Lie brackets we get :

$$\left[\frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial x^{j}}\right] = 0 \, ;$$

$$\left[\frac{\partial}{\partial x^{i}}, N_{\alpha}\right] = C^{j}_{i\alpha}\frac{\partial}{\partial x^{j}} \, ;$$

$$\left[N_{\alpha}, N_{\beta}\right] = C^{j}_{\alpha\beta}\frac{\partial}{\partial x^{j}}$$

where

$$C^{j}_{i\alpha} = -\frac{\partial t^{j}_{\alpha}}{\partial x^{i}},$$

$$C^{j}_{\alpha\beta} = \frac{\partial t^{j}_{\alpha}}{\partial x^{\beta}} - \frac{\partial t^{j}_{\beta}}{\partial x^{\alpha}} + t^{k}_{\alpha} C^{j}_{k\alpha} - t^{k}_{\alpha} C^{j}_{k\beta},$$

$i,j,k, = 1,\cdots,\ell$ and $\alpha,\beta = \ell+1,\cdots,m$.

Note that the tangential geometry of a foliation "depends" upon the choice of a horizontal bundle for the foliation under consideration. There is no canonical way of choosing such a bundle, unless we introduce more structure on the underlying manifold [e.g., a Riemannian metric].

Assume that the manifold M is equipped with a Riemannian metric g_{M}. If this is the case, the horizontal bundle $\mathcal{H}$ can be [uniquely] chosen to be the orthogonal complement [with respect to g_{M}] of the vertical bundle $\mathcal{V}$ of the foliation :

$$\mathcal{H} := \mathcal{V}^{\perp}.$$

Locally [i.e., in distinguished local coordinates], we have :

$$0 = g_M\left(\frac{\partial}{\partial x^i}, N_\alpha\right) = g_M\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^\alpha}\right) - t^j_\alpha\, g_M\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) =$$
$$= g_{i\alpha} - t^j_\alpha\, g_{ij}$$

and therefore

$$g_{i\alpha} = t^j_\alpha\, g_{ij}\,; \quad i,j, = 1,\cdots,\ell \text{ and } \alpha = \ell{+}1,\cdots,m.$$

Let $X,X' \in \mathfrak{X}(M)$. We have

$$g_M(X,X') = g_M(X^v{+}X^h, X'^{\,v} + X'^{\,h}) = g_M(X^v,X'^{\,v}) + g_M(X^h,X'^{\,h}).$$

Set

$$g_V(X,X') := g_M(X^v,X'^{\,v})$$

The mapping

$$g_V : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \Omega^o(M),\ (X,X') \longmapsto g_V(X,X')$$

is a symmetric $\Omega^o(M)$–bilinear form on [the module] $\mathfrak{X}(M)$, whose kernel is $\mathcal{H}$. g_V is called the *vertical metric* [induced by g_M] of the foliation $\mathcal{F}$.

So, any Riemannian metric on M defines an associated vertical metric. Conversely, let g_V be a verical metric for $\mathcal{F}$ [i.e., a symmetric $\Omega^o(M)$–bilinear form on $\mathfrak{X}(M)$, whose kernel is $\mathcal{H} = \mathcal{V}^{\perp}$]. Then there exist Riemannian metrics on M that have g_V as their associated vertical metric.

Just consider an arbitrary Riemannian metric g'_M on M and set

$$g_M(X,X') := g_V(X,X') + g'_M(X^h,X'^{\,h}).$$

This defines a Riemannian metric on M which admits g_V as its vertical metric.

Locally [i.e., in distinguished local coordinates], we have :

$$g_V(X,X') = g_M(X^v,X'^{\,v}) = g_M\left((X^i + t^i_\alpha X^\alpha)\frac{\partial}{\partial x^i}, (X'^{\,j} + t^j_\beta X'^{\,\beta})\frac{\partial}{\partial x^j}\right) =$$
$$= (X^i + t^i_\alpha X^\alpha)(X'^{\,j} + t^j_\beta X'^{\,\beta})\, g_M\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) =$$

$$= g_{ij}(X^i + t^i_\alpha X^\alpha)(X^j + t^j_\beta X'^\beta) = (g_{ij}\, \theta^i \otimes \theta^j)(X,X')$$

and therefore

$$g_V = g_{ij}\, \theta^i \otimes \theta^j ,$$

where $(\theta^i, \theta^\alpha)$ is the $\mathcal{H}$–adapted basis associated to the distinguished chart (U,φ).

Notice that a vertical metric g_V on $(M, \mathcal{F}, \mathcal{H})$ is a *metric along the leaves* : its restriction to each leaf L of $\mathcal{F}$ is a Riemannian metric on the manifold $L \subset M$.

2.2 HORIZONTAL VECTOR FIELDS

Let TM be the tangent bundle of M, let $\Omega^\bullet(M)$ be the algebra of real valued forms on M, and let $\mathfrak{X}(M)$ be the $\Omega^0(M)$–module of vector fields on M.

Let $\mathcal{H}$ be a fixed horizontal bundle for the foliation $\mathcal{F}$. This is a vector bundle over M with typical fibre $\mathbb{R}^n$.

(2.2.1) **DEFINITION.** A section Z of the horizontal bundle $\mathcal{H}$ will be called an *$\mathcal{H}$–horizontal vector field* [or a horizontal vector field with respect to $\mathcal{H}$] on the foliated manifold $(M, \mathcal{F})$.
The set

$$\mathfrak{X}_H \equiv \mathfrak{X}_H(M, \mathcal{F}, \mathcal{H})$$

of all $\mathcal{H}$–horizontal vector fields on $(M, \mathcal{F})$ is a submodule of $\mathfrak{X}(M)$. A vector field Z on the foliated manifold $(M, \mathcal{F})$ is $\mathcal{H}$–horizontal if and only if its value Z_x at each point of M belongs to the horizontal subspace $\mathcal{H}_x$ at x.

(2.2.2) **PROPOSITION.** *Let Z be a vector field on the foliated manifold* $(M, \mathcal{F})$ *and let* $\mathcal{H}$ *be a horizontal bundle for* $\mathcal{F}$. *The following statements are equivalent :*

(i) *Z is $\mathcal{H}$–horizontal.*

(ii) *In every distinguished chart (U,φ) with distinguished local coordinates (x^i,x^α) Z has the form :*

$$Z\Big|_U = Z^i \frac{\partial}{\partial x^i} + Z^\alpha \frac{\partial}{\partial x^\alpha}$$

where the coefficients satisfy the condition :

$$Z^i + t^i_\alpha Z^\alpha = 0,$$

with t^i_α uniquely defined by :

$$\left(\frac{\partial}{\partial x^\alpha}\right)^v = t^i_\alpha \frac{\partial}{\partial x^i}$$

PROOF. Let Z be a vector field on M. In distinguished coordinates Z has the form :

$$Z\Big|_U = Z^i \frac{\partial}{\partial x^i} + Z^\alpha \frac{\partial}{\partial x^\alpha}.$$

Now, Z is $\mathcal{H}$–horizontal if and only if it can be written as follows :

$$Z\Big|_U = \bar{Z}^\alpha N_\alpha,$$

that is

$$Z\Big|_U = \bar{Z}^\alpha \left(\frac{\partial}{\partial x^\alpha} - t^i_\alpha \frac{\partial}{\partial x^i}\right).$$

So

$$\bar{Z}^i = t^i_\alpha Z^\alpha,$$
$$\bar{Z}^\alpha = Z^\alpha.$$

This means that the coefficients must satisfy the condition :

$$Z^i + t^i_\alpha Z^\alpha = 0.$$ □

Let $(M,\mathcal{F})$ be a foliated manifold and let $\mathcal{H}$ be a horizontal bundle for $\mathcal{F}$. $\mathcal{F}$ is said to be *$\mathcal{H}$–flat* if the horizontal bundle $\mathcal{H}$ is integrable. That means that the module $\mathcal{X}_H(M, \mathcal{F}, \mathcal{H})$ of $\mathcal{H}$–horizontal vector fields on $(M, \mathcal{F})$ is a Lie subalgebra of $\mathcal{X}(M)$. We get immediately the following :

integrability condition :

$$\text{(I)}\quad C^{i}_{\alpha\beta} = 0 \; ; \quad i = 1,\cdots,\ell \text{ and } \alpha,\beta = \ell{+}1,\cdots,m.$$

Assume that the manifold M is equipped with a Riemannian metric g_M. In this case there is a canonical way of choosing a horizontal bundle $\mathcal{H}$ for $\mathcal{F}$:

$$\mathcal{H} := \mathcal{V}^{\perp}.$$

For X,X′ ∈ $\mathfrak{X}$ (M), define a tensor field A, of type (1,2) on M, by

$$A_X X' := (\nabla^M_{X^h} X'^{v})^{v} + (\nabla^M_{X^h} X'^{v})^{h},$$

where ∇^M is the Riemannian covariant differentiation on M.

This tensor field A is called the *O'Neill tensor* associated to $(M, \mathcal{F}, \mathcal{H})$.
A is essentially the *integrability tensor* of the horizontal bundle $\mathcal{H}$:

$$A_Z Z' = \tfrac{1}{2}\,[Z,Z']^{v} \text{ for } Z,Z' \in \mathfrak{X}_H(M,\mathcal{F},\mathcal{H}),$$

cf. [O].

REMARK. The geometry of $\mathcal{F}$ is locally determined by two tensor fields : the O'Neill tensor and the so–called *second fundamental tensor* [of the leaves] (see, e.g., [O], [JWZ] as well as [KW], [KT3], [KT4]).

2.3 SHEETS AND TANGENTIAL HOLONOMY

Let $\mathcal{H}$ be a fixed horizontal bundle for the foliation $\mathcal{F}$ on M.
A [piecewise smooth] curve c :[0,1] ⟶ M is said to be *$\mathcal{H}$–horizontal* [or horizontal with respect to $\mathcal{H}$] if its tangent vector field lies in $\mathcal{H}$ [i.e.,

$$c'(t) \in \mathcal{H}_{c(t)} \text{ for all } t\,].$$

A *vertical curve* for $\mathcal{F}$ is a [piecewise smooth] curve which lies entirely in one leaf of $\mathcal{F}$.

(2.3.1) DEFINITION. Let x be a point of M. The set S(x) of all points of M that can be joined to x by an $\mathcal{H}$–horizontal curve is called the *$\mathcal{H}$–sheet* of $\mathcal{F}$ passing through x. Clearly, the sheets of $\mathcal{F}$ constitute a partition of M. Under certain geometric conditions the sheets are immersed submanifolds of M (see, e.g., [BH3],[BH4]).

For example, if $\mathcal{F}$ is a foliation on a [connected] manifold M and $\mathcal{H}$ is an *Ehresmann connection* for $\mathcal{F}$ preserving a complete parallelism of the leaves, then the $\mathcal{H}$–sheets S(x) are [immersed] submanifolds foliating M. All the leaves of this foliation $\mathcal{G}$ are mutually diffeomorphic and the closures of these leaves give a foliation $\bar{\mathcal{G}}$ on M. Moreover, if M is compact, the leaves of $\bar{\mathcal{G}}$ are the fibres of a [locally trivial] fibre bundle $M \longrightarrow W$ (cf. [BH2],[BH3]).

Note that, roughly speaking, $\mathcal{H}$ is an Ehresmann connection for $\mathcal{F}$ if every $\mathcal{H}$–horizontal [resp. vertical] curve can be "transported" along vertical [resp. $\mathcal{H}$–horizontal] curves.

For every $\mathcal{H}$–horizontal curve $c : [0,1] \longrightarrow M$ there exists a family of diffeomorphisms

$$\varphi_t : D_0 \longrightarrow D_t \,, \; t \in [0,1]$$

such that :

(EH–1) D_t is a neighbourhood of c(t) in the leaf $L_{c(t)}$ of $\mathcal{F}$ passing through c(t) for all t ;

(EH–2) $\varphi_t(c(0)) = c(t)$ for all t ;

(EH–3) For $x \in D_0$, the curve

$$t \longmapsto \varphi_t(x)$$

is $\mathcal{H}$–horizontal.

(EH–4) φ_0 is the identity mapping of D_0.

Such a family of diffeomorphisms is called an *element of holonomy* along c (cf.[BH1] ; see also [BH2],[BH3]).

The elements of holonomy along the $\mathcal{H}$–horizontal curve c is unique [in the sense that two such families of diffeomorphisms must agree in a neighbourhood of c(0)] (cf. [BH1],[BH2]).

Assume that the foliation $\mathcal{F}$ admits an Ehresmann connection $\mathcal{H}$. Thus every $\mathcal{H}$–horizontal curve uniquely determines germs of local diffeomorphisms from one leaf to another. When the leaves of the foliation have a geometric structure, we say that *$\mathcal{H}$ preserves the geometry of leaves* if the element of holonomy along each $\mathcal{H}$–horizontal curve is a family of local isomorphisms of the geometric structure under consideration.

The family of local isomorphisms thus induced on each leaf L generates a pseudogroup $Hol_{\mathcal{H}}(L)$ which, up to equivalence of pseudogroups (see, e.g., [H4]), is independent of the leaf L. This pseudogroup is called the *tangential holonomy pseudogroup* [of the foliation]. Clearly, the orbits of $Hol_{\mathcal{H}}(L)$ are just the intersections of L with the $\mathcal{H}$–sheets of $\mathcal{F}$.

By construction, $Hol_{\mathcal{H}}(L)$ is the "dual" of the *transverse holonomy pseudogroup* $Hol_{tr}(L)$ (see, e.g., [H3],[H4],[Wo]). In particular, it determines the *tangential holonomy groups* of all the leaves : the tangential holonomy group $\mathrm{Hol}_{\mathcal{H}}(L,x)$ of L at x naturally projects onto $Hol_{tr}(L,x)$ and is independent of the particular Ehresmann connection used to define it (cf. [BH2],[BH3]).

2.4 TANGENTIAL FUNCTIONS

Let $\Omega^0(M)$ be the ring of functions on M, let $\mathcal{H}$ be a fixed horizontal bundle for $\mathcal{F}$, and let $\mathfrak{X}_H(M, \mathcal{F}, \mathcal{H})$ be the $\Omega^0(M)$–module of $\mathcal{H}$–horizontal vector fields on $(M, \mathcal{F})$.

(2.4.1) DEFINITION. A function f on M is called *$\mathcal{H}$–tangential* [or tangential with respect to $\mathcal{H}$] if

$$Zf = 0 \quad \text{for all } Z \in \mathfrak{X}_H(M, \mathcal{F}, \mathcal{H}).$$

The set

$$\Omega^o_{tg} \equiv \Omega^o_{tg}(M, \mathcal{F}, \mathcal{H})$$

of all $\mathcal{H}$–tangential functions on $(M, \mathcal{F})$ is a subring of $\Omega^o(M)$.

(2.4.2) PROPOSITION. *Let* f *be a function on* M. *The following statements are equivalent :*

(i) f *is $\mathcal{H}$–tangential.*

(ii) f *is constant on each $\mathcal{H}$–sheet.*

(iii) *In every distinguished chart* (U,φ) *with distinguished local coordinates* (x^i, x^α) f *satisfies the condition :*

$$\frac{\partial f}{\partial x^\alpha} = t^i_\alpha \frac{\partial f}{\partial x^i} \, ; \; \alpha = \ell+1, \cdots, m$$

where the t^i_α*'s are completely determined by*

$$\left(\frac{\partial}{\partial x^\alpha} \right)^v = t^i_\alpha \frac{\partial f}{\partial x^i} .$$

PROOF. (i) $\Leftrightarrow$ (ii). f if $\mathcal{H}$–tangential $\Leftrightarrow$ $Zf = 0$ for all $Z \in \mathfrak{X}_H(M, \mathcal{F}, \mathcal{H}) \Leftrightarrow$ $(Zf)(x) := \frac{d}{dt}\Big|_{t=o} (f \circ c_x)(t) = 0$ for all $x \in M$ and for all $\mathcal{H}$–horizontal curve c_x passing through $x \Leftrightarrow f(c_x(t)) = $ const. for all $x \in M$ and for all $\mathcal{H}$–horizontal curve c_x passing through $x \Leftrightarrow f|_{S(x)} = $ const. for all $x \in M \Leftrightarrow$ f is constant on each $\mathcal{H}$–sheet.

(i) $\Leftrightarrow$ (iii). In distinguished coordinates a horizontal vector field Z has the form

$$Z = Z^i \frac{\partial}{\partial x^i} + Z^\alpha \frac{\partial}{\partial x^\alpha} ,$$

where $Z^i = - t^i_\alpha Z^\alpha$.

So, f is $\mathcal{H}$–tangential $\Leftrightarrow$ Zf = o for all $Z \in \mathfrak{X}_H(M, \mathcal{F}, \mathcal{H})$

$\Leftrightarrow (-t^i_\alpha \frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial x^\alpha})\, Z^\alpha = 0$ for all Z^α

$\Leftrightarrow \frac{\partial f}{\partial x^\alpha} = t^i_\alpha \frac{\partial f}{\partial x^i}$ for all $\alpha = \ell+1,\cdots,m$ □

2.5 VERTICAL AND TANGENTIAL VECTOR FIELDS

Let $\Omega^o(M)$ be the ring of functions on M, let $\mathfrak{X}(M)$ be the $\Omega^o(M)$–module of vector fields on M, and let $\mathcal{V}$ be the verticle bundle of the foliation $\mathcal{F}$. Let $\mathcal{H}$ be a fixed horizontal bundle for $\mathcal{F}$ and let $\mathfrak{X}_H(M, \mathcal{F}, \mathcal{H})$ be the $\Omega^o(M)$–module of $\mathcal{H}$–horizontal vector fields on $(M, \mathcal{F})$.

(2.1.5) DEFINITION. A section Y of the vertical bundle $\mathcal{V}$ will be called a *vertical vector field.*

The set

$$\mathfrak{X}_V \equiv \mathfrak{X}_V(M, \mathcal{F})$$

of all vertical vector fields is a submodule of $\mathfrak{X}(M)$.

A vector field X on M is vertical if and only if its value X_x at each point x of M belongs to the *vertical subspace* $\mathcal{V}_x$ at x.

REMARK. Let (U,φ) be a distinguished chart with distinguished local coordinates (x^i, x^α). The vertical subspace $\mathcal{V}_x$ is generated by the derivatives

$$\left.\frac{\partial}{\partial x^i}\right|_x , \ x \in U \ ; i = 1,\cdots,\ell.$$

Recall that the foliation $\mathcal{F}$ is described locally [in U] by the equations :

$$x^\alpha = \text{const.}, \ \alpha = \ell+1,\cdots,m.$$

In other words, the vertical bundle $\mathcal{V}$ is described locally by the 1–forms dx^α [i.e., the 1–forms dx^α constitute a local basis for the annihilator $\mathcal{V}^o$ of $\mathcal{V}$]. Clearly, the following result holds.

(2.5.2) PROPOSITION *Let* X *be a vector field on* M. *The following statements are equivalent* :

(i) X *is vertical.*

(ii) *In every distinguished chart* (U,φ) *with distinguished local coordinates* (x^i,x^α) X *has the form* :

$$X\Big|_U = X^i \frac{\partial}{\partial x^i}$$

(2.5.3) DEFINITION. A vertical vector field Y is called *$\mathcal{H}$–tangential* [or tangential with respect to $\mathcal{H}$] if

$$[Y,Z] \in \mathfrak{X}_H(M,\mathcal{F},\mathcal{H}) \text{ for all } Z \in \mathfrak{X}_H(M,\mathcal{F},\mathcal{H}).$$

The set

$$\mathfrak{X}_{tg} \equiv \mathfrak{X}_{tg}(M,\mathcal{F},\mathcal{H})$$

of all $\mathcal{H}$–tangential vector fields is a Ω^0_{tg}–submodule of $\mathfrak{X}_V(M,\mathcal{F})$.

[This follows immediately from the formula

$$[fY,Z] = f[Y,Z] - (Zf)Y \text{].}$$

Moreover, $\mathfrak{X}_{tg}(M,\mathcal{F},\mathcal{H})$ is a Lie subalgebra of $\mathfrak{X}_V(M,\mathcal{F})$.

[This follows immediately from the Jacobi identity].

By calculating the Lie bracket for $Y = Y^i \dfrac{\partial}{\partial x^i}$ and $Z = Z^\alpha N_\alpha$, we get :

$$[Y,Z] = \left(t^i_\alpha \frac{\partial Y^j}{\partial x^i} - \frac{\partial Y^j}{\partial x^\alpha} - \frac{\partial t^j_\alpha}{\partial x^i} Y^i \right) Z^\alpha \frac{\partial}{\partial x^j} + \frac{\partial Z^\alpha}{\partial x^i} X^i N_\alpha .$$

(2.5.4) PROPOSITION. *Let* Y *be a vertical vector field. The following statements are equivalent* :

(i) Y *is $\mathcal{H}$–tangential.*

(ii) *The* 1*–parameter group of transformations of* M *associated to* Y *preserves the horizontal bundle* $\mathcal{H}$.

(iii) *The* 1*–parameter group of transformations of* M *associated to* Y *preserves the* $\mathcal{H}$ *sheets of* $\mathcal{F}$.

(iv) Y *commutes with every local* $\mathcal{H}$*–horizontal foliate vector field.*

(v) *In every distinguished chart* (U,φ) *with distinguished local coordinates* (x^i,x^α) Y *has the form* :

$$Y\big|_U = Y^i \frac{\partial}{\partial x^i},$$

where the coefficients satisfy the condition :

$$\text{(T)}\quad \frac{\partial Y^i}{\partial x^\alpha} = t^j_\alpha \frac{\partial Y^i}{\partial x^j} - \frac{\partial t^i_\alpha}{\partial x^j}$$

with the t^i_α*'s completely determined by*

$$\left(\frac{\partial}{\partial x^\alpha}\right)^v = t^i_\alpha \frac{\partial}{\partial x^i}.$$

PROOF. Let (φ_t) be the 1–parameter group of transformations associated to Y.

(i) ⇔ (ii) This is an immediate consequence of the following relation

$$[Y,Z](x) = \lim_{t\to 0} [\, Z_x - (\varphi_t)_{*,x} Z_x \,].$$

(ii) ⇔ (iii) Assume that (φ_t) preserves the horizontal bundle $\mathcal{H}$ [i.e.,

$$(\varphi_t)_{*,x}(\mathcal{H}_x) \subset \mathcal{H}_x \text{ for all } x \in M,$$

where $(\varphi_t)_{*,x} : T_xM \longrightarrow T_{\varphi_t(x)}M$ is the tangent mapping of φ_t at x].

Let $c : [0,1] \longrightarrow M$ be an $\mathcal{H}$–horizontal curve . It suffices to show that $\varphi_t \circ c$ is also an $\mathcal{H}$–horizontal curve. We have

$$\begin{aligned}(\varphi_t \circ c)'(\tau) &= (\varphi_t \circ c)_*(\tau,1) = (\varphi_t)_*(c_*(\tau,1)) \\ &= (\varphi_t)_*(c'(\tau)) \in \mathcal{H}_{\varphi_t(c(\tau))}.\end{aligned}$$

Using similar arguments, we can easily show that the converse is also true.

(i) $\Longrightarrow$ (iv) Recall that a vector field X is foliate if

$$[X,Y] \in \mathfrak{X}_V(M,\mathcal{F}) \text{ for all } Y \in \mathfrak{X}_V(M,\mathcal{F}).$$

The implication follows straightforwardly.

(iv) $\Longrightarrow$ (v) Recall that locally [i.e., in distinguished local coordinates] a foliate vector field X has the form

$$X = X^{\alpha}\frac{\partial}{\partial x^{\alpha}},$$

where the X^{α}'s satisfy

$$\frac{\partial X^{\alpha}}{\partial x^{i}} = 0 \ ; \ i = 1,\cdots,\ell \text{ and } \alpha = \ell+1,\cdots,m.$$

So, for every local $\mathcal{H}$–horizontal foliate vector field Z, we have :

$$0 = [Y,Z] = \left(t^{i}_{\alpha}\frac{\partial Y^{j}}{\partial x^{i}} - \frac{\partial Y^{j}}{\partial x^{\alpha}} - \frac{\partial t^{j}_{\alpha}}{\partial x^{i}}\,Y^{i}\right)Z^{\alpha}\frac{\partial}{\partial x^{j}}.$$

As a result

$$t^{i}_{\alpha}\frac{\partial Y^{j}}{\partial x^{i}} - \frac{\partial Y^{j}}{\partial x^{\alpha}} - \frac{\partial t^{j}_{\alpha}}{\partial x^{i}}\,Y^{i} = 0.$$

(v) $\Longrightarrow$ (i) It is straightforward. □

REMARK. Let Y be a tangential vector field on (M, $\mathcal{F}$). If in every distinguished chart (U,φ) with distinguished local coordinates (x^{i},x^{α}) the following condition holds :

$$\frac{\partial t^{i}_{\alpha}}{\partial x^{j}} = 0 \ ; \ i,j = 1,\cdots,\ell \text{ and } \alpha = \ell+1,\cdots,m$$

[i.e., the t^{i}_{α}'s are local basic functions], then the coefficients Y^{i} of the vector field Y are [local] tangential functions.

2.6 VERTICAL AND TANGENTIAL FORMS

Let $\Omega^\bullet(M)$ be the algebra of real valued forms on M, let $\mathfrak{X}(M)$ be the $\Omega^o(M)$–module of vector fields on M, and let $\mathcal{V}$ be the vertical bundle of the foliation $\mathcal{F}$. Let $\mathcal{H}$ be a fixed horizontal bundle for $\mathcal{F}$ and let $\mathfrak{X}_H(M,\mathcal{F},\mathcal{H})$ be the $\Omega^o(M)$–module of $\mathcal{H}$–horizontal vector fields on $(M,\mathcal{F})$.

(2.6.1) DEFINITION. A r–form η on M is called *vertical* if

$$\iota_Z\eta = 0 \text{ for all } Z \in \mathfrak{X}_H(M,\mathcal{F},\mathcal{H}).$$

The set $\Omega^r_V \equiv \Omega^r_V(M,\mathcal{F},\mathcal{H})$ of all vertical r–forms is a submodule of $\Omega^r(M)$.

The direct sum

$$\Omega^\bullet_V \equiv \Omega^\bullet_V(M,\mathcal{F},\mathcal{H}) := \bigoplus_{r=0}^{m} \Omega^r_V(M,\mathcal{F},\mathcal{H}),$$

equipped with the exterior product, is the *algebra of vertical forms* of the foliation.

(2.6.2) PROPOSITION. *Let η be a 1–form on* M. *The following statements are equivalent* :

(i) *η is vertical.*

(ii) *In every distinguished chart* (U,φ) *with distinguished local coordinates* (x^i,x^α) *η has the form* :

$$\eta\big|_U = \eta_i\, dx^i + \eta_\alpha\, dx^\alpha$$

where the coefficients satisfy the condition

$$\eta_\alpha = t^i_\alpha\, \eta_i\,.$$

PROOF. Let η be a 1–form on M. In distinguished coordinates η has the form :

$$\eta = \eta_i\, dx^i + \eta_\alpha\, dx^\alpha.$$

Now, η is vertical iff it can be written as follows :

$$\eta = \bar{\eta}_i \, \theta^i,$$

that is

$$\eta = \eta_i \, (dx^i + t^i_\alpha \, dx^\alpha).$$

So,

$$\eta_i = \bar{\eta}_i ,$$

$$\eta_\alpha = t^i_\alpha \, \bar{\eta}_i .$$

This means that the coefficients must satisfy the condition

$$\eta_\alpha = t^i_\alpha \, \eta_i$$

□

REMARK. In local distinguished coordinates a vertical 1–form η has the form :

$$\eta = \eta_i \, \theta^i,$$

where

$$\theta^i = dx^i + t^i_\alpha \, dx^\alpha.$$

Consider now the general case. $\eta \in \Omega^r(M)$. In distinguished local coordinates η can be written as

$$\eta = \eta_1 + \eta_2 ,$$

where

$$\eta_1 = \frac{1}{r!} \eta_{i_1 \dots i_r} \, \theta^{i_1} \wedge \cdots \wedge \theta^{i_r}$$

and η_2 belongs to the ideal generated by the 1–forms dx^α.

Notice that

$$\eta_1(X_1, \cdots, X_r) = \eta(X_1^v, \cdots, X_r^v) \text{ for } X_1, \cdots, X_r \in \mathfrak{X}(M).$$

Indeed, one has

$$\eta(X_1^v, \cdots, X_r^v) = \eta\left((X_1^v)^{i_1} \frac{\partial}{\partial x^{i_1}}, \cdots, (X_r^v)^{i_r} \frac{\partial}{\partial x^{i_r}}\right) =$$

$$= \eta_{i_1 \dots i_r} (X_1^v)^{i_1} \cdots (X_r^v)^{i_r} = \eta_{i_1 \dots i_r} \, \theta^{i_1}(X_1) \cdots \theta^{i_r}(X_r) =$$

$$= \frac{1}{r!} \, \eta_{i_1 \dots i_r} \, \theta^{i_1} \wedge \cdots \wedge \theta^{i_r} (X_1, \cdots, X_r) = \eta_1 (X_1, \cdots, X_r).$$

According to the definition, η is vertical iff $\eta_2 \equiv 0$.

Consequently, in distinguished local coordinates a vertical r–form η has the form :

$$\eta = \frac{1}{r!}\, \eta_{i_1 \cdots i_r}\, \theta^{i_1} \wedge \cdots \wedge \theta^{i_r},$$

where $\theta^i = dx^i + t^i_\alpha\, dx^\alpha$.

(2.6.3) PROPOSITION. *Let η be a r–form on M. The following statements are equivalent :*

(i) *η is vertical ;*

(ii) *In every distinguished chart (U,φ) with distinguished local coordinates (x^i,x^α) η has the form :*

$$\eta\big|_U = \sum_{i_1 < \cdots < i_r} \eta_{i_1 \cdots i_r}\, dx^{i_1} \wedge \cdots \wedge dx^{i_r} +$$

$$+ \sum_{k=0}^{r-1} \sum_{i_1 < \cdots < \alpha_r} \eta_{i_1 \cdots i_k\, \alpha_{k+1} \cdots \alpha_r}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge \cdots \wedge dx^{\alpha_r},$$

where the coefficients satisfy the conditions :

$$\eta_{i_1 \cdots i_k \alpha_{k+1} \cdots \alpha_r} = t^{i_{k+1}}_{\alpha_{k+1}} \cdots t^{i_r}_{\alpha_r}\, \eta_{i_1 \cdots i_r},\ k = 0,\cdots,r-1.$$

PROOF. In distinguished local coordinates η can be written as

$$\eta = \eta_1 + \eta_2\,,$$

where

$$\eta_1 = \frac{1}{r!}\, \eta_{i_1 \cdots i_r}\, \theta^{i_1} \wedge \cdots \wedge \theta^{i_r}$$

and η_2 belongs to the ideal generated by the 1–forms dx^α.

We have :

$$\text{(a)}\quad \eta_1 = \frac{1}{r!}\,\eta_{i_1\cdots i_r}\,\theta^{i_1}\wedge\cdots\wedge\theta^{i_r} = \sum_{i_1<\cdots<i_r}\eta_{i_1\cdots i_r}\,\theta^{i_1}\wedge\cdots\wedge\theta^{i_r} =$$

$$= \sum_{i_1<\cdots<i_r}\eta_{i_1\cdots i_r}\,dx^{i_1}\wedge\cdots\wedge dx^{i_r} +$$

$$+ \sum_{k=0}^{r-1}\ \sum_{i_1<\cdots<\alpha_r}\eta_{i_1\cdots i_r}\,t^{i_{k+1}}_{\alpha_{k+1}}\cdots t^{i_r}_{\alpha_r}\,dx^{i_1}\wedge\cdots\wedge dx^{\alpha_r},$$

and

$$\text{(b)}\quad \eta_2 = \sum_{k=0}^{r-1}\ \sum_{i_1<\cdots<\alpha_r}\left(\eta_{i_1\cdots i_k\,\alpha_{k+1}\cdots\alpha_r} - \eta_{i_1\cdots i_r}\,t^{i_{k+1}}_{\alpha_{k+1}}\cdots t^{i_r}_{\alpha_r}\right)$$

$$dx^{i_1}\wedge\cdots\wedge dx^{\alpha_r}.$$

The r–form η is vertical iff $\eta_2 \equiv 0$, that is :

$$\eta_{i_1\cdots i_k\,\alpha_{k+1}\cdots\alpha_r} = t^{i_{k+1}}_{\alpha_{k+1}}\cdots t^{i_r}_{\alpha_r}\,\eta_{i_1\cdots i_r},\ k=0,\cdots,r-1.$$

□

A vertical r–form is an r–form on M which is non–zero only on the vertical vector fields. One can identify such a form with a section of the vector bundle $\overset{r}{\wedge}\mathcal{V}^*$ [i.e., an alternate r–linear mapping of $\mathcal{X}_V \times \cdots \times \mathcal{X}_V$ into $\Omega^0(M)$]. So, a vertical r–form of $\mathcal{F}$ is a *form along the leaves* : its restriction to each leaf L of $\mathcal{F}$ is a r–form on the submanifold L.

If one prefers, a vertical r–form of $\mathcal{F}$ can be identified with an *$\mathcal{F}$–foliated r–form* [i.e., an element of the quotient space $\Omega^r(M)/_{\Omega^r_{\mathcal{F}}(M)}$, where $\Omega^r_{\mathcal{F}}(M)$ is the subspace of *relative* forms : $\eta \in \Omega^r_{\mathcal{F}}(M)$ if $\eta(X_1,\cdots,X_r) = 0$ for all $X_1,\cdots,X_r \in \mathcal{X}_V$]. (For details see [A1],[HMS]).

One has an exact sequence of $\Omega^o(M)$–modules

$$0 \longrightarrow \Omega^{\bullet}_{\mathcal{F}}(M) \longrightarrow \Omega^{\bullet}(M) \xrightarrow{\pi_V} \Omega^{\bullet}_V(M,\mathcal{F}) \longrightarrow 0.$$

The mapping π_V will be called the *natural projection* of $\Omega^{\bullet}(M)$ onto $\Omega^{\bullet}_V(M,\mathcal{F},\mathcal{K})$. One has $\pi_V\eta(X_1,\cdots,X_r):= \eta\,(X_1^v,\cdots,X_r^v)$ *for* $X_1,\cdots,X_r \in \mathfrak{X}\,(M)$.

Notice that for $r > \ell$ [$\ell = \dim\ \mathcal{F}$]

$$\Omega^r_V\,(M,\mathcal{F}) = 0.$$

The exterior differentiation d in $\Omega^{\bullet}(M)$ induces a differential operator

$$d_V := \pi_V \circ d$$

in $\Omega^{\bullet}_V(M,\mathcal{F},\mathcal{K})$, called the *vertical* [or *foliated*] *exterior differentiation.*

Let $\eta \in \Omega^r_V(M,\mathcal{F},\mathcal{K})$. We have :

$$d_V\eta(X_1,\cdots,X_{r+1}) = \sum_{i=1}^{r+1} (-1)^{i+1}\, X_i\cdot \eta(X_1,\cdots,\hat{X}_i,\cdots,X_{r+1}) +$$

$$+ \sum_{i<j} (-1)^{i+j}\, \eta([X_i,X_j],\, X_1,\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_{r+1})$$

for $X_1,\cdots,X_{r+1} \in \mathfrak{X}_V(M,\mathcal{F})$.

In local distinguished coordinates, if

$$\eta = \frac{1}{r!}\,\eta_{i_1\cdots i_r}\,\theta^{i_1}\wedge\cdots\wedge\theta^{i_r},$$

then

$$d_V\eta = \frac{1}{r!}\,\frac{\partial\ \eta_{i_1\cdots i_r}}{\partial x^i}\,\theta^{i}\wedge\theta^{i_r}\wedge\cdots\wedge\theta^{i_r}.$$

In particular, for $r = 0$ [i.e., η is a function on M] :

$$d_V\eta = \frac{\partial\eta}{\partial x^i}\,\theta^i.$$

Now, let $X \in \mathfrak{X}\,(M)$.

Clearly

$$\iota_X(\,\Omega^{\bullet}_V\,(M,\mathcal{F},\mathcal{K})) \subseteq \Omega^{\bullet}_V\,(M,\mathcal{F},\mathcal{K}).$$

Therefore, the interior product ι_X in $\Omega^\bullet(M)$ induces a *skew–derivation*, also denoted by ι_X, *of degree* -1, in $\Omega^{\bullet}_{V}(M,\mathcal{F},\mathcal{H})$. That is ι_X is a linear mapping of $\Omega^{\bullet}_{V}(M,\mathcal{F},\mathcal{H})$ into itself such that :

(i) $\iota_X(\eta\wedge\omega) = (\iota_X\eta)\wedge\omega + (-1)^r\eta\wedge\iota_X\omega$ for $\eta\in\Omega^{r}_{V}(M,\mathcal{F},\mathcal{H})$ and $\omega\in\Omega^{\bullet}_{V}(M,\mathcal{F},\mathcal{H})$.

(ii) $\iota_X(\Omega^{r}_{V}(M,\mathcal{F},\mathcal{H}))\subseteq\Omega^{r-1}_{V}(M,\mathcal{F},\mathcal{H})$.

Let $\eta\in\Omega^{r}_{V}(M,\mathcal{F},\mathcal{H})$. We have :

$$\iota_X\eta\,(X_2,\cdots,X_r) = \eta(X,X_2,\cdots,X_r)$$

for $X\in\mathfrak{X}(M)$ and $X_2,\cdots,X_r\in\mathfrak{X}_V(M,\mathcal{F})$.

In local distinguished coordinates, if

$$\eta = \frac{1}{r!}\,\eta_{i_1\cdots i_r}\,\theta^{i_1}\wedge\cdots\wedge\theta^{i_r}$$

and

$$X = X^i\frac{\partial}{\partial x^i} + X^\alpha\frac{\partial}{\partial x^\alpha},$$

then

$$\iota_X\eta = \frac{1}{(r-1)!}\,\eta_{ii_2\cdots i_r}\,(X^i + t^i_\alpha X^\alpha)\,\theta^{i_2}\wedge\cdots\wedge\theta^{i_r}.$$

The Lie differentiation $\mathcal{L}_X$ in $\Omega^\bullet(M)$ induces a *derivation*

$$\theta_X := \pi_V\circ\mathcal{L}_X$$

of degree 0 in $\Omega^{\bullet}_{V}(M,\mathcal{F},\mathcal{H})$, that is θ_X is a linear mapping of $\Omega^{\bullet}_{V}(M,\mathcal{F},\mathcal{H})$ into itself such that :

(i) $\theta_X(\eta\wedge\omega) = (\theta_X\eta)\wedge\omega + \eta\wedge(\theta_X\omega)$ for $\eta,\omega\in\Omega^\bullet(M)$;

(ii) $\theta_X(\Omega^r_V) \subseteq \Omega^r_V.$

Let $\eta \in \Omega^r_V(M,\mathcal{F},\mathcal{H})$. One has :

$$\theta_X\eta(X_1,\cdots X_r) = X\cdot\eta(X_1,\cdots,X_r) - \sum_{i=1}^{r} \eta(X_1,\cdots,[X^v,X_i],\cdots,X_r)$$

for $X \in \mathfrak{X}(M)$ and $X_1,\cdots,X_r \in \mathfrak{X}_V(M,\mathcal{F})$.

In local distinguished coordinates, if

$$\eta = \frac{1}{r!}\,\eta_{i_1\cdots i_r}\,\theta^{i_1}\wedge\cdots\wedge\theta^{i_r}$$

and

$$X = X^i\frac{\partial}{\partial x^i} + X^\alpha\frac{\partial}{\partial x^\alpha},$$

then

$$\theta_X = \frac{1}{r!}\left(X(\eta_{i_1\cdots i_r}) + \sum_{h=1}^{r}\eta_{i_1\cdots i_{h-1}ki_{h+1}\cdots i_r}\left(\frac{\partial X^k}{\partial x^{i_h}} + t^k_\alpha\frac{\partial X^\alpha}{\partial x^{i_h}}\right)\right)\theta^{i_1}\wedge\cdots\wedge\theta^{i_r}.$$

(2.6.4) DEFINITION. A vertical r–form η of the foliation $\mathcal{F}$ is called *tangential* if

$$\theta_Z\eta = 0 \quad \text{for all } Z \in \mathfrak{X}_H(M,\mathcal{F},\mathcal{H}).$$

The set

$$\Omega^r_{tg} \equiv \Omega^r_{tg}(M,\mathcal{F},\mathcal{H})$$

of all tangential r–forms is a Ω^0_{tg}–submodule of $\Omega^r_V(M,\mathcal{F},\mathcal{H})$.

The direct sum

$$\Omega^\bullet_{tg} \equiv \Omega^\bullet_{tg}(M,\mathcal{F},\mathcal{H}) := \bigoplus_{r=0}^{\ell}\Omega^r_{tg}(M,\mathcal{F},\mathcal{H}),$$

equipped with the exterior product, is the *algebra of tangential forms* of the foliation $\mathcal{F}$.

REMARK A tangential form is a form along the leaves which is invariant in the direction of the horizontal bundle $\mathcal{H}$.

Let $X \in \mathfrak{X}(M)$ and $\eta \in \Omega^{1}_{V}(M, \mathcal{F}, \mathcal{H})$. In local distinghushed coordinates one has :

$$X = X^{i} \frac{\partial}{\partial x^{i}} + X^{\alpha} \frac{\partial}{\partial x^{\alpha}}$$

and

$$\eta = \eta_{i}\, \theta^{i},$$

where $\theta^{i} = dx^{i} + t^{i}_{\alpha}\, dx^{\alpha}$.

We have

$$\theta_{X}\eta = \left(X(\eta_{i}) + \eta_{k} \left(\frac{\partial X^{k}}{\partial x^{i}} + t^{k}_{\alpha} \frac{\partial X^{\alpha}}{\partial x^{i}} \right) \right) \theta^{i}.$$

Now consider $Z \in \mathfrak{X}_{H}(M, \mathcal{F}, \mathcal{H})$.

We get :

$$\theta_{Z}\eta = \left(Z(\eta_{i}) - \eta_{k} \frac{\partial t^{k}_{\alpha}}{\partial x^{i}} Z^{\alpha} \right) \theta^{i}.$$

The vertical 1–form η is tangential iff

$$Z(\eta_{i}) = \eta_{k} \frac{\partial t^{k}_{\alpha}}{\partial x^{i}} Z^{\alpha} \quad \text{for all } Z,$$

that is

$$\frac{\partial \eta_{i}}{\partial x^{\alpha}} = \eta_{k} \frac{\partial t^{k}_{\alpha}}{\partial x^{i}} + t^{j}_{\alpha} \frac{\partial \eta_{i}}{\partial x^{j}}.$$

Additionally, if we assume that the t^{i}_{α}'s are [local] basic functions, it follows that η is tangential iff its coefficients η_{i} are [local] tangential functions.

REMARK. ([C2]). The following relations holds :

(i) $d_{V}(\Omega^{\bullet}_{tg}) \subseteq \Omega^{\bullet+1}_{tg}$.

(ii) $\iota_{X}(\Omega^{\bullet}_{tg}) \subseteq \Omega^{\bullet-1}_{tg}$ for all $X \in \mathfrak{X}_{tg}$.

(iii) $\mathscr{L}_{X}(\Omega^{\bullet}_{tg}) \subseteq \Omega^{\bullet}_{tg}$ for all $X \in \mathfrak{X}_{tg}$.

Note that for $X \in \mathfrak{X}(M)$ and $\eta \in \Omega^r_V(M, \mathcal{F}, \mathcal{H})$ the following condition holds :

$$\pi_V(\iota_X\eta) = \iota_X(\pi_V\eta).$$

As a result, it follows immediately that

$$\theta_X = \iota_X \circ d_V + d_V \circ \iota_X$$

[on vertical forms only].

Indeed, we have :

$$\begin{aligned} &\theta_X - (\iota_X \circ d_V + d_V \circ \iota_X) = \pi_V \circ \mathscr{L}_X - (\iota_X \circ d_V + \\ &+ d_V \circ \iota_X) = \pi_V \circ (\iota_X \circ d + d \circ \iota_X) - (\iota_X \, \pi_V \circ d + \\ &+ \pi_V \circ d \circ \iota_X) = (\pi_V \circ \iota_X - \iota_X \circ \pi_V) \circ d = 0. \end{aligned}$$

(2.6.5) PROPOSITION. Every vertical form η which satisfies the condition

$$d_V \, \eta = 0$$

is tangential.

PROOF. Let $Z \in \mathfrak{X}_H(M, \mathcal{F}, \mathcal{H})$. Then $\theta_Z\eta = \iota_Z(d_V \, \eta) + d_V(\iota_Z \, \eta) = 0$.

□

2.7 THE VERTICAL FRAME BUNDLE

Let $\mathcal{V}$ be the vertical bundle of the foliation $\mathcal{F}$.

A *vertical frame* at a point x of M is an ordered basis

$$u = (X_{1x}, \cdots, X_{\ell x})$$

of the vertical subspace $\mathcal{V}_x$, or equivalently, a linear isomorphism

$$u : \mathbb{R}^\ell \longrightarrow \mathcal{V}_x.$$

Let $B^1_V \equiv B^1_V(M, \mathcal{F})$ be the set of all vertical frames at the different points of M, and let π^1_V be the mapping of B^1_V onto M which to a vertical frame at x associates x.

The Lie group $GL(\ell ; \mathbb{R})$ acts by right translations on B^1_V : if $u = (X_{ix})$ is a vertical frame at x and $a = (a^i_j) \in GL(\ell ; \mathbb{R})$ then we denote the composition $u \circ a$ by $R_a(u) = ua$; one has :

$$ua = (a^i_j X_{ix}).$$

It is clear that $GL(\ell ; \mathbb{R})$ acts freely on B^1_V and

$$\pi^1_V(u') = \pi^1_V(u) \text{ iff } u' = ua$$

for some $a \in GL(\ell ; \mathbb{R})$. If (U,φ) is a distinguished chart for $\mathcal{F}$ with distinguished local coordinates (x^i,x^α), then the correspondance

$$x \longmapsto \left(\frac{\partial}{\partial x^1}\bigg|_x , \ldots , \frac{\partial}{\partial x^\ell}\bigg|_x \right)$$

defines a *field of natural vertical frames* s_V associated to the local chart. This gives a "*local trivialization*"

$$(\pi^1_V)^{-1}(U) = U^1_V \longrightarrow U \times GL(\ell ; \mathbb{R})$$

via the correspondance :

$$s_V(x)\, a \longmapsto (x,a).$$

We equip $B^1_V(M, \mathcal{F})$ with the unique manifold structure for which the local trivializations thus defined are local diffeomorphisms.

Thus $B^1_V(M, \mathcal{F})$ becomes a principal fibre bundle over M with structure group $GL(\ell ; \mathbb{R})$, called the *vertical frame bundle* of $(M, \mathcal{F})$.

The *fundamental form* of $B^1_V(M, \mathcal{F})$ is the $\mathbb{R}^\ell$–valued 1–form θ^1_V defined by :

$$\theta^1_V(X_u) := u^{-1}((\pi^1_{V*,u} X_u)^v)$$

for $u \in B^1_V$ and $X_u \in T_u B^1_V$.

[$v : TM \longrightarrow \mathcal{V}$ is the vertical projection].

Let (U,φ) be a distinguished chart for $\mathcal{F}$ with distinguished local coordinates (x^i,x^α). Every vertical frame u at a point $x \in U$ can be uniquely expressed by

$$u = s_V(x)a = \left(a^i_j \frac{\partial}{\partial x^i}\Big|_x \right), \text{ where } a = (a^i_j) \in GL(\ell\,;\mathbb{R}).$$

Let $a^{-1} = (\tilde{a}^i_j)$ be the inverse matrix of a. We shall express the fundamental form θ^1_V in the local chart (U^1_V,φ^1_V) associated to (U,φ) :

$$\varphi^1_V(u) = (x^i,x^\alpha,X^i_j) \in \mathbb{R}^{m+\ell^2}, \text{ where } X^i_j = a^i_j.$$

Let $(e_1,\cdots,e_\ell)$ be the natural basis of $\mathbb{R}^\ell$ and set :

$$\theta^1_V = (\theta^1_V)^i e_i.$$

Consider

$$X_u \in T_u B^1_V,\ u \in U^1_V .$$

If

$$X_u = a^i \frac{\partial}{\partial x^i}\Big|_u + a^\alpha \frac{\partial}{\partial x^\alpha}\Big|_u + A^i_j \frac{\partial}{\partial X^i_j}\Big|_u$$

so that

$$\pi^1_{V*,u} X_u = a^i \frac{\partial}{\partial x^i}\Big|_x + a^\alpha \frac{\partial}{\partial x^i}\Big|_x$$

and

$$\left(\pi^1_{V*,u} X_u \right)^v = (a^i + t^i_\alpha a^\alpha) \frac{\partial}{\partial x^i}\Big|_x ,$$

then :

$$\theta^1_V(X_u) = u^{-1} \left((a^i + t^i_\alpha a^\alpha) \frac{\partial}{\partial x^i}\Big|_x \right) = \tilde{X}^i_j (a^j + t^j_\beta a^\beta) e_i.$$

Consequently

$$(\theta^1_V)^i = \tilde{X}^i_j\, \theta^j, \text{ where } \theta^j = dx^j + t^j_\beta\, dx^\beta.$$

Now, let $\mathcal{H}$ be a fixed horizontal bundle for foliation $\mathcal{F}$ and let $\pi^1_V : B^1_V \longrightarrow M$ be the vertical frame bundle of $\mathcal{F}$.

Let $\mathcal{F}^1_V := (\pi^1_V)^* \mathcal{F}$ be the pull–back of F by π^1_V and let $\mathcal{V}(B^1_V, \mathcal{F}^1_V)$ be the vertical [i.e., tangent] bundle of $\mathcal{F}^1_V$. The foliation $\mathcal{F}^1_V$ will be called the *pull–back of* $\mathcal{F}$ *to* $B^1_V(M, \mathcal{F})$.

Notice that :

(i) $\dim \mathcal{F}^1_V = \ell(\ell+1)$ [where $\ell = \dim \mathcal{F}$] ;

(ii) $\operatorname{codim} \mathcal{F}^1_V = \operatorname{codim} \mathcal{F} = n$.

The horizontal bundle $\mathcal{H}(B^1_V, \mathcal{F}^1_V)$ can be defined from the natural lifts to B^1_V of local foliated horizontal vector fields of $\mathcal{F}$.

Let u_0 be a vertical frame at $x_0 = \pi^1_V(u_0)$. Let Z be a foliated horizontal vector field on U and let (φ_t) be the local flow generated by Z. Since (φ_t) preserves $\mathcal{F}$ we have that for each $u \in (\pi^1_V)^{-1}(U)$, $(\varphi_t)_*(u)$ is a curve in $(\pi^1_V)^{-1}(U)$. This flow in $(\pi^1_V)^{-1}(U)$ generates a vector field Z^1_V on $(\pi^1_V)^{-1}(U)$. Z^1_V is the *natural lift* of Z to B^1_V.

Define

$\mathcal{H}(B^1_V, \mathcal{F}^1_V)_{u_0} := \{Z^1_{Vu_0} \mid$ Z is a foliated horizontal vector field defined on a neighbourhood of $\pi^1_V(u_0)\}$.

Clearly, $\mathcal{H}(B^1_V, \mathcal{F}^1_V)_{u_0}$ is a subspace of $T_{u_0} B^1_V$ and so we obtain a distribution on B^1_V.

(2.7.1) PROPOSITION ([BH3]). *Let* u_0 *be a vertical frame at* x_0. *Then* π^1_{V*,u_0} *maps* $\mathcal{H}(B^1_V, \mathcal{F}^1_V)_{u_0}$ *isomorphically onto* $\mathcal{H}(M, \mathcal{F})_{x_0}$ *and*

$$T(B^1_V) = \mathcal{V}(B^1_V, \mathcal{F}^1_V) \oplus \mathcal{H}(B^1_V, \mathcal{F}^1_V).$$

REMARK. ([C1]). The horizontal bundle $\mathcal{H}(B^1_V, \mathcal{F}^1_V)$ of $\mathcal{F}^1_V$ depends only on the decomposition $TM = \mathcal{H}(M,\mathcal{F}) \oplus \mathcal{V}(M,\mathcal{F})$, that is

$$\mathcal{H}(B^1_V, \mathcal{F}^1_V) = \{X \in \mathfrak{X}(B^1_V) \mid \iota_X \theta^1_V = 0 \text{ and } \iota_X d\theta^1_V = 0\}.$$

2.8 VERTICAL CONNECTIONS AND VERTICAL G–STRUCTURES

VERTICAL CONNECTIONS

Let $\mathcal{V}$ be the vertical bundle of the foliation $\mathcal{F}$ and let $\mathfrak{X}_V(M,\mathcal{F})$ be the module of vertical vector fields of $\mathcal{F}$. Let $\mathcal{H}$ be a fixed horizontal bundle for $\mathcal{F}$ and let $\mathfrak{X}_H(M,\mathcal{F},\mathcal{H})$ be the $\Omega^0(M)$–module of $\mathcal{H}$–horizontal vector fields on $(M,\mathcal{F})$.

(2.8.1) DEFINITION. A *connection* [or *covariant derivation*] on $\mathcal{V}$ is a mapping

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}_V(M,\mathcal{F}) \longrightarrow \mathfrak{X}_V(M,\mathcal{F}),$$

written $\nabla(X,Y) = \nabla_X Y$, such that

(i) $\nabla_{X_1+X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$;

(ii) $\nabla_X(Y_1+Y_2) = \nabla_X Y_1 + \nabla_X Y_2$;

(iii) $\nabla_{fX} Y = f \cdot \nabla_X Y$;

(iv) $\nabla_X(fY) = f \cdot \nabla_X Y + X(f) \cdot Y$

for $X, X_1, X_2 \in \mathfrak{X}(M)$, $Y, Y_1, Y_2 \in \mathfrak{X}_V(M,\mathcal{F})$ and $f \in \Omega^0(M)$.

REMARK. $\nabla_X Y$ is a kind of directional derivative which differentiates vertical vector fields of $\mathcal{F}$ in X–directions.

(2.8.2) PROPOSITION ([Bo], *p.*23). *There always exist connections* ∇ *on* $\mathcal{V}$.

(2.8.3) DEFINITION. Let ∇ be a connection on $\mathcal{V}$.

(i) The *torsion* of ∇ is the mapping T_∇, defined by :

$$T_\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}_V(M, \mathcal{F})$$

$$(X, X') \longmapsto T_\nabla(X, X') = \nabla_X(X'^v) - \nabla_{X'}(X^v) - [X, X']^v.$$

T_∇ is a $\mathcal{V}$–valued 2–form on M.

(ii) The *curvature* of ∇ is the mapping R_∇, defined by :

$$R_\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}_V(M, \mathcal{F}) \longrightarrow \mathfrak{X}_V(M, \mathcal{F}),$$

$$(X, X') \longmapsto R_\nabla(X, X') = \nabla_X(\nabla_{X'} Y) - \nabla_{X'}(\nabla_X Y) - \nabla_{[X, X']} Y.$$

REMARK. T_∇ [resp. R_∇] is a *tensor field along the leaves*, of type (1,2) [resp. (1,3)] : its restriction to each leaf L of $\mathcal{F}$ is a tensor held on the submanifold L.

(2.8.4) DEFINITION. A connection ∇ on V is called *vertical* if

$$\nabla_Z Y = [Z, Y]^v \text{ for all } Z \in \mathfrak{X}_H(M, \mathcal{F}, \mathcal{H}) \text{ and } Y \in \mathfrak{X}_V(M, \mathcal{F}).$$

(2.8.5) PROPOSITION. *There always exist vertical connections on* $\mathcal{V}$.

PROOF. Choose an arbitrary connection ∇' on $\mathcal{V}$. Define ∇ on $\mathcal{V}$ by :

$$\nabla_X Y = \nabla'_{X^v} + [X^h, Y]^v,$$

for $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}_V(M, \mathcal{F})$.

Thus if $X \in \mathfrak{X}_H(M, \mathcal{F}, \mathcal{H}) \subset \mathfrak{X}(M)$, $X^v = 0$ and we obtain the previous formula for $\nabla_X Y$.

It is trivial to check that this defines a connection on $\mathcal{V}$.

□

(2.8.6) DEFINITION. A vertical connection ∇ on $\mathcal{V}$ is called *tangential* if

$$R_{\nabla}(Z,X) = 0 \quad \text{for all } Z \in \mathfrak{X}_H(M,\mathcal{F},\mathcal{H}) \text{ and } X \in \mathfrak{X}_V(M,\mathcal{F})$$

REMARKS.

(i) A fortiori, a tangential connection ∇ on $\mathcal{V}$ is vertical.

(ii) In general, the existence of a tangential connection is not assured.

Let $\mathcal{F}$ be a foliation on (M,g_M), with induced metric g_V on $\mathcal{V}$ and let ∇^M denote the Riemannian connection of g_M [i.e., the Levi–Cività connection on (M,g_M)].

A connection ∇ on $\mathcal{V}$ can be defined by the formula :

$$\nabla_X Y := (\nabla^M_X Y)^v \quad \text{for } X \in \mathfrak{X}(M) \text{ and } Y \in \mathfrak{X}_V(M,\mathcal{F}).$$

(2.8.7) PROPOSITION. *The connection* ∇ *on* $\mathcal{V}$ *is metric [with respect to the induced metric* g_V *].*

$$\nabla_X g_V = 0 \quad \textit{for all } X \in \mathfrak{X}(M)$$

[*i.e.,* $X\cdot g_V(Y,Y') = g_V(\nabla_X Y,Y') + g_V(Y,\nabla_X Y')$ *for all* $X \in \mathfrak{X}(M)$ *and* $Y,Y' \in \mathfrak{X}_V(M,\mathcal{F})$].

PROOF. Using succesively that ∇^M is metric [with respect to g_M] and the sum $\mathcal{V} \oplus \mathcal{H}$ orthogonal, one has :

$$\begin{aligned} X\cdot g_V(Y,Y') &= X\cdot g_M(Y,Y') = g_M(\nabla^M_X Y,Y') + g_M(Y,\nabla^M_X Y') = \\ &= g_M((\nabla^M_X Y)^v,Y') + g_M(Y,(\nabla^M_X Y')^v) = g_V(\nabla_X Y,Y') + \\ &+ g_V(Y,\nabla_X Y'), \end{aligned}$$

which proves that ∇ is metric [with respect to g_V]. □

Notice that the connection ∇ on $\mathcal{V}$ is vertical [i.e., $\nabla_Z Y = [Z,Y]^v$ for all $Z \in \mathfrak{X}_H(M,\mathcal{F},\mathcal{H})$ and $Y \in \mathfrak{X}_V(M,\mathcal{F})$] if and only if

$$g_M(\nabla^M_Z Y - [Z,Y], Y') = 0$$

for $Y, Y' \in \mathfrak{X}_V(M, \mathcal{F})$ and $Z \in \mathfrak{X}_H(M, \mathcal{F}, \mathcal{H})$, which is equivalent to the condition that $\mathcal{F}$ is a *totally geodesic foliation* (See Annexe B).
Moreover, in this case, the vertical connection ∇ is tangential (see [JN]) and is called the *tangential Levi–Cività connection* of $\mathcal{F}$.

REMARKS.

(i) ∇ depends on the Riemannian metric g_M [and, consequently, on the resulting choice of the horizontal bundle $\mathcal{H}$].

(ii) The torsion T_∇ of ∇ does not vanish.

(iii) Restricted to each leaf L of $\mathcal{F}$, the tangential Levi–Cività connection is the usual induced connection for the submanifold L.

It is convenient to consider the same situation also from the principal bundle point of view.

There is a canonical equivalence between connections ∇ on $\mathcal{V}$ and connections ω on $B^1_V(M,\mathcal{F})$.

(2.8.8) DEFINITION. A [infinitesimal] *connection* on $B^1_V(M,\mathcal{F})$ is a $\mathfrak{gl}(\ell\,;\mathbb{R})$–valued 1–form ω on $B^1_V(M,\mathcal{F})$ such that :

(i) $\omega(A^*) = A$ for every $A \in \mathfrak{gl}(\ell\,;\mathbb{R})$;

(ii) $\omega(R_{a*}X) = \mathrm{ad}(a^{-1})\cdot\omega(X)$ for every $a \in GL(\ell\,;\mathbb{R})$ and every $X \in \mathfrak{X}(B^1_V(M,\mathcal{F}))$.

[A^* is the fundamental vector field corresponding to A and ad denoted the adjoint representation of $GL(\ell\,;\mathbb{R})$ in $\mathfrak{gl}(\ell\,;\mathbb{R})$.]

(2.8.9) DEFINITION. Let ω be a connection on $B^1_V(M, \mathcal{F})$.

(i) The *curvature* of ω is the $gl\,(\ell\,;\mathbb{R})$–valued 2–form Ω^1_V, defined by :

$$\Omega^1_V := d\omega + \tfrac{1}{2}\,[\omega,\omega],$$

where $[\omega,\omega](X,Y) = 2[\omega(X),\omega(Y)]$.

(ii) The *torsion* of ω is the $\mathbb{R}^\ell$–valued 2–form Σ^1_V, defined by :

$$\Sigma^1_V := d\theta^1_V + \omega \wedge \theta^1_V,$$

where $(\omega \wedge \theta^1_V)(X,Y) = \omega(X)\cdot \theta^1_V\,(Y) - \omega(Y)\cdot \theta^1_V(X)$.

(2.8.10) DEFINITION. A connection ω on $B^1_V(M, \mathcal{F})$ is called *vertical* if the horizontal bundle $\mathcal{H}\,(B^1_V, \mathcal{F}^1_V)$ of the foliation $\mathcal{F}^1_V$ is vertical with respect to

$$\omega(Z) = 0 \text{ for all } Z \in \mathfrak{X}_H(B^1_V, \mathcal{F}^1_V, \mathcal{H}).$$

REMARK. The connection ω is vertical if ω is a vertical $gl\,(\ell\,;\mathbb{R})$–valued 1–form of $\mathcal{F}^1_V$.

(2.8.11) PROPOSITION. *Let ω be a connection on $B^1_V(M, \mathcal{F})$. The following statements are equivalent :*

(i) *ω is vertical.*

(ii) *The torsion Σ^1_V of ω satisfies :*

$$\iota_Z\,\Sigma^1_V = 0 \textit{ for all } Z \in \mathfrak{X}_H(B^1_V, \mathcal{F}^1_V, \mathcal{H}).$$

PROOF. From the definitions of $\mathcal{H}\,(B^1_V, \mathcal{F}^1_V)$ and Σ^1_V one has :

$$\iota_Z\,\Sigma^1_V = \omega(Z)\cdot \theta^1_V \text{ for all } Z \in \mathfrak{X}_H(B^1_V, \mathcal{F}^1_V, \mathcal{H}).$$

One deduces immediately the equivalence between (i) and (ii)

□

(2.8.12) DEFINITION. A connection ω on $B^1_V(M, \mathcal{F})$ is called *tangential* if ω is a tangential $gl\,(\ell\,;\mathbb{R})$–valued 1–form of $\mathcal{F}^1_V$.

REMARK. A fortion, a tangential connection ω on B^1_V is vertical.

(2.8.13) PROPOSITION. *Let ω be a connection on B^1_V with curvature Ω^1_V and torsion Σ^1_V. The following statements are equivalent:*

(i) *ω is tangential.*

(ii) $\iota_Z\Omega^1_V = \iota_Z\Sigma^1_V = 0$ *for all* $Z \in \mathfrak{X}_H(B^1_V, \mathcal{F}^1_V)$.

PROOF.

(i) ⟹ (ii). If ω is tangential, then the forms Ω^1_V and Σ^1_V are vertical for $\mathcal{F}^1_V$; hence the condition (ii).

(ii) ⟹ (i). If ω satisfies (ii), then ω is vertical. It follows that one has

$$\iota_Z\omega = 0 \quad \text{for all } Z \in \mathfrak{X}_H(B^1_V, \mathcal{F}^1_V).$$

But then

$$\iota_Z d\omega = 0$$

which implies $\theta_Z\omega = 0$ and so the form ω is tangential for $\mathcal{F}^1_V$.

□

VERTICAL G–STRUCTURES

Let G by a Lie subgroup of $GL(\ell\,;\mathbb{R})$ with Lie algebra $\mathfrak{g}$ and let E be a subbundle of $B^1_V(M, \mathcal{F})$ with structure group G, that is E^1_V is a reduced bundle of $B^1_V(M, \mathcal{F})$. In other words, E^1_V is a submanifold of $B^1_V(M, \mathcal{F})$ such that :

(i) for all $u \in E^1_V$, $ua \in E^1_V$ iff $a \in G$;

(ii) the restriction of π^1_V to E^1_V is a surjective submersion onto M.

If one prefers, E^1_V is a reduced bundle of $B^1_V(M, \mathcal{F})$ if $B^1_V(M, \mathcal{F})$ is the natural extension of E^1_V under the change of groups from G to $GL(\ell\,;\mathbb{R})$ [i.e.,

$$B^1_V(M, \mathcal{F}) \cong E^1_V \times_G GL(\ell\,;\mathbb{R}).]$$

If ω is a connection on $B^1_V(M, \mathcal{F})$, we will say that ω is *adapted* to E^1_V if its restriction to E^1_V is $\mathfrak{g}$–valued 1–form $\omega_{E^1_V}$.

(2.8.14) DEFINITION. A G–reduction E^1_V of $B^1_V(M, \mathcal{F})$ is called a *vertical* G–*structure* on $(M, \mathcal{F})$ if for every $u \in E^1_V$, the horizontal subspace $\mathcal{H}\,(B^1_V, \mathcal{F}^1_V\,)_u$ at u is tangent to E^1_V [i.e., for every $u \in E^1_V$

$$\mathcal{H}\,(B^1_V, \mathcal{F}^1_V\,)_u \subset T_u\, E^1_V\,].$$

(2.8.15) PROPOSITION. ([C1]). *If* E^1_V *is a* G–*reduction of* $B^1_V(M, \mathcal{F})$, *then* E^1_V *is a vertical* G–*structure iff there exist on* $B^1_V(M, \mathcal{F})$ *a vertical connection that is adapted to* E^1_V .

(2.8.16) DEFINITION. A vertical G–structure E^1_V on $(M, \mathcal{F})$ is said to be *integrable* if for every point x of M there exists a distinguished chart (U,φ) for $\mathcal{F}$ with distinguished coordinates (x^i,x^α), whose domain U contains x, such that the section $\left(\dfrac{\partial}{\partial x^i}\right)$ of $B^1_V(M, \mathcal{F})$ over U [i.e., the field of natural vertical frames associated to the local chart] is a section of E^1_V over U.

We shall call such a local chart *admissible* with respect to the given vertical G–structure E^1_V.

If (U,φ) and (U',φ') are two admissible local charts, then the "tangential block" J_1 of the Jacobian matrix

$$\mathrm{Jac}(\varphi' \circ \varphi^{-1}) = \begin{pmatrix} J_1 & J_2 \\ 0 & J_3 \end{pmatrix}$$

belongs to G at each point of $U \cap U'$.

EXAMPLES OF VERTICAL G–STRUCTURES

(i) If $G = \{e\}$, then a vertical G–structure on $(M, \mathcal{F})$ is called a *tangential parallelism.* A tangential parallelism on $(M, \mathcal{F})$ is given by a set $(X_1, \cdots X_\ell)$ of ℓ tangential vector fields that are linearly independent at each point of M.

A *tangentially parallelizable foliation* is a foliation is a foliation equipped with a tangential parallelism.

(ii) If $G = O(\ell\ ;\ \mathbb{R})$ is the group of orthogonal matrices, then a vertical G–structure on $(M, \mathcal{F})$ is called a *tangential Riemannian structure.*

The vertical $O(\ell\ ;\ \mathbb{R})$ on $(M, \mathcal{F})$ are in a natural 1–1 correspondence with fibre metrics g_V on $\mathcal{V}$ such that

$$\mathscr{L}_Z g_V = 0 \text{ for all } Z \in \mathfrak{X}_H(M, \mathcal{F}, \mathcal{H}).$$

A *tangentially Riemannian foliation* is a foliation equipped with a tangential Riemannian structure. Such a foliation is also called *totally geodesic* (see [KT2],[To] ; also Annexe B).

(iii) If $\ell = 2k$ and if $G = Sp(k\ ;\ \mathbb{R})$ is the group of symplectic matrices over $\mathbb{R}^{2k}$, then a vertical G–structure on $(M, \mathcal{F})$ is called a *tangential almost symplectic structure.*

Recall that $Sp(k\ ;\ \mathbb{R})$ is the group of linear transformations of $\mathbb{R}^{2k}$ leaving the standard symplectic form

$$\sigma_0 = dx^1 \wedge dp_1 + \cdots + dx^k \wedge dp_k$$

on $\mathbb{R}^{2k}$ invariant. In other words,

$$Sp(k\ ;\ \mathbb{R}) = \{a \in GL(2k\ ;\ \mathbb{R}) \mid {}^t a\, J a = J\},$$

where $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$.

Let E^1_V be a tangential almost symplectic structure on $(M, \mathcal{F})$.

Let $x \in M$, let $\mathcal{V}_x$ be the vertical subspace at x, and let u be a point of E^1_V above x [i.e., a basis of $\mathcal{V}_x$].

Define

$$\sigma_x : \mathcal{V}_x \times \mathcal{V}_x \longrightarrow \mathbb{R}$$

by the formula :

$$\sigma_x(Y_x, Y'_x) := \sigma_0(u^{-1}(Y_x), u^{-1}(Y'_x)),$$

where σ_0 is the canonical symplectic form on $\mathbb{R}^{2k}$.

The mapping σ_x is well defined : the RHS does not depend on the choice of u. Indeed, let u′ be another point of E^1_V above x. So, $u' = ua$, where $a \in Sp(k\,;\mathbb{R})$. This means that

$$u'^{-1} = a^{-1} \circ u^{-1}.$$

One has :

$$\sigma_0(u'^{-1}(Y_x), u'^{-1}(Y'_x)) = \sigma_0(a^{-1} \circ u^{-1}(Y_x), a^{-1} \circ u^{-1}(Y'_x)) =$$
$$= \sigma_0(u^{-1}(Y_x), u^{-1}(Y'_x)).$$

If $Y, Y' \in \mathfrak{X}_V(M, \mathcal{F})$, then we denote by $\sigma(Y,Y')$ the smooth function whose value at x is

$$\sigma(Y,Y')(x) = \sigma_x(Y(x), Y'(x)).$$

This defines a skew–symmetric $\Omega^0(M)$–bilinear mapping

$$\sigma : \mathfrak{X}_V(M, \mathcal{F}) \times \mathfrak{X}_V(M, \mathcal{F}) \longrightarrow \Omega^0(M)$$

[i.e., a vertical 2–form on $(M, \mathcal{F})$].

Note that σ satisfies :

$$\sigma^k \neq 0 \text{ at all points of } M.$$

We will say that σ is the vertical 2–form associated to the tangential almost symplectic structure.

Clearly, the vertical Sp(k;$\mathbb{R}$)–structure E_V^1 is entirely determined by the vertical 2–form σ. Consequently, the vertical Sp(k ; $\mathbb{R}$)–structures on $(M, \mathcal{F})$ are in a natural 1–1 correspondence with the vertical 2–forms σ of $\mathcal{F}$ that are non–degenerate along the leaves :

$$\sigma^k \neq 0 \text{ at all points of } M.$$

Let us consider now the integrability problem for vertical Sp(k ; $\mathbb{R}$)–structures. Recall the following integrability criterion :

(2.8.17) PROPOSITION. ([K]). Let E_V^1 be a vertical Sp(k ; $\mathbb{R}$)–structure on $(M, \mathcal{F})$ and let σ be the vertical 2–form associated to E_V^1. Then E_V^1 is integrable if and only if for each point x of M there exists a distinguished chart (U,φ) for $\mathcal{F}$, whose domain U contains x, such that the components of σ are constant functions on U.

Using standard arguments (see [AG],[AM],[K],[LM] or [S]) it follows that the integrability of a tangential almost symplectic structure on $(M, \mathcal{F})$ is characterized by the condition :

$$\text{(IC)} \qquad d_V\sigma = 0.$$

(2.8.18) DEFINITION. A *tangentially symplectic foliation* is a foliation $\mathcal{F}$ equipped with a *tangential symplectic structure* [i.e., an integrable tangential almost symplectic structure].

Such a foliation [on M] will be denoted by $(M, \mathcal{F}, \sigma, \mathcal{H})$ or simply by $(\mathcal{F}, \sigma)$.

Chapter 3

BASIC PROPERTIES OF TANGENTIALLY SYMPLECTIC FOLIATIONS

A *tangentially symplectic foliation* is a foliation equipped with a *tangential symplectic structure* [i.e., an integrable tangential almost symplectic structure].

Such a structure can be described in terms of a

(i) tangentially symplectic atlas ;
(ii) symplectic vertical form ; or
(iii) regular Poisson structure.

In this chapter I investigate these three essentially equivalent approaches and some basic properties of tangentially symplectic foliations.

Throughout, M is an m–dimensional connected manifold and $\mathcal{F}$ is a 2k–dimensional foliation on M ; m = 2k+n.

3.1 TANGENTIALLY SYMPLECTIC FOLIATED ATLASES

Let $\mathcal{H}$ be a fixed horizontal bundle for the foliation $\mathcal{F}$ on M.
Let Sp(k;$\mathbb{R}$) [also denoted by Aut($\mathbb{R}^{2k}$,σ_0)] be the group of all symplectic transformations of the standard symplectic manifold ($\mathbb{R}^{2k}$,σ_0) onto itself [i.e., the group of symplectic matrices over $\mathbb{R}^{2k}$]. The symplectic group Sp(k;$\mathbb{R}$) acts analytically, transitively and effectively on $\mathbb{R}^{2k}$.

(3.1.1) DEFINITION. A foliated atlas [defining the foliation $\mathcal{F}$]

$$\mathcal{A} = \{(U_\mu, \varphi_\mu)\}_{\mu \in \mathcal{M}}$$

of dimension 2k and codimension n on M is said to be *tangentially symplectic* if :

(TS–1) For any two overlapping foliated charts [i.e., distinguished charts for $\mathcal{F}$] (U_μ,φ_μ) and (U_ν,φ_ν), the coordinate transition map

$$\varphi_{\nu\mu} = \varphi_\nu \circ \varphi_\mu^{-1} : \varphi_\mu(U_\mu \cap U_\nu) \longrightarrow \varphi_\nu(U_\mu \cap U_\nu)$$

$$(x^i,x^\alpha) \longmapsto (\varphi^1_{\nu\mu}(x^i,x^\alpha), \varphi^2_{\nu\mu}(x^\alpha))$$

has the property that the map $\varphi^1_{\nu\mu}(\cdot\,,x^\alpha)$ belongs to Sp(k;$\mathbb{R}$) for every fixed $(x^\alpha) \in \mathbb{R}^n$.

(TS–2) The symplectic structure "along the leaves" [naturally induced on leaves by the standard symplectic structure of $\mathbb{R}^{2k}$] is invariant in the direction of $\mathcal{H}$.

REMARK. According to condition (TS–2), the symplectic structure "along the leaves" is "locally projectable along the $\mathcal{H}$–sheets of the foliation". In other words, the symplectic structure is invariant by sliding along the $\mathcal{H}$–sheets [or, if one prefers, by sliding along $\mathcal{H}$–horizontal curves].

As usual, we say that two tangentially symplectic foliated atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are *equivalent* if their union is again a tangentially symplectic foliated atlas.

(3.1.2) DEFINITION. A *tangential symplectic structure* [in the second sense] on M is an equivalence class of tangentially symplectic foliated atlases [for the first sense, see (2.8.18)].

REMARK. A tangential symplectic structure [in the second sense] on M will be given by choosing a particular tangentially symplectic foliated atlas $\mathcal{A}$ on M. Without loss of generality one can assume that $\mathcal{A}$ is maximal [or complete]. The corresponding tangential symplectic structure is denoted by $\hat{\mathcal{A}}$.

Note that the model foliation $\mathcal{F}_0$ on $\mathbb{R}^m = \mathbb{R}^{2k} \times \mathbb{R}^n$ has a natural tangential symplectic structure [in this case, the horizontal bundle $\mathcal{H}$ is the trivial bundle $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$].

A tangential symplectic structure [in the first sense] as introduced in (2.8.18) is clearly equivalent to a tangential symplectic structure [in the second sense] as defined in (3.1.2).

So, a tangentially symplectic foliation is a foliation such that the ambient space [i.e., the underlying manifold] is covered by a collection of distinguished charts for which the coordinate transformations are symplectic in the direction tangent to the leaves and such that the symplectic structure "along the leaves" is invariant in the direction of some given horizontal bundle for the foliation [i.e., a complement of the vertical bundle].

3.2 SYMPLECTIC VERTICAL FORMS

Let $\mathcal{H}$ be a fixed horizontal bundle for the foliation $\mathcal{F}$ on M. I have adapted the following definition from [HMS].

(3.2.1) DEFINITION. A vertical 2–form σ of $\mathcal{F}$ is called a *symplectic vertical form* if :

(SV–1) It is non–degenerate along the leaves :

$$\sigma^k \neq 0 \text{ at all points of M.}$$

(SV–2) It is d_V–closed :

$$d_V\sigma = 0.$$

As we have already seen, the symplectic vertical forms on (M, $\mathcal{F}$) are in a natural 1–1 correspondence with the tangential symplectic structures [in the first sense] on (M, $\mathcal{F}$) [cf. (2.8)].

So, a tangentially symplectic foliated manifold $(M, \mathcal{F}, \sigma, \mathcal{H})$ is a foliated manifold $(M, \mathcal{F})$ equipped with a symplectic vertical form σ.

Let σ be a vertical 2–form on $(M, \mathcal{F})$. We write

$$\sigma^{\flat} = \iota_{v}\sigma,$$

where $v \in \mathcal{V}_x \subset T_xM$.

In other words, $\sigma^{\flat}(v)$ denotes the unique element of $\mathcal{V}_x^*$ such that, for every $w \in \mathcal{V}_x$:

$$\sigma^{\flat}(v)(w) = -\sigma(v,w).$$

the associate map

$$\sigma^{\flat} : \mathcal{V} \longrightarrow \mathcal{V}^* \ , \ v \longmapsto \sigma^{\flat}(v)$$

is a vector bundle morphism.

Using standard arguments (see, e.g., [LM]) it follows that :
the vertical 2–form σ is non–degenerate along the leaves iff the associate map $\sigma^{\flat}$ is an isomorphism.

Let σ be a symplectic vertical form of the foliation $\mathcal{F}$.

REMARKS.

(i) In every distinguished chart (U,φ) with distinguished local coordinates (x^i,x^α), σ has the form :

$$\sigma\Big|_U = \frac{1}{2}\,\sigma_{ij}(x^1,\cdots,x^m)\ \theta^i \wedge \theta^j,$$

where $\theta^i = dx^i + t_\alpha^i dx^\alpha$; $i = 1,\cdots,2k$ and $\alpha = 2k+1,\cdots,m$.

(ii) As any d_V–closed vertical form on $(M, \mathcal{F})$ is tangential [see (2.6)] it follows that, actually, σ is a tangential 2–form of the foliation $\mathcal{F}$.

Consequently, σ can be identified [see (2.6)] with an alternate bilinear mapping of $\mathfrak{X}_V(M,\mathcal{F}) \times \mathfrak{X}_V(M,\mathcal{F})$ into $\Omega^0_{tg}(M,\mathcal{F},\mathcal{H})$.

The following result holds :

(3.2.2) THEOREM. (The Darboux theorem for tangentially symplectic foliations). *Let* $(M, \mathcal{F}, \sigma, \mathcal{H})$ *be a tangentially symplectic foliated manifold. Then for each point* x *of* M *there is a distinguished chart* (U,φ) *about* x, *with distinguished coordinates* (x^i,x^α) *and such that* σ *has the form* :

$$\sigma\Big|_U = d\theta^{k+1} \wedge d\theta^1 + \cdots + d\theta^{2k} \wedge d\theta^k,$$

where

$$\theta^i = dx^i + t^{\,i}_{\alpha} dx^\alpha$$

We call such a chart a *distinguished Darboux chart* for $\mathcal{F}$.

REMARK. A similar theorem is stated for Lagrangian fibrations in [AG] ; see also [I], [Li2], [W3], as well as [HMS].

3.3 REGULAR POISSON MANIFOLDS AND TANGENTIALLY SYMPLECTIC FOLIATIONS

Let $\mathcal{F} \equiv \mathcal{F}_\Lambda$ be the characteristic foliation of a regular Poisson manifold (M, Λ) of symplectic dimension 2k [i.e., the dimension of $\mathcal{F}$ is 2k]. Every leaf L of $\mathcal{F}$ possesses a unique symplectic structure σ_L such that the inclusion map

$$i_L : L \longrightarrow M$$

is a Poisson morphism.

Let σ be the 2–form on M defined by :

$$\sigma(X_f,X_g) = \Lambda\,(df,dg).$$

Recall that $\mathcal{F}$ is generated by the Hamiltonian vector fields of (M, Λ).

The restriction of σ to each leaf L coincides with the symplectic form σ_L :

$$\sigma\big|_L = \sigma_L.$$

One deduces that :

(i) σ is vertical.

(ii) $d_V\sigma = 0$.

(iii) $\sigma^k \neq 0$ at all points of M, that is σ is a symplectic vertical form.

Consequently, the characteristic foliation $\mathcal{F}_\Lambda$ of a regular Poisson manifold (M, Λ) has a natural tangential symplectic structure.

Conversely, let $(\mathcal{F},\sigma)$ be a tangentially symplectic foliation of dimension 2k on the manifold M.

To any function $f \in \Omega^0(M)$ one can associate a unique vector field $X_f \in \mathfrak{X}_V(M, \mathcal{F})$ such that :

$$\iota_{X_f}\sigma = -\, d_V f.$$

More general, to any 1–form $\eta \in \Omega^1(M)$ one can associate an unique vector field $X_\eta \in \mathfrak{X}_V(M, \mathcal{F})$ such that :

$$\iota_{X_f}\sigma = -\,\pi_V\eta.$$

Define a morphism

$$\Lambda^{\#} : T^*M \longrightarrow TM$$

such that :

$$\Lambda^{\#} \circ \eta = X_\eta \quad \text{for } \eta \in \Omega^1(M).$$

By construction :

(i) $\operatorname{Ker} \Lambda^{\#} = \nu^* \mathcal{F}$;

(ii) $\Lambda^{\#}(T^*M) = T\mathcal{F}$.

The tensor field Λ defined by :

$$\Lambda\ (df,dg) = \sigma(X_f,X_g)$$

is a Poisson tensor on M.

Note as well that the vector fields X_f are Hamiltonian vector fields of (M, Λ) and

$$\mathcal{F}_{\Lambda} = \mathcal{F}.$$

Consequently, to any tangentially symplectic foliation ($\mathcal{F}$,σ) on the manifold M there corresponds a regular Poisson structure, such that the characteristic foliation $\mathcal{F}_{\Lambda}$ is $\mathcal{F}$.

So, we have proven the following

(3.3.1) PROPOSITION. *The tangentially symplectic foliations on a manifold* M *are in a natural* 1–1 *correspondence with the regular Poisson structures on* M.

3.4 AUTOMORPHISMS AND INFINITESIMAL AUTOMORPHISMS

Let (M, $\mathcal{F}$, σ, $\mathcal{H}$) be a tangentially symplectic foliated manifold [with horizontal bundle $\mathcal{H}$].

(3.4.1) DEFINITION. A diffeomorphism φ of M is called an *automorphism* of (M, $\mathcal{F}$, σ, $\mathcal{H}$) if it is an automorphism of $\mathcal{F}$ and if

$$\varphi^* \sigma = \sigma.$$

The set Aut(M, $\mathcal{F}$, σ, $\mathcal{H}$) of all such automorphisms is a group.
Let

$$\Lambda \equiv \Lambda_{\mathcal{F}}$$

be the associated Poisson tensor.
Clearly,

$$\mathrm{Aut}(M,\mathcal{F},\ \sigma,\mathcal{H}) = \mathrm{Aut}(M,\mathcal{F}) \cap \mathrm{Aut}(M,\Lambda).$$

REMARK. In general, the group Aut$(M, \mathcal{F}, \sigma, \mathcal{H})$ is not transitive on M.

(3.4.2) DEFINITION. Let $(M, \mathcal{F}, \sigma, \mathcal{H})$ be a tangentially symplectic foliated manifold. A vector field X on M is called an *infinitesimal automorphism* if the local 1–parameter group $(\varphi_t)_{|t| < \delta}$ associated to X has the following property: for all t, the local diffeomorphism φ_t preserves the foliation and the tangential symplectic structure [i.e., the symplectic vertical form r].

Straightforwardly, we have

(3.4.3) PROPOSITION. *Let* X *be a vector field on the tangentially symplectic foliated manifold* $(M, \mathcal{F}, \sigma, \mathcal{H})$. *The following statements are equivalent* :

(i) X *is an infinitesimal automorphism of* $(M, \mathcal{F}, \sigma, \mathcal{H})$.

(ii) X *is a foliate vector field which is a Poisson infinitesimal automorphism of* (M, Λ).

The set Aut$(M, \mathcal{F}, \sigma, \mathcal{H})$ of all infinitesimal automorphism of $(M, \mathcal{F}, \sigma, \mathcal{H})$ is a Lie algebra.

Chapter 4

EXAMPLES

The purpose of this chapter is to illustrate the importance of tangentially symplectic foliations in mathematics and mathematical physics. I have already indicated in section (3.3) that the characteristic foliation of a regular Poisson manifold inherits a natural tangential symplectic structure. Thus from the point of point of view of foliation theory regular Poisson manifolds may be regarded as a special case in tangentially symplectic foliations.

I will now indicate three further examples that illustrate the breadth of application of tangentially symplectic foliation theory :

(i) canonical manifolds : a canonical manifold is naturally equipped with a codimension 1 tangentially symplectic foliation.

(ii) symplectic fibre bundles : a symplectic fibre bundle determines, in a natural way, a tangentially symplectic foliation.

(iii) regular Poisson–Riemann structures : a regular Poisson–Riemann structure is a special type of tangential symplectic structure.

Note further that *F. Alcalde Cuesta* recently produced important results on symplectic integration of Poisson manifolds [A1],[A2], where the concept of tangentially symplectic foliations plays an important role, and that the subject is also relevant in the theory of geometric quantization.

I will now briefly outline how the three examples enumerated above fall within the framework of tangentially symplectic foliation theory.

4.1 CANONICAL MANIFOLDS

The concept of a canonical manifold was introduced by André Lichnerowicz [Li1] in 1976, as a generalization of the type of structure that is used in the description of time–dependent [i.e., non–autonomous] dynamical systems in Hamiltonian mechanics. The time coordinate in such a system may be interpreted as a [global] regular function on the "canonical" phase manifold for this system. The canonical transformations of analytical dynamics may then be interpreted as the automorphisms of this "canonical" structure (see, e.g., [Li1],[FLS]).

Let M be a connected (2k+1)–dimensional manifold and let $\Omega^o(M)$ be the algebra of all functions on M.

(4.1.1) DEFINITION. A *canonical structure* on M is given by a Poisson tensor Λ on M, of rank 2k, and a regular function $\hat{t}$ on M such that the characteristic foliation $\mathcal{F}_\Lambda$ is defined by $\hat{t}$.

A manifold M equipped with such a structure is called a *canonical manifold* and will be denoted by $(M, \Lambda, \hat{t})$.

REMARKS.

(i) The regular [global] function $\hat{t} \in \Omega^o(M)$ is called the *time function* of the canonical manifold $(M, \Lambda, \hat{t})$. Note that regularity means

$$d\hat{t} \neq 0.$$

In other words, $\hat{t}$ is a submersion.

(ii) The characteristic foliation $\mathcal{F}_\Lambda$ consists of the connected components of the inverse images by $\hat{t}$ of the points of $\mathbb{R}$. Note that the leaves are closed imbedded submanifolds of dimension 2k.

On each leaf L of $\mathcal{F}_\Lambda$ there exists a symplectic 2–form σ_L, defined by :

$$\sigma_L(x)(v,w) := \{f,g\}(x)$$

where $x \in L$, $v,w \in T_xL$ and $f,g \in \Omega^0(M)$ such that

$$X_f(x) = v, \ X_g(x) = w.$$

(iii) For any $t \in \mathbb{R}$, the set

$$M_t := \{x \in M | \hat{t}(x) = t\}$$

is a 2k–dimensional submanifold of M. Each component of M_t is a leaf of the foliation $\mathcal{F}_\Lambda$.

Therefore M_t has, just as the leaves have, a symplectic 2–form σ_t. These 2–forms fit together into a symplectic vertical form σ.

In other words, the characteristic foliation $\mathcal{F}_\Lambda$ inherits a natural tangential symplectic structure (see [HMS],[Li2],[W3]).

(iv) The (2k+1)–form on M

$$\eta = d\hat{t} \wedge \sigma_{\hat{t}(x)}$$

is a volume element attached to the canonical structure. So, any canonical manifold is orientable ([Li1]).

EXAMPLE. Let (N, σ_N) be a 2k–dimensional symplectic manifold, and

$$M = \mathbb{R} \times N.$$

Any point of M is a pair (t,y) with $t \in \mathbb{R}$, $y \in N$. Let $\hat{t}$ denote the first projection

$$M = \mathbb{R} \times N \longrightarrow \mathbb{R},\ (t,y) \longmapsto t.$$

when f and g are two functions on M, we may consider t as a parameter, and then for each fixed $t \in \mathbb{R}$,

$$y \longmapsto \hat{f}_t(x) := f(t,y)$$

and

$$y \longmapsto \hat{g}_t(y) := g(t,y)$$

are two functions on N.

We define the Poisson bracket {f,g} by

$$\{f,g\}(t,y) := \{\hat{f}_t,\hat{g}_t\}(y).$$

Endowed with this structure, $M = \mathbb{R} \times N$ is a canonical manifold, called a *product canonical manifold.*

4.2 SYMPLECTIC FIBRE BUNDLES

The concept of a symplectic fibre bundle was introduced by M.J. Gotay, R. Lashof, J. Sniatycky and A. Weinstein [GLSW] in 1983.

(4.2.1) **DEFINITION.** A *symplectic fibre bundle* is a fibre bundle $E \xrightarrow{\pi} M$ over M with typical fibre F, whose structure group G preserves a symplectic structure on F. The subbundle

$$\operatorname{Ker} \pi_* \subset TE$$

carries a field of bilinear forms, called the *symplectic structure along the fibres*, and denoted by σ.

Note that σ is a symplectic vertical form for the [simple] foliation $\mathcal{F}$, where leaves are the connected components of the fibres of E.

The fibres of a symplectic fibre bundle constitute a tangentially symplectic foliation. This structure seems particularly well suited to mechanics and further investigations seem worthwhile.

4.3 POISSON–RIEMANN STRUCTURES

The concept of a Poisson–Riemann structure was introduced by Pierre Molino [Mo9] in 1990. Some results have been obtained by P. Molino and C.M. Diop [DM],[D], but the theory is far from complete.

Let (M,g) be a [connected] Riemannian manifold of dimension m and let $\mathcal{F}$ be a 2k–dimensional foliation on M ; $m = 2k+n$. Let $\mathcal{H}$ denote the orthogonal complement [with respect to the metric g] of the tangent bundle of $\mathcal{F}$.

We assume that :

(PR–1) $\mathcal{F}$ is a tangentially symplectic foliation [this implies that M is equipped with a regular Poisson tensor Λ].

(PR–2) g is a bundle–like metric for $\mathcal{F}$ [i.e., $\mathcal{F}$ is a Riemannian foliation ; see, e.g., Annexe C].

(PR–3) If $\mathcal{T}(M, \Lambda, \mathcal{H})$ denotes the Lie algebra of all [global] vector fields tangent to $\mathcal{F}$ [i.e., vertical vector fields on $(M, \mathcal{F})$] that preserve the Poisson tensor Λ and the horizontal bundle $\mathcal{H}$, then the subset
$$\mathcal{T}_c(M, \Lambda, \mathcal{H}) \subset \mathcal{T}(M, \Lambda, \mathcal{H})$$
consisting of complete vector fields is transitive on the leaves.

REMARK. Clearly, if M is compact,
$$\mathcal{T}_c(M, \Lambda, \mathcal{H}) = \mathcal{T}(M, \Lambda, \mathcal{H}).$$

(4.3.1) DEFINITION. Such a structure on M will be called a *regular Poisson–Riemann structure* and is denoted by $(M, \Lambda, \mathcal{H})$.

Note that, consequently, $\mathcal{F}$ is a [regular] *almost–isometric foliation* for the metric g (see [D],[DM],[Mo9]).

Chapter 5

DISCUSSION AND CONCLUSION

The principal accomplishment of this thesis is the clarification, formulation, systematization and development of the concept of tangential geometry.

The key concept in this respect is that of the horizontal bundle. In transversal geometry, the fundamental concept is invariance in directions tangent to the leaves. These directions lie in a vector bundle naturally defined by the foliation [i.e., the vertical bundle]. In tangential geometry the analogous fundamental concept is invariance in directions transverse to the leaves. There is, however, no suitable vector bundle naturally defined by the foliation itself. It is therefore necessary to supply such a bundle as a fundamental part of the structure. This is the horizontal bundle introduced in section (2.1).

The concepts of tangential functions, vector fields and forms now follow naturally from that of the horizontal bundle, together with that of horizontal vector fields. These I defined and developed in sections (2.2) to (2.6), with the exception of section (2.3) on sheets and tangential holonomy, which is already treated in the literature.

The most important accomplishment of this work is the definition of vertical connections and vertical G–structures in section (2.8). The necessary infrastructure for these concepts is that of the vertical frame bundle of section (2.7).

The notion of a vertical G–structure makes possible the definition of an interesting new category of foliations with tangential geometry. The case of totally geodesic foliations [i.e., "tangentially Riemannian"] is already well known, though the methods used in its study are not those of the tangential geometry presented here. The structure group in this case is the orthogonal group. Other cases may now be generated by suitable choice of the structure group G. A remarkable particular case is that with structure determined by the symplectic group. This gives rise to tangentially symplectic foliations studied in chapter three. From this investigation it appears that the most tractable form of such a foliation is that defined by a symplectic vertical form, discussed in section (3.2). Its principal advantage is the substitution of the degenerate contravariant 2–tensor characteristic of the [regular] Poisson manifold by a special type of 2–form on the manifold, which is non–degenerate on the leaves and also closed in the directions tangent to the leaves.

The applications of the methods of tangential geometry presented in this thesis are still rudimentary. The examples enumerated in chapter four are all promising for applications, in particular in the study of dynamical systems. Especially interesting is the third enumerated case, that of Poisson–Riemann structures, which from the point of view of foliation theory, display a richer and more complex structure than the first two.

Annexe A

NOTIONS OF EQUIVALENCE FOR FOLIATIONS

Let M be a manifold of dimension m and let $\mathcal{F}_0$ and $\mathcal{F}_1$ be two ℓ–dimensional foliations on M.

(1) $\mathcal{F}_0$ and $\mathcal{F}_1$ are said to be *conjugate* if there exists a diffeomorphism φ of M which preserves the foliations [i.e., which maps the leaves of $\mathcal{F}_0$ onto the leaves of $\mathcal{F}_1$]. In other words,

$$\varphi_*(\mathrm{T}\,\mathcal{F}_0) = \mathrm{T}\,\mathcal{F}_1$$

[φ_* : $\mathrm{TM} \longrightarrow \mathrm{TM}$ is the tangent mapping of φ]

or

$$\varphi^*\,\mathcal{F}_1 = \mathcal{F}_1$$

[$\varphi^*\,\mathcal{F}_1$ is the pull–back of $\mathcal{F}_1$ by φ].

If φ is isotopic to the identity the foliations are called *completely equivalent.*

(2) $\mathcal{F}_0$ and $\mathcal{F}_1$ are said to be *homotopic* if there is a continuous family of integrable plane fields $\mathcal{T}_t$, $t \in [0,1]$, such that :

$$\mathcal{T}_0 = \mathrm{T}\,\mathcal{F}_0 \text{ and } \mathcal{T}_1 = \mathrm{T}\,\mathcal{F}_1$$

(3) $\mathcal{F}_0$ and $\mathcal{F}_1$ are said to be *integrably homotopic* if there exists a foliation $\mathcal{F}$ on the product $M \times [0,1]$ such that :

(a) $\mathcal{F}$ is transverse to the slice $M \times \{t\}$, for each $t \in [0,1]$;

(b) $\mathcal{F}$ induces the foliation $\mathcal{F}_0$ on $M \times \{0\}$ and $\mathcal{F}_1$ on $M \times \{1\}$ [by intersection of these slices with the leaves of $\mathcal{F}$].

REMARKS.

(i) If $\mathcal{F}_0$ and $\mathcal{F}_1$ are integrably homotopic, they are homotopic. The converse is not true.

(ii) If M is compact, then $\mathcal{F}_0$ and $\mathcal{F}_1$ are integrably homotopic if and only if they are completely equivalent.

Annexe B

TOTALLY GEODESIC FOLIATIONS

Let $\mathcal{F}$ be an ℓ–dimensional foliation on the Riemannian manifold (M, g_M). Let g_V be the induced metric on the vertical bundle $\mathcal{V}$ of $\mathcal{F}$ and let ∇^M denote the Riemannian connection of g_M.

The foliation $\mathcal{F}$ is called *totally geodesic* if each leaf L of $\mathcal{F}$ is a totally geodesic submanifold of M ; that is, any geodesic tangent to L at some point must lie within L.

The following conditions are equivalent :

(i) $\mathcal{F}$ is totally geodesic.

(ii) $g_M(\nabla^M_X X', Z) = 0$
for all $X, X' \in \mathfrak{X}_V(M, \mathcal{F})$ and $Z \in \mathfrak{X}_H(M, \mathcal{F}, \mathcal{H} = \mathcal{V}^{\perp})$.

(iii) $\mathcal{L}_Z g_V = 0$
for all $Z \in \mathfrak{X}_H(M, \mathcal{F}, \mathcal{H} = \mathcal{V}^{\perp})$.

(iv) $g_M(\nabla^M_Z X - [Z,X], X') = 0$
for all $X, X' \in \mathfrak{X}_V(M, \mathcal{F})$ and $Z \in \mathfrak{X}_H(M, \mathcal{F}, \mathcal{H} = \mathcal{V}^{\perp})$.

The study of these foliations was initiated by D.L. Johnson and L.B. Whitt [JW1],[JW2].

More generally, a foliation $\mathcal{F}$ is said to be *geodesible* if there exists a Riemannian metric on the ambient manifold for which $\mathcal{F}$ is totally geodesic.

REMARK. Any foliation transverse to a Riemannian foliation is geodesible. For further information concerning totally geodesic foliations see, e.g., [BH1],[C1],[C2],[C3],[C4],[CG],[G1],[Go],[To].

Annexe C

RIEMANNIAN FOLIATIONS

Let $\mathcal{F}$ be an ℓ–dimensional foliation on the manifold M.
A Riemannian metric g_M on M is called a *bundle–like metric* for $\mathcal{F}$ if it has one [and therefore all] of the following equivalent properties :

(M–1) For any open set U and for all vector fields Z, Z′ on U that are foliate and perpendicular to the leaves, the function $g_M(Z, Z')$ is basic [on U].

(M–2) In any distinguished chart (U, φ) with distinguished local coordinates (x^i, x^α) the metric can be written

$$g_{ij}(x^1, \cdots, x^m)\ \theta^i\theta^j + g_{\alpha\beta}(x^{\ell+1}, \cdots, x^m)dx^\alpha dx^\beta$$

(M–3) The transverse bundle $\nu\,\mathcal{F}$ is *totally geodesic* [i.e., each geodesic which is perpendicular to the leaves remains perpendicular to the leaves at all of its points].

(M–4) The leaves are locally equidistant.

(M–5) The corresponding *transverse metric* g_Q on the transverse bundle $Q = \nu\mathcal{F}$ is locally projectable along the leaves [i.e.,

$$\mathscr{L}_X g_Q = 0$$

for all vector fields X tangent to the leaves].

EXAMPLES.

(i) Transversally parallelizable foliations are Riemannian. In particular, Lie foliations and foliations of codimension 1 defined by a nowhere zero 1–form are Riemannian.

(ii) Compact foliations [i.e., foliations with all the leaves compact] with finite holonomy are Riemannian.

(iii) Transversally homogeneous G/H – foliations [where H is a compact Lie subgroup of G] are Riemannian. In particular, Euclidian, elliptic and hyperbolic foliations are Riemannian.

(iv) A foliation defined by the locally free action of a Lie group of isometries is Riemannian.

For further information concerning Riemannian foliations see, e.g., [Go],[He1],[He2],[Mo6],[Mo7],[Re4],[To].

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