EXPLORING GRADE 11 LEARNERS' MATHEMATICAL PROBLEM-SOLVING SKILLS USING POLYA'S MODEL DURING THE LEARNING OF EUCLIDEAN GEOMETRY.

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By

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DECLARATION

I, Ratham Hlupeni, hereby declare that the work in this thesis is my own, and where ideas from other writers have been used, they are acknowledged in full using referencing according to the Rhodes University Education Guide to References. I further declare that the work in this thesis has not been submitted to any university for degree purposes.

13/07/2023

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ABSTRACT

The skill of Problem-solving in Mathematics is very imperative. Poor performance by most South African learners in schools and international tests such as the Trends in International Mathematics and Science, calls for emphasis to be placed on problem-solving in the teaching and learning of Mathematics. Euclidean Geometry is perceived, especially by learners, to be one of the difficult components of Mathematics. Thus, the aim of this study was to explore and develop the mathematical problem-solving and geometric skills of Grade 11 learners in Euclidean Geometry. Polya's model of problem-solving was employed in geometric skills development as a tool for intervention. The concepts Geometry and problem-solving formed the conceptual framework of the study, while the social cognitive theory constituted the theoretical framework. A case study was used as the main research method following a mixed method approach within an interpretivist paradigm. Purposive and convenience sampling methods were used in the selection of both the Mathematics class and the six learners whose work was further observed and analysed. Data about the geometric skills displayed by the learners was gathered using a moderated pre-intervention test; observations; document analysis; a moderated post-intervention test; and focus group interviews. Data was analysed quantitatively using descriptive statistics and qualitatively using thematic analysis.

In the pre-intervention test, learners did not bring with them expected geometric skills to the classroom before they were introduced to grade11 Geometry content and when doing problem-solving during intervention, the four stages of model used were not necessarily following each other in a linear sequence with most of the learners not applying the fourth stage "look back". In the post intervention test, the frequency of use and application of most geometric skills improved in comparison to the pre-intervention test; the frequency of correct and inappropriate application of the skills increased at the expense of incorrect application. Learners appreciated the four stages model and gave their views related to the challenged faced during the use of the four stages model and the challenges revealed include: practice related challenges, challenges specific to certain learners, concept related challenges, curriculum-related challenges, model application challenge, and context related challenges. The study concludes that the effective use of Polyas' four stages model can yield great results in developing learners' geometric and problem-solving skills. The study recommends that teachers give more attention to prior geometric knowledge, teaching of geometric theorems, teaching of geometric problem-solving, and the learning environment.

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Publications from this study

Two publications have resulted from this study to date. These are a given below:

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ABBREVIATIONS AND ACRONYMS

CAPS	Curriculum and Assessment Policy Statement
DBE	Department of Basic Education
HoD	Head of Department
NCTM	National Council of Teachers of Mathematics
NSC	National Senior Certificate
TIMSS	Trends in International Mathematics and Science Study
FET	Further Education and Training
DHET	Department of Higher Education and Training
GET	General Education and Training
COVID	Corona Virus
GCE	General Certificate of Education
SAQA	South African Qualification Authority
OECD	Organisation for Economic Cooperation and Development
POA	Program of Assessment
ATP	Annual Teaching Plan
RTP	Required to Prove
APA	American Psychology Association
TIMSS	Trends in International Mathematics and Science Study
FET	Further Education and Training
Pre-cor	Pre-test correct application
Pre-inc	Pre-test incorrect application
Pre-ina	Pre-test inappropriate application
Pos-cor	Post-test correct application
Pos-inc	Post-test incorrect application
Pos-ina	Post-test inappropriate application

CHAPTER 1

BACKGROUND AND CONTEXT OF STUDY

1.1.BACKGROUND OF THE STUDY

Geometry forms an important component in both elementary and high school Mathematics curricula. At high school level, it is often perceived as one of the most complex part of the curriculum (Patkin & Levenberg, 2012). The study of Geometry contributes to learners' development of visualisation, critical thinking, intuition, problem-solving, conjecturing, deductive reasoning, logical argument and proofs (Jones, 2002). Thus, geometric representations can be used to help learners make sense of other areas of Mathematics and apply the learnt skills to other curriculum areas like science, geography, art, design and technology (Jones, 2002).

In the National Senior Certificate (NSC) examinations, Grade 12 learners write a Mathematics examination comprising two papers, and it is in Paper 2 where learners' Euclidean Geometry knowledge and skills are examined. According to the Department of Basic Education's (DBE) (2019) Internal Moderators Report, the annual Grade 12 post- apartheid pass rate in Mathematics is still low (ranging between 45.7% and 59%) with Euclidean Geometry reported as one of the most failed areas. The DBE (2019) reported several cases of no attempts by learners in many questions in Geometry, yet Geometry contributes, on average, one-third of Grade 12 Mathematics Paper 2, and In Paper 2 of the Mathematics NSC examination, Geometry also contributes the majority of higher order questions, which are problem-solving in nature (Buthelezi & Sithole, 2019). Research has shown that learners' general mathematical competencies are closely linked to their geometric understanding (Alex & Mammen, 2018; Inam, 2016). This shows the importance of Geometry in the school Mathematics curriculum, and further explains why Geometry assumes a dominant place in South Africa's, and many other countries', school Mathematics curricula.

In South Africa, Mudaly (2007) compiled research findings that showed that learners have had poor geometric understanding over decades, and cited reasons that include teaching geometric concepts to learners who are not at the appropriate Van Hiele level. Poor learning of Geometry has continued to be a problem (Alex & Mammen, 2018) and is propagated by several factors. These factors include promotional requirements in the Further Education and Training (FET) phase in South Africa, that allow learners to progress to the next grade while having only attained a minimum pass mark in Mathematics. According to the promotional requirements, a learner can be progressed to the next grade after passing four subjects, and considering other factors such as number of years in a phase (Republic of South Africa Department of Basic Education, 2015). This means that learners are likely to move to the next grade without having mastered basic and necessary problem solving skills resulting in topics like Geometry, considered as difficult at secondary school level (Luneta, 2015). In another study by Naidoo and Kapofu (2020).it was found that learners in Mathematics classes in South Africa consider Geometry to be problematic. Learner perceptions revealed that Mathematics was considered difficult, confusing, stressful, and time-consuming, and that Euclidean Geometry was difficult and confusing because it was all about proofs.

It is evident that the numerous interventions implemented by the DBE, such as developing and providing learners and teachers with good quality textbooks, the radical 1+4 Intervention Model that advocates professional learning communities, and self-study guides are not bringing significant change to learners' mathematical conceptual understanding (DBE, 2018). Many learners struggle with the subject (DBE, 2018). This study proposes a rigorous investigation on problem-solving to promote geometric skills development in the learning of Geometry. Problem-solving is defined as a mental process requiring one to think critically and creatively, to look for alternative ideas and specific steps in order to cope with any hindrances or flaws (Makhsin, 2006). Since one of the aims of education is to equip learners to solve problems, problem-solving in the Mathematics curriculum contributes to the generic skill of problem-solving in the South African Curriculum Framework (DBE, 2018). Polya (1945) has one of the widely used frameworks on problem-solving that identifies what a problem is and what mathematical problem-solving entails (Carifio, 2015). He identifies four basic steps of problem-solving, which are: understanding and exploring the problem; finding a strategy; using the strategy to solve the problem; and looking back and reflecting on the solution. The purpose of this study was to explore and promote the geometric skills development and problem-solving skills of learners in Geometry using Polya's model.

1.2.CONTEXT

South Africa's education system is divided into two departments: the Department of Basic Education (DBE) and the Department of Higher Education and Training (DHET). The DBE is responsible for primary and secondary schools, while the DHET is for tertiary education and vocational training. In the whole system of education, there are three levels involved: General Education and Training (GET); Further Education and Training (FET); and Higher Education and Training (HET) (South Africa Info, 2011). The GET band consists of learners from Grade 8 to 9 and the FET band consists of learners from Grade 10 to Grade 12. The HET band comprises of students registered for a degree, diploma, or certificate. The first common national examination to be written is the matriculation examination at grade 12, and passing this examination is a passport for admission to tertiary or higher education.

South Africa's education system was divided among ethnic and racial lines before democracy, which led to unequal distribution of resources in school systems, especially human and material resources (Badat & Sayed, 2014). This situation led to underdevelopment and perpetual poor achievement in Mathematics, science and technology education in black secondary schools (Badat & Sayed, 2014). As a solution to this imbalance caused by the apartheid government, the education curriculum had to undergo several reforms after democracy in 1994 (Badat & Sayed, 2014). Currently the South African government is still grappling with the issue of poor learner underachievement, financial barriers, and inequitable access to better and quality (Graven, 2014).

South Africa has Mathematics education as one of its national priorities, seen in the call by the office of the Presidency, Republic of South Africa (DBE Info, 2015) for an increase in the number of learners achieving 50% in literacy and Mathematics respectively. The nation acknowledges that, in the over two decades of democracy, the apartheid legacy of inferior Mathematics education provided for the majority of learners is still prevalent in most public schools (Awuah, Philosophy & Education, 2018).

1.3.PROBLEM STATEMENT

Several benchmark tests like Trends in Mathematics and Science Study report that conceptual understanding of Mathematics among learners in South African schools, particularly in Geometry (geometric skills), is still poor (Mullis, Martin & Kelly, 2020). This is so despite various initiatives to support learning, like the provision of textbooks, revision guides, past examination papers and memorandums, extra classes, involvement of lead educators, peer teaching, digital lessons, and WhatsApp and Facebook support groups. This is a growing concern for all the stakeholders in the education system, especially educators, parents, and government.

As a classroom teacher, I have realised that most learners at my school, who are promoted and progressed to Grade 11, still lack problem-solving skills in Mathematics. At the beginning of topics like Geometry, the teacher is compelled to re-teach Grade 10 content before introducing Grade 11 concepts. This seems to be the same situation prevailing in several schools in my area, as many teachers report the same, especially in circuit and district teacher development workshops.

It is against this background that there was need for an intervention into mathematical problem-solving and geometric skills development. Therefore, my study aims to develop mathematical problem-solving and geometric skills of Grade 11 learners using Polya's model during the learning of Geometry.

1.4.PURPOSE AND SIGNIFICANCE OF THE STUDY

First, it is my intention that this study will inform and enrich my own classroom teaching practices since I am a high school teacher. The study will also provide recommendations and guidelines to fellow Grade 11 Mathematics teachers on the best ways to develop problem solving and geometric skills during teaching and learning of Euclidean Geometry, and how associated challenges faced by learners can be addressed. Any other stakeholders intending to intervene in geometric and problem-solving development can also make use of information from this study.

1.5.AIM AND OBJECTIVES OF THE STUDY

This study aims to explore and develop the mathematical problem-solving and geometric skills of Grade 11 learners in Euclidean Geometry. The study endeavours to:

- identify selected learners' geometric skills prior to learning Grade 11 Euclidean Geometry content.
- observe how Polya's model can be used in a selected Grade 11 Mathematics class to learn Euclidean Geometry for problem solving and geometric skill development; and
- identify selected Grade 11 learners' Geometry learning challenges experienced when solving Euclidean Geometry problems using Polya's model for geometric skills development.

1.6.RESEARCH QUESTIONS

- What geometric skills are demonstrated by selected learners prior to learning Grade 11 Geometry content?
- How can the use of Polya's model enable or constrain the development of problem solving and geometric skills in a selected Grade 11 Mathematics class?

- What Geometry learning challenges are experienced by selected Grade 11 learners when solving problems in Geometry?

1.7.RESEARCH METHODS

The research method adopted for this study was an interpretive study of grade 11 learners' problem-solving skills in Euclidean Geometry. Aspects relating to research methods such as the research design, suitability to a qualitative framework, development and reliability of data collection tools are explained in detail in chapter 3. I used five data collection tools which are:

- Pre-intervention test
- Observation
- Document analysis
- Post-intervention test
- Interviews.

The reliability of the instruments and the triangulation of data are key aspects that feature in subsequent chapters. The pre-intervention test and post-intervention test were moderated to:

- check for their relevance and suitability to participants,
- ensure relevant generation of data,
- enhance analysis and interpretation of data gathered, and
- ensure validity of instruments and reliability of findings is possible.

1.8.RATIONALE FOR THE STUDY

The researcher's interest in carrying out this research study was prompted by several factors, a few of which are presented in this section. One is that most learners at the school where the researcher taught were not performing well in Mathematics generally, and in Geometry in particular. The researcher realised, from circuit and district subject content developmental workshops, that the same was prevailing in most of the schools in the district. Mamali (2015) notes that Euclidean Geometry is among the topics in the Mathematics curriculum that teachers find difficult to teach. This prompted the researcher to do an investigation into the problem-solving skills of Grade 11 learners by making use of Polya's model. The researcher believed this would be of help to stakeholders in improving learner performance in the topic.

1.9.LIMITATIONS OF THE STUDY

This research study was conducted in the classroom which makes it innately constrained. However, this was beyond the control of the researcher. The limitations of this study are as follows:

- The problem under exploration concentrated on the learner and not on the teacher.
- A single grade 11 class was selected at a particular school in the district, and such a small sample could be questioned for external validity of the findings since this study was conducted within the paradigm of qualitative research.
- The data was analysed based on the researcher's personal interpretation. This could result in bias in the findings. No matter how impartial an observer attempts to be, there is always some element of bias present (Wienekus, 2015).
- A camera was used to record parts of some lessons, and the presence of the camera could have affected the sincerity of learners' responses. However, as the intervention progressed, learners became more accustomed to the presence of the camera in the classroom and could have given more authentic responses then.
- This study was carried out during the COVID-19 era, and this had some effect on the teaching methodology of the researcher. It was not easy to employ group workings during teaching and learning as per researcher's plan. Some methods like flipped classroom were employed instead.

In this study, it was not easy to overcome the above limitations. However, it is important that I acknowledge such limitations and attempt to minimise their influence on the research process in one way or another.

1.10. OUTLINE OF CHAPTERS

The chapters that make up this research thesis are outlined below.

Chapter One: Introduction

This was an introductory chapter providing the background and context of the study and describing the educational system of South African schools. The chapter provides a statement of the problem, the research questions, significance of the study, the aim and objectives, and a brief definition of terms used as well as the structure of the thesis.

Chapter Two: Literature Review and theoretical Framework

This chapter presents the conceptual structure of the study and reviews some relevant literature. The literature sets the groundwork for the main themes of the study: Geometry; origin of Geometry and history of Geometry in South African schools; Geometrics skills and geometric skills development; problem-solving and problem-solving skills; problem-solving models and Polya's model of problem-solving as well as Social Cognitive Learning Theory.

Chapter Three: Research Methodology

The methods used in the study, the research design, how samples were collected, the data collection instruments and the development of the instruments, as well as the procedures of data collection, are all presented in this chapter. This chapter also presents the case study and the ethical issues that arose from the study.

Chapter Four: Findings and Discussion of Findings

Highlighted in this chapter are the data analysis methods and procedures. The results of the data analysis applied to evaluate the findings of the study and answer the research questions are also presented here. In this chapter, the findings of the study are deliberated together with their inferences.

Chapter Five: Summary, Conclusion and Recommendations

This chapter is all about the summary of the study as well as drawing conclusions upon which recommendations are made.

1.11. CONCLUSION

This chapter was the introductory chapter where the orientation of the study was established, and the contextual view of the project highlighted. Research questions and significance of the study, as well as the aim and objectives, problem statement, definition of terms and outlines of other chapters of the study, were addressed.

CHAPTER 2
LITERATURE REVIEW

2.1 INTRODUCTION

The engagement of learners in Geometry problem-solving is supported by social cognitive learning theory that sees learner behaviour as learnt through observation, imitation, and modelling. It is critical that all theories, concepts, and terms that are used in this study are defined and explained. This chapter looks at Geometry as a concept and what the process of problem-solving entails. Review of literature focuses on local and international studies done on problem-solving in Geometry and other fields. Models of problem-solving are discussed in depth, including Polya's model of problem-solving since it forms the basis for problem-solving in this study. The analytical framework, based on Polya's stages of problem solving and Nur & Nurvitasari (2017) geometric skills, used to analyse the problem-solving process in a social cognitive environment, is also discussed.

2.2. CONCEPTUAL FRAMEWORK

2.2.1. *What is Geometry?*

Geometry is a branch of Mathematics concerned with the configurations of points, lines, angles, circles, and the most basic of these figures (Brannan, Esplan & Gray, 1999). Güven and Kosa, (2008) define Geometry as the study of space and shape in which more excellent reasoning capability is necessary to understand the concept well. According to Kotzé (2007), Euclidean Geometry is a body of knowledge consisting of statements justified by proofs, which depend on mathematical axioms and an underlying logic. Euclidean Geometry deals with axioms and proofs of theorems through deductive thinking (Mamali, 2015). Considering these definitions of Geometry, it shows that when dealing with Geometry, focus, is on identifying shapes, measuring different aspects of those shapes, and categorising such shapes according to their properties as well as proving theorems and their application. The aims for the teaching Geometry can be summarised as follows: to offer a range of geometrical familiarities in 2 and 3 dimensions; to develop spatial awareness and geometrical insight; to advance familiarity and knowledge of, and the capability to, use geometrical properties and theorems; to encourage the development and use of conjecture, deductive reasoning and proof; to develop skills of applying Geometry through modelling and problem-solving in real world settings; to generate a positive attitude to Mathematics; and to develop an awareness of the historical and cultural heritage of Geometry in society, and of the modern applications of Geometry.

The teaching and learning of Euclidean Geometry are critical in this study. According to Jones (2002), the understanding of Geometry was accumulated and ordered in a writing named Euclid's Elements. Euclid's Elements represents the axiomatic-deductive method, and maintains that several theorems were proven by deductive logic (Pereyra, 2022). Jones (2002) further states that in the nineteenth century, there were changes in the concepts of Geometry when Euclid introduced abstract Geometry. However, according to Sir Zeeman, Geometry is composed of Mathematics that calls on visual intuition to perceive reality, to remember theorems, understand proofs, stimulate conjecture, and give universal or inclusive insight (Zeeman, 1976). In this study, Euclidean Geometry is viewed as comprising both Euclid and Zeeman's opinions. We, therefore, believe that Geometry perhaps started with perceiving and recognising the immediate world concretely, formulating conjectures about the environment; and applying the accumulated perceived and recognised knowledge as well as pre-conceived conjecture to seek approval through existing axiomatic-deductive knowledge. Hence, inductive, and deductive logic are viewed as supplementing each other in the teaching and learning of Euclidean Geometry.

2.2.2. *The history and foundation of Geometry*

Geometry is one of the most ancient forms of Mathematics, and is associated with the ancient Greeks (Scott & Coxeter, 1926). The word 'Geometry' comes from the Greek words '*geo*', meaning the '*earth*', and '*metrein'*, meaning 't*o measure*', and appears to have originated from the need for measuring land (Jones, 2002). According to Jones (2002)Geometry originated from several ancient cultures, including Indian, Babylonian, Egyptian, and Chinese cultures, which used Geometry to measure lengths, heights, angles, directions, and distances. The story of axiomatic Geometry therefore, begins with Euclid, the most famous mathematician in history (Story, 2011). The British mathematician, Sir Christopher Zeeman, saw Geometry as that which comprises those branches of Mathematics that exploit visual intuition, the most dominant of our senses, requiring remembering of theorems, understanding of proofs, inspiring of conjectures, perceiving of reality, and giving global insight (Jones, 2002). According to Tachie (2020), those who are geometrically inclined tend to have a high level of memory restoration, and can easily link or relate events and occurrences. This is one of the reasons why Euclidean Geometry was introduced as a core topic in the South African school curriculum (Tachie, 2020).

2.2.3. *Teaching and learning of Euclidean Geometry in South Africa*

Euclidean Geometry requires educators to assist learners in linking new knowledge to existing knowledge and develop instructional techniques that would facilitate cognitive growth and change during teaching and learning (Kotzé, 2007). Research has proven that Geometry plays a very significant role in the teaching of Mathematics all over the world. Euclidean Geometry provides a rich source of visualisation for understanding arithmetic as well as algebraic and statistical concepts (Binti et al., 2003). These are some of the reasons why the importance of Geometry in the lives of people cannot be underestimated. However, many teachers find it challenging to teach the concept in South Africa due to the lack of subject content knowledge and lack of cognitive skills (Tachie, 2020). According to Adolphus (2011), one of the problematic ideas that causes teachers to struggle to teach Mathematics is Euclidian Geometry, because of the technicalities related to the language, visualisation of objects for better identification of properties, and inadequate conceptual understanding.

The teaching and learning of Mathematics must be connected to reality, meaning that in the teaching of Geometry, teachers should try to make use of available materials at their disposal to convey the meaning of geometrical concepts (Howson & Freudenthal, 1992). Some researchers have disputed the notion that a teaching approach which connects to reality helps learners develop and apply Mathematics to problems (Tachie, 2020). Sanni (2007) discovered that the instruction of Geometry concepts was more of ordered Mathematics classroom practices such as checking homework, followed by teacher lecture and demonstration, followed by learner practice in a sequence of classroom instructional activities. The South African Euclidean Geometry curriculum does not recommend such an approach since it does not consider Van Hiele's (1986) five levels of Geometry thinking, which serve as a rational basis for the teaching of Geometry in schools (DBE, 2011). According to Vojkuvkova (2012), the Van Hiele Level 1 is recognition or visual level, where learners recognise figures by their appearance, and level 2 is descriptive or analytic level, where learners recognise and analyse figures by their properties. Level 3 is the abstract or relational or ordering level, where learners distinguish between necessary and sufficient conditions for a concept, as well as form abstract definitions and classify figures by elaborating on their interrelationships. Level 4 is the formal deduction level at which learners establish theorems within an axiomatic system and recognise the difference between undefined terms, definitions, axioms, and theorems, as well as construct original proofs. Level 5 is the last level, which is mathematical rigor. At this level, learners understand the relationship between various systems of Geometry and can describe the effect of adding or deleting an axiom on a given geometric system, and can compare, analyse, and create proofs under different conditions. Van Hiele (1986) believes that these five sequential geometric levels are needed for identification and sorting; developing critical thinking and problem-solving skills; formulating conjectures; intuition, postulation, deductive reasoning, and logical arguments; and for proof formulation (Tachie, 2020). McIntyre (2007), states that Van Hiele's theory of geometrical thinking is the theory behind the teaching of Euclidean Geometry in schools. However many educators still prefer using the traditional teaching approaches in the teaching of Geometry in schools (Tachie, 2020).

South Africa has experienced numerous curriculum reforms, as every time the Minister of Education changes a new curriculum was introduced (Gumede & Biyase, 2016). That has affected the teaching of Euclidean Geometry in schools (Tachie, 2020). Curriculum reform has also affected other countries such as Zimbabwe and China. Gladys (2018) confirms that Geometry is one of the most affected fields in Mathematics, and it was only in the previous five years that its status had improved in Zimbabwe. Learners with this qualification had a deficient level of understanding of Euclidean Geometry as it was not comprehensively taught, which led to challenges being experienced in its teaching in Zimbabwean secondary schools.

Xei (2005) regards China as one of the countries that underwent curriculum reforms leading to a great deal of change in its Mathematics curriculum changing Mathematics content taught to elementary and middle school learners in the People's Republic of China. Xei (2005) explains that the reforms brought both favorable experiences and frustrating ones as their cultural identity and national conditions were compromised. They needed to examine the superiority and limitation of Chinese Mathematics education by looking at the trends in international Mathematics education so that China's curriculum could be recognized internationally. The Chinese Ministry of Education made Euclidean Geometry compulsory for all learners. The reason for this was that it had been observed that in the traditional Chinese culture, logical thinking was rare, and Mathematics-related training, especially on Euclidean Geometry, was a great need. They underscored the importance of deductive reasoning, which is extractable from the teaching of Euclidean Geometry for a nation doing it as a core topic in Mathematics. However, many teachers faced challenges in teaching Euclidean Geometry as it had been newly introduced into the school curriculum.

A similar issue has been observed in South African schools, where Euclidean Geometry was seen as having no weight at some tertiary institutions resulting in education graduates having not been educated in the topic of Euclidean Geometry, and Some teachers have already qualified as teachers yet have not undergone training on some of the newly-introduced topics such as Euclidean Geometry (Tachie, 2020). In 2012, South Africa experienced the introduction of a new curriculum for Grades R–12, which was an amendment of the existing National Curriculum Statement (NCS), which did not offer Euclidean Geometry as a core topic for Grades R-12 Curriculum and Assessment Policy Statement (CAPS). Teachers who had not studied this topic in their secondary schooling or their teacher education programs found themselves teaching the subject with little knowledge and understanding. Such teachers' lesson preparations were negatively affected since teaching Mathematics depends on how one understands the Mathematics content itself. Brodie (2010) found that there is always a significant decline in the cognitive levels of Mathematics content whenever a teacher is not comfortable with the topic, thereby compromising knowledge transfer. The same situation occurred with the teaching of Euclidean Geometry (Brodie, 2010).

2.2.4. *My choice of Geometry in the Mathematics curriculum*

In the National School Certificate (NSC), the CAPS document recommends seven content areas for paper 1, and four content areas for paper 2 Matric final examinations. Geometry is examined in paper 2 under Analytical and Euclidean Geometry. Of the eleven content areas prescribed, this study focused on Euclidean Geometry for the perceived role in learner mental skills development. According to Jones (2002), the study of Geometry contributes to development of vital learning skills of visualisation, critical thinking, insight, perspective, problem-solving, conjecturing, deductive reasoning, logical argument and proof. The Euclidean Geometry content, as defined in the CAPS document (DBE, 2011), includes the concept of measurement that can be tested in the context of optimisation in calculus, as well as composite shapes that can be formed by combining a maximum of two stated shapes. Proofs of theorems are examinable. Examples include the line drawn from the Centre of a circle perpendicular to a chord bisects the chord; the angle subtended by an arc at the center of a circle is double the size of the angle subtended by the same arc on the circumference; the opposite angles of a cyclic quadrilateral are supplementary; the angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle subtended by the chord in the alternate segment; a line drawn parallel to one side of a triangle divides the other two sides proportionally and equiangular triangles are similar. Corollaries derived from the theorems and axioms are necessary in solving riders and these include: an angle in a semi-circle is a right angle; equal chords subtend equal angles at the circumference; the exterior angle of a cyclic quadrilateral is equal to the interior opposite angle of the quadrilateral; and tangents drawn from a common point outside the circle are equal in length up to the point of contact.

Geometric representations can assist learners to make sense of other areas of Mathematics like fractions and multiplication in arithmetic, the relationships between the graphs of functions, and graphical representations of data in statistics (Jones, 2002). Geometric spatial reasoning is important in other areas of the curriculum such as science, art, design, geography, and technology. This shows that Geometry is not only learnt for itself, but for a bigger purpose in learner development. Geometry also acts as a framework within which to teach and learn Mathematics and well taught and learnt Geometry concepts promote exploration and interest. The interest can develop learners' attitudes towards Mathematics, hence their learning. According to Jones (2002), as learners deliberate on problems in Geometry, and communicate their ideas and develop well thought-out points of view to support their opinions, they develop better communication skills and appreciation of the importance of proof. Even though Geometry is not the only basis for developing ideas of proofs, it is a very rich source of opportunities to develop the ideas.

According to Bursill-Hall (2002), Geometry has been studied because it has been held as the most exquisite, perfect, paradigmatic truth available to us outside divine revelation; and it is the surest, clearest way of thinking available to us. He continues to argue that studying Geometry reveals – in some way – the deepest true essence of the physical world and its teaching trains the mind in clear and meticulous thinking. All this implies that, if Geometry is taught well, it could mean empowering more learners to become successful in Mathematics and other areas of study. Euclidean Geometry acts as a tool for providing logical thinking and in the CAPS document, has more weight than other Mathematics topics at FET. Thus, this study examined the Mathematics policy document to check how the topic is arranged and treated. About 33% of the marks in Paper 2 come from Euclidean Geometry, which means that if a learner is well versed with the topic, he/she will be in a good position to pass the paper since the pass mark is 30%. For the purposes of accruing marks in grade twelve Mathematics Paper 2, learners need to do well in Euclidean Geometry. Even on time, in the Further Education and Training (FET) band, Geometry is allocated 16 weeks of which nine weeks are for Euclidean Geometry constituting 56.25% of the total time reserved for Geometry.

2.2.5. *The importance of Geometry*

Mathematics instruction programs should pay attention to Geometry and spatial sense so that all learners, among other things, use visualization and spatial reasoning to solve problems both within and outside of Mathematics (Hekimoglu & Sloan, 2005). This is one of the motives behind the inclusion of Geometry in South African school curriculum. There are many demands and problems of the 21st century which can be solved geometrically. Health industry needs Geometry designs. Dentist, plastic surgeries and many operations are performed by employing Geometry strategy (Makhubele, 2012). Geometry also plays an important role in terms of transport. All car designs and other modes of transport are geometrically designed. Geometry involves the study of shapes. Shapes play an important role in our everyday life. From home improvements to architecture, we see and use geometric shapes in a variety of ways. According to Srinivasan (2005), this makes Geometry an important strand of Mathematics as it is an essential part of our lives.

In food production, Ivester (2008) argues that Geometry technological advancement has led to dramatic improvement in the production of food and the processing of food. There is a huge demand for clothes because of the growing population. Very few of us living in this century still clothe ourselves with garments that are made entirely by hand. Most 21st century citizens are dependent on factory-made garments. Most modern garments are designed through the use of advanced technologies that employ basic geometric concepts (Makhubele, 2012). In housing, Geometry plays an important role when it comes to plans and building quality houses. The possibility of a sudden rise in water levels at coastal areas, as well as unstable surfaces because of flooding and earthquakes, has heightened the necessity of dealing with housing geometrically (Makhubele, 2012).

The above explanation of the roles of Geometry in our society justifies the inclusion of Geometry in the school curriculum. Geometry has an important place in school Mathematics curricula. It develops learners' spatial ability, logical reasoning skills, and the ability to solve real-world problems in which geometrical terminology and properties occur (Riastuti, Mardiyana & Pramudya, 2017). According to the National Council of Teachers of Mathematics (1989), Geometry is an important school subject because it provides perspectives for developing learners' deductive reasoning abilities and spatial awareness (Crosswhite, 1989). This is alluded to by Clements and Battista (1994) who argue that understanding Geometry is an important mathematical skill since the world in which we live is inherently geometric. Improving learners' geometric thinking levels is one of the major aims of Mathematics education since geometric thinking is very important in many specific, technical, and occupational areas (Olkun, 2005).

Generally, Geometry is used in many fields including physics**,** engineering, architecture, space exploration, and anything that requires measurement. According to Leopold (2006), Geometry is the most useful of all the sciences. The understanding of Geometry helps in the learning of all other sciences, and no other science can be learned unless you know something about Geometry (Vojkuvkova, 2012). Studying Geometry will make people's eyes quick in seeing things, and hands steadier in doing things. People can draw better, write better, cut clothes, make boots and shoes, work at any mechanical trade, or learn any art. Geometry provides learners with the means to analyse, understand and describe their world, and to deepen their understanding while adding to the ability to solve real-world problems (Jones, 2002). As such, Geometry is important for the economy and nation building. According to Leopold (2006), the rationale for including formal Geometry in the school curriculum is twofold; it is seen as a vehicle for teaching and learning deductive thinking (proof), and also as a first encounter with a formal axiomatic system. According to Brown, Jones and Taylor (2003), the important objectives in teaching Mathematics at the secondary school level include developing knowledge and understanding of, and the ability to use, geometrical properties and theorems; encouraging the development and use of conjecture, deductive reasoning and proof. One of the rationale of Mathematics teaching as suggested by the Southampton/Hampshire Group of mathematicians and Mathematics educators is for designing and developing suitable teaching materials that support the teaching and learning of geometrical reasoning (Brown et al., 2003). Geometry is a natural avenue for developing learners' reasoning and justification skills. All this explains the importance of Geometry that explains the South African government's inclusion of Geometry in the CAPS as a compulsory subject.

2.2.6. *Geometry skills development*

Geometry focuses on the development and application of spatial concepts through which learners learn to represent and make sense of the world (Tait-McCutcheon, 2014), and it is an essential part of the South African Mathematics curriculum (Alex & Mammen, 2016). Evidence from a variety of sources makes it clear that learners at junior school level are not learning Geometry concepts appropriately to prepare them for success in their high school Geometry course (Scholarsarchive &

Genz, 2006). However, Geometry is regarded as important in school because it provides perspectives for developing learners' deductive reasoning abilities and the development of spatial awareness (Crosswhite, 1989). In addition to that, the level of geometric thinking has relevance to learners' ability to demonstrate geometric skills (Nur & Nurvitasari, 2017). Developing problem-solving skills related to Geometry is one of how learners can solve Geometry problems. Geometry skills include visual, verbal, drawing, logical and applied skills.

- *Visualisation skills* in Geometry involve the ability to identify geometric structures based on observations that can be seen directly (Karapınar & Alp İlhan, 2018). Visual skills refer to the ability to recognise a variety of figures and space; identify the center of the object and the interconnection of a part with the other; recognise the symmetry and classification of an object; and envision a model (Thomas, Druck & Huilet, 2015). Thus, visualisation skills include recognition, observation of properties, interpretation of maps, imaging, and recognition from different angles (Nasir, Ghazali & Razak, 2014).
- *Verbalisation skill* is the ability to define a form based on characteristics or attributes attached to the form (Nurwijayanti & Fitriana, 2018). This skill denotes the capability to classify objects of Geometry by name, visualise them with their verbal descriptions, define them appropriately, show relationships between objects, formulate generalisations and abstractions, and appreciate the logical structure of oral problems. Further, the skill includes the correct use of terminology and accurate communication in describing spatial concepts and relationships (Thomas, Druk & Huilet, 2015).
- *Drawing skill* refers to the capacity to abstract a form based on the nature and characteristics it possesses (Nurwijayanti & Fitriana, 2018). This involves the ability to sketch a plane and label certain points, sketch according to verbal descriptions, construct picture based on its properties, add useful elements to an image, identify the limitations of the image sketch, and construct a geometric model. Actually, this skill is all about communicating through drawing, representation of geometric shapes, making scale diagrams, and sketching isometric figures (Searcóid, 2002).
- *Logical skill* encompasses the ability to understand the principles of conservation of geometric shapes and spatial reasoning (Nurwijayanti & Fitriana, 2018). On this note, one must recognise differences and similarities between geometric objects, classify them by their properties, understand and apply the essential properties of definitions, exhibit logical consequences of data, develop logical proofs, and recognise the role and limitations of

deductive methods. This skill is more about classification, recognition of essential properties as criteria, discerning patterns, formulating and testing hypothesis, making inferences, and using counter-examples (Tripathi, 2020).

- *Applied skill* includes the ability to connect other mathematical concepts in Geometry and develop a given model to solve a problem (Nurwijayanti & Fitriana, 2018). This skill involves recognition of the physical model of the object, construction of the Geometry model based on its physical object, application of the properties of the Geometry model to the physical object and development of geometric models for natural phenomena, and their application in problem-solving. The applied skill also includes real-life applications using geometric results learnt, and real uses of Geometry, as for designing packages (Nurwijayanti & Fitriana, 2018).

These five basic skills are interconnected and should not be viewed in isolation from one another. For example, a learner will be able to apply a concept of Geometry to daily life problems if he/she is able to develop his/her geometrical logical thinking as well as explain it verbally. This is because the verbal and visual skills have to develop the basic logical skill and those basic geometrical skills are important and need to be balanced with the development of problem-solving, which needs an integration of the skills (Nurwijayanti & Fitriana, 2018). These five geometrical basic skills are regarded as a whole unit that the learners must use to solve Geometry problems.

2.2.7. *Problem-solving*

Mathematical problem solving is defined as an abstract and complicated process and involves human thinking and reasoning (Inam, 2016). It is stated that problem solving is an approach to solve a certain problem (Maricic, 2010). Thus, problem solving can be defined as a method employed to solve a particular problem that involves individual reasoning on the specific steps to be taken in the process.

As one of the aims of education is to equip learners to solve problems (DBE, 2015), problem-solving in the Mathematics curriculum contributes to the generic skill of problem-solving. Problem-solving is the foundation of various Mathematics activities in all Mathematics subject areas (Reys, Lindquist, Lambdin, Smith & Suydam, 2004). Skills in problem-solving contribute to self-actualisation and lead to greater opportunities for employment as well as to economic growth (Hanushek, Jamison & Wößmann, 2008). Due to its significance, there have been calls for the teaching of problem-solving, as well as the teaching of Mathematics through problem-solving to be included in the South African Mathematics curriculum (Liljedahl, Trigo, Malaspina & Bruder, 2016).

Problem solving is one of the important aspects of Mathematics learning that must be mastered by learners in the learning of Mathematics (Capecchi, 2017). These facts must be noted in problem-solving: problem-solving is not based on the possessed memory; each problem possesses unique strategies; various approaches should be learned and understood to result in appropriate problem-solving (In'am, 2014).

Problem solving in any field is recognised as a prerequisite for any advancement in knowledge (Giannakopoulos & Kakoma, 2015). In South Africa it is one of the seven critical outcomes of education, together with critical thinking (SAQA, 2000). With the introduction of the Curriculum and Assessment Policy Statement in South Africa, it is stated that the learner should be able to identify and solve problems and make decisions using critical and creative thinking (DBE, 2011).

2.2.8. *Problem-solving skills*

Problem-solving skills have been identified through much research as one of the major requirements in the job markets in most economies. There are calls for the inclusion of problem-solving in most curricula in today's classroom (OECD, 2012). Different terms have been used to describe problem-solving skills in literature. To describe the measure of a learner's problem-solving, Learners, Case, Walter, and Bester (2014) used the terms proficiency. Scherer and Beckmann (2014) used competence, Renkl and Atkinson (2003) used capabilities to mean the same. Awuah and Ogbonnaya (2018) used skills for the same purpose. Like Makhubele (2012) in this study, I used the term problem-solving skills, defined as a measure of one's ability to solve problems from intellectual domains such as Mathematics. This study measured learners' problem-solving skills by studying their skills in solving geometric problems.

2.2.9. *Research into learners' problem-solving skills*

Awuah, (2018) investigated the problem-solving skills of Grade 12 learners in probability in South Africa. A total of 490 Grade 12 learners from seven schools, categorised under four quintiles (socio-economic factors) were purposefully selected for the study (Awuah, 2018). A mixed method research methodology was employed in the study. Bloom's taxonomy and the aspects of probability enshrined in the Mathematics Curriculum and Assessment Policy Statement (CAPS) document of 2011 were used as a framework for analysis. Results of the study showed that, except for Bloom's taxonomy's synthesis level, learners in Quintile 4 (fee-paying schools) had statistically significant higher achievement scores than learners in Quintiles 1 to 3, (i.e., non-fee-paying schools) at the levels of knowledge, comprehension, application, analysis and evaluation of Bloom's taxonomy. Contrary to expectations, it was revealed that the achievement of the learners in probability in this study decreased from Quintile 1 to Quintile 3 in all but the synthesis level. Based on these findings, the study argued that the quintile ranking of schools in South Africa may be a useful but not a perfect means of categorisation to help improve learner achievement.

Chirinda and Barmby (2017) used a mixed method approach, based on the constructivist model, to research how the improvement of the mathematical problem-solving skills of Grade 8 learners would enhance achievement in Mathematics in a secondary school in Gauteng province of South Africa. A teaching and learning environment, in which the presentation and solution of problems were the main deductive mathematical activities, was created by the author. Data was collected through a questionnaire, the compilation of a register of mathematical problem-solving skills, direct observation and questioning of participants, semi-structured interviews, learner journals, mathematical tasks, and written tests including pre- and post-multiple choice and a word problem test. Data was analysed using descriptive analysis and it was found that participants in the experimental group had obtained and improved mathematical problem-solving skills on conclusion of the intervention. The qualitative results also indicated that the improvement of mathematical problem-solving skills enhanced learners' performance and achievement in Mathematics.

Tigere, (2014) evaluated the learner problem-solving competency in respect of physical sciences in the Highveld Ridge East and West circuits in Mpumalanga province of South Africa. The study collected data from learners from three different schools using random sampling. Learners were required to write a stoichiometric achievement test administered by the physical sciences teachers at their respective schools. A memorandum was used to score the test. The study investigated, among other things, the relationship between conceptual problem-solving proficiency and algorithmic proficiency of Grade 12 physical sciences learners. It also attempted to determine the capacity for problem-solving in respect of physical sciences in stoichiometry, according to problem-solving strategies and the weakness that existed in stoichiometry problem-solving that could be reflected during teaching. The results revealed that learner proficiency in both algorithms and conceptual problem solving was low. Algorithm and conceptual problem-solving proficiency were found to be weakly correlated. The percentage of problems correctly solved was the lowest (26.78%) for algorithmic solutions and 5.46% for conceptual solutions, in comparison with the percentage of incorrect solutions (42.27%) and problems not attempted (18.65%). The findings also revealed that there were no Grade 12 learners with high algorithmic and high conceptual abilities. The study provides a basis for further research in that it did not include rural, farm and private schools.

Mugisha (2012) studied learner performance in Mathematics in open distance learning settings. The author investigated the problem-solving skills of the University of South Africa learners in calculus module MAT112. Data was collected from the end-of-year examination scripts of the learners between 2006 and 2009. The study reassessed the work done in the end-of-year Calculus examinations, by both looking at the distribution of marks awarded and assigning new scores based on an assessment rubric adapted for the problem at hand. Further assessment of qualitative dimensions that is important for problem solving in Calculus is developed from the data obtained from the assessment rubric. Using factor analysis, a hesitation factor, transfer-of-knowledge factor, as well as ingenuity factor, was identified in successful Calculus problem solving. The study proposes two conceptual models; the first is to guide learners in solving Calculus problems while the second one is meant to assist lecturers in the assessment of learners of Calculus. The researcher developed a qualitative questionnaire from the quantitative data to give more meaning to the quantitative data. Due to the mode of data collection, one may not be able to guarantee the appropriateness of the data for the purpose since it was secondary data as Saunders (1997) is of the view that secondary data might not be qualitative enough to measure what it intended to measure, and might be outdated or might have been used in previous studies for the same purpose. This limitation characterising Mugisha’s (2012) study is solved by making use of primary data in the current study.

Dhlamini (2012) studied the effect of implementing a context-based problem-solving instruction on learners’ performance. The study employed cognitive load theory as the framework. Participants were 783 learners and four Grade 10 teachers from Gauteng province in South Africa. Data was collected by making use of a standardised functional Mathematics achievement test. The study used a non-equivalent control group design, consisting of a pre- and post-measure. Classroom and semi-structured interviews were conducted with teachers and learners. The teachers used normal problem-solving lessons in the four control schools while the researcher deployed context-based problem-solving instruction in the five experimental schools. The design of the context-based problem-solving instruction was such that the learners in experimental schools were at ease with the basic context-based problem-solving tasks that were posed to them through the worked-out examples. A standardised functional Mathematics achievement test was used as the data collection instrument. A pre-test was given to learners to determine their initial problem-solving status, after which the intervention followed. The one-way analysis of covariance and the analysis of variance and other statistical techniques were employed for the data analysis. The findings revealed that the learners in the experimental schools outperformed those in the control group. This was confirmation that the context-based problem-solving instruction was an effective tool to improve the problem-solving abilities of Grade 10 learners in Mathematics. The study challenged researchers to further examine what learners are thinking during the process of developing problem-solving skills in Mathematics. This, according to the researcher, would explain how learners advance from basic skills to advanced problem-solving skills in Mathematics. The researcher urged that more studies be done on how instruction can be tailored effectively to improve the problem-solving skills of South African learners. The current study investigated learners' problem-solving skills on a topic in Mathematics, in Geometry by making use of Polya's model. The findings would demonstrate the problem-solving skills of learners hence bridging the gap of knowledge on what Dhlamini (2012) studied.

Lim and Noraini Idris (2006) conducted a research study on the algebraic problem-solving ability of Form 4 learners in Malaysia, revealing that 62% of the learners had less than a 50% chance of success. The authors used the Structure of Observing Learning Outcome (SOLO) model as a theoretical framework for assessing the learner's abilities in using linear equations. The content that was investigated included linear pattern direct variation concepts of functions and arithmetic sequences. It was found that the learners had problems in the use of algebraic symbols to generalise their thought processes. From an analysis of the qualitative data, it was revealed that the participants seemed to be more proficient in searching for recurring linear patterns to identify the linear relationship between variables than in other areas. According to the study, they could coordinate all the information supplied in the questions to form algebraic expressions and linear equations. The low-ability learners showed that they could master the drawing and counting method though they did not understand the algebraic concepts. The study provides evidence that learners have challenges in the algebraic solving of problems and thus need assistance in the concepts of algebra.

In a study done by Zanzali & Nam (1992) evaluating learners' levels of problem-solving abilities in

Mathematics in Malaysia, 242 Form 4 science and non-science learners from four urban schools were selected from a Malaysian secondary school. The respondents were made to solve mathematical problems. The tasks comprised content questions, multiple choice questions, and structured questions. Their levels of ability in using basic knowledge, the standard procedure, as well as their problem-solving skills, were evaluated. The evaluation was done by applying Polya's model for solving problems. The findings revealed that the learners were unable to use the correct and appropriate mathematical symbols and vocabulary to provide the reasoning behind certain problem-solving procedures. The study further showed that learners had a good command of the basic knowledge and skills required for problem-solving, although they did not demonstrate the use of problem-solving strategies as expected. The study provided a basis for further research, particularly in knowing learners' specific geometric problem-solving skills, as the study was silent on that aspect. The current study investigated the specific geometric problem-solving skills of learners.

In Turkey, Yigiter (2013) conducted an investigation into the problem-solving skills, self-esteem and preferences of university learners regarding sport and social activity. The study comprised 500 learners in an English preparatory programme from different departments of the university. The study concluded that problem-solving skills were not associated with gender or age, nor did they differ according to gender and preferences in relation to sport and social activity. But problem-solving skills are associated with some demographic variables such as age. However, sport was found to have a positive effect on self-esteem but not on the problem-solving skills of learners of the university. The study provides evidence that the problem-solving skills of learners are independent of gender but dependent on certain factors such as recreational physical activities. Syafii and Yasin (2013) used a quasi-experimental non-equivalent pre-test and post-test control group design to investigate biology learners' problem-solving skills at high school level in Indonesia. The study employed a problem-based module in teaching the experimental group of learners. The treatment group applied the problem-based module in learning problem-solving skills under the guidance of teachers, and the problem-based module was used as a framework in their teaching and learning sessions. The learners used task sheets indicating the main steps in implementing the problem-based module to guide their learning activities. The teachers acted only as facilitators. The learner learning activities were structurally observed, utilising observation sheets to monitor the learning progress in respect of problem-solving skills. The control group was taught biology through conventional teaching methods, with the teacher guided by the available lesson plan. The teaching methods only involved lectures and drills. Learners were not exposed to the problem-based module. Their progress was monitored through observation. The following were used as indicators for problem-solving abilities: ability to identify problems; ability to gather data; ability to plan the solution; ability to execute the plan and solve the problem; and the ability to evaluate the problem-solving process (Syafii & Yasin, 2013). The post-test was conducted after the teaching and learning (T and L) session to measure the ability of learners from both groups.

The learners' problem-solving skills were analysed using descriptive analysis and inferential analysis through SPSS 18 software. The performance of the experimental and control groups in mastering the concepts and learning products was tested to compare the problem-solving skills of the two groups. The problem-solving skills of the experimental group were measured and found to be 85.47 % (very good); far better than that of the control group which was measured and found to be only 25.12% (low). Average achievement in the experimental group was measured and found to be 84.26% (good), while that of the control group was 79.08% (moderate). Overall, the average of the product of the learning was 89.89% in the case of the experimental group compared to 52.10% in the control group. The findings clearly indicated that problem-based modules can improve learners' problem-solving skills in biology. The findings provided evidence that learners performed better when exposed to a problem-based module. However, the study was silent on the specific problem-solving stages at which learners performed. It may happen that learners performed better at some stages in the process of problem-solving. Thus, this study measured learners' problem-solving skills at different stages in the process.

Tang and Pham (2017) studied Vietnamese learners' problem-solving skills in learning about the error of measurements. The study was conducted following two parallel approaches, firstly by analysing academic materials according to phraseological organisation in relation to components of problem-solving skills to clarify the formation and development of problem-solving skills through the evaluation system. Secondly, the other approach was by building an experiment of processing errors using statistical tools that learners had learned in the curriculum and evaluating the degree of problem-solving skills of a sample of 201 learners randomly chosen in Ho Chi Minh City. The study was evaluated based on four phases. Phase 1 presented how the learners identified a solution to the problems. The results showed that 57%, representing 115 learners provided an incorrect solution; 69 of the learners representing 33% provided partially correct solutions; and 10% provided the correct solution. According to the study, few learners were able to reach Phase 2 of the study. The results showed that processing errors by using statistics, as presented in Mathematics and physics textbooks did not enable most learners to propose a solution in the experiment. The research questions, the practical teaching of teachers and teacher training in pedagogical universities, as well as providing an evaluation of the reality of the current educational system, contributed to the development of the Mathematics curriculum and composition after 2018. Based on the findings of the study, the authors recommended that teacher training focus on the formation and development of problem-solving, and on providing an understanding that is deep in terms of major knowledge (mathematical statistics) and broad in terms of relevant sciences (theory of errors).

2.2.10. *Evaluation of problem-solving skills*

From the literature review, different problem-solving models, educational strategies, and methods have been engaged to assess learners' problem-solving skills. For instance, Tang and Pham (2017) evaluated learner problem-solving skills by analysing academic materials and also building an experiment researching processing errors by using the statistical tools that the learners had learned. Dimitriou-Hadjichristou and Ogbonnaya (2015) used Bloom's taxonomy to evaluate learners' problem-solving skills. These methods have been used to evaluate problem-solving across disciplines. For this study, Polya's model is discussed because of its extensive use in the literature on this subject. Several studies have used Polya's method in evaluating problem-solving skills. Examples of such studies include Brijlall (2015) who used the model to evaluate learners' problem-solving skills during collaborative learning in studying the mathematical topic, fractions. The study was a small-scale action research that used 47 Grade 10 learners in a South African middle school as participants. The participants from two classes attempted to solve tasks involving the concept of fractions. In one class, the learners were assigned to work together in groups while participants in the other class would work individually. Qualitative methods were used for data collection and analysis. Most of the stages of Polya's problem-solving model were discernible in the groups working together. The study helped to identify the stages in the model that promoted effective problem-solving. The study made recommendations to Mathematics classroom practitioners regarding problem-solving. The study recommended that learners be viewed as intelligent and creative individuals whose questions are important, and that more time be provided for discussion.

Lupahla (2014) used Polya's problem-solving model to document the level of attainment of problem-solving skills among Grade 12 learners from the Oshana Region in Northern Namibia. A computer-aided algebraic problem-solving assessment program was used. A mixed method triangulation design was employed, while the data collection instruments consisted of a knowledge-based diagnostic test and an algebraic problem-solving achievement test, as well as an item analysis matrix, used to evaluate the alignment of examination content with the curriculum and its assessment objectives. Questionnaire instrument was administered on a purposively selected learner sample after which interviews were conducted and analyzed. It was reported that 83% of the learners performed below the Trends in Mathematics, Science and Technology Education (TIMSS) Level 2 (low) in mathematical problem-solving skills, while there was a correlation of r = 0.5 (Pearson correlation coefficient) between the achievement in the knowledge base and the problem-solving test. The correlation between the learners' achievement in the algebraic problem-solving test and their achievement in the final Namibian senior secondary school certificate examination of 2010 was r = 0.7. Polya's first step was identified as presenting problems for learners in respect of the reading as well as the understanding of the problem. The study found that the algebraic approach was the most successful solution strategy.

In a similar study, Learners, Case, Walter and Bester (2014) measured learners' level of problem-solving by making use of Polya's four stages of problem-solving as a departure point. The study used 128 second year marketing learners offering quantitative techniques from Walter Sisulu University in South Africa. The instruments utilised to collect data included a written test and questionnaire. A profile of participants' problem-solving was constructed. Learners' strengths and weaknesses, as demonstrated in problem-solving, were investigated. The findings of the study revealed that 72.29% achieved the highest marks in understanding a problem. Participants who were able to solve the problem comprised 73.77% of the learners. The findings revealed that 29.38% of the participants of the study could interpret their results. The study recommended that the curriculum be revised to include course material on problem-solving to improve learners' proficiency in problem-solving. The findings of the study, however, did not conform to those of Lim & Noraini Idris (2006) or the findings from the report of the TIMSS study. Learners performed better in the use of algebraic strategy in solving problems, according to Lupahla (2014). However, the reason for learners' poor performance in the TIMSS study can be attributed to the use of a purposive sampling technique instead of a random sampling technique. Polya's problem-solving model has been used extensively to evaluate learners' problem-solving skills.

2.2.11. *Problem-solving models*

A problem-solving model is defined as an algorithmic and deductive approach to solving a problem (Nfon, 2013). This implies that there are steps that one must follow to arrive at the desired solution. Various problem-solving models are well documented in the literature. Literature review has revealed that there are several models that are used to evaluate mathematical problem-solving. Among these are the models developed by Schoenfeld (1987), Galbraith and Stillman (2006), Zakaria and Yusoff (2009), Yimer and Ellerton (2010), Gibson, Marriott and Davies (2007), Wu and Adams (2006), and Polya (1945).

Schoenfeld's problem-solving model

Schoenfeld (1987) believes that becoming a good mathematical problem solver or a good thinker in any domain may be as much a matter of acquiring the habits and dispositions of interpretation and sense-making as of acquiring any set of skills, strategies, or knowledge. Thus, he came up with a problem-solving model with four categories of skills that are needed to be successful in problem-solving. The first category is of resources, which is the proposition and procedural knowledge of Mathematics. The second is heuristics, which represents strategies and techniques for problem solving such as working backwards or drawing figures. Thirdly, is control, that is, decisions about when and what resources and strategies are to be used. The fourth category is beliefs. These are regarded as mathematical worldviews that determine how someone approaches a problem. Schoenfeld (1987) is well known for his work on metacognition, positing that, for one to be able to solve a problem successfully, one should have known about the question. After knowing about the question, the problem solver should be able to analyse and explain the question. Correct analysis and explanation indicate the understanding of the problem, hence, the ability to solve the problem. Here, the problem solver would be able to come up with a formula or strategy that would assist in solving the problem. Implementation is very critical at this point, involving substitution and evaluation to identify the unknown variables in the question. Ultimately, the problem solver should link the question to the answer by giving a vivid explanation of the solution.

The major challenge with this model regards the teacher. The teacher must perceive the implications of learners' different approaches, whether they may be fruitful and, if not, what might make them so (McIntosh, Jarrett & Peixotto, 2000). McIntosh et. al (2000) argued that the problem with Schoenfeld model is the teacher must always decide when to intervene, and what suggestions will help the learners, while leaving the solution essentially in their hands, and carrying this through for each learner. The teacher will, at times, be in the position of not knowing yet to work well without knowing all the answers requires experience, confidence, and self-awareness that some teachers might not have in some cases.

Marriott, Davies and Gibson's problem-solving model

Gibson, Marriott and Davies (2007) developed a problem-solving model, referred to as the problem-solving approach, with four phases which are: specify the problem and plan, collect data, process, and represent, and interpret and discuss. These authors believe the person solving the problem must be able to identify the problem and come up with a plan. After reading the problem, the problem-solver should be able to extract the information given in the question and identify the unknown. Subsequently, the problem solver would have to process the information and represent the information in the form of diagrams. The problem solver then should come up with a formula to solve it. The last step involves the interpretation of results and discussions. I strongly believe that problem-solving approach can also be useful when teaching learners how to solve problems in Geometry.

Stillman and Galbreath's problem-solving model

Galbraith and Stillman (2006) applied a model based on information processing with four phases, which are: information gathering, information representation, search and information processing, and information validation. This model contributes to the planning of teaching, in particular the identification of prerequisite knowledge and skills, preparation of interventions for introduction at key points (if required), and the scaffolding of significant learning episodes. Other potential uses of the framework are in the design of modelling tasks, and in assessment in which taking account of the activities in the respective transitions should enable tasks to be designed so that all phases of the modelling cycle are adequately represented (Galbraith & Stillman, 2006). The framework may be useful as an instrument in classroom-based research. Task design has been less of a focus, and this is an area that changes the importance of the model depending on the purpose for which the model is to be used.

Yimer and Ellerton's PS model

Yimer and Ellerton (2010) proposed a model that helped learners to develop their ability to monitor their own problem-solving activities and the model consists of five stages. Empirical data from a study of pre-service teachers doing mathematical problem-solving that was not of a routine nature was used. The five phases used were engagement, transformation, formulation, implementation, evaluation and internalisation (Yimer & Ellerton, 2010). Yimer and Ellerton (2010) showed aspects of mathematical problem-solving by pre-service teachers. Their model had a reflection at each phase of the problem-solving process and considers internalisation as a separate phase. This was the main difference between their model and other models. This makes the model too time-consuming as one must make a reflection at every stage of problem-solving.

Wu and Adams's PS model

Wu and Adams (2006) problem-solving model linked problem-solving and cognitive processes. The model was used to identify specific weaknesses of learners in solving mathematical problems. The model has four stages consisting of extracting information from the question, a real-life and common sense approach to solving problems, reasoning and finally applying standard computational skills, and carefulness in carrying out computations (Wu & Adams, 2006). Wu & Adams (2006) model served to solve the cognitive processes found in a task which can be used as a problem-solving skill profile of learners. The only problem with this mode is that it is a framework that does not provide for the assessment of reflection, or the interpretation of the results learners may encounter.

Zakaria and Yusoff's Problem-Solving model.

Zakaria and Yusoff (2009) model of problem-solving was developed to assess skills in algebra based on Mayer's (1992) model. The model consists of problem translation and integration, solution planning and monitoring, and solution execution (Zakaria & Yusoff, 2009). The model's first three stages align well with the first three phases of Polya. It is my view that the lack interpretation stage of the results in this model is a limitation.

Many different scholars have proposed different problem-solving models, and it should be noted that all these models require the problem solver to have knowledge about the question being solved. It proposes that one's ability to solve a problem depends, in one way or the other, on the individual's existing knowledge on the said concept as the collection of data and thought of the appropriate method required to solve the problem is found in the skills required in solving a problem. One model of problem-solving I discuss in more detail is by Polya (1945) because it is central to this study.

Polya’s Problem Solving model.

Polya's (1945) model of problem-solving is one of the oldest known models and is considered one of the most important ones (Carifio, 2015). The Polya model has been implemented to solve Mathematics problems at primary, secondary and tertiary levels of learning Mathematics. The model consists of four stages; the first of which is ***understanding the problem***. Understanding the problem has to do with understanding the language of the problem statement, knowing what has been asked, being able to restate the problem in one’s own words, being able to come up with a picture or diagram to represent what is being asked and to acknowledge whether there is enough data to propose a solution. In other words, visual, verbal and drawing skills are very important at this stage.

The second stage of the model is ***devising a plan***. In the second stage, Polya (1945) advocates that there be more than one strategy to solve a problem. Therefore, the skill of choosing the appropriate strategy is necessary, and this can best be learned by solving many problems. Some of the known strategies include looking for the pattern, drawing a picture, using a formula to estimate and check, and many more. The third stage is ***carrying out the plan***. This step generally involves care and patience, particularly when one has the skill. More important, it is also in this stage where one must be alert to when a particular selected plan is not working out, for the change to another plan. Logical and applied skills are the most active skills at this stage. The last stage of Polya’s model is ***looking back***. This involves a critical examination of the solution obtained to ascertain if the result is correct or whether the plan can be used to solve another problem. This stage still demands logical and applied skills.

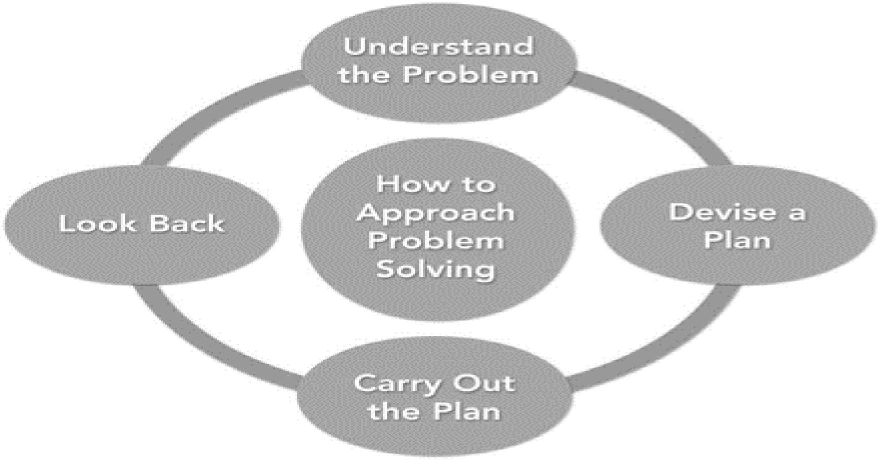


Figure 1: Diagrammatic representation of Polya’s problem-solving cycle

2.2.12. *Why Polya's model*

The researcher used Polya's model of problem-solving in this study for several reasons. Polya's model is a model of problems and problem-solving by young adults such as Grade 11 and 12 learners, to adult problem solver (Carifio, 2015). Polya's model is not about elementary problems and problem-solving, but rather, is about more sophisticated problems that are more complex in form. According to (Carifio, 2015), Polya's model is still ranking very high in use and as a referenced model. This is because Polya's problem-solving process provides learners with opportunities to develop their abilities to adapt and change methods to fit new situations that they find themselves in (Nfon, 2013). Further, the model provides learners participating in the mathematical learning process with communication, representation, modelling and reasoning skills (Lesh, Cramer, Doerr & Zawojewski, 2003). Another reason why Polya's model is favoured is its use in helping learners develop their own thinking and reflecting abilities (Nfon, 2013) and in aiding learners in their interpretations of problems (Lesh et al., 2003).

There are several other effective problem-solving models developed after Polya's model. However, it is important to note that most of these models were developed and modified from Polya's work to suit specific scenarios. For example, the model of Mathematics teaching and learning according to the Mathematics instruction framework of South Africa is closely linked to Polya's model (DBE, 2018). This study thus used Polya's model in the learning of Geometry in Grade 11 to develop problem solving and geometric skills of the learners.

Polya's model is useful in the problem-solving process as learners solve Mathematics problems and it is also very useful in the teaching process (Ortiz, 2016). For example, as the teacher assesses the learner's problem-solving process, he or she might notice that the learner is working at devising a plan stage of the process. In such a case, the teacher could facilitate the process by providing help at that stage though s/he should not get involved with the carrying out the plan stage, which is the next stage.

2.2.13. *Critics of Polya model*

Polya's model assumes that problems are solved in stages and steps along solution paths that may be fruitful, tolerant, and reviving, or twisted, dead-ended, frustrating and even distressing. Polya's critical point here is that solution stages and solution paths have positive or negative outcomes to some extent, and thus, they do not follow one another in a straight line necessarily to a solution. Solving a true problem and true problem solving tends most often to be circuitous and non-linear and riddled with disappointments and failures, as well as elations and success (Carifio, 2015). Many researchers still contend that the phases of the model are often presented in linear steps and thus, there is a need to use a framework that emphases the dynamic and cyclic nature of genuine problem-solving (Hasni & Lodhi, 2011).

We need to keep in mind that learning this four problem-solving steps might not be sufficient to become a better problem solver or mathematician, as this approach is mainly a working framework for problem solving (Ortiz, 2016). From experience, I realised that an effective problem solver could work on a problem with flexibility, and following a linear four-step approach might not work all the time. In some circumstances, one might start to solve a problem without a complete understanding of the problem, and that does not stop one from trying to find a solution. From my own observations, most textbook problems are found already neatly set up for you, yet most real-life problems are presented haphazardly. In such cases, one might try to solve the problem, and in the process, one starts to understand the problem better.

Sometimes the four-step problem solving process is more useful when you start to organise your arguments and intuitively, one might feel he or she knows the solution to a problem, but still need to convince others that he or she have the correct method or answer (Ortiz, 2016). Thus, Polya's (1945) steps can be used to present an acceptable and convincing argument.

The four-step approach can become too methodical or too linear, and might prevent learners from being more creative, and think out of the box (Ortiz, 2016). As indicated by Schoenfeld (1987), it is possible to teach learners to use general strategies such as those suggested by Polya, but that is insufficient. It might take several revolutions through the four-step process before finding a solution to a problem.

2.2.14. *Summary of problem-solving models*

A summary of the problem-solving models studied during this research is given in Table 1 below. Most of the models in Table 1 have minor variations from Polya’s model as they all show progressive stages of a problem-solving process. Wu & Adams (2006), Zakaria & Yusoff (2009), and others, end their problem-solving processes with the execution stage. These models do not include the reflection stage. I believe that this is a limitation of the models highlighted by the fact that researchers like Schoenfeld, 1987; Yimer & Ellerton, 2006; and others developed models that include the reflection stage in the problem-solving process.

Table 1: Summary of problem-solving models

MODEL	**Stage1**	**Stage2**	**Stage3**	**Stage4**	**Stage5**
Polya (1945)	*Understand*	*Devise a plan.*	*Carry out the plan.*	*Look back.*	
Schoenfeld (1987)	*Knowledge*	*Analysis and Exploration*	*Implementation*	*Interpretation*	
Stillman & Galbraith (2006)	*Collect information*	*Information representation*	*Processing*	*Validation*	
Wu & Adams (2006)	*Extract information*	*Plan*	*Mathematisation and carry out the plan*		
Marriott, Davies & Gibson (2007)	*Specify problem and plan*	*Collect data.*	*Process and represent*	*Interpret and discuss*	
Zakaria & Yusoff (2009)	*Problem translation and Problem integration*	*Solution planning and monitoring*	*Solution execution*		
Yimer & Ellerton (2010)	*Engagement*	*Transformation Formulation*	*Implementation*	*Evaluation*	*Internalisation*

2.3. THEORETICAL FRAMEWORK – Social Cognitive Learning Theory

2.3.1. *Introduction*

This study is informed by the social cognitive theory as envisaged by Albert Bandura (1985). Bandura (1985) social cognitive theory explains individual learning, development, acquisition of knowledge, and self-regulated competency within a social context in which parents, peers, and teachers play a significant role as models. Social cognitive learning theory provides a framework for learning that takes into account the social environment, the personal factors such as affect and cognition of the learner, and behaviour (Bandura, 1985). Bandura's social cognitive learning theory assumes that the environmental, personal, and behavioural factors are influenced by inactive and observational learning from one's environment, personal motivation and the ability to self-regulate. This study links to the observational learning assumptions of the social cognitive learning theory.

2.3.2. *The theory*

Social cognitive learning theory focuses on the idea that much of human learning takes place in a social environment. By observing others, people acquire knowledge of rules, skills, strategies, beliefs, and attitudes. According to social cognitivism, people actively participate in their lives rather than passively react to changes in their brains brought about by environmental circumstances. People use their sensory, motor, and cognitive systems as tools to carry out the tasks and accomplish the goals that give their lives direction and significance (Harré & Gillet, 1994). Active social engagement (Xu, 2022) has a role in how we see the environment and in making learning sustainable. Social cognition emphasizes learning as a process rather than as a result (Glasersfeld, 1995). Learners are committed lifelong students capable of handling challenges in the real world as professionals (Glasersfeld, 1995). The development of geometric problem-solving skills of grade 11 learners was investigated in this study by integrating the elements of social cognitive theory.

2.3.3. *Social cognitive theory in problem-solving*

Social cognition's strength can be harnessed in educational practices to facilitate problem-solving and learning in the classroom. Finding creative ways to capitalize on social-cognitive strengths in the classroom is valuable for improving the problem-solving and academic achievement of learners (Butler, 2013). Butler (2013) developed a socially based curriculum to teach reading comprehension strategies. Compared to 3rd graders randomly assigned to a more traditional curriculum, students in the socially based curricular group did better on tasks that required application, making inferences and visualizing (Lucariello, Butler & Tine, 2012). Palincsar & Magnusson (2001) employed students' social-cognitive skills to benefit learning and the findings suggest that children's social-cognitive strengths may be recruited to improve academic performance (Klahr, 2001). The current study investigates whether the integration of such strength necessitates the development of geometric problem-solving skills. Reliance on activities that are collective, goal-directed, and entail artifacts is an approach that utilises the social/interpersonal domain in math education (Saxe & Guberman, 1998). The suggestion here is that Mathematics that is embedded within a socially based problem-solving context may capitalize learner's strength in social reasoning. At the same time, learners who are motivated to succeed in Mathematics are likely to show higher achievement (Atit et al., 2020). Furthermore, from the findings by Atit et al. (2020), the role of motivation in students' Mathematics achievement does not differ for students who demonstrate different levels of spatial skills.

2.3.4. *Social cognitive view of learning*

Bandura (1986)'s social cognitive learning theory assumes that learning is an internal mental process that may or may not be reflected in immediate behavioral change. The theory has several tenets, namely: people learn from one another; people learn from observation; people learn from imitation; people learn from modeling; and there's an interaction between cognitive, behavioral, and environmental influences. Essentially, this means we learn socially. We observe, we imitate, we observe other people, who are our models, and then there is an intersection of what is going on cognitively as well as behaviorally and environmentally. Within a social cognitive instructional framework, learners are provided opportunities to interact with their peers for the purpose of discussing, generating, and sharing knowledge (Amineh & Asl, 2015). In social cognitive learning theory, learners can learn from making observations of others, and through these observations, learners learn many things, including how to perform a specific task (Bandura, 2007). Through observational learning, a learner acquires new patterns of behaviour that they may not have had a chances to learn previously (Bandura, 1985). He proposes four observational processes involved in observational learning. These include attentional processes, retention processes, reproduction processes, and reinforcement and motivational processes.

Attentional Processes- learners cannot learn through observation alone if they do not attend to, or accurately recognise the key features of the modelled behaviour (Bandura, 1971). Therefore, attentional processes form an essential component of learning by example. This process enables learners to decide what elements of the modelled behaviour to observe and what is gained from these experiences. In social groups, certain learners will command greater attention than other members, and the effectiveness of the behaviours used by these learners will have a significant impact on whether their behaviour will be closely observed and modelled or will be ignored (Thomas, Morgan & Kerry, 2014). Interpersonal attraction is very important as models with desired behavioural qualities will be identified and those without discounted. Therefore, learners' capability to process information will influence the degree and level of benefit to be gained from observational learning.

Retention Processes – The second key component of observational learning is the long term retention of activities that have been modelled on one occasion or another (Bandura, 1971). This means that although learners may observe a model's behaviour, they need to recall these actions to be influenced by them. In other words, if a learner is to reproduce a model's behaviour at a later date without the presence of the latter, these patterns need to be embodied in symbolic form through the imaginal and verbal representational systems (Thomas, Morgan & Kerry, 2014). They further argue that the images of modelled performances can be accessed following repeated exposure to the modelling motivations when physical activity has been repeatedly observed.

Observational learning and retention are further supported by verbal coding of a significant amount of information in an easily stored structure. Once modelled activities are changed into images with accompanying verbal symbols, these act as memory codes to reproduce the desired behaviour (Bandura, 1971). He argued that alongside symbolic coding, mentally rehearsing or performing modelled patterns of behaviour are also key aspects in the process of retaining knowledge efficiently. It has been my observation that, although I must give examples of problems on the board, it is only through repeating the process over and over that a student maintains what is learned. According to social cognitive theory, learning and performance are distinct processes. If we can learn a tremendous amount by just observing, it is perhaps the perceived need that we may have to apply what we have learned to reinforce the observed skill. Organising and rehearsing modelled behaviour mentally, and then practicing these actions results in high levels of observational learning.

Reproduction Processes – The third element of modelling consists of transforming symbolic representations into motor behaviours (Bandura, 1971). Bandura highlights that these motor reproductions can be split into four parts: cognitive organisation of responses, their instalment, monitoring, and refinement based on informative feedback given. Starting at the cognitive level even if the modelled activities are acquired and retained, learners can only reproduce these behaviours if they could execute the required skills. If these elements are missing, the complex skill must be developed first through modelling and practice. Even when the person could produce the skill, physical limitations may restrict the ability to reproduce the required behaviour. According to

Bandura, in most learning situations people usually produce an uneven likeness to the newly developed skills through modelling, and it is only through observation and feedback from performance that refinement and adaption of behaviours occur.

Reinforcement and Motivational Processes – The final key element of learning through observation is that a person needs to be motivated to apply the modelled behaviour (Bandura, 1971). This is a crucial aspect, and in as much as people can have the capability to execute the modelled behaviour, they do not always perform or act on everything they learn. Modelled behaviour is more likely to be copied if it results in favourable outcomes for learners as opposed to adverse outcomes. People also evaluate reactions to their own behaviours in order to decide which learned behaviours to apply in specific situations (Thomas, Morgan & Kerry, 2014).

It is important to note that the learner is not a passive learner. They do not just receive information. Instead, learning is reciprocal. Cognition, environment, and behaviour all influence one another. This leads us to reciprocal determinism and the role it plays in social learning theory and collaborative teaching and learning. Reciprocal determinism posits that, not only is the student influenced by the learning environment, teacher, and peers, but also influences the learning environment, teacher, and peers. Not only does the teacher influence the learner, but also influences the teacher and how he or she teaches. Essentially, social cognitive learning says that learning takes place in social contexts, and that collaborative teaching and learning is directly connected to the social learning theory, especially through collaboration. Within collaborative settings, learning can occur through observation, modelling, or reciprocal teaching.

2.3.5. *Social cognitive view of an educator*

While it is one thing to study social learning, it is another to really incorporate it into one's classroom in-order to ensure the learning theory achieves positive results. In social cognitivism, the educator is no longer the central figure in the learning process, as learners take an active role. According to Amineh & Asl (2015), educators who are facilitators in social cognitivism first provide support and help for learners, and this support is gradually decreased as learners begin to learn independently. In addition to that, the educator's role as a facilitator of group work and a manager of the shared learning space that enables argumentative talk, is to ensure that ground rules encourage learners to interact and inter-think (Rahmi, Nadia, Hasibah & Hidayat, 2017). This means that the educator will give the main idea then the learners will get the details. In this thinking, the educator does not teach the detail that will make learners find it difficult to find an understanding of the details. The educator should promote helping learners create a "geometrical mindset" that celebrates mistakes as steps toward improved solutions (Boaler, 2015).

This study believes that the interaction between the learners and the educator is important, and the intensity of communication should be increased during the completion of group and pair tasks. This confirms that learning flourishes in a social environment where conversations take place between learners. This study is also of the view that learners cannot be left alone all the time, and an expert or more capable person must be available to help with misconceptions or stumbling blocks that may surface during learners' interaction. As previously stated, one of the prerequisites for observational learning to be successful is for the observer's attention to be focused on the behaviour. In this study, before demonstrating or modelling something to the learners, it was considered paramount to have their full attention to ensure the learners were engaging as much as possible. Retention of the behavior or information modelled was also considered key to successful learning. Because individual learners learn in a variety of ways, one of the ways I could help learners to retain information and behaviors was to incorporate as many different activities into the lesson as possible. A multisensory approach to learning was used to help learners to increase retention while teaching a lesson verbally. I could use visual aids as well to help reinforce the information. According to Bandura, for observational learning to be successful, the observer must be motivated to reproduce the behavior. I could also motivate the learners extrinsically through positive reinforcement and rewards.

2.3.6. *Social cognitive view of teaching*

The Social cognitive view of teaching in a Geometry classroom says that the educator should avoid leading learners to a mental cul-de-sac (Denhere, Chinyoka, & Mambeu, 2013). For example, the tasks given should be challenging to such an extent that mediation by the knowledgeable educator or peer is needed. Social cognitive teaching approaches emphasize reciprocal teaching, peer collaboration, cognitive apprenticeships, anchored instruction, and other methods that involve learning with others. Social influences like educators, parents, and peers have a strong impact, especially in the adolescent years (Hsiao & Nova, 2016). After the learner has attained mastery of the concept with the assistance from others, he/she can do the task independently. This is the point where the learner can start to consolidate what he understands of the concept, and then practice.

2.3.7. *Social cognitive view of a Geometry learner*

The learners learn Geometry through the active construction of the meaning of geometric concepts and they learn through individual re-organization, representation, and re-construction with peers, elders, and educators (Belbase, 2016). Instead of learners being receivers of ready-made Geometry, they are treated as active participants, thereby compromising knowledge transfer (Van den Heuvel-Panhuizen & Drijvers, 2014). The social environment is the source of knowledge where people construct knowledge when they interact with each other or with their experiences. Discussion can be promoted by the presentation of specific concepts, problems or scenarios, and is guided by means of directed questions, the introduction and clarification of concepts and information, and references to previously learned material (Fujii, 2016). According to social cognitivism, social interaction plays an elementary role in the process of cognitive development. Learners need to be given a chance to reflect on their correct and incorrect solutions. It is significant that educators do not dismiss wrong or incorrect solutions but should rather allow the learners to explain and reflect on how they arrived at their solutions. Effective learning of Geometry requires that learners understand what they know and need to learn, and this motivation will help them learn more (Bhowmik, 2015).

2.3.8. *Conclusion*

The social cognitive theoretical framework and the conceptual framework underpinning this study were discussed. I believe that interaction between the learners and the environment is very important as learning takes place that way. I am also of the view that an expert must be available to help the learners with misconceptions or uncertainties that they come across during their interaction. Peer collaboration, flipped classrooms, and gamification are some of the necessary components of modern classroom practice used in this study. The next chapter explores the methodology associated with this research, along with ethical considerations, and the role of the researcher in the research.

CHAPTER 3

RESEARCH METHODOLOGY

3.1. INTRODUCTION

The research orientation, research methodology and research design guiding this research study are presented in this chapter. This study investigated Grade 11 learners' problem-solving skills in Euclidean Geometry, and focused on the following three research questions:

1. What geometric skills are demonstrated by selected learners prior to learning Grade 11 Geometry content?
2. How can the use of Polya's model enable or constrain the development of problem solving and geometric skills in a selected Grade 11 Mathematics class?
3. What Geometry learning challenges are experienced by selected Grade 11 learners when solving problems in Geometry?

The methodology adopted was founded on the objectives of the research as postulated in the first chapter. The sampling technique employed, the data collection, ethical and validity instruments, the procedure for the data collection, and data analysis methods are discussed. Then reliability issues considered in carrying out this study are presented as well.

3.2. RESEARCH ORIENTATION- Interpretivism

This study is oriented in the interpretivist paradigm. The paradigm posits that methods used to understand knowledge related to human and social sciences cannot be the same as that in physical sciences because humans interpret their world and act based on such interpretation (Lan, 2018). There are five research paradigms in which research can be conducted: Interpretivist, Positivist, Critical Realist, Critical Theory and Feminist paradigms. The selection of a paradigm is influenced, mostly by the type of research conducted. In this study, I opted for an interpretive paradigm that supports the view that there are several truths and realities. The interpretive paradigm posits that a single phenomenon may have multiple interpretations rather than a single truth that can be determined by a process of measurement (Lan, 2018). The interpretivist perspective believes that researchers tend to gain a deeper understanding of a phenomenon and its complexity in its unique context instead of trying to generalise the base of understanding for the whole population (Creswell, Hanson, Clark Plano & Morales, 2007). Different people have diverse insights, needs and experiences. Thus, I adopted an interpretivist paradigm since the core purpose of the study was not to generalise the findings but to describe the geometric skills and problem-solving skills of the selected learners in their learning of Euclidean Geometry.

The interpretive paradigm is concerned with understanding the world as it is from the subjective experiences of individuals (Guba & Lincoln, 1989). The interpretive paradigm is reinforced by observation and interpretation, where observation would be used to collect data and interpretation to make meaning of the data. In this study, observation was used to collect data on how learners applied the four stages in the solving of geometric problems as well as how social cognitive learning principles supported learning during the study. The interpretive paradigm assisted the collection of data that relates to the participants' behavior, perceptions, experiences, and challenges on the use of the four stages in geometric problem-solving. In this study, I did not only use descriptions but also got involved in the social context and the study was conducted in a normal setting.

3.3. RESEARCH METHODOLOGY

In this section, the research method is described, and justification is provided on why the research method was chosen. Research methodology explains the methods by which one may proceed with a research and research methods involve conduct of experiments, tests, surveys and the like (Sam, 2012).

3.3.1. *Approach – Mixed Methods (Qualitative and Quantitative)*

This study follows a mixed methods approach that, according to Slootman (2018), involves collecting, analysing, and integrating quantitative and qualitative data in the same study. The mixed methods approach is one in which I based my knowledge claims on logical grounds. This involved collecting quantitative and qualitative data either concurrently or in sequence, to best understand the research problems. The integration of the two approaches provided a better understanding of the research problems than either qualitative or quantitative approach could provide. By mixing both quantitative and qualitative research methods, I sought to gain the breadth and depth of learners' problem-solving and geometric skills by offsetting the weaknesses inherent in using one approach by itself as noted by Slootman (2018). Where quantitative research is weak in interpreting the context or setting in which people behave, qualitative research is well suited for such situations. For example, there are instances where it was not easy to further describe the behaviour of learners during problem-solving quantitatively hence a complement by qualitative descriptions had to be employed.

3.3.1.1. Qualitative research

In the case of this study, the qualitative component of the research was used to identify the themes that would be used for thematic analysis. According to Rabionet (2009), qualitative methodology is designed to study people and understand their social and cultural phenomena involving an interpretive and naturalistic approach to make sense and interpret the phenomena. Thus, the themes that emerged from literature and some that emerged during data collection were used to develop the qualitative component of this study. These themes included the geometric skills displayed by learners in the pre-, during, and post-intervention test. Other themes came from the problem-solving stages as suggested by Polya, as well as from the principle of observational learning of the social cognitive learning theory. According to Thomas (2006). qualitative methodology is usually supported by interpretative research paradigm, where reality is socially constructed as researchers interact with participants to get their perceptions or experiences on a particular aspect. It was suitable for me to adopt the qualitative methodology for this study to learn from participants about their perceptions or experiences in their own settings. As noted by Daniel (2016), the system through which data is retrieved in qualitative research approach is unique and relies on the collection of non-numerical primary data like words and pictures. The researcher serves as an instrument himself, which makes qualitative research well-suited for providing the existence of objective truths and facts which need to be sought in a paradigm. During the intervention, I collected some data by noting learners' behaviour during geometric problem-solving. I used the four stages model in a social cognitive learning environment for planning subsequent lessons.

Qualitative research, however, is seen as vulnerable to biased interpretations lacking in terms of generalising findings to a large group as compared to quantitative research. This was the challenge with the qualitative component of this study whereby the data collected would be unique to my study and not generalisable to other groups. Non-use of numbers by qualitative researchers makes it difficult to simplify findings and observations (Daniel, 2016). In comparison to the quantitative component, I realised that it was more difficult to deal with only qualitative data without incorporating quantitative data. Thus, I integrated the quantitative component as it does not have the same weakness that qualitative methodology has therefore the two would complement each other (Awuah, 2018).

3.3.1.2.Quantitative methodology

In this study, the data collected can be divided into categories or put into ranks, or measured in terms of units of measurement hence, the need for the quantitative approach. Quantitative research is a form of research that relies on the methods of natural sciences, which produces numerical data and hard facts aimed at establishing cause and effect relationship between two variables by using mathematical, computational and statistical methods (Ahmad, Wasim, Irfan & Gogoi, 2019). One of the reasons for incorporating this method is that graphs and tables of raw data could be constructed with the help of quantitative research, making it easier for me to analyse the results.

In this study, researcher detachment from the participants was a weakness of the quantitative research approach as it meant that the researcher was an "observer" or an "outsider looking in". With this type of researcher/ participant relationship, it is extremely difficult to understand in-depth the phenomena within its natural settings as advised by Daniel (2016). He also believed that when considering the existence of social differences in society and schools in particular, the quantitative research approach is not well suited to examining the complex and dynamic contexts of public education in its forms, sites, and variations. Thus, the quantitative methodology was used to complement the qualitative methodology in this study, thereby utilising a mixed methods approach.

3.3.2. *Method – Case Study*

In this study, a case study method was used because of its great advantage of providing an opportunity of gathering rich data through an in-depth study of a bounded system such as an activity, event, process, or individual. A case study is an empirical inquiry that investigates a contemporary phenomenon within its real-life context, especially when the boundaries between phenomenon and context are not clearly defined (Ebneyamini, Reza & Moghadam, 2018). A case study is used to describe a phenomenon and the real-life context in which it occurred (Yazan, 2015). The case in this study was the Mathematics Grade 11 class at a selected school. The unit of analysis was learners' geometric and problem-solving skills during learning. I used a case study to understand selected grade 11 Geometry learners' actions, thoughts, experiences, and other behaviour related to geometric problem-solving in a social cognitive learning environment. It was also because of the belief that case studies provide rich, detailed, in-depth real-life accounts and examinations of the phenomena of interest.

3.3.2. *Why mixed methods study*

Though employing mixed methods has some shortcomings like the need for much investment of time, energy and resources that can pose main obstacles behind not undertaking mixed methodology as research design, there are several advantages I considered. According to Awuah (2018), mixed methodology provides a more complete and comprehensive understanding of the research problem than qualitative or quantitative research alone. The use of the mixed method allows terms, pictures, and narratives to be used to add additional information to numbers and utilises numbers to add precise quantitative data to words, pictures, and narratives. The nature of my research demanded a mixed methodology since it had a qualitative and a quantitative component.

Qualitative and quantitative research methods do not conflict. They actually work better. Ahmad, Wasim, Irfan, Gogoi, Srivastava and Farheen (2019) posit that qualitative research is almost always the starting point when you seek to discover new problems and opportunities to do deeper research later, while quantitative data gives you measurements to confirm each problem or opportunity. I used the mixed methods approach because of its properties such as completeness, developmental, expansion, compensation and diversity (Alasmari, 2020). The completeness of this approach was useful in this study for achieving a holistic view of the phenomenon studied by integrating findings from the quantitative and qualitative component. This study amalgamated the outcomes from the qualitative and quantitative components into one complete project. The expansion property of this approach helped me to expand the findings derived from the first phases of the investigation. This approach was also compensatory in nature when I managed to balance the shortcomings of one method utilising the supremacy of the other method. I made use of its diversity feature to analyse the varied depictions of the same phenomenon using the mixed methodology. In the next section, the research design employed in this study is reflected on.

3.4. RESEARCH DESIGN

Research design is a plan for the study that provides the overall framework for collecting data (Leedy, 1997). A research design has procedures for research from broad assumptions to detailed methods of data collection and analysis. This study was interpretative, and data collection and analysis were predominantly determined in relation to the contextual setting and perspectives of the learners. A descriptive research design was employed in this study. Considering Leedy (1997) definition, research design provides the whole structure of how data is collected and how the results provided can be judged as reliable. The research design of this study included administration of the pre-intervention test, the intervention program, observations, post-intervention test, as well as interviews. The research process was carried out in five phases as illustrated by Figure 2 below.

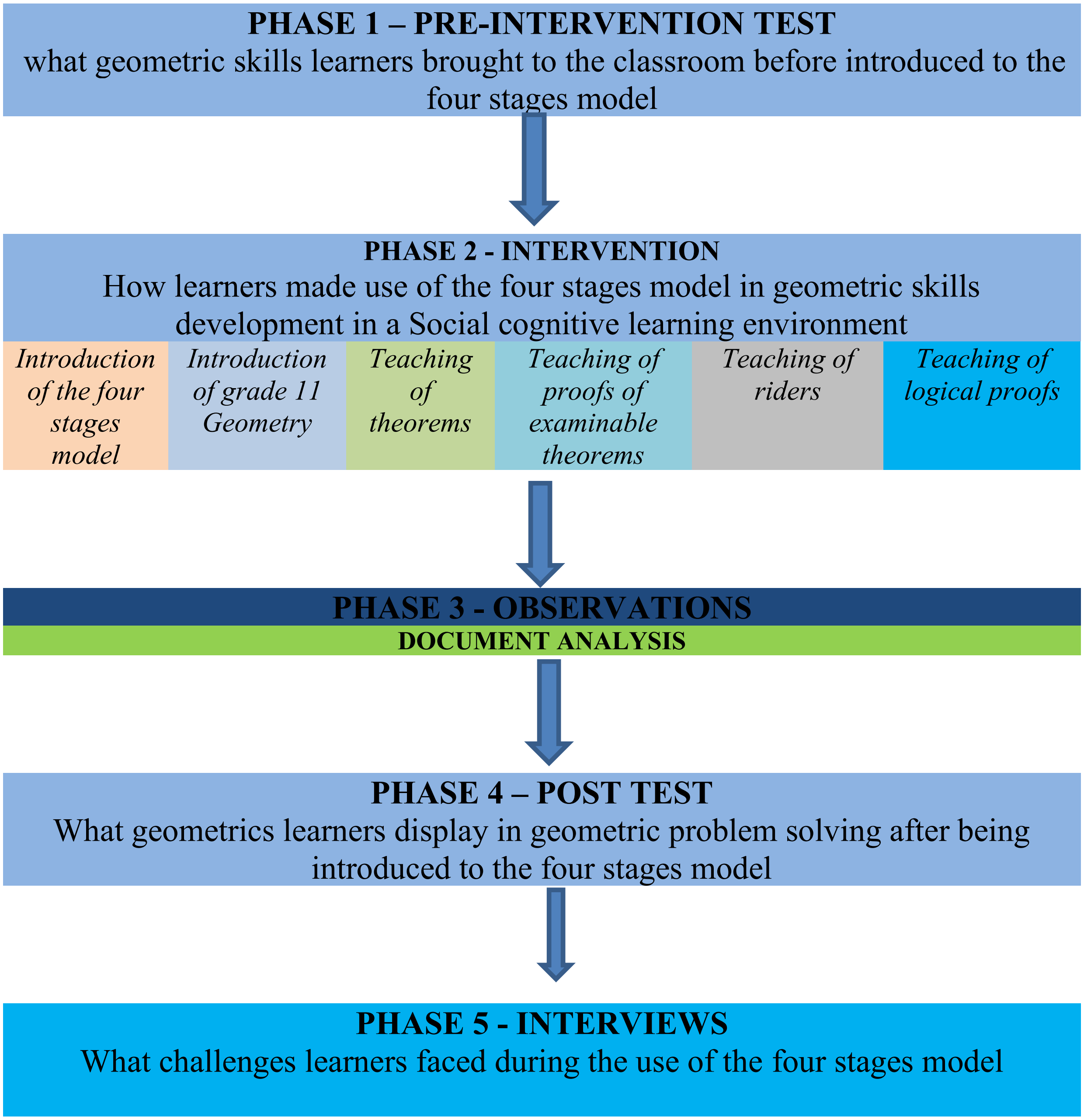


Figure 2: Research process flow diagram

Phase 1: Pre-Intervention Test on Euclidean Geometry.

The Pre-Intervention Test based on Grade 10 Euclidean Geometry work (Appendix I) was administered to the selected Grade 11 class. The test was timed and written under an invigilated condition for 1 hour. The test was designed to assess the geometric skills of learners before they were exposed to Polya's model and introduced to Grade 11 Euclidean Geometry content, therein. The researcher engaged the peer educators, subject head and the school Head of Mathematics and Sciences Department to moderate the Pre-Intervention Test. The Pre-Intervention Test was given to all the Grade 11 learners in the Mathematics class as part of the normal learning process. The Pre-Intervention Test with a total of 50 marks was used to assess the learners' geometric skills prior to teaching and learning of Grade 11 content. Learners' marks were recorded for analysis. Data obtained in this phase was used to answer the first research question.

Phase 2: The Intervention-Learning of Grade 11 Euclidean Geometry.

The intervention program content for this study consisted of two units; one focusing on the utilisation of Polya's model in geometric problem-solving, and the other on the elements of social cognitive learning theory supporting the development of problem-solving skills. The intervention program was aligned to the DBE Mathematics Program of Assessment (POA) and Annual Teaching Plan (ATP) covering the same Euclidean Geometry content taught in Grade 11 Mathematics (for further details about the content of see (DBE, 2011). Learners were taught and assessed using individual class activities, short informal tests, and homework.

The learning styles used during the program were discussed with the learners, and these were all social cognitivism. The learning approaches were closely monitored to ensure the following elements of social cognitive learning were not compromised:

- Attention: To learn from observation, students must pay attention to, and accurately perceive, the significant features of the models. Therefore, lessons were monitored to ensure that they always engaged learners sufficiently to hold their attention.
- Retention: Learners should be able to remember what they have seen or heard. This can be encouraged through rehearsals and more practice activities. Therefore, an informed pairing for learners was done to encourage more practice and modelling.
- Reproduction: Learners should be given time to practise observed behavior. I had to monitor the process and give guidance where applicable to assist productive actions. Participation in presentations and whole class discussions were encouraged.
- Motivation: Learners must be able to see the benefit of new behaviour for long term assimilation. This was accomplished through reinforcements and rewarding of behavior.

The learning practices included the pairing of learners for peer collaboration, teaching, and coaching;

verbal reward and punishment as part of almost every lesson to encourage good practice and discourage bad practice; modelling; and gamification and rewarding. Pair work was the most common learning method as group work was totally prohibited due to COVID 19 protocols. The learners were video recorded with the camera not focusing on the learners but on their work on the use of the four stages model to capture their interaction in a social cognitive learning environment during geometric problem-solving. This helped me to obtain information to answer the second research question.

Table 2: How social cognitivism was used with Polya's model in geometric problem solving

	CONCEPTS	**Teaching and Learning Practices most applied**	**Most targeted Element(S) of Social Cognitivism**
1	**POLYA'S FOUR STAGES MODEL**	Modelling	Attention Retention
2	**EUCLIDEAN GEOMETRY** (Theorems and their Proofs)	Modelling Flipped classrooms.	Attention Reproduction
3	**EUCLIDEAN GEOMETRY** *(Riders)*	Peer collaboration Peer coaching Gamification	Retention Motivation
4	**EUCLIDEAN GEOMETRY** (Logical Geometric proofs)	Peer collaboration Peer coaching	Retention Motivation

The intervention was done in such a way that every concept had teaching and learning practices targeting different elements of the Social cognitive observational learning. As illustrated in Table 2, during the introductory part of the intervention whereby learners were introduced to both the four stages model and Geometry, the most common teaching and learning style was more of modelling. There, the teacher and other selected learners used demonstrations, whole class discussions on most part of the lessons. At this point attention and retention were the most targeted elements of observational learning.

Introduction of the four stages model

In this study, Polya's problem solving model was introduced first in isolation and later during

Euclidean Geometry learning soon after the pre-test was administered. The model was taught in detail to make learners understand the sequence of steps involved when using the model, as well as what each stage entails. I made copies of the four stages model for the learners to paste in their exercise books so that they could refer to them when need arises. Each lesson was then planned and taught following Polya's model of problem-solving. Learners were expected to show how they had executed the four stages of Polya in the process of solving problems. I encouraged learners to make use of the Polya's model during problem-solving, and the whole class had to be observed while doing classwork and other related activities. The teacher also made use of learners' errors and misconceptions to inform the course of the class activities during teaching and learning. As clarified in Table 2, during the introductory part of the intervention, learners were introduced to the four stages model by Polya, the most common teaching and learning style was more of modelling, in which the teacher and other chosen learners had to use demonstrations, whole class discussions on most parts of the lessons. At this point, attention and retention were the most targeted elements of observational learning.

Introduction of grade 11 Geometry- geometric theorems

After introducing the four stages model, I began teaching learners the theorems of Euclidian Geometry. This was after reminding the learners of the grade 10 geometric theorems and axioms. We discussed the theorems and axioms from Grade 10 and used the key geometric words to group the theorems. The key words used included the geometric terms center, tangent, cyclic quadrilateral, chords, diameter, parallel, equal, isosceles, radius etc. This phase of teaching and learning was meant to make learners comprehend the theorems so that they can state the theorems and make sketches of diagrams of those theorems. Individual class activities were given to check if learners were now able to state theorems in words as well as make sketches of them. This was meant to build a foundation for the application of the model since it is a requirement in the planning and implementation stage of the four stages model one must know to state and make sketches of theorems. Teaching learners the ways to comprehend, recall and sketch theorems was also done in preparation of the next phase of teaching and learning which was the proving of examinable theorems. Since in the process of proving theorems, drawing skill and application of other theorems and axioms was a requirement, it was wise to start by making learners familiar with drawing and understanding theorems. In Geometry, it is a policy requirement that theorems be proven before use. This is because proof explains how the concepts are related and proof is highly connected to problem solving. At this phase of teaching and learning, learners were taught how to do proofs of the four examinable theorems. At this point, learners were expected to have basic drawing and construction skills, and application skills to be able to prove the theorems. Just like during the introduction of the four stages model, attention and retention were the most targeted elements of observational learning.

Examinable geometric proofs

The learners were taught to do proofs of the four examinable theorems which are: a line from the center perpendicular to a chord bisects the chord; an angle at the center is twice one on the circumference; opposite angles of a cyclic quadrilateral are supplementary; as well as the Tan-chord theorem. These theorems are the core theorems from which other theorems were deducted, and I suppose that is the reason why learners must know how to do proofs of these four theorems. It is at this phase of teaching and learning that learners are supposed to start applying all the four stages of the model, which are reading for understanding, planning, implementation, and reflection on every question. I therefore used these specific stages to monitor how the learners utilised the model. A sample of the stages to be followed and the structure of how the four stages were to be applied were given to the learners, and I demonstrated to them how the model had to be applied. At this stage of the intervention, the most common teaching and learning styles were modelling and flipped classroom, which were mainly meant to target learner attention and reproduction elements of observational learning since a major part of the proofs involved some set standards to be followed in doing these proofs.

Geometric Riders

After examinable geometric proofs, there followed the teaching of riders. In this case, I taught the learners how to use the four stages of the model in solving of geometric riders whereby one is expected to use given information to find missing angles with support of geometric theorems. On the question paper, learners were supposed to show that reading for understanding has taken place. This could be done both on the statement and diagram. On the statement, one was supposed to identify the key words like center, tangent, perpendicular, etc. Markings were also expected to be done on the diagram to show lines to be perpendicular, radii to be equal, and so forth. Going onto the answer sheet, planning had to be done by listing all the theorems that talk about the key words identified. Still on planning, one had to go on the diagram and check for the applicability of the listed theorems and add more information on the diagram. By the end of the planning stage, most, if not all, the needed answers were already there on the diagram. Once the planning was completed, one had to do implementation, which was all about presenting the answers logically and supporting every statement with a reason. Now most answers needed to be reflected on. This is when one must check for the correctness of the answer. However, it is not every answer that had to be reflected on since some of the answers could be found by simply applying a theorem. For example, an angle subtended from the diameter is a right angle. Such an answer does not need any reflection. The learners were encouraged to use their discretion to see any situation that needed reflection. Peer collaboration and peer coaching were the most used teaching and learning styles, and the solving of riders was at times treated as a rewarded game to increase learner retention and motivation.

Logical geometric proofs

Logical geometric proofs were taught last. Such problems are meant for higher order reasoning. Logical geometric proofs consist of questions that demand one to prove lines to be parallel, a line to be a tangent, a line to be a diameter, a line to bisect an angle or another line, sides, or angles to be equal, a triangle to be isosceles, a quadrilateral to be cyclic, and so forth. From content workshop, work experience, and Mathematics paper 2 internal moderators' report (DBE, 2019) , I was aware that most learners do not even attempt such questions that demand logical proofs in examination settings. I had to teach how to apply Polya's four stages model to solve such geometric problems so that one could attain, at least some marks in such questions. Just like in the solving of riders, peer collaboration and coaching were the most used teaching and learning methods meant to ensure retention.

Table 3: Summary of the lessons held during teaching of Geometry

LESSON	**TOPIC**	**CURRICULUM STATEMENT**	**DURATION**	
			From	To
1 **2**	**Polya's four stages model**	- State the four stages of Polya's model. - Explain what each stage of Polya's model entails. - Apply each stage in each context	29/07/21	30/07/21
3 **4** **5** **6** **7**	**EUCLIDEAN GEOMETRY** *(Theorems And Their Proofs)*	- Apply results established in earlier grades as axioms and that a tangent to a circle is perpendicular to the radius, drawn to the point of contact. - Investigate and prove the following theorems of the Geometry of circles, assuming results from earlier grades: ✓ **The line drawn from the centre of a circle perpendicular to a chord bisects the chord.** ✓ The perpendicular bisector of a chord passes through the centre of the circle. ✓ **The angle at the centre of a circle is double the size of the angle at the circle.** ✓ Angles subtended by a chord of the circle, on the same side of the chord, are equal. ✓ **The opposite angles of a cyclic quadrilateral are supplementary**. ✓ Two tangents drawn to a circle from the same point outside the circle are equal in length. ✓ **The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.** NB: **The proofs of the four theorems printed in bold above can be asked in examinations.**	02/08/2021	10/08/21
8 **9** **10**	**EUCLIDEAN GEOMETRY** *(Riders)*	- Solve circle Geometry problems, providing reasons for statements	**11/08/22**	13/08/ 22
11 **12** **13** **14**	**EUCLIDEAN GEOMETRY** (Logical Geometric proofs)	- Make proofs of the following: ✓ parallel lines and bisection of an angle or line ✓ tangent and diameter ✓ cyclic quadrilateral or isosceles triangle Etc.	17/08/22	20/08/22

***Phase 3: Observations**.*

Observations were done through document analysis whereby the classwork exercise books for the six identified learners were regularly monitored and observed to see whether the learners were applying the four stages learnt in the classroom. The observation was also meant to see the common errors that the learners were making, both in the application of the four stages as well as in the concept itself. This could then be considered in lesson planning for the following lessons. Observation notes were frequently developed, analysed and utilised. The six selected learners' written class work was further tracked, and the learners were also given the opportunity to explain their problem-solving processes during the follow-up discussions that I would carry with each one of them individually. Follow-up discussions and interviews were recorded.

***Phase 4: Post-Intervention Test**.*

A Post-Intervention Test (Appendix I) was administered at the end of the teaching of Euclidean Geometry. The test was written under invigilated conditions. The test had the same structure as the pre-intervention test, intending to test the same skills as those that the pre-intervention test was testing. I engaged and asked the peer educator, Mathematics subject head and the departmental head of Mathematics and sciences to moderate the Post-Intervention Test. The Post-Intervention Test was intended to evaluate the learners' geometric skills learnt during teaching and learning. The Post-Intervention Test was recorded for the purpose of further analysis. Data from the Post-Intervention Test would complement data from phase 2 and 3 in answering the second research question. The researcher used the Post-Intervention Test to describe the geometric skills learners can present at the end of Geometry teaching and learning. The post-intervention test was given to the whole class and both learners' answer-sheets collected, as well as any scribblers used and question papers. These helped to trace and check for evidence of reading for understanding (stage 1) and planning (stage 2) during their answering of geometric test questions. The other two stages (implementation and reflection) would be checked from the learners' solutions. The test scripts would be marked, and those of the six selected learners evaluated using Polya's stages of problem solving. Data from this phase answered the second research question.

***Phase 5: Focus Group Interviews**.*

At the end of the teaching process, the six selected learners were interviewed using semi-structured interview questions to find out their experiences and perceptions during the teaching and learning of

Euclidean Geometry. The focus was on the challenges learners faced when solving Euclidean Geometry problems. Learners were given a chance to write at the end of activities reflecting on their experiences and their errors would be used as stimulants in the interviews. I used interviews as they could provide opportunities for me to probe and follow-up on aspects of interest that the learners would have raised. As such, focus group interviews were expected to provide a better response rate (Fox & Midlands, 2009). The interview schedule (see Appendix F) was used as a guide during focus group interviews. This phase provided data for the third research question of my study.

3.5. PARTICIPANTS

The inclusion or exclusion criteria should be clearly stated in a research study. According to (Polit and Beck (2020), a population or target population is the total number of people or elements that fit the specific set specifications of the study. According to Ingham-broomfield (2017), it would be ideal to include every relevant subject in a study but this is usually impossible, for example because of the economics related to size, time and cost. In this study, the target population was all Grade 11 learners of 2021 Mathematics class at one of the secondary schools in Nondweni cluster, Nquthu circuit, Umzinyathi district, Kwa-Zulu Natal province, South Africa. The inclusion criterion for selecting the learners was based on their willingness to participate as well as availability to attend afternoon classes when this study was carried out. The exclusion criterion involved anyone not from the grade 11 Geometry class, or anyone from the class but not willing to participate and unavailable during research times. The study involved Euclidean Geometry, a topic in the Grade 11 curriculum. Thus, all Grade 11 Mathematics learners in the selected class were requested and encouraged to take part in the study. All 20 grade 11 Mathematics learners agreed to take part in the study.

Sample Selection and Criteria

In this study, six learners from the study population were selected purposively and conveniently. Purposive sampling, also called judgment sampling, is the deliberate choice of a participant due to the qualities the participant possesses (Etikan, 2016). In other words, it is a non-random technique that does not need underlying theories or a set number of participants. Convenient sampling was also used in this study. Convenient sampling (also known as Haphazard Sampling or Accidental Sampling) is a type of nonprobability or non-random sampling, where members of the target population that meet certain practical criteria, such as easy accessibility, geographical proximity, availability at a given time, or the willingness to participate, are included for the purpose of the study

(Etikan, 2016). In fact, this sampling method refers to the researching subjects of the population that are easily accessible to the researcher.

The criteria and process used to select the six learners were:

- The six learners were divided into three groups: 2 top performers, 2 middle performers and 2 underperformers. In this study, the top performers were those learners who had scored an overall mark of 70% and above in their Grade 10 end of year results. Middle performers were learners who had scored between 40% and 69% and under performers below 40%.
- The six learners were also selected on the basis that they consistently attended school and Mathematics lessons, regularly wrote tests, and did classwork, and were willing to be participants in the study.

Purposive sampling was useful in this case as the purposively selected participants managed to provide the required data without spending much time and resources. I could manage to get how learners' geometric skills were developing during the use of the four stages model from different learners from different extremes of mathematical abilities in the group. Results from the study were available in real-time. Convenience sampling brought some benefits as well to the study. It proved to be an affordable way of collecting data and provided for the qualitative component of this research study. I also noted that data was immediately available when using this type of sampling method since there was no need to travel distances to obtain the data. It was easy to note potential biases, and this improved the validity of the study outcomes. Thus, the sample selection methods employed in this study helped me in the development of data collection instruments. The section that follows deliberates on the data collection instruments.

3.5. RESEARCH INSTRUMENTS

It is imperative for the researcher to justify the use of the selected instrument. The rationale may clearly state the advantages and disadvantages of using one tool rather than another and the literature search should also have commented on the use of particular instruments in previous studies (Polit & Beck, 2020). Since case studies normally have numerous data sources, they include multiple data collection techniques. The inclusion of many data collection techniques and sources reinforced the integrity of the outcomes of the data analysis. It also validated the results of the study such that stakeholders who make use of the research findings can take informed decisions (Bans-Akutey & Tiimub, 2021). In this study, five research instruments were used: Observation Schedule, Pre-intervention test; Post-Intervention Test; and Interview Schedule.

3.5.1. *Observations schedule*

Observation schedules were utilized in this study during observations to develop some observation Notes. An observation is a qualitative method with roots in traditional ethnographic research, whose objective is to help researchers learn the perspectives held by study populations (Bans-Akutey & Tiimub, 2021). Through observations, researchers may also detect patterns in people's behaviours that they are not consciously aware of and cannot share with researchers in an interview, focus group, or survey (Brancati, 2018). Thus, observations were crucial for this study. Observation schedules were used in three different types of observations done in this study as follows:

Whole class use of the four stages model - I could observe learners during the use of the four stages model to see how they were applying the four stages model in geometric problem-solving. The observations were done during teaching and learning, and an observation schedule (see Appendix J) was used to record learner observable behaviour. The focus was on checking whether the learners could read for understanding, could do planning, could do implementation, and do reflection. I then note down any observed errors and misconceptions for use in the planning of succeeding lessons. The whole group was observed, though much focus was on the six selected learners.

Elements of Social cognitive learning – the learning styles supporting social cognitivism were also a point of focus. In this study, the participants were encouraged and observed closely if they were observing the learning styles required of the research study based on the elements of social cognitivism. All learning activities were backed with a learning style promoting observational learning in one way or another.

Document analysis and follow-up discussions – in this case observations were done, and observation notes developed as well as observation of the classwork exercise books of the six selected learners. The exercise book observations were also supported by follow-up discussions that were regularly carried out with the six learners. The observations were done to collect data to answer the second and third research question of this study. Inferences drawn from the observations helped me to interpret whether participants' behaviour answered my research question. Lessons were video recorded, and I watched the recordings afterwards to observe how learning unfolded. It also provided permanent records of what happened in the classroom. I could go through the observed lessons repeatedly, picking up non-verbal information that I could have missed during lesson observation.

3.5.2. *Pre-Intervention Test*

A pre-test can be used at beginning of a course or an intervention. In this study a pre-intervention test (Appendix I) was the preliminary test given to learners at the beginning and before teaching of the topic Euclidean Geometry. The test was given before the four stages model was introduced to the learners to see the problem-solving skills that the learners were bringing with them to the classroom. This was a test with questions meant to test their prior knowledge on the application of visual, verbal, drawing, application, and logical skills. The test comprised 50 marks (see Appendix I).

Administering pre-tests before the lecture would increase the attentiveness, curiosity, eagerness to listen to the lecture, among the learners (Shivaraju, Manu, Savkar & Madhav., 2017). I realised that this had an advantage over simply using class responses to judge background since some learners may already have understood certain topics, giving the false impression that the class had a greater understanding of the material than was the case. The pre-intervention test also helped me in designing the intervention program accordingly.

However, I realised there were some drawbacks that I came across when using the pre-intervention test. Some learners seemed to be demoralised after getting very low marks in the pre-intervention test which would have affected their attitude to the topic. I had a task to get some time to encourage the learners and make them aware that the marks they got were only to be used for research purposes and were not supposed to make them feel demoralised in any way. I also encouraged them not to lose hope and promised that the strategy we were going to use would make them succeed in Geometry. The other serious concern for this methodology was the time factor; both the amount of class time used for the pre-tests and the time to grade them. I sacrificed some extra time together with learners to recover time that would have been used for preparing and administering the pre-intervention test.

3.5.3. *Post-Intervention Test*

After the completion of a course or an intervention, participants are normally given a post-test. The post-test is normally meant to answer the same set of questions, or a set of questions of comparable difficulty (Board, 2018). Board (2018)posits that comparing participants' post-test scores to their pre-test scores enables you to see whether the training was successful in increasing participant knowledge of the training content. In this study, a post-Intervention Test (Appendix I) was a test given to all the learners to write after the topic of Euclidean Geometry and the four stages model has been completely taught. The post-intervention test was of the same structure as the pre-intervention test testing for the same skills. The general performance of all the learners in the class were recorded as marks for general analysis, and the six observed learners' work were further analysed thematically.

In this study the post-intervention test was desirable because it was relatively simple to implement. A post-intervention test provides better evidence of the effectiveness of a program compared to prior designs. I also understood that the post-intervention test could provide valuable information about the long-term impact of an intervention. The challenge I noticed in using a post-intervention test was that meaningful pre-test post-test comparisons required that participants be present at the start and end of the program, but consistent attendance can be difficult to obtain.

3.5.4. *Interview Schedule*

Interviews were part of this research study and were done at the end of the intervention. An interview is a qualitative research technique that involves conducting intensive individual dialogues with a small number of respondents to explore their perspectives on a particular idea, program, or situation (Jacobvitz, Deborah, Curran, Melissa, Moller & Naomi., 2002). Jacobvitz et al (2002) say that interviews are often used to provide context to other data (such as outcome data), offering a more complete picture of what happened in the program and why. This made this strategy an ideal technique to get some information on the feelings and challenges that the learners faced during the use of the four stages model in geometric problem-solving. I used the interviews as a way of collecting data at the end of the intervention program, i.e., after Geometry had been taught and the four stages model used during problem solving. The interviews were audio recorded and an interview schedule with a set of questions to guide the researcher in the process of interviewing the learners was used. An interview schedule (Appendix F) is a list containing a set of structured questions that have been prepared, to serve as a guide for interviewers, researchers, and investigators in collecting information or data about a specific topic or issue.

Six selected learners who had been observed were interviewed at the end of the intervention program. By making use of the interviews, I managed to explore the lived experiences of the participating learners. The interviews were meant to understand my research participants' learning and cognitive processes. At this point, I succeeded in collecting data and producing information that could not be directly observed during the intervention. The interviews helped me to get information to answer the third research question which sought to understand the participants' experiences, perceptions, and challenges during the use of the four stages model.

This data collection tool allowed me to prompt interviewees during the interviews and get an in-depth understanding of the opinions and perceptions by the interviewees. The structured environment helped to reduce nervousness on the side of interviewees hence, they were able to disclose most of the necessary information.

I also faced challenges when making use of the interview schedule. At times the interviewees could go off-topic when responding and I could notice that the method had potential to go longer than the allocated time. In some instances, it was difficult to report the findings and compare data due to various responses from the interviewees. I could therefore only focus on that information that had relevance to the study.

3.6. ANALYSIS

Data analysis involves the breaking up of data into manageable themes, patterns, trends, and relationships, allowing the researcher to organise the data into smaller sections, so that any obvious repetitions or errors may be easily noticed. According to Richmond (2006), data analysis includes ways of working with information (data) to support the work, goals and plans of a program. In this study, data analysis was carried out at five levels where descriptive statistics was used for quantitative data and thematic analysis for qualitative data, with the analytical frameworks informed by Polya's stages of problem solving, Nur & Nurvitasari (2017) geometric skills as well as the elements of the observational learning of social cognitivism. In the first and fourth level, data analysis included a qualitative and qualitative component. Quantitatively, the data was analysed using descriptive statistics and qualitatively in the form of thematic analysis, where geometric skills displayed by learners formed the themes. Level two, three and five involved qualitative analysis in form of thematic analysis in which learners' behaviour as well as perceptions determined the themes to be analysed. Details of what happened at every level are as follows:

Level 1- Pre-Intervention test analysis

The geometric skills of Grade 11 learners demonstrated in the Pre-Intervention Test were analysed before the learners were introduced to the topic of Euclidean Geometry and to Polya's model of problem-solving. Data from Pre-Intervention Test was first analysed quantitatively using descriptive statistics. The record sheet developed from the analytical tool (see Appendix G) was used to identify the geometric skills learners bring with them to class prior to Grade 11 content learning. The Pre-Intervention Test was then qualitatively analysed using the analytical tool shown in Appendix G. The five geometric skills formed the five themes that were used to qualitatively analyse data. Emerging themes were noted and discussed as well. The emerging themes involved re-planning skills, required to prove (RTP) finding skills and computational skills.

Level 2- Use of the four stages model for geometric problem-solving in social cognitive learning environment

The data from lesson observations were analysed using the analytical tool designed from Polya's work (see Appendix G). The four stages of Polya's model formed the themes for analysis. The themes include stage 1 (Reading for understanding), stage 2 (Planning), stage 3 (Implementation) and stage 4 (Looking back). These themes were analysed based on the elements of observational learning of the social cognitive learning theory. The principles of observational learning of social cognitivism that were applied in this study are attention, retention, reproduction, and motivation. In this case, the analysis focused on how the teaching and learning activities involved during intervention supported these elements of observational learning. This analysis was meant to explore whether learners utilised Polya's model, and if used, how it was used during the learning process. Video records and photo shots were used to support the analysis of the observed findings at this point.

Level 3 – Observations (Document analysis and follow up discussions)

Analysis of observations was first done on classwork exercise books in form of document analysis. The daily Mathematics school work of the six selected learners was further analysed for the application of the four stages model. Thematic analysis was employed where themes were classified as those in level 2.

Level 4 – Post intervention test

The Post-Intervention Test was analysed and used to check if any learning and development of geometric skills had taken place during teaching and learning of the topic of Geometry. The bigger part of the analysis was done in comparison to the Pre-Intervention Test. The results were analysed using analytical tools in Appendices. The skills that emerged and were applied in the post-intervention test were part of the analysis. The post-intervention test results analysis was done both quantitatively using descriptive statistics, and qualitatively using thematic analysis. Analysis was also done on general geometric skills and further on specific geometric skills under every general skill. For example, the drawing skill was evaluated as a general skill and specific skills falling under drawing like "adding useful information on the diagram" were further analysed. The specific skills falling under the general skills formed the themes that were used for thematic analysis contributing towards the qualitative component of data analysis for this study.

Level 5- Interviews

Data from interviews were analysed to identify the learner's perceptions, experiences and challenges faced during the process of problem solving. Analysis was based on both the themes developed from literature and those that emerged. Analytical tools in the appendices were used to analyse data from the interviews. Learners' challenges and perceptions associated with understanding and use of the four stages model in geometric problem-solving were looked at. At this level, I was more interested in the geometric skills of learners during and after the learning process. These included visual, verbal, logical, and others as shown in Appendices.

3.7. VALIDITY AND RELIABILITY

Validity and reliability were considered throughout this research study. Validity of an assessment is the degree to which it measures what it is supposed to measure, whereas reliability is the extent to which an assessment tool gives results that are highly consistent (Mohajan, 2017). Efforts were made to ensure that the instruments used were reliable and valid. Reliability is a requirement for validity (Awuah, 2018). For an instrument to be content valid, it is expected to adequately cover all the content that it should with respect to the variable being measured, that is, the instrument should cover the entire domain related to the variable it is supposed to measure (Heale & Twycross, 2018). Content validity answers the question of how well an assessment measures what it is intended to measure (Awuah, 2018). Content validity can be determined by a panel of judges who are experts in the field to rate the item regarding content congruence according to laid down criteria (Zamanzadeh, Ghahramanian, Rassouli, Abbaszadeh, Alavi-Majd & Nikanfar, 2015). In this study, the tests used as pre-test and post-test were set by extracting different questions from grade 10 previous examination question papers and common tests. Questions from textbooks were also used in the setting of the question papers. The test was first administered to three Grade 11 learners from different schools. These learners were part of my tutoring group, and this was done before the main participants wrote it to increase reliability. The ultimate questions were then produced after some adjustments were made, based on the pilot study. The purpose of doing the trial was to distinguish whether the test instrument can be used to determine understanding. The use of the pilot study was based on the argument that pre-testing what they intend investigating can help to refine and improve the test instrument (Babbie, Mouton, Vorster, & Prozesky, 2009).

Curriculum experts, Mathematics educators and subject advisors can be used to determine the content validity of such instruments. In this study, the content validity of the tests used was determined by three professionals in the field of Mathematics education including a peer Mathematics educator with more than ten years' experience in Mathematics teaching and marking, school Mathematics subject head and the departmental head of Mathematics and sciences. They were requested to moderate if the content does align with what is stipulated in the curriculum assessment guidelines, checking as well on the mark allocated to each question, the language used, and the content covered. Upon completion of moderation, the departmental head indicated his approval by putting a signed school departmental stamp where no changes were made on the original question papers and marking guidelines. In addition to that, I kept on consulting my supervisor and peers concerning emerging findings.

During the study, the examination of participants' documents, and recorded notes and audios from interviews contributed to validity. Throughout the research, I used self-examination to add credibility to the research. According to McMillan and Schumacher (2010) by posing difficult questions to himself or herself, a researcher assumes that he or she cannot be neutral, objective or detached. Therefore, there was an unavoidable element of subjectivity in the research. This study was validated using methodological and data triangulation. Methodological triangulation is beneficial in providing confirmation of findings, more comprehensive data, increased validity, and enhanced understanding of the studied phenomenon (Casey & Murphy, 2009). According to Casey and Murphy, (2009), methodological triangulation uses more than one kind of method to study a phenomenon. Thus, in the context of this study, the researcher used learners' workbook observations together with semi-structured interviews for methodological triangulation. Data triangulation was also used to validate the outcome of this study by involving the use of a variety of data sources, including time, space and persons, in the study (Patton, 1999). The findings of a study can be corroborated, and any weaknesses compensated for by the strengths of other data sources. This in turn increases the validity and reliability of the results. The data triangulation approach can be used in many sectors to strengthen conclusions about findings as well as reduce the risk of false interpretations (Patton, 1999).

In my research, triangulation contributed to the validity of the study as themes and ultimate conclusions were based on multiple sources of data. The techniques (interviews, observations, pre-test & post-test) used in my research assisted in ensuring triangulation. Member checking is one of the most crucial technique for establishing credibility (Guba & Lincoln, 1989). This was done to all the participants who were given their scripts after marking to check how they performed and if there were any errors in marking. The six learners interviewed were also given a chance to investigate the observation schedules and interview scripts. The learners confirmed the correctness of what they checked. I also used direct words from the participant without correcting grammar or tense to support validity of data.

3.8. ETHICAL ISSUES

It is unethical to place participants in a situation where they might be at risk or harmed either physically or psychologically because of their participation in a particular study. According to Resnik (2014) ethics is a method, procedure or perspective for deciding how to act, and for analysing complex problems and issues. Some ethical issues were observed to ensure the study was void of any unethical issues.

In the first place, permission was sought from the Department of Basic Education. The DBE issued a certificate (Appendix B) to allow the research to be carried out in their schools. The certificate instructed the researcher to submit a copy to the district director and the principal. The school principal, upon receiving a copy of the certificate from the department of education, granted me an approval letter (Appendix C) to proceed with my research study. The university Ethics Committee responsible for ethical clearances cleared the work and an ethical clearance certificate (Appendix A) was issued to allow the work to be done in an ethical manner. All concerned learners were given consent forms (Appendix E) for their parents to sign, allowing them to participate in the study. This process ensures that the rights, values, needs, and desires of each informant are respected, as argued by Resnik (2014). The parents of learners who consented to them taking part in the study also signed and returned consent letters indicating that they did accept and allow their children to take part in the research project. The purpose of the research was clearly explained to participants and their right to withdraw was communicated. They were made to realise that it was not obligatory for them to take part in the study. There was no form of falsification of the evidence, findings, and conclusions. For the anonymity of participants, pseudonyms were assigned to them. Raw data was kept and saved with the researcher for a minimum of five years in accordance with the American Psychology Association (APA) requirements (2011). The researcher would undertake all responsibilities regarding the conduct of the research, and this gave participants the confidence to respond freely.

3.9. SUMMARY TABLE OF THE RESEARCH PROCESS

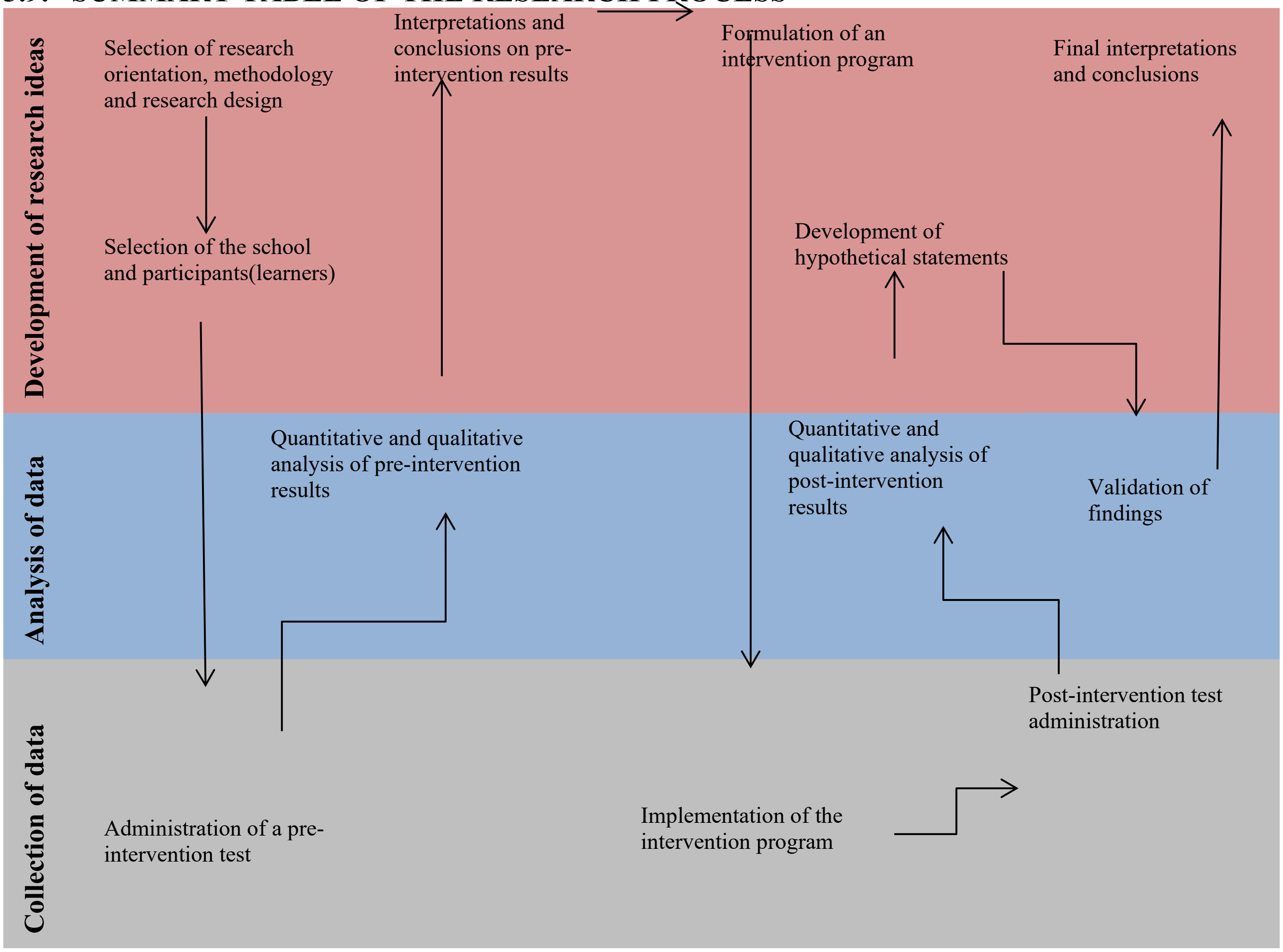


Figure 3: Methodological processes tree diagram

3.10. CONCLUSION

This chapter outlined the research design and methodology, namely, the case study approach; the methods of data collection employed; and detailed the research processused in this study. As the procedure was both qualitative and quantitative in nature, the methods of data collection used were consistent with the tenets of qualitative and quantitative research. Issues pertaining to reliability and validity of this research, as well as ethical considerations that guided me in the process were discussed in relation to the research site, the nature of participation and data collection methods. In the next chapter a detailed description of the findings and data analysis is presented.

CHAPTER 4

DATA ANALYSIS AND PRESENTATION

4.1. INTRODUCTION

The current chapter presents, interprets, and analyses data collected**.** The aim of this study was to investigate Grade 11 learners' problem-solving skills in Geometry under social cognitive conditions. In this study, learners' problem-solving skills were explored in different aspects of Geometry and geometric skills executed in the process of problem-solving. Descriptive statistics was used for quantitative analysis and thematic analysis was used for qualitative analysis. Data analysis was carried under three major headings: "learners' geometric skills demonstrated prior to intervention", "problem-solving and geometric skills development" and "learners' perceptions, experiences and challenges faced during the process of problem solving". The chapter provides necessary information and evidence to answer the three main research questions of this study.

4.2. LEARNERS' GEOMETRIC SKILLS PRIOR TO INTERVENTION

The pre-intervention test was designed to uncover learners' geometric skills prior to the intervention program. As such, it sought to find answers to the first research question; ***what geometric skills are demonstrated by selected learners prior to learning Grade 11 Geometry content?*** The data were analysed quantitatively using descriptive statistics and qualitatively using thematic analysis. The themes used in this study were predetermined and some emanated from the data.

4.2.1. General Performance of learners

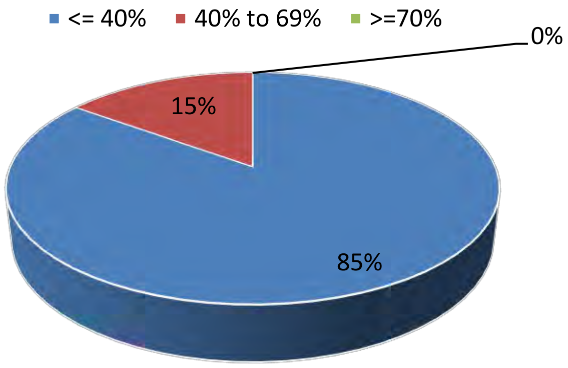


Figure 4: Pie chart showing percentages of learners per performance interval

This data provided me with an understanding of the kind of geometric skills learners possessed before being introduced to the four stages model of problem-solving. The criteria that were used in the selection of the six learners were also used to group learner performance into top-performance (>=70%), middle performance (40% to 69%) and under performance (< 40%). As indicated in Figure 4, very few (15%) of the learners who wrote the test managed to fall into at least the middle performers' category, while the majority (85%) of the learners fell in the category of underperformance. I regarded the under-performance by most of the learners as an indication that something was not right with the learners' geometric problem-solving skills hence it was necessary to look deeper into the learners' current geometric problem-solving skills. Thus, I further analysed the pre-intervention test results looking at different specific geometric skills as discussed next.

4.2.2. Pre-test results analysis on specific geometric skills

Here, the focus was on specific geometric skills that participating learners possessed prior to the intervention. The analysis was still centered on the performance of learners in the pre-intervention test by looking at different geometric problem-solving skills separately.

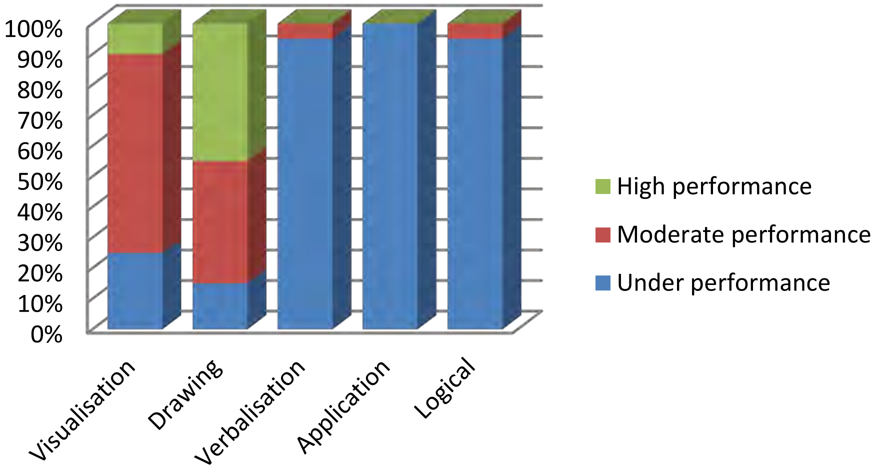


Figure 5: Composite bar chart showing percentage pre-test results on specific geometric skills

From Figure 5, the application skill was the worst skill to be displayed in the pre-intervention test as all the learners under-performed in applying this skill. Visualisation skills, however, showed outstanding application and moderate performance as most of the learners (65%) performed moderately. On high performance was the drawing skill that outweighed all other skills in terms of its application, as 45% of the learners highly performed under the drawing skill as compared to the other skills where high performance constituted at most 10%. However, under examination settings, drawing and visualisation skills are used to an end, whereby they can be applied but no marks awarded for their application. Marks are awarded for verbalisation, application, and logical skills applications. This shows that the results were a concern as the learners performed better in the skills where no marks are awarded, and performed poorly in the application of those skills where marks are awarded. This further confirmed the need for an intervention in the development of the geometric skills of the learners.

4.2.3. Learners' Geometric Skills as Displayed in the Pre-intervention test.

In this section, I looked at the work of the six learners that were selected from the class. I qualitatively analysed their scripts to identify how specific geometric skills were displayed in the pre-intervention test. In every instance, I could use extracts of learners' work from different categories (under-performers, moderate performers & high performers) unless both learners from a particular category did not display that skill.

Visual Skills

Visualisation skills include leaners' ability to recognise theorems or axioms, recognise properties of shapes, recognise different parts and lines in a diagram, as well as recognise different angles.

Learners' ability to recognise a theorem or axiom: Learners managed to apply the skill of recognition of theorems and axioms in different ways, even though none of the underperformers and one of the moderate performers managed to show the ability to recognise theorems and axioms in the test. Only three learners were able to recognise theorems and axioms. From Figure 6 below, Learner 2 (one of the moderate performers) and learner 7 (one of the high performers) recognised theorems and axioms when answering question 5. Learner 2 recognised theorems and axioms inappropriately four times. She could recognise theorems and axioms on the diagram, but these theorems were not used appropriately to answer the question at hand as indicated in Figure 6. Same applies to learner 7 who, on question 5, could recognise theorems or axioms correctly once and twice inappropriately. Inappropriate applications occurred when she could recognise some theorems and axioms but then apply them in a wrong context i.e., she could actually recognise a theorem or axiom right but without answering the question at hand. Just like learner 7, other learners did the same in recognition of theorems and axioms on question 5, as they could recognise theorems and axioms inappropriately more times than correctly. Like Makhubele (2012), I realised that there were many cases of inappropriate application of recognition of theorems and axioms, which was an indication that learners were finding it difficult to apply the knowledge of angles and theorems in solving riders (Makhubele, 2012). This also implied that the learners had an idea of the theorems and axioms but only had challenges with how to apply the skill of recognising theorems and axioms. Learners also had a tendency of forgetting the content learnt in previous grades as was observed by Gafoor and Kurukkan (2015) who note that the factors that make Mathematics difficult for learners to learn included difficulty in remembering the content learnt in the previous classes. Extracts from some learners' work were used to show how the learners were applying the skill of ability to recognise theorems and axioms as shown in figure 6. All in all, the skill was poorly applied, and this information equipped me with knowledge of how much intervention is needed to address the problem related to the application of the skill of recognising theorems and axioms.

Learner 02 **Learner 07**

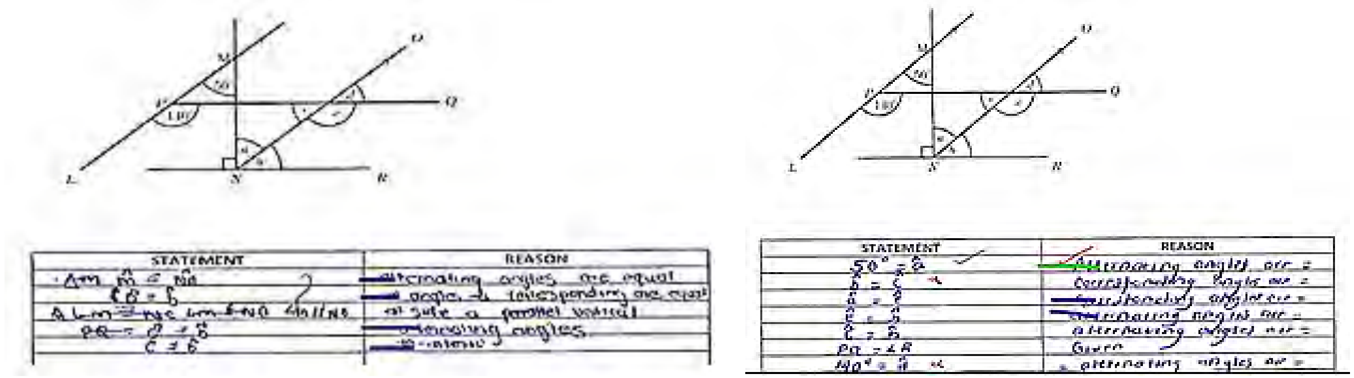
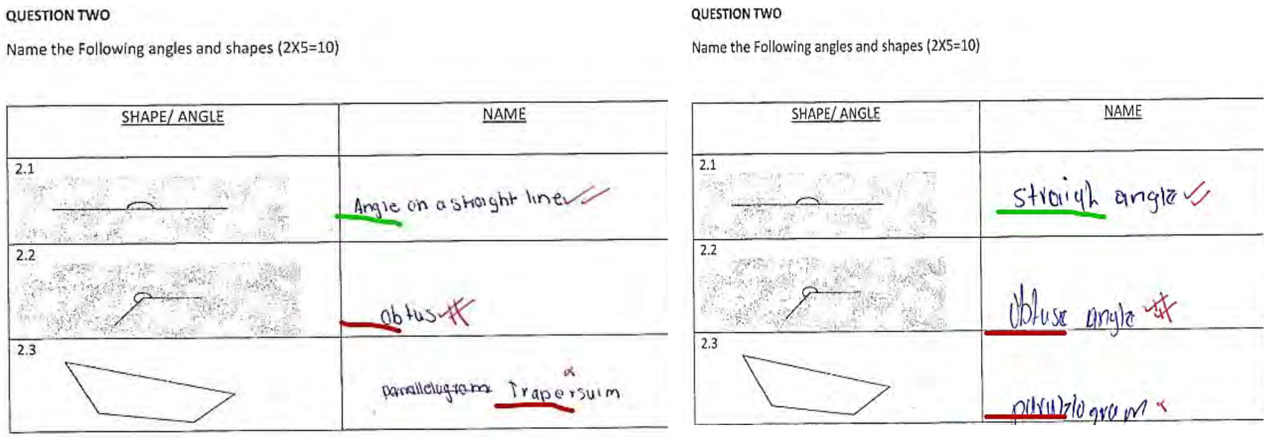


Figure 6: Extracts of learners' work on recognition of a theorem or axiom

Learners' ability to recognise properties of shapes: All the six learners could make attempts in recognising properties of shapes. Recognition of properties of shapes was realised in answering question 2 of the pre-test, whereby one was supposed to consider properties of a shape to be able to name the shape. As illustrated in Figure 7 below, Learner 2 recognised properties of shapes twice correct and once incorrect. In this case, the learner managed to recognise properties of two shapes correctly and managed to name them correctly, whereas in one case, she made a wrong judgment and ended up giving the shape a wrong name. Learner 12 could also recognise properties of shapes three times, one of which was a correct application and in two cases, the learner failed to give correct names to the shapes. To address the same issue, other learners could make accurate judgment in several cases and managed to name the three shapes correctly. Some managed to recognise properties of shapes very well in comparison to inappropriate and incorrect applications. All the learners could not identify a reflex angle and it shows they confused a reflex angle with an obtuse angle as most of the learners named the angle as obtuse.

This shows that there were more cases of correct recognition of properties of shapes than incorrect recognition, which implies that learners possessed some better knowledge on properties of shapes. Just like Makhubele (2015) who discovered in his study that learners were able to identify particular properties of shapes but not in logical order, I also realised that, in as much as learners could correctly recognise properties of shapes, they could, at times, confuse properties of shapes, ending up giving a shape an incorrect name.



Learner 02

QUESTION TWO

Name the Following angles and shapes (2X5=10)

SHAPE/ ANGLE	NAME
2.1	Angle on a straight line ✓✓
2.2	obtus ✗
2.3	parallelugram Trapersuim

Learner 12

QUESTION TWO

Name the Following angles and shapes (2X5=10)

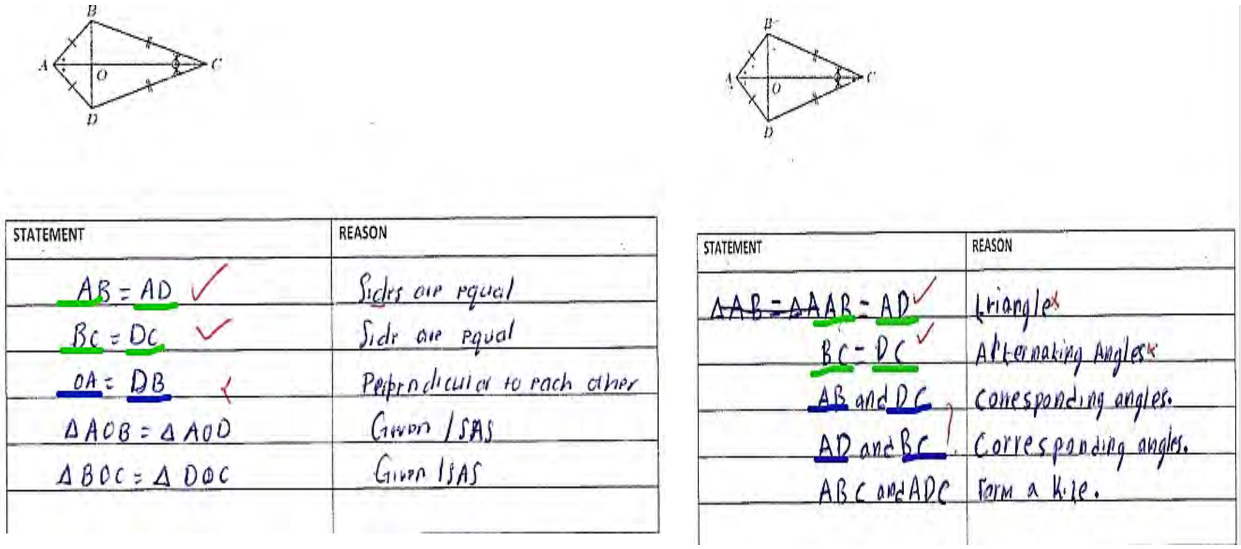
SHAPE/ ANGLE	NAME
2.1	straigh angle ✓✓
2.2	obtuse angle ✗
2.3	parallelogram ✗

Figure 7: Extracts of learners' work on recognition of properties of shapes.

From Figure 7, Learner 2 and 12's work was extracted to elucidate the ability of learners to recognise properties of shapes. Overall, the skill of recognition of properties of shapes was averaged and this gave me an idea that several things still needed to be done to make learners able to apply the skill of recognition of properties of shapes effectively. The information helped me to address issues of misconceptions when planning for the intervention such as the one mentioned above of the difference between obtuse and reflex angle.

Learners' ability to recognise different parts and lines in a circle or diagram: Learners from the underperforming category never showed any ability to recognise parts and lines in a circle or diagram. From Figure 8, Learner 7 recognised parts and lines twice correctly and thrice inappropriately. She could recognise two sets of equal lines on question 4 as she was attempting the question. She could also identify some lines from the diagram even though the effort could not help in answering the question. On the same question, Learner 17 managed to recognise eight lines on the diagram, four of which he used correctly in answering the question and the other four he misplaced in the process. Most learners could identify lines in the diagram but were having challenges to use them correctly to answer the question. Extracts from learner 7 and 17 (see Figure 8) were used to show how the learners displayed their ability to recognise lines and parts in a circle and diagram.

Learner 07 **Learner 17**



STATEMENT	REASON
AB = AD ✓	Sides are equal
BC = DC ✓	Side are equal
OA = DB	Perpendicular to each other
ΔAOB = ΔAOD	Given / SAS
ΔBOC = ΔDOC	Given / SAS

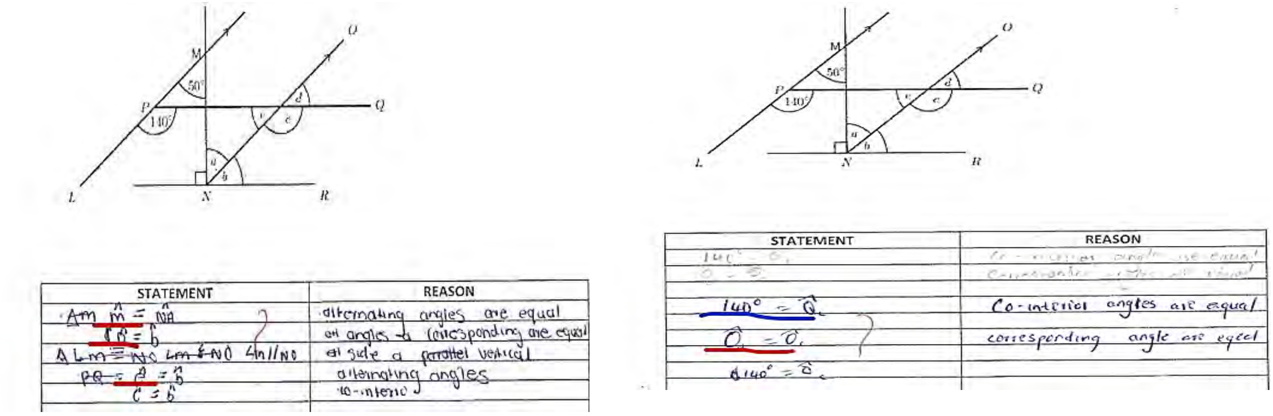
STATEMENT	REASON
~~ΔAB = ΔA~~AB = AD ✓	triangle ×
BC = DC ✓	Alternating Angles ×
AB and DC	corresponding angles.
AD and BC	Corresponding angles.
ABC and ADC	form a kite.

Figure 8: Extracts of learners' work on Recognition of different parts and lines in a diagram

Although in some cases learners could identify parts and lines on diagrams, they were having challenges to utilize the information, for example, Learner 7 on question 4 who managed to recognise line AB to be equal to line AD, which was a correct observation, but the reasoning given in answering the question to justify why AB is equals to AD was not valid. Such a problem was common in most of the learners. The skill of recognition of parts and lines in a circle and diagrams was not well displayed. This gave me a clue that the learners still needed some interference to address the areas that still needed attention in the application of the skill of recognition of lines and parts in circles and diagrams. The information also helped me to identify how learners fail to utilise the information and ideas they must do problem-solving in general and particularly in Geometry.

Learners' ability to recognise different angles: The skill of recognition of different angles was applied by all the learners to answer some questions in the pre-test. Figure 9 shows that Learner 2 was able to recognise four angles in her answering even though she used them incorrectly in answering question 2 and question 5. Learner3 as well, identified four different angles but used one inappropriately and three incorrectly, to answer the questions. Learner 11 applied the skill eight times, four of which were correctly applied, once inappropriately and in three cases it was applied incorrectly. Other learners also made use of the skill of recognition of different angles in the pre-test, and they utilised the skill in more incorrect cases than correct and inappropriate ones. Learner 17 applied the skill once correctly, once inappropriately and three times incorrectly. Learner 2's and 3's work was used to illustrate how the skill of recognition of different angles has generally been applied by learners in the pre-test (see Figure 9).

Learner 02 **Learner 03**



Learner 02

STATEMENT	REASON
ΔAm $\hat{m} = \hat{NA}$	alternating angles are equal
$\hat{B} = \hat{b}$	co angles to corresponding are equal
ΔLm = NO ~~Lm~~ ≠ NO Ln//NO	co side a parallel vertical
~~PQ~~ = $\hat{a} = \hat{b}$	alternating angles
$C = \hat{b}$	co-interio

Learner 03

STATEMENT	REASON
140° = O	Co-interior angles are equal
O = O	Corresponding angles are equal
140° = $\hat{Q}$	Co-interior angles are equal
$\hat{O} = \hat{O}$	corresponding angle are eqeal
140° = $\hat{O}$	

Figure 9: Extracts of learners' work on Recognition of different angles

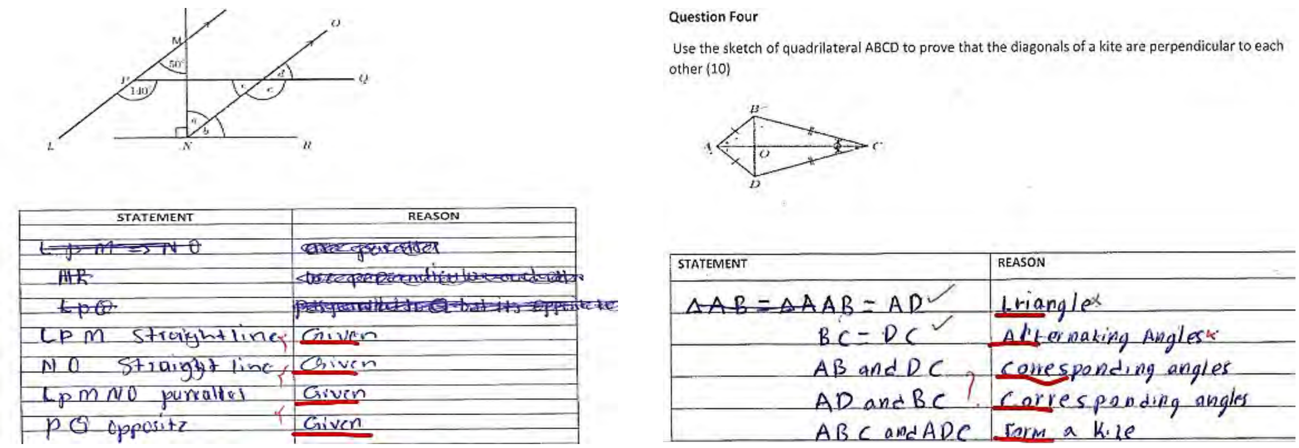
As illustrated in Figure 9, the learners exhibited some ability to recognise angles from diagrams and relate the angles using theorems and axioms even though some learners could use them incorrectly to answer the questions. This was indicated by a few cases when learners were displaying correct applications compared to incorrect and inappropriate applications. Since there were more cases of incorrect and inappropriate applications than correct application, it was an indication that there was still a gap that needed to be filled in as far as recognising angles is concerned. Just like Makhubele (2012), I realised the learners were having a challenge in applying knowledge of angles in answering questions and this provided me with background information as well on how I was going to plan my intervention on this section.

Verbal Skills

In Geometry, one kind of helpful verbalisation skill when doing problem-solving is stating a rule or giving a reason for an action. Verbal skills involve giving verbal descriptions, naming objects and giving definitions.

Learners' verbal description skills: Ability to verbally describe geometric concepts is a skill that was applied by all learners and was required to answer question 5 of the pre-test. As shown in Figure 10, Learner 12 incorrectly applied the verbal description skill eight times. In every case the learner tried to apply the skill he could do it incorrectly. Learner 17 likewise did apply the same skill incorrectly in nine situations (see Figure 10). The other learners made use of the skill and there were more incorrect and inappropriate cases of application than correct application just like is illustrated in Figure 10 below.

Learner12 **Learner 17**



STATEMENT	REASON
~~∠PM = NO~~	~~[illegible]~~
~~MR~~	~~[illegible]~~
~~∠PQ~~	~~[illegible] that its opposite to~~
∠PM Straightline ✓	Given
NO Straight line ✓	Given
∠PM NO parallel	Given
PQ opposite ✓	Given

Question Four

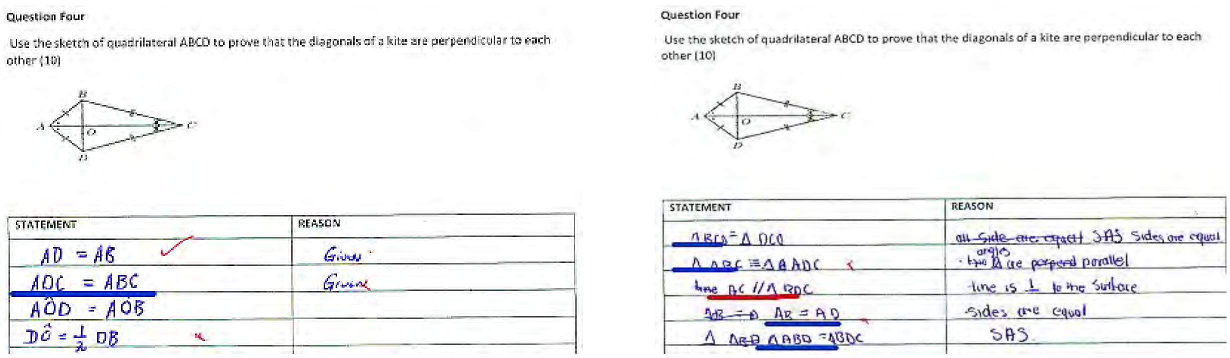
Use the sketch of quadrilateral ABCD to prove that the diagonals of a kite are perpendicular to each other (10)

STATEMENT	REASON
~~ΔAB = ΔA~~AB = AD ✓	triangle ✗
BC = DC ✓	Alternating Angles ✗
AB and DC	corresponding angles
AD and BC	Corresponding angles
ABC and ADC	form a kite

Figure 10: Extracts of learners' work on Learners' verbal descriptions skills

As displayed in Figure 10, the skill of verbal descriptions was poorly displayed as there were too many cases of incorrect and inappropriate applications in comparison to correct applications. This was a sign that learners were having challenges in making verbal descriptions in problem-solving. The learners knew that it was a requirement to support every statement with a reason in geometric problem-solving, but it was unfortunate that the reasoning was not valid in most cases. This called for my involvement in the ways learners were supposed to reason in Geometry when supporting geometric statements. This information also helped me to realise that more attention was needed on the geometric foundation of the learners as geometric foundation is based on theorems and axioms, which are basically used for verbal descriptions in geometric problem-solving.

Learners' skills of naming shapes, objects, and parts: This skill was applied in several cases by all the learners in answering question 2. Learner 11(see Figure 11) applied the skill four times, of which three applications were correctly done and one was inappropriate. Learner 2 also applied the skill of naming shapes and parts in eight cases. She named two shapes correctly and six shapes were named inappropriately. This skill was also made use of by other learners moderately in relation to other applications.



Learner 11

Question Four

Use the sketch of quadrilateral ABCD to prove that the diagonals of a kite are perpendicular to each other (10)

STATEMENT	REASON
AD = AB ✓	Given
ADC = ABC	Given
$A\hat{O}D = A\hat{O}B$	
$D\hat{O} = \frac{1}{2} DB$	

Learner 2

Question Four

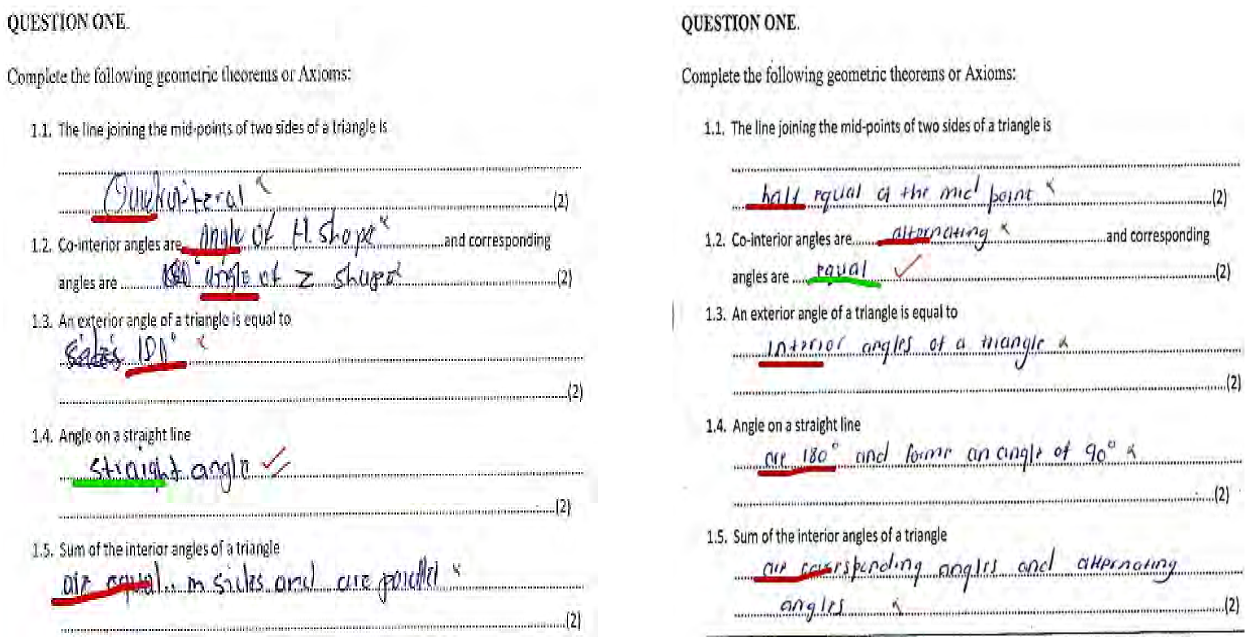
Use the sketch of quadrilateral ABCD to prove that the diagonals of a kite are perpendicular to each other (10)

STATEMENT	REASON
ΔBCA = Δ DCO	~~all Side are equal~~ SAS Sides are equal
Δ ABC ≡ Δ A ADC	· two angles Δs are ~~perpend~~ parallel
line BC // Δ BDC	line is ⊥ to the Surface
~~AB = A~~ AB = AD	Sides are equal
Δ ~~ABD~~ ΔABD = ΔBDC	SAS.

Figure 11: Extracts of learners' work on Naming of shapes, objects, and parts

The skill of naming shapes, objects, and parts (see Figure 11) was moderately applied as there were a reasonable number of correct applications as compared to incorrect and inappropriate applications. This skill was better displayed when the questions needed a direct name of a shape like rectangle, than when a question required one to name a shape using mathematical symbols like ΔABC meaning triangle ABC. Such information equipped me with an idea of the knowledge gaps that learners have in the skill of naming shapes and objects, which I used for planning the intervention. This information also shows that the learners needed more attention in using geometric ways of naming shapes than using general and direct names that were used in lower grades like in primary and lower secondary grades.

Learners' Defining and Stating in own words skills: Definitions and stating in own words was a skill that was needed to address question 5 in the pre-test. Every learner employed this skill of defining and stating in their own words except one learner. As shown in figure 12, Learner 12 made some theoretical definitions six times but only one was done correctly, and the rest were incorrect. It was the same case for Learner 7. Other learners also applied the skill of defining and stating in their own words many times, but they could apply the skill for a very limited number of times correctly and for several cases incorrectly.



Learner12

QUESTION ONE.

Complete the following geometric theorems or Axioms:

1.1. The line joining the mid-points of two sides of a triangle is

Quadrilateral ✗ (2)

1.2. Co-interior angles are angle of H shape ✗ and corresponding angles are 180° angle of Z shape (2)

1.3. An exterior angle of a triangle is equal to

Sides 180° ✗ (2)

1.4. Angle on a straight line

straight angle ✓ (2)

1.5. Sum of the interior angles of a triangle

are equal in sides and are parallel ✗ (2)

Learner 7

QUESTION ONE.

Complete the following geometric theorems or Axioms:

1.1. The line joining the mid-points of two sides of a triangle is

half equal at the mid point ✗ (2)

1.2. Co-interior angles are alternating ✗ and corresponding angles are equal ✓ (2)

1.3. An exterior angle of a triangle is equal to

interior angles of a triangle ✗ (2)

1.4. Angle on a straight line

are 180° and form an angle of 90° ✗ (2)

1.5. Sum of the interior angles of a triangle

are corresponding angles and alternating angles ✗ (2)

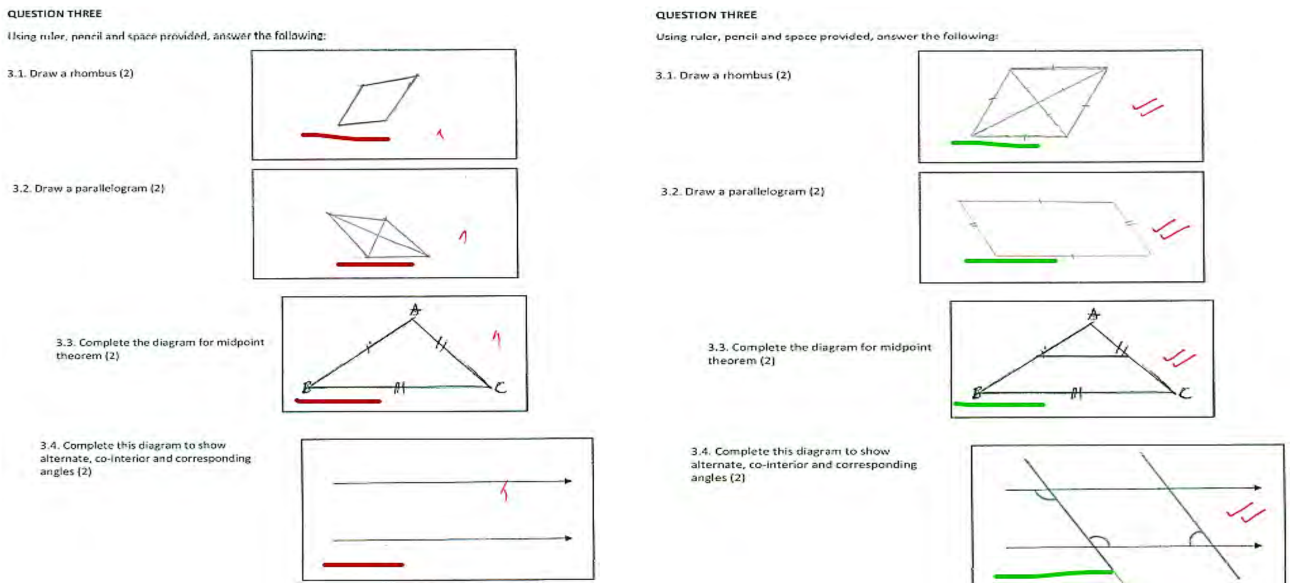
Figure 12: Extracts of learners' work on Definitions and Stating in own words

As displayed in Figure 12, definitions and stating in own words as a skill was poorly applied. The poor application might be because the learners had already forgotten the geometric content they had learnt from previous grades, or it was a recurring problem from previous grades. Learners could of course put in their own words but incorrectly in most cases. There were very few cases of correct application of the skill of defining and stating in own words as compared to incorrect applications. Some of the incorrect definitions showed that some learners had no clue on geometric theorems and axioms. This information pointed back to the geometric foundation of the learners that it needed more devotion, thus, helping me in my planning of the intervention, especially of foundation Geometry. This helped me much in that more attention was demanded to address the gap on pre-knowledge on the aspect of definitions and stating in own words during geometric problem-solving.

Drawing Skills

Through drawing, one can group together information that is needed and avoid searching for the elements needed to make problem-solving interpretations and decisions. Drawing skill displayed by learners in the pre-test was only the construction of diagrams based on properties.

Learners' skills on constructing diagrams based on its properties: The skill was applied by all the learners in question 3, with Learner 3 (see Figure 13) applying the skill only once correctly, and four times incorrectly. As well, Learner 11 applied the skill four times correctly and once incorrectly on the same question. The skill was applied in the same manner by other learners, where some could apply the skill correctly in more cases than the incorrect cases, and others had it the other way round. Learner 3's and 11's work was used to represent how learners were normally applying the skill on constructing diagrams based on its properties.



Learner 3 **Learner 11**

QUESTION THREE

Using ruler, pencil and space provided, answer the following:

3.1. Draw a rhombus (2)

3.2. Draw a parallelogram (2)

3.3. Complete the diagram for midpoint theorem (2)

3.4. Complete this diagram to show alternate, co-interior and corresponding angles (2)

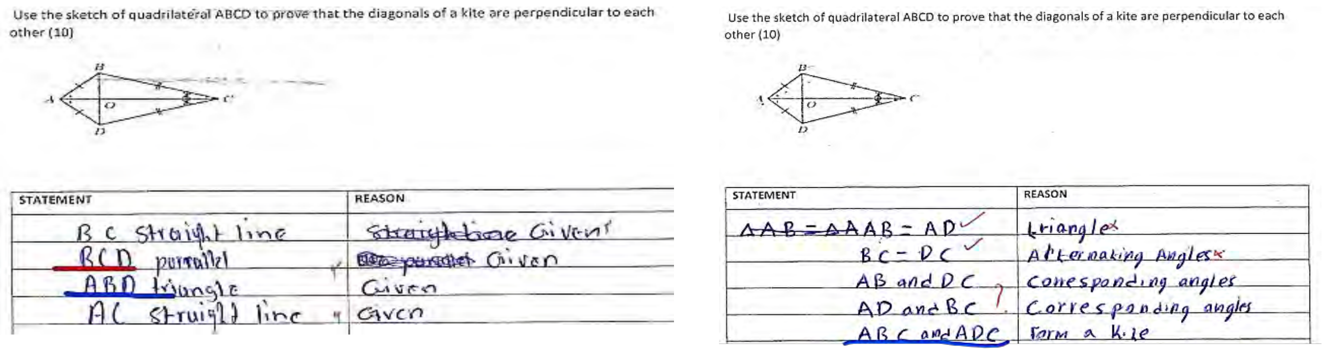
Figure 13: Extracts of learners' work on constructing diagrams based on its properties.

The skill of constructing diagrams based on its properties as shown in Figure 13 was averagely applied as there were a reasonable number of correct application cases of the skill. Some learners were showing ideas of properties of shapes while others were showing to be struggling. This might be because some learners had forgotten the properties of different geometric shapes. This helped me much in the groundwork I did for the intervention as I had obtained some information concerning the degree to which the learners could apply the skill of constructing diagrams based on their properties. The skill of constructing diagrams based on its properties was better displayed by learners compared to other skills. Even though the learners could fail to apply the skill correctly in every part of the question, all the parts of the question managed to be correctly answered by at least one learner. This shows the learners still had the ability to apply the skill of constructing diagrams based on its properties even though there were some factors that could have affected their application skills in one way or another. The information equipped me with ideas that not much was needed in the intervention to address the skill of constructing diagrams based on its properties especially on pre-knowledge.

Logical Skills

Logical skills normally involve one’s ability to relate situations and make logical conclusions, especially from unfamiliar situations. The skills applied by learners from this dimension include recognition of differences between geometric objects, classification of shapes based on their properties, and developing logical proofs.

Learners’ Recognition of differences and similarities between geometric objects skills: Question 4 was demanding for the application of this skill whereby one needed to be able to recognize the differences and similarities between triangles in-order to identify congruent triangles in the process. Learner 12 (see Figure 14) tried to apply the skill but only to find that he applied it once inappropriately and once incorrectly. Learner 17 as well applied the skill inappropriately twice and once. Application of the skill of recognition of differences and similarities between geometric objects was also witnessed with other learners and most cases of application were incorrect and inappropriate. Extracts of Learner 12’s and 17’s work on Recognition of differences and similarities between geometric objects was used to demonstrate instances when the skill could be applied (see Figure 14)



Learner 12

Use the sketch of quadrilateral ABCD to prove that the diagonals of a kite are perpendicular to each other (10)

STATEMENT	REASON
B C straight line	~~straight line~~ Given
BCD parallel	~~[illegible] parallel~~ Given
ABD triangle	Given
AC straight line	Given

Learner 17

Use the sketch of quadrilateral ABCD to prove that the diagonals of a kite are perpendicular to each other (10)

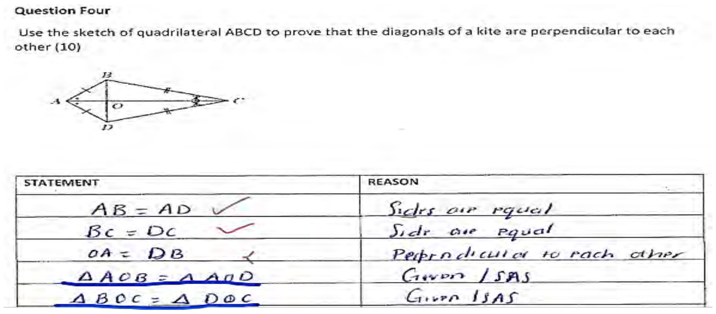
STATEMENT	REASON
~~ΔAB = Δ~~AAB = AD ✓	triangle ✗
BC = DC ✓	Alternating Angles ✗
AB and DC	Conesponding angles
AD and BC ?	Corresponding angles
ABC and ADC	Form a Kite

Figure 14: Extracts of learners’ work on Recognition of differences and similarities between geometric objects

Figure 14 above shows that learners were generally not able to correctly apply the skill of recognition of differences and similarities between geometric objects. In other words, the skill of recognition of differences and similarities between geometric objects was poorly displayed. This was indicated through incorrect application of the skill with only inappropriate and incorrect applications. This information helped much in preparing my intervention as the skill of recognition of differences and similarities between geometric objects proved to be one of the problematic areas and demanding more attention in the intervention. I understood that some learners could identify the geometric objects from a given diagram and at times could relate them correctly but do that in vain as they failed to answer the question at hand. This, therefore, needed attention in the intervention to avoid learners using their effort unfruitfully.

Learners' Classification of shapes by their properties skills: Only one learner succeeded in displaying this skill in question four of the pre-test. As indicated in Figure 15, Learner 7 applied the skill twice inappropriately. That is to say, the learner had the ability to classify geometric objects but without answering the question and this happened in two cases. For example, the learner could correctly identify two congruent triangles, but this argument was put in the wrong context to answer an inappropriate question. Such information helped me in preparing the intervention by noting areas that needed more attention such as the skill of classification of shapes by their properties.

Learner 07



Question Four

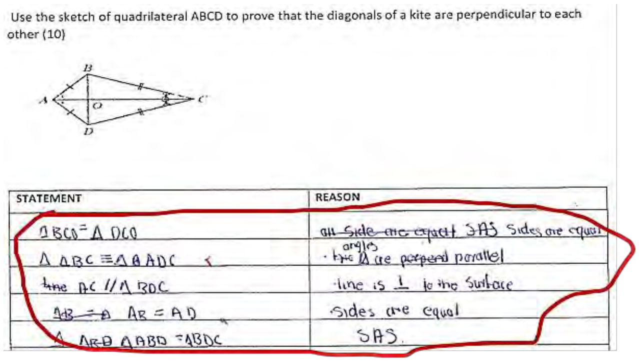
Use the sketch of quadrilateral ABCD to prove that the diagonals of a kite are perpendicular to each other (10)

STATEMENT	REASON
AB = AD ✓	Sides are equal
BC = DC ✓	Side are equal
OA = DB	Perpendicular to each other
ΔAOB = ΔAOD	Given / SAS
ΔBOC = ΔDOC	Given / SAS

Figure 15: Extracts of learner's work on Classification of shapes by their properties

Learners' Development of logical proofs skills: Logical proofs were required of learners in question 4. Just like is shown in Figure 16, all the learners made one attempt to make a logical proof to answer question 4 but none of them managed to make a correct logical proof. All the proofs made were incorrect. As was also discovered by Chigonga (2016) that learners are possibly operating at Van Hiele's level 3 or less of Geometry thinking, and as a result, proof development continues to be a problematic area for learners. I also comprehended that geometric proof development was a serious problem to the learners hence more responsiveness was needed on this area of geometric problem-solving from my side as well as learners' side.

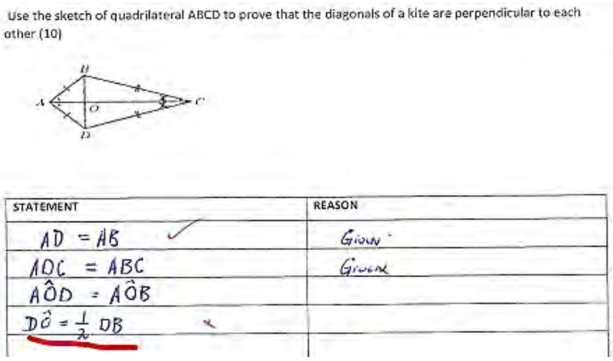
Learner 02



Use the sketch of quadrilateral ABCD to prove that the diagonals of a kite are perpendicular to each other (10)

STATEMENT	REASON
ΔBCD = Δ DCO	~~all Side are equal~~ SAS Sides are equal
Δ ΔBC ≡ Δ ~~B~~ ADC	· the angles Δ are ~~perpend~~ parallel
~~line~~ AC // Δ BDC	· line is ⊥ to the Surface
~~AB = Δ~~ AB = AD	· Sides are equal
Δ ~~ABD~~ ΔABD = ΔBDC	SAS.

Learner 11



Use the sketch of quadrilateral ABCD to prove that the diagonals of a kite are perpendicular to each other (10)

STATEMENT	REASON
AD = AB ✓	Given
ADC = ABC	Given
$A\hat{O}D = A\hat{O}B$	
$D\hat{O} = \frac{1}{2} DB$ x	

Figure 16: Extracts of learners' work on Development of logical proofs

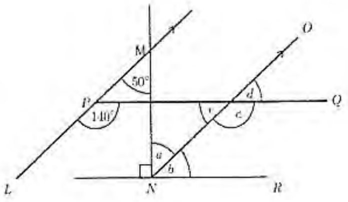
Application Skills

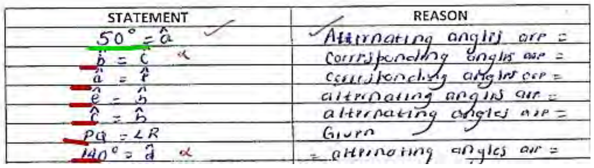
The skill of application is all about being able to use known ideas correctly from one situation into another different situation as a problem-solving means. The skill of ability to apply theorems was commonly used by learners in the pre-test. The skill included by the learners under application skills was the ability to apply theorems and axioms in problem-solving.

*Learners' ability to apply theorems and axioms in problem-solving***:** Learners observed prospered in displaying this skill of applying theorems and axioms in problem-solving in answering question 5. Learner 12 (see Figure 17) applied theorems & axioms in eight incorrect cases. Learner 17 also correctly applied the theorems and axioms twice, once inappropriately and seven times incorrectly. This skill to apply theorems and axioms in problem-solving was also displayed by other learners, and they were all applying the skill more incorrectly and inappropriately than correctly.

From Figure 17 below, extracts from Learner 7 and 17 are displayed and it is clear that the learners were failing to apply theorems in problem-solving. The learners were showing that they have little, if any clue about theorems, let alone their application. This is shown by the number of incorrect as well as inappropriate applications of the skill of theorem and axiom application. Most of the learners had an idea of how to present their work in statement and reason format but most of the statements and reasons were incorrect and invalid. This therefore shows that the skill was poorly applied, hence the need for mediation to address the issue. The information from the analysis helped me to identify areas of need in the application of theorems and axioms by learners. I appreciated, from the analysis that the axioms and theorems learnt from the previous grades still needed to be addressed before the current content is taught, as learners showed to be almost clueless of the theorems and axioms together with their application. The information also helped me in planning my intervention on this skill with its application. Such a situation was encountered by (Ndlovu & Mji, 2012) in their research, where some learners ironically did not have the presence of mind to apply theorems to show equal angles. This information helped me to plan a strong intervention as much needed to be done to address such issues.

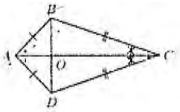
Learner 7

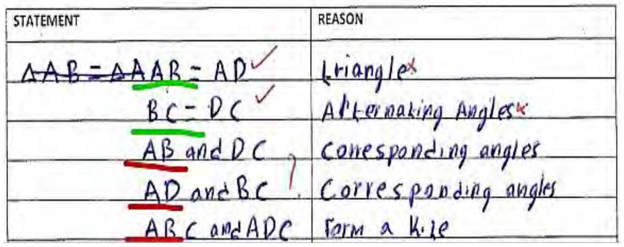




STATEMENT	REASON
$50° = \hat{a}$ ✓	✓ Alternating angles are =
$\hat{b} = \hat{c}$ ✗	Corresponding angles are =
$\hat{a} = \hat{f}$	corresponding angles are =
$\hat{e} = \hat{b}$	alternating angles are =
$\hat{c} = \hat{b}$	alternating angles are =
PQ = ∠R	Given
$140° = \hat{d}$ ✗	= alternating angles are =

Learner 17





STATEMENT	REASON
~~ΔAB = ΔA~~ AB = AD ✓	triangle ✗
BC = DC ✓	Alternaking Angles ✗
AB and DC	Conesponding angles
AD and BC	Corresponding angles
ABC and ADC	form a kite

Figure 17: Extracts of learners' work on application of theorems and axioms in problem-solving.

Summary of pre-test findings

The pre-test was used to check which skills learners had before the intervention and this was meant to answer the research question "***What geometric skills are demonstrated by selected learners prior to learning Grade 11 Geometry content?"*** The research question was answered through the analysis of the pre-test results, where I looked at whether the skill has been applied or not and if applied, I had to look at the degree and nature of its application.

There are three major concerns noted from this test that informed me to design an intervention to address the geometric problem-solving skills of the learners. First, I understood that some skills were not applied at all, and in all the attempts made by the learners in applying different geometric skills, most of the learners applied the skills incorrectly and inappropriately. In other words, most of the skills were poorly applied as has been alluded to by the analysis above. Just like the findings of Adolphus (2011), I discovered that there was an element of learners having poor foundation in geometric problem-solving. This indicated that something needed to be done to ensure the correct application of skills in problem-solving. The second concern was that some learners failed to even attempt to apply the skills that other learners could apply in problem-solving. The third concern was that most of the correct attempts made were more on drawing and visualisation skills than on verbalisation, application, and logical proofs. This was of much concern since, in summative assessment activities in Geometry, visualisation and drawing are used to an end whereby marks are only awarded for verbalisation, application and logic. Hence, I had confirmation that truly, an intervention was needed, especially in developing the geometric skills of the learners, with the aim to increase the number of cases in which the learners could correctly apply the skills as well as improving the number of attempts by everyone in solving geometric problems. The other reason to balance the correct application of skills across the board, i.e. making sure learners do correct applications of skill in drawing, visualisation, verbalisation, application, as well as in logical proofs.

4.3. USING POLYA'S MODEL IN PROBLEM-SOLVING AND GEOMETRIC SKILLS DEVELOPMENT

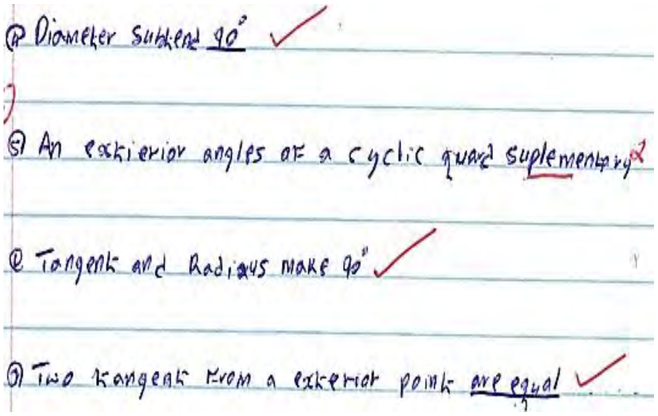
This section presents the data that was collected and analysed through observations of the selected grade 11 class when using Polya's model in problem-solving and geometric skills development. Document analysis was made of the exercise books of 6 purposively selected learners from this grade 11 class. The data was used together with the data from the post-intervention test to answer the second research question: ***How can the use of Polya's model enable or constrain the development of problem solving and geometric skills in a selected Grade 11 Mathematics class?*** The four stages model was used to teach different concepts of Geometry in a social cognitive classroom as indicated below. The use of the model in addressing different geometric concepts was done following this sequence:

Part 1- Introduction to the Geometry theorems,
Part 2 - Proving of examinable theorems,
Part 3 - Solving of riders, and
Part 4 - Dealing with logical geometric proofs.

After the model was utilised, a post- intervention test was given to see if the model had enabled or restricted the problem-solving and geometric skills of the learners. The post-intervention test was structured in the same way as the pre-intervention test, testing geometric skills constituting the 5 groups of skills which are visualisation, verbalisation, drawing, application, and logical skill. Due to COVID 19 restrictions, 2 learners were the maximum that could make a group. Therefore, the learners were logically paired from the beginning to enhance peer collaboration. In most cases, a learner was recommended to sit alone so we could only make pairs when need arises during peer collaborations.

Part 1: the social cognitive teaching and learning practices that were common at this stage of the intervention were flipped classroom and modelling. These practices were meant to promote attention and retention by the learners. The learners were tasked to research on the theorems of Geometry before the theorems were discussed in the classroom. From the 10 pairs, we managed to get all the theorems that are learnt at Grade 11 plus those learnt at Grade 10 and lower classes. Socratic questioning strategy could then be applied to check for understanding i.e. to identify misconceptions and provide corrective feedback and to invite dialogue, to help students develop a better understanding of themselves and their progress, share their deeper thinking and to make deeper connections within the content. The teacher was compelled to use lectures and demonstrations at some point during this introductory phase for clarifications and illustrations. Through a thorough discussion of the theorems, most learners were able to write the theorems using their own words in an individual short test given as illustrated in Figure 18. The geometric skills that were most developed at this phase of the intervention are verbalisation, visualisation and drawing skills. This formed the foundation for geometric problem-solving, since in geometric problem-solving every statement must be supported by a reason which, in other words, is a theorem. At the same time, learners were supposed to be able to identify theorems in the form of diagrams so that in any given geometric diagram, they can disintegrate the diagram into different separate theories for problem-solving.

Learner 2



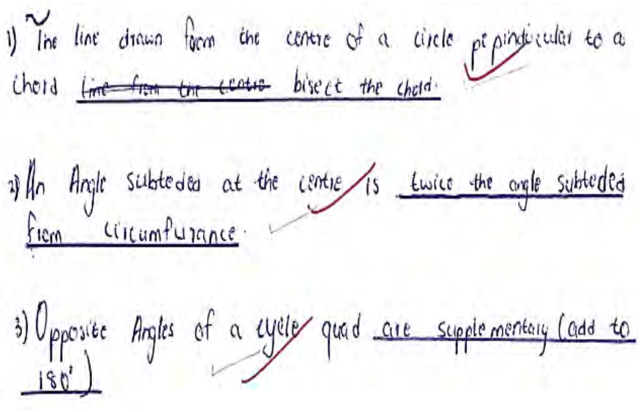
Diameter subtend 90°

An exterior angles of a cyclic quad suplementary

Tangent and Radius make 90°

Two tangent from a exterior point are equal

Learner 3



1) The line drawn form the centre of a circle perpindicular to a chord ~~line from the centre~~ bisect the chord.

2) An Angle subteded at the centre is twice the angle subteded from circumfurance.

3) Opposite Angles of a cycle quad are supplementary (add to 180°)

Figure 18: Learners work on writing and completing theorems in own words

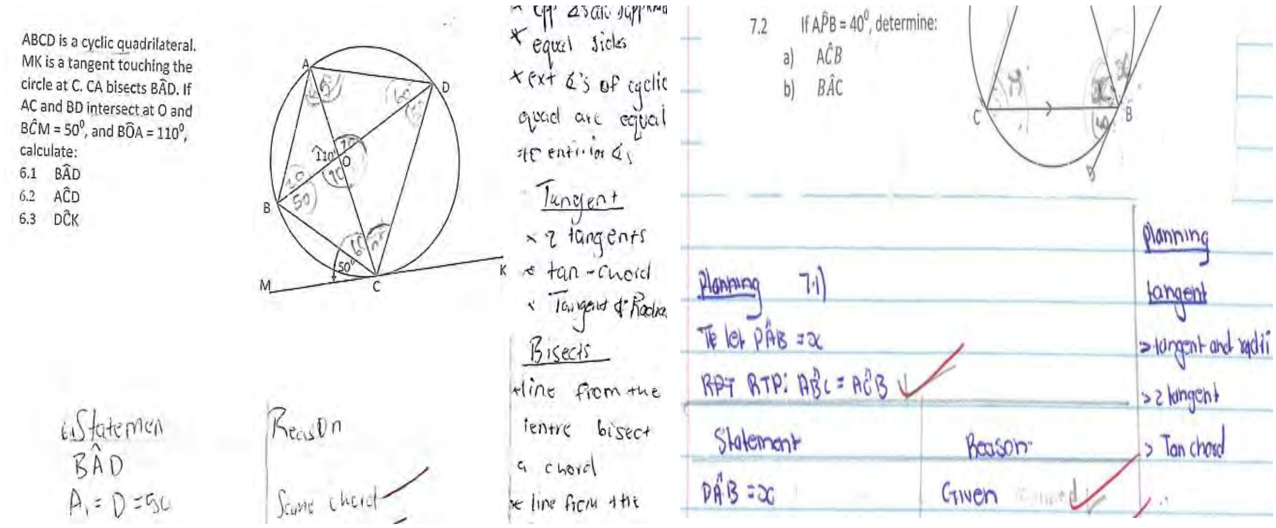
Part 2: According to the Mathematics CAPS document, there are some theorems that the learners must be able to prove (DBE, 2011). The teacher used a similar approach to the teaching and learning of this concept respecting the elements of Social cognitive theory. Different pairs of learners were given separate theorems to go and research on how to prove them. Presenters from different groups were welcome to make presentations on how to make proof of different theorems. The floor had to ask for clarifications and the other group members were supposed to support their peer – the presenter. The teacher could only come in when the class got stuck at a certain point or where a need for deeper clarifications and demonstrations arises. In this case, the teacher was focusing much on learner attention and reproduction, since they were supposed to reproduce the same steps involved in proving the theorems. After a thorough discussion of the concept, an individual activity was given, where the teacher could move around looking and noting down some misconceptions. Drawing, visualisation, verbalisation, application, and logical skills were all learnt during this phase. In proving of theorems, learners were expected to: make some constructions and put additional information on diagrams (drawing skills); identify different theories in each diagram (visualisation skills); support every statement with a reason (verbalisation skills); apply different theories to support different statements as well as any other mathematical operations (application skills); and make logical deductions (logical skills). During problem-solving, the learners were following the four stages model. However, many learners were not applying the fourth stage (reflection) of the model in most cases.

All the theorems for proving listed in the CAPS Document were learnt and tested for in the activity. Minor misconceptions were addressed as the teacher was moving around. Two major misconceptions noted were:

- Most learners failed to state the Required to prove (RTP) once the letters are changed in the diagram which is evidence of lack of mutual understanding of the concept of RTP.
- Most learners were unable to give the last supporting reason after proving a theorem which was supposed to be the converse of the Required to prove (RTP).

The teacher had to go an extra mile elucidating on the two aspects of proofs. A further individual activity was given as remedial work to those who did not do well in that area and others were given an extra activity of the same theorems by just changing the letters and how the diagram is presented.

Part 3: at this point most of the learners were shown to have gained some interest in the topic as was indicated by the learners during the follow-up interviews with the six learners. The solving of riders proved to be the most interesting section of the topic. The most common social cognitive teaching and learning practices during this phase involved peer coaching and collaboration as well as gamification. These practices were meant to instill retention and motivation in the learners. Just like in part 2, all the geometric skills were required in solving of riders. Advanced verbalisation, application and logical skills were demanded at this point. By the end of this phase, most of the learners had developed much of the geometric and problem-solving skills. During problem-solving, just like in part 2, the learners were making use of the four stages model in problem solving. The only concern picked as the teacher was making observations and during document analysis is that most learners were ignoring the fourth stage of the four stages, which is the "look back" stage.



Learner 12

ABCD is a cyclic quadrilateral. MK is a tangent touching the circle at C. CA bisects $B\hat{A}D$. If AC and BD intersect at O and $B\hat{C}M = 50^0$, and $B\hat{O}A = 110^0$, calculate:

6.1 $B\hat{A}D$

6.2 $A\hat{C}D$

6.3 $D\hat{C}K$

6.1 Statemen | Reason

$B\hat{A}D$

$A_1 = D = 50$ | Same chord

opp ∠s are supp
equal sides
ext ∠'s of cyclic quad are equal
to entrior ∠s

Tangent
2 tangents
tan - chord
Tangent & Radius

Bisects
line from the centre bisect a chord
line from the

Learner 2

7.2 If $A\hat{P}B = 40^0$, determine:

a) $A\hat{C}B$

b) $B\hat{A}C$

Planning 7.1)

let $P\hat{A}B = x$

RTP: $A\hat{B}C = A\hat{C}B$

Statement	Reason
$D\hat{A}B = x$	Given

Planning

tangent

> tangent and radii

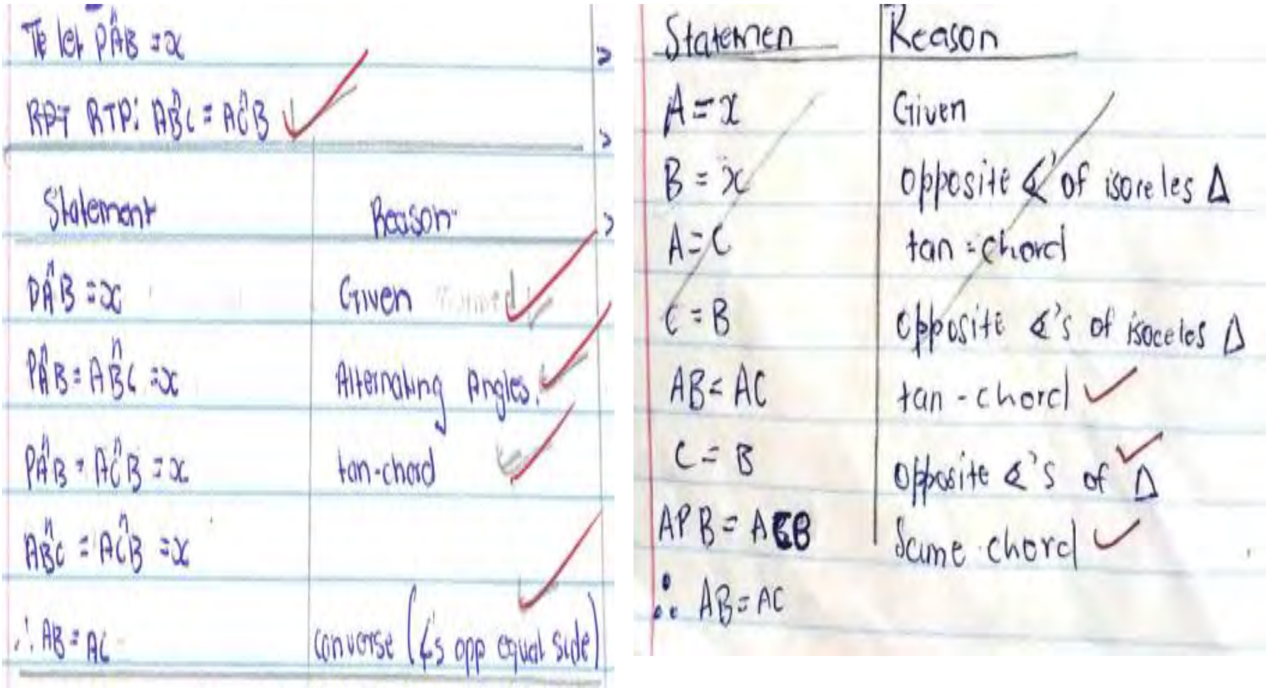
> 2 tangent

> Tan chord

Figure 19: Learner's work using four stages model

As shown in Figure 19, most learners could apply all the other three stages of the model but ignored the “look back” stage.

Part 4: this was the last concept to be addressed and it proved to be the most complicated aspect of the topic. This was a part dealing with geometrical logical proofs. It is at this point when a very critical skill emerges, which is required to prove (RTP) Finding. It was very rare for a learner to make a logical deduction without defining the RTP. Peer coaching and collaboration were the most utilised social cognitive learning practices aiming much at retention and motivation. Just like in other phases, the learners were making use of the four stages model with stage 4 still scarcely applied. All the geometric skills were demanded during this phase. Some learners were still having challenges to come up with a correct RTP and to support the last statement with the converse reason for the RTP.



Learner 2

Te let $P\hat{A}B = x$

~~RPT~~ RTP: $A\hat{B}C = A\hat{C}B$ ✓

Statement	Reason
$P\hat{A}B = x$	Given ✓
$P\hat{A}B = A\hat{B}C = x$	Alternating Angles ✓
$P\hat{A}B = A\hat{C}B = x$	tan-chord ✓
$A\hat{B}C = A\hat{C}B = x$	✓
∴ AB = AC	Converse (∢s opp equal side)

Learner 12

Statemen	Reason
~~A = x~~	~~Given~~
~~B = x~~	~~opposite ∢ of isoceles Δ~~
~~A = C~~	~~tan - chord~~
~~C = B~~	~~opposite ∢'s of isoceles Δ~~
AB = AC	tan - chord ✓
C = B	opposite ∢'s of Δ ✓
APB = ACB	same chord ✓
∴ AB = AC	

Figure 20: Learner’s work making logical deductions

As illustrated in Figure 20, learner 2 could define her RTP well and support her final answer with a converse of the theorem as a reason. The case was different for learner 12 who could not define his RTP as well as the reason to support the final statement. After this concept was done, an intensive revision was done before the post-intervention test was given.

4.3.1. Geometric Skills Displayed by Learners in the Post- Intervention Test

The post-intervention test was designed to be a part of the answer to the second research question targeting development of geometric skills. The data were analysed quantitatively using descriptive statistics and qualitatively using thematic analysis. The themes used in this study emanated from literature and some themes emerged.

4.3.1.1. *Learner Performance in the Post-intervention test*

The post-intervention test was written by all the 20 learners ($n = 20$) from the class. According to the findings of the pre-intervention test, the average percentage mark was 55.8%. In other words, out of the 50 marks of the paper, an average learner got around 28 marks. The criteria that were used in the selection of the 6 learners were used as well here to determine top-performance, middle performance and under performance. From the post-intervention test analysis, 90% managed to get at least a moderate pass and 10% underperformed. The figure below displays the pre-intervention test and post-intervention test results to see if there are any noticeable differences.

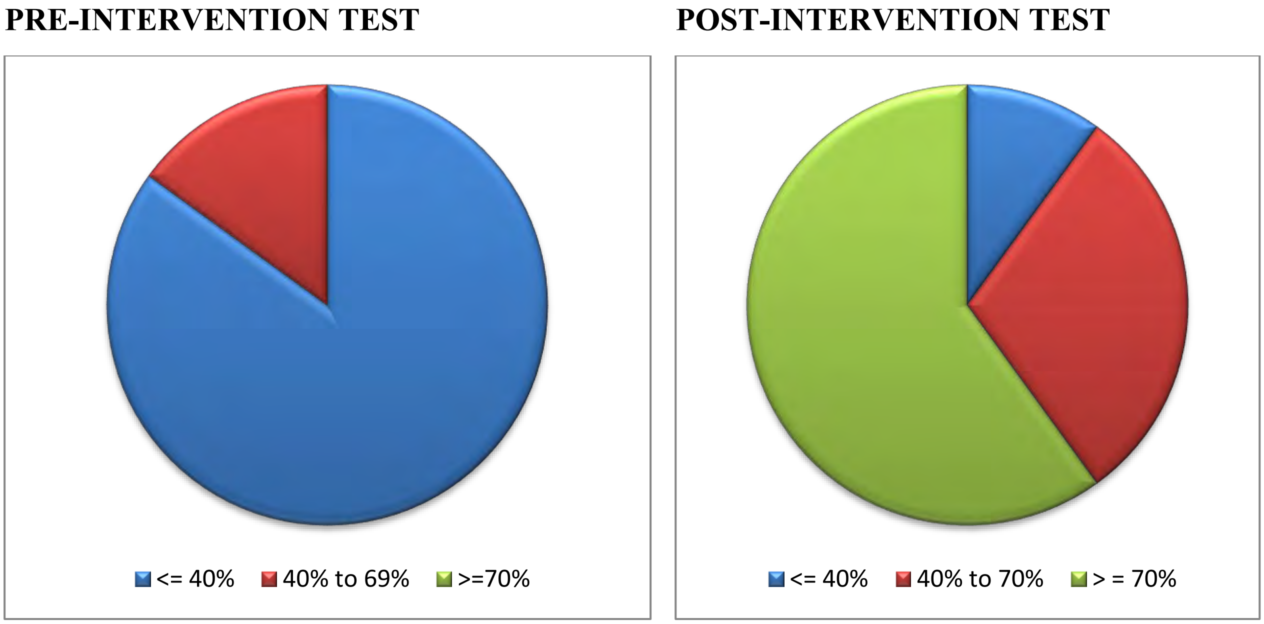
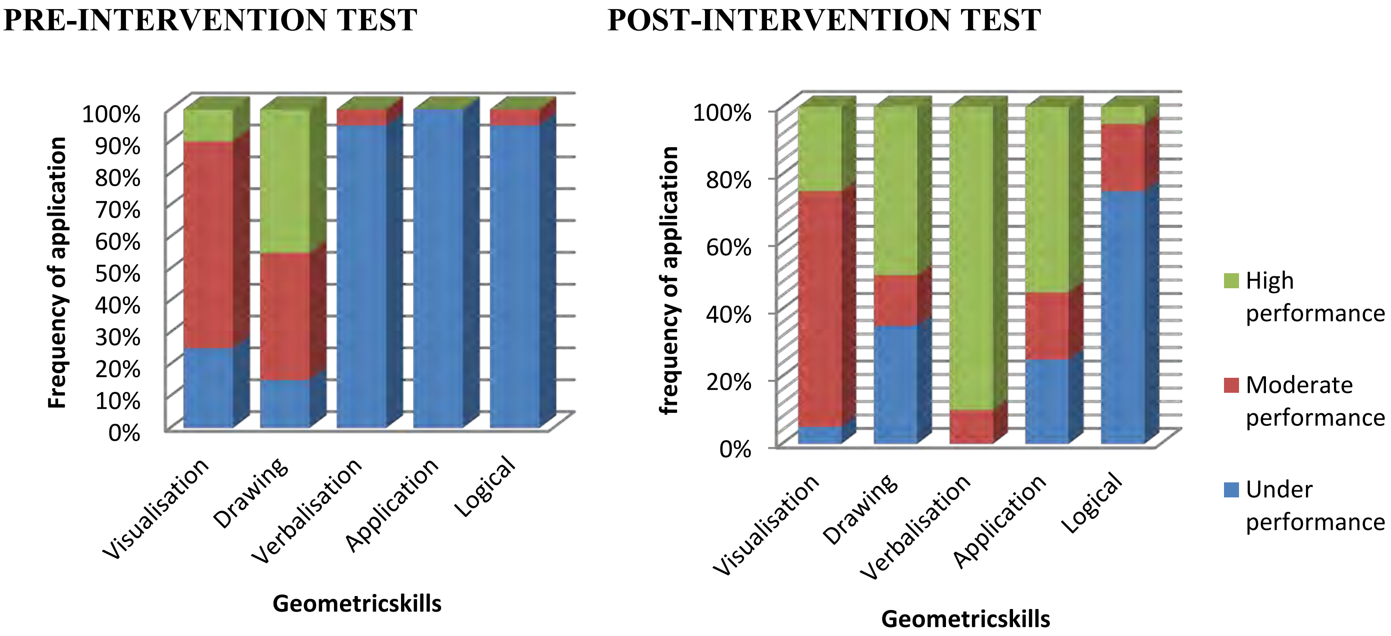


Figure 21: Pre-intervention test vs post-intervention test overall results

As indicated in Figure 21 above, very few (10%) learners who wrote the test underperformed as most (60%) of the learners fell in the category of moderate performance which at least showed some development in their geometric skills has taken place. This I say because in the pre-intervention test most learners (85%) fell in the underperformers' category. On the same footing, 30% of the learners who managed to perform highly in the post-intervention test unlike in the pre-intervention test where none of the learners managed to perform that far. However, even though there showed an element of improvement in several ways in the post-intervention test, there were still some (10%) who were showing to have challenges with their problem-solving and geometric skills. This kind of data provided me with understanding of what kind of geometric skills learners possessed after utilizing the four stages model. The results helped me to plan also for some interviews to find out on the challenges learners had during the use of the Polya's four stages of problem solving during the development of the learners' geometric skills.

4.3.1.2. *Post-intervention test results analysis on specific geometric skills*

The questions on the post-intervention test were grouped in such a way that every question will test geometric skill falling under the 5 major skills. Results from the tested learners in this study indicated what geometric skills the learners were now possessing, and the analysis is shown in Figure 22 below. Learner's specific geometric skills results were categorized using the same criteria that were used in this study to determine underperformance, moderate performance, and high performance.


PRE-INTERVENTION TEST

POST-INTERVENTION TEST

Figure 22: Analysis of pre-intervention test vs post-intervention test results on specific geometric skills

Figure 22 shows that there was a performance shift in all the skills in every aspect of performance. On visualisation question, a noticeable change is seen on underperformance and high performance in the pre-intervention test and post-intervention test. From the pre-intervention test, the number of underperformers dropped whilst the number of high performers increased in the post-intervention test. Verbalization was a skill that all the learners did well, and this is shown by the shift of under-performance to 0% in the post-intervention test. However, drawing and logical skills were found to have no big shifts especially with the underperformers. In drawing skills, the number of under-performers even increased though by a small margin. Generally, the results show a positive shift in the skills by learners from underperformance towards higher performance. Most learners did not do well in logical skills as compared to other skills. Ultimately, the 5 main groups of skills were further subdivided into specific skills and analysed to see which exact skills were displayed by the learners in the post-intervention test after being exposed to Polyas's four stages model.

4.3.2. Geometric Skills Displayed by Selected Learners in the Post- Intervention Test

The work of the same 6 learners selected from the class was observed closely to see which specific geometric skills learners displayed in the post-intervention test after Polya's model has been used in the classroom. I looked at the pre-intervention test, classroom work and post-intervention test to see if there was any evident improvement in geometric as well as problem-solving skills. I looked at the nature of application, whether the learner applied the geometric skills correctly, incorrectly, or inappropriately. All cases where learners applied particular skills were discussed, except when no skill was displayed. Thus, the themes that were discovered from literature and those that emerged were used in the analysis. These themes include visualisation, drawing, verbalisation, application, and logical skills.

According to Williams (2014) emergent themes are a basic building block of inductive approaches to qualitative social science research, and are derived from the life worlds of research participants. In this study, I discovered three emerging themes that developed using Polya's model, and these themes included re-planning skill, finding of required to prove (RTP) skill and computational skill. All these themes are discussed looking at how they were developed in the classroom as well as how the learners applied them in the post-intervention test. Pre-intervention test work was also referred to see if there was any noticeable change in learners' ability to apply these skills.

Most of the learners' work presented here focused on different comparisons of work from pre-intervention, during intervention and after intervention. Such presentation and analysis were done to see if there was any relationship of events, leading to an improvement, decline or no shift in learner performance. Therefore, the frequency of application of a particular skill correctly, inappropriately, and incorrectly was used in presentation and analysis to make conclusion.

Visualisation Skills

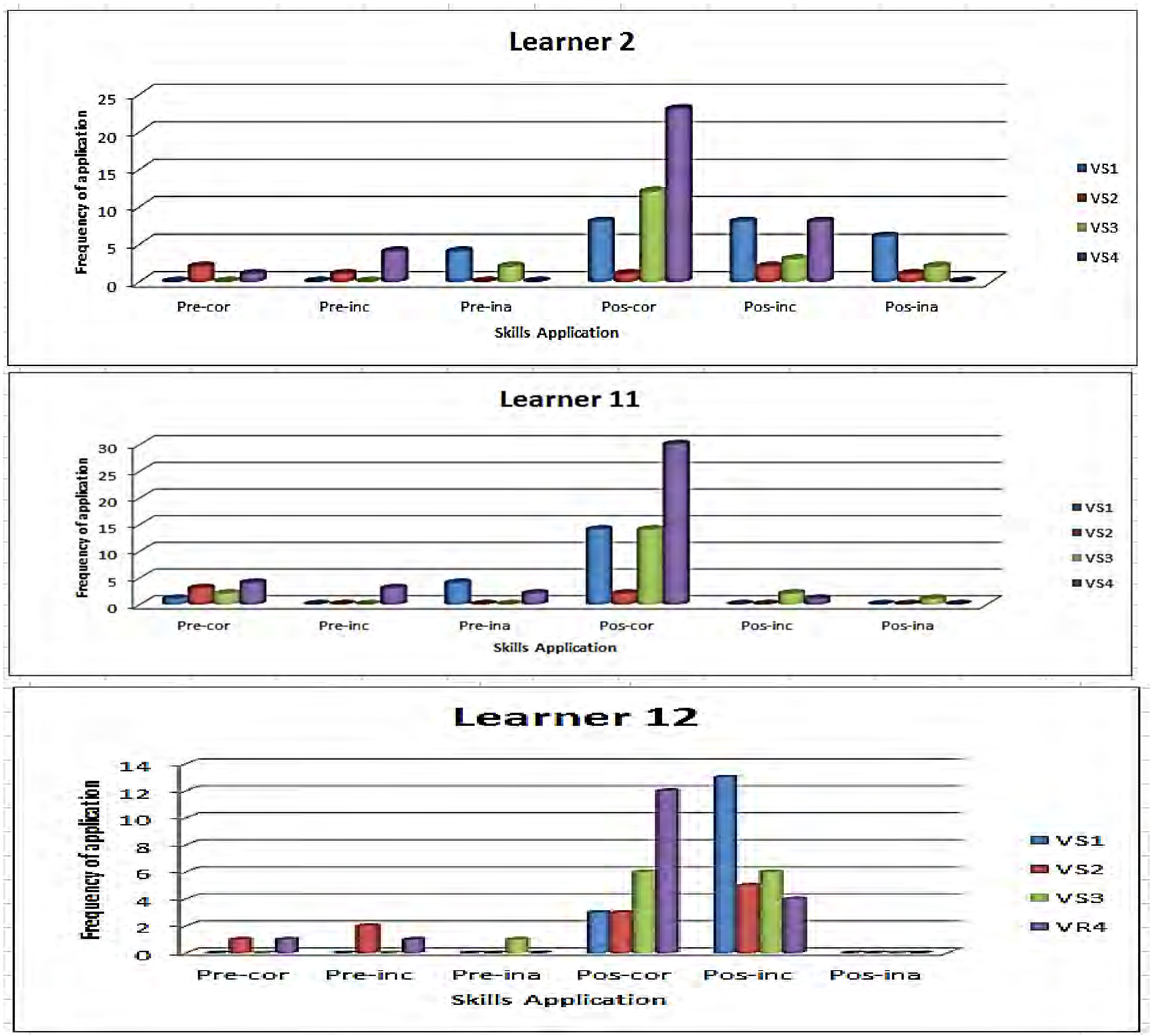


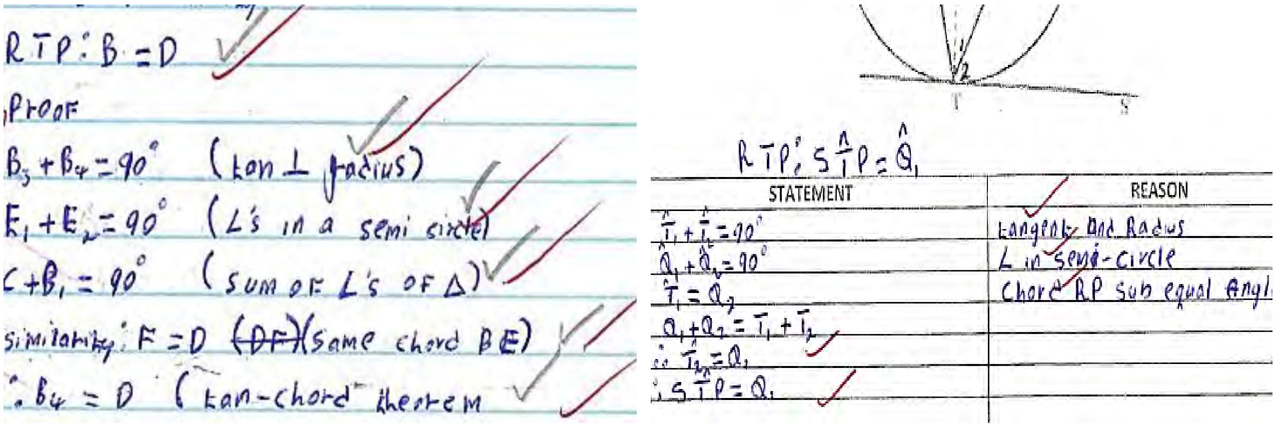
Figure 23: Visual skills analysis

Learners' recognition of a theorem or axiom skill

Visualisation is a strong tool for searching mathematical problems, giving meaning to mathematical concepts and the relationships between them (Yilmaz & Argun, 2018). The skill of recognition of theorems and axioms is one of the skills contributing to the theme of visualisation which focuses on visual construction by learners in geometric problem-solving. The 6 observed learners could apply the skill of recognition of a theorem or axiom though they were applying it in diverse ways in the post-intervention test (see Figure 24). This skill was applicable to question 3, 4 and 5 whereby one was supposed to recognise a theorem to support his/her arguments in problem-solving. As shown in

Figure 23, the frequency of occurrence of the application of the skill of recognition of a theorem or axiom in the post-intervention test in comparison with that of pre-intervention test is as follows: Learner 2 applied the skill 8 times correctly, 8 times incorrectly and 6 times inappropriately in the post-intervention test yet in the pre-intervention test applied inappropriately in 4 cases. Learner 11 applied the skill 14 times correctly in the post-test while in the pre-intervention test, once correctly and 4 times inappropriately. Learner 12 could apply the skill 3 times correctly and 13 times incorrectly in the post-intervention test, whereas in the pre-intervention test, they never applied the skill. In the post-intervention test, Learner 3 used the skill of recognition of a theorem or axiom 12 times correctly, yet in the pre-intervention test applied once inappropriate and once incorrect as shown in Figure 23. For Learner 3, the skill of Recognition of a theorem or axiom was poorly employed in the post–intervention test, and a change for the better started to be witnessed in the classroom during class activities. Learner 7 in the pre-intervention test applied the skill correctly in one case, and inappropriately in two cases, whereas in the post-intervention test, correctly used the skill in 7 cases, once incorrectly and once inappropriately. In the post-intervention test, Learner 17 utilised the skill 5 times correctly and 3 times incorrect yet in the pre-intervention test had applied twice inappropriately and twice incorrectly.

Work from Learner 3 and 11 were used to illustrate how the skill was applied by learners from different categories of performance. The work from pre-intervention test and post-intervention test displayed in Figure 24 were used for this purpose. As shown in Figure 24, and the presented information in Figure 23, I understood that there was a noticeable positive change in the application of the skill in all the three categories. The shift noticed was on the number of correct applications of the skill, as well as the general regularity of application of the skill. The frequency of correct applications generally improved at the expense of the inappropriate and incorrect application of the skill. Most learners were making very few mistakes from the classwork as far as recognition of theorems is concerned, as was also noticed during document analysis. The reason might be that the learners were given time to discuss in pairs and sometimes as a whole class before an individual class activity could be administered.



CLASSWORK (learner 17)

RTP: B = D

Proof

$B_3 + B_4 = 90°$ (tan ⊥ radius)

$E_1 + E_2 = 90°$ (∠'s in a semi circle)

$C + B_1 = 90°$ (sum of ∠'s of △)

similarity: F = D ~~(DF)~~ (same chord BE)

∴ $B_4 = D$ (tan-chord theorem

POST-INTERVENTION TEST (learner 17)

RTP: $S\hat{T}P = \hat{Q}_1$

STATEMENT	REASON
$\hat{T}_1 + \hat{T}_2 = 90°$	tangent And Radius
$\hat{Q}_1 + \hat{Q}_2 = 90°$	∠ in semi-circle
$\hat{T}_1 = Q_2$	Chord RP sub equal Angl
$Q_1 + Q_2 = T_1 + T_2$	
∴ $\hat{T}_2 = Q_1$	
∴ $S\hat{T}P = Q_1$	

Figure 24: Recognition of theorems in classwork vs post-intervention test

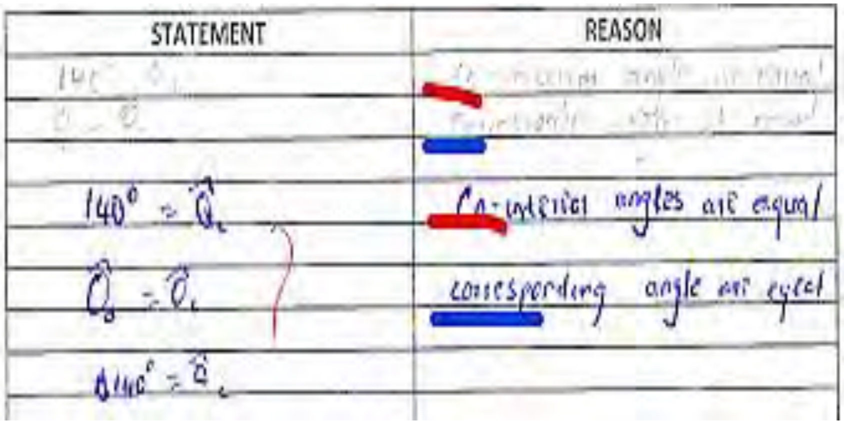
The other reason why there was some con-cordial agreement in the positive application of the theorem in classwork and post-intervention test as shown in Figure 24 might be that most of the class activities were given to individual learners soon after a concept has been taught and discussed, which might have given the learners an advantage to get the problems solved well.

The advantage with the class activities was that the teacher was moving around looking at learner situations and helped learners to come back to track if lost. The teacher could also have dialogue with learners and help them with their misunderstandings and misconceptions. Inability by learners to recognise theorems was common when it came to logical proofs such as: prove a line to be a tangent; prove a quadrilateral to be cyclic; prove two lines to be parallel; and so forth.

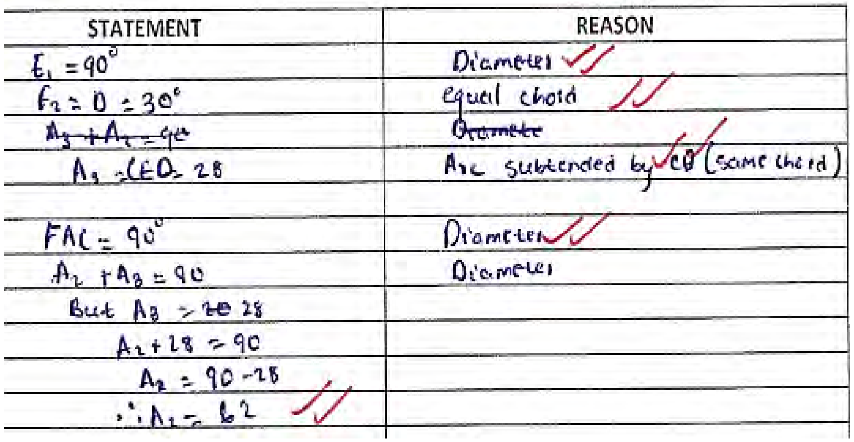
I understood that the frequency of application of this theorem on the post-intervention test was less compared to what the learners were doing in the class activities. The major reason discovered is that some of the learners were not following the procedures used in the classroom for problem-solving. For example, some learners were not indicating the key words to show reading for understanding in the post-intervention test even though their answers were showing they had read for understanding. Thus, the frequency of evidence of recognition of theorems and axioms was compromised. As shown in Figure 25, Learner 11 showed evidence of reading for understanding by noting down theorems involved but Learner 3 did not do the same, yet the solution was correct.

PRE-INTERVENTION TEST WORK **POST-INTERVENTION TEST WORK**

Learner 03

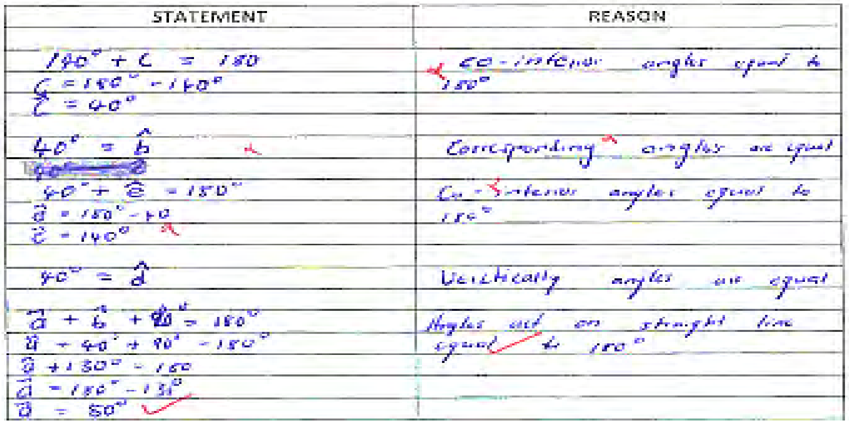


STATEMENT	REASON
[illegible]	[illegible]
$140^\circ = \hat{Q}_2$	Co-interior angles are equal
$\hat{Q}_3 = \hat{Q}_1$	corresponding angle are equal
$140^\circ = \hat{Q}_2$	



STATEMENT	REASON
$E_1 = 90^\circ$	Diameter ✓✓
$F_2 = D = 30^\circ$	equal chord ✓✓
~~$A_3 + A_2 = 90$~~	~~Diameter~~
$A_3 = CED = 28$	Arc subtended by ✓CD ✓(same chord)
$FAC = 90^\circ$	Diameter ✓✓
$A_2 + A_3 = 90$	Diameter
But $A_3 =$ ~~20~~ 28	
$A_2 + 28 = 90$	
$A_2 = 90 - 28$	
$\therefore A_2 = 62$ ✓✓	

Learner 11



STATEMENT	REASON
$140^\circ + C = 180$	Co-interior angles equal to 180°
$C = 180^\circ - 140^\circ$	
$C = 40^\circ$	
$40^\circ = \hat{b}$	Corresponding angles are equal
[illegible]	
$40^\circ + \hat{e} = 180^\circ$	Co-interior angles equal to 180°
$\hat{e} = 180^\circ - 40$	
$\hat{e} = 140^\circ$	
$40^\circ = \hat{d}$	Vertically angles are equal
$\hat{a} + \hat{b} + \hat{d} = 180^\circ$	Angles act on straight line equal ✓ to 180°
$\hat{a} + 40^\circ + 90^\circ = 180^\circ$	
$\hat{a} + 130^\circ = 180$	
$\hat{a} = 180^\circ - 130^\circ$	
$\hat{a} = 50^\circ$ ✓	

STATEMENT	REASON
4.1 $E_1 = 90^\circ$ ✓	✓ Diameter
4.2 $\hat{F}_2 = \hat{D} = 30^\circ$ ✓	Equal chords (AB = AC)
4.3 $A_3 = \hat{E}_2 = 28^\circ$ ✓	Same chord (C;D)
~~6.4 $F\hat{A}C = 90^\circ$~~	~~Diameter~~
4.5 $\hat{A}_2 + \hat{A}_3 = 90$	✓ Diameter
$\hat{A}_2 + 28^\circ = 90^\circ$	
$\hat{A}_2 = 90^\circ - 28^\circ$ ✓	
$\hat{A}_2 = 62^\circ$	
4.4 $F\hat{A}C = 90^\circ$ ✓	✓ diameter
$\hat{A}_2 + \hat{A}_3 = 90^\circ$	
$A_2 + 28^\circ = 90$	
$A^2 = 90^\circ - 28$	
$A^2 = 62^\circ$	

Centre

Line from the centre perpen cur bisect

Line from the centre bisect perpendicula

∠ at centre twice

62+28=90

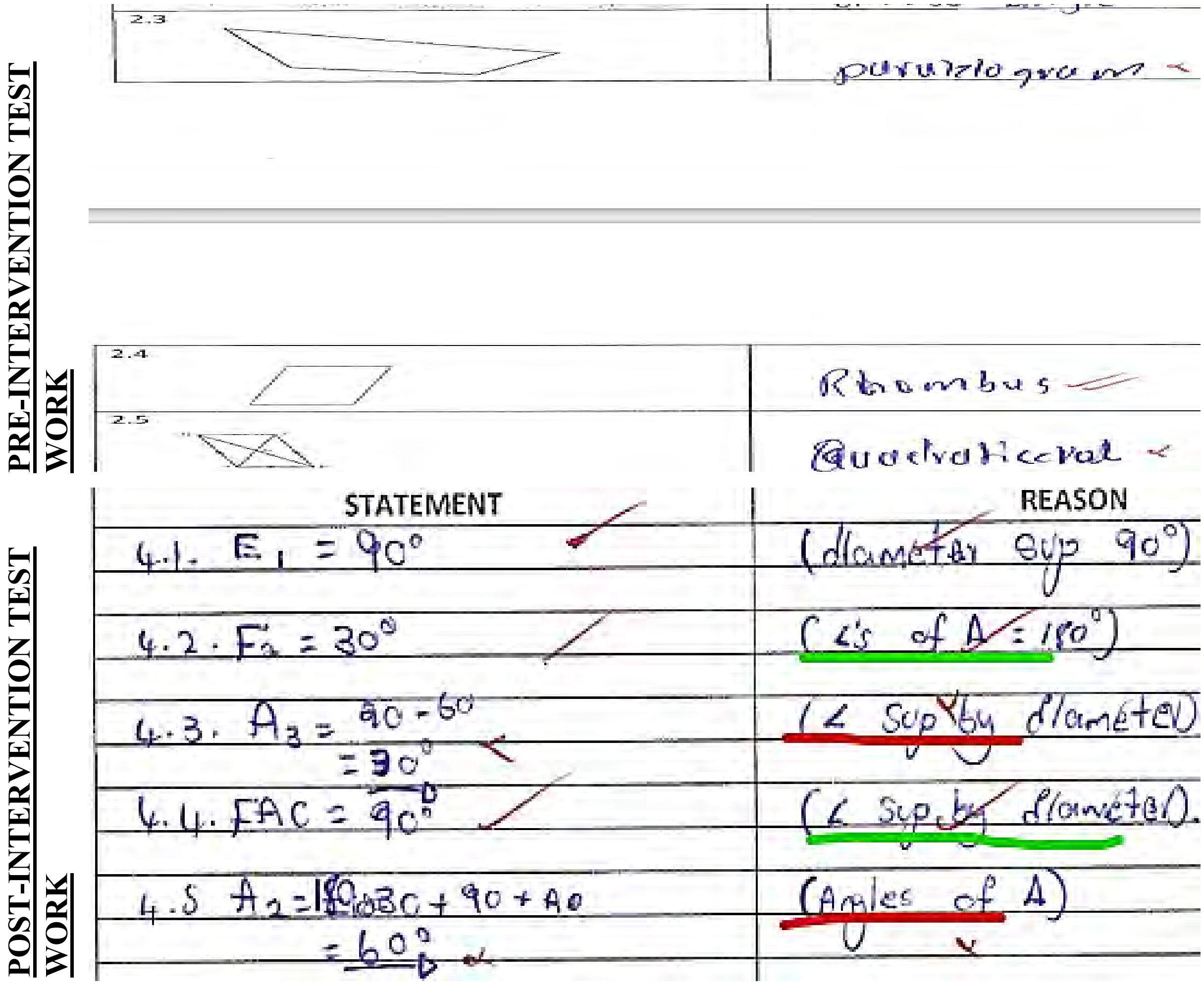
Figure 25: Skill of recognition of theorems and axioms displayed by learners

Learners' recognition of properties of shapes skill.

The skill of recognition of properties of shapes is one of the skills targeting the visual construction by the learners. This could be applied anywhere in the question paper if there was a demand for recognition of properties of shapes in a diagram to be able to name it or to identify applicability of a particular theorem. Shapes like triangles and cyclic quadrilateral were the commonly identified shapes. Figure 23 shows the frequency of application of the skill and Figure 26 displays the way learners were applying the skill in the pre-test and in the post-intervention test. The frequency of application of the skill of recognition of properties of shapes is as follows: Learner 2 applied the skill of recognition of properties of shapes once correct, once inappropriate and twice incorrect in the post-intervention test, while in the pre-intervention test, it was applied correctly twice and once incorrectly, Learner 11 applying it twice correctly in the post-intervention test but in the pre-intervention test had used it thrice correctly. Learner 12 applied it thrice correctly, and five times incorrectly in the post-intervention test; yet in the pre-intervention test, had applied the skill of recognition of properties of shapes once correctly and twice incorrectly. Learner 3 utilised the skill twice correctly in the post-intervention test yet in the pre-intervention test applied once correctly and twice incorrectly. Learner 7 applied the skill of recognition of properties of shapes once correctly in the post-intervention test, yet in the pre-intervention test had applied it twice correctly and once incorrectly. Learner 17 used it once correctly and twice inappropriately in the post-intervention test, while in the pre-intervention test, had applied the skill twice incorrectly (see Figure 23).

The way the learners displayed the skill of recognition of properties of shapes both in the pre-intervention test and post-intervention test was almost the same. An illustration is given below of an extract from the work by Learner 7 and Learner 12. Recognition of properties of shapes skill was not widely used even in the class activities. Just like the findings by Ismail (2020), maybe the difficulties faced by the learners are because of the inability to imagine the shapes (imagination). The other reason could be that the skill is not directly tested in FET phase as in the GET phase. The application skill is only applied when one must know properties of a particular shape to be able to apply theorems applicable in that scenario. This might have limited the learners to apply the skill in the classroom and in the post-intervention test. The skill is expected to have been developed during stage 1, 2 & 3 of Polya's model of problem-solving. The name of the shape in a statement acts as a key word and one is expected to highlight the word in stage 1 of the model. Properties of the shape are to be listed in stage 2 and used as reasons to support statements in stage 3 of the model.

LEARNER 12



2.3 paruzlogram

2.4 Rhombus

2.5 Quadratiecral

STATEMENT	REASON
4.1. $E_1 = 90^\circ$	(diameter sup 90°)
4.2. $F_2 = 30^\circ$	(∠'s of Δ = 180°)
4.3. $A_3 = 90 - 60$ $= 30^\circ$	(∠ sup by diameter)
4.4. $FAC = 90^\circ$	(∠ sup by diameter).
4.5 $A_2 = 180 - 30 + 90 + A0$ $= 60^\circ$	(Angles of Δ)

Figure 26: Recognition of properties of shapes analysis

Learners' skill of recognition of different parts and lines in a circle or diagram

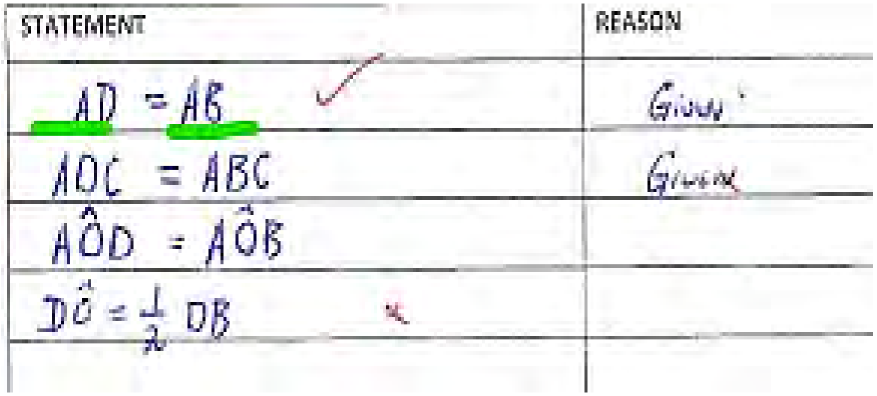
I believe that the ability to recognise different parts and lines in a circle or diagram could aid the understanding of the concept of problem-solving in Geometry. This skill of recognition of different parts and lines in a circle or diagram was most applicable to questions 1, 3, 4 and 5. This skill was the cornerstone for stage 2 of Polya's model. When doing planning on the diagram, one is expected to know different parts of the diagram or circle such as tangents, chords, diameter, center and so forth. This skill was taught and learnt much during the first days of the introduction of Geometry and the four stages model. As displayed in Figure 23, the rate of application of the skill in the post-intervention test vs pre-intervention test is as follows: Learner 2 applied this skill of recognition of different parts and lines in a circle or diagram 12 times correctly, twice incorrectly and 3 times inappropriately in the post-test, yet in the pre-intervention test applied inappropriately in two cases. Learner 11 applied the skill of recognition of different parts and lines in a circle or diagram 14 times correctly, twice incorrectly, and once inappropriately in the post-intervention test; yet in the pre-intervention test used twice correctly. Learner 12 applied the skill 6 times correctly and 6 times incorrectly in the post-intervention test but in the pre-intervention test applied once inappropriately. Learner 3 used the skill 15 times correctly, twice incorrectly and twice inappropriately in the post-intervention test whilst in the pre-intervention test never applied the skill of recognition of different parts and lines in a circle or diagram. Learner 7 applied the skill 10 times correctly, thrice incorrectly and once inappropriately in the post-intervention test, while in the pre-intervention test, employed the skill 4 times correctly and inappropriately. Learner 17 used the skill of recognition of different parts and lines in a circle or diagram 8 times correctly and once inappropriately in the post-intervention test, whereas in the pre-intervention test applied 4 times correctly and 4 times inappropriately.

From the information obtained (see Figure 27), cases where learners applied the recognition skill correctly were very limited in the pre-intervention test. Thus, there is a positive shift in the quantity of the number of cases as well as correct application of the skill. It may be concluded that the visualisation skills used in the abstraction processes significantly aided the understanding of the concept of congruent figures and strengthened the visual images of the participants. Yilmaz & Argun's (2018) study made this same observation. Figure 27 illustrates that this was also evidenced during document analysis where most learners were seen recognising different parts and lines in a circle or diagram.

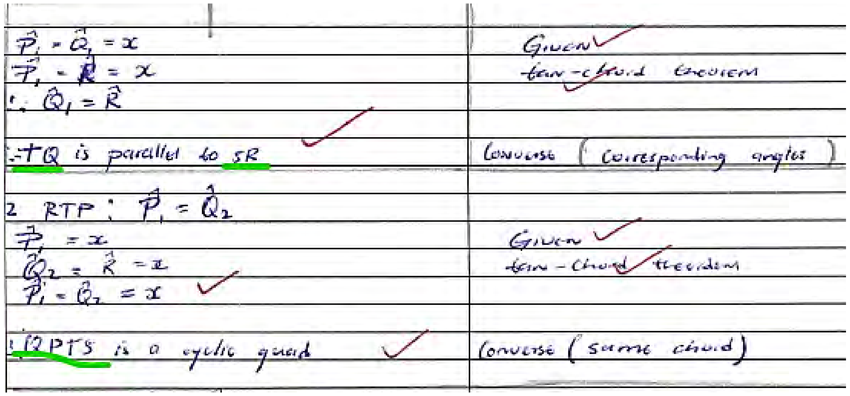
PRE-INTERVENTION TEST WORK

POST-INTERVENTION TEST WORK

Learner 11

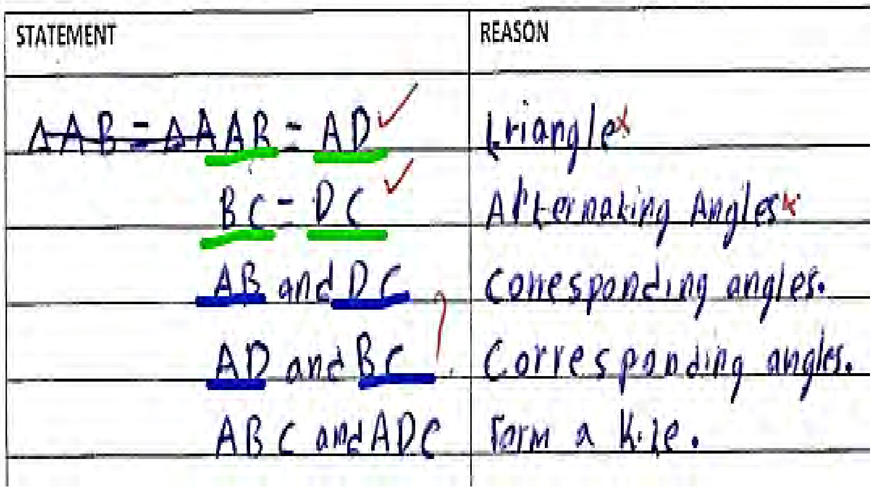


STATEMENT	REASON
AD = AB ✓	Given
ADC = ABC	Given
$A\hat{O}D = A\hat{O}B$	
$D\hat{O} = \frac{1}{2} DB$ x	

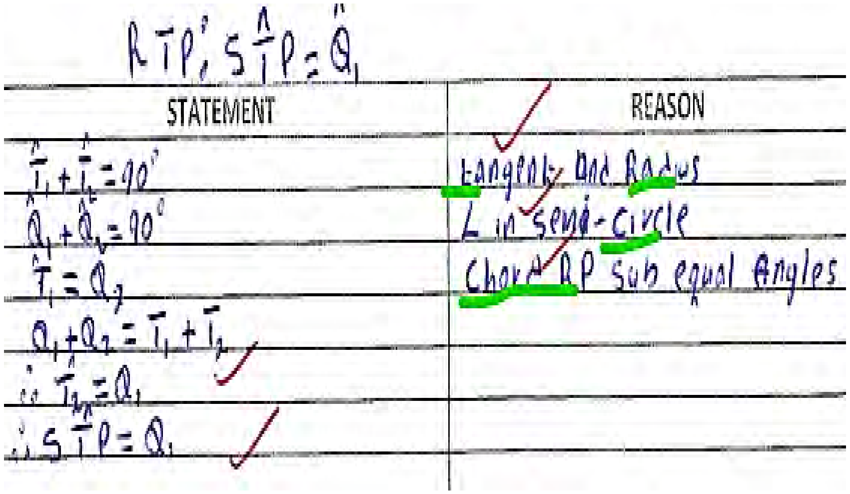


$\hat{P}_1 = \hat{Q}_1 = x$	Given ✓
$\hat{P}_1 = \hat{R} = x$	tan-chord theorem ✓
$\therefore \hat{Q}_1 = \hat{R}$	
$\therefore$ TQ is parallel to SR ✓	Converse (Corresponding angles)
2 RTP: $\hat{P}_1 = \hat{Q}_2$	
$\hat{P}_1 = x$	Given ✓
$\hat{Q}_2 = \hat{R} = x$	tan-chord ✓ theorem
$\hat{P}_1 = \hat{Q}_2 = x$ ✓	
$\therefore$ QPTS is a cyclic quad ✓	Converse (same chord)

Learner 17



STATEMENT	REASON
~~ΔAB = ΔAAB~~ = AD ✓	triangle x
BC = DC ✓	Alternating Angles x
AB and DC	corresponding angles.
AD and BC ?	Corresponding angles.
ABC and ADC	form a kite.



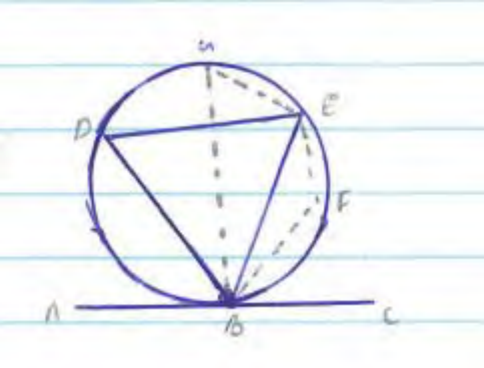
RTP; $S\hat{T}_1P = \hat{Q}_1$

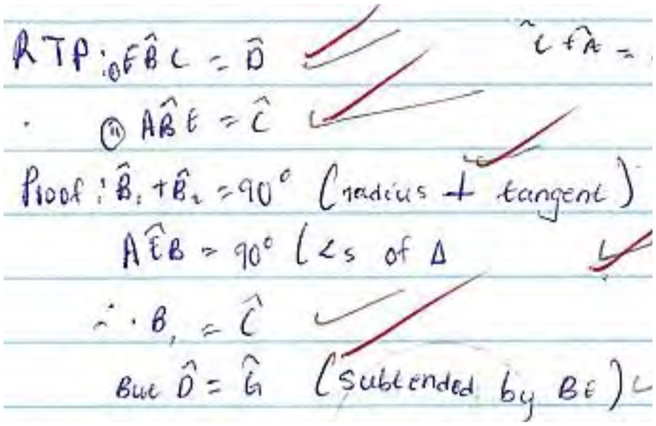
STATEMENT	REASON
$\hat{T}_1 + \hat{T}_2 = 90°$	✓ tangent and Radius
$\hat{Q}_1 + \hat{Q}_2 = 90°$	✓ ∠ in semi-circle
$\hat{T}_1 = \hat{Q}_2$	✓ Chord RP sub equal Angles
$Q_1 + Q_2 = T_1 + T_2$ ✓	
$\therefore \hat{T}_2 = Q_1$	
$\therefore S\hat{T}_1P = Q_1$ ✓	

Figure 27: Analysis of recognition of different parts and lines in a circle or diagram

From document analysis and during classroom activities, most of the learners displayed their ability to recognise parts of shapes in proving examinable theorems, as well as other problems in Geometry as indicated in Figure 28. Recognition of parts of a circle could be learnt unknowingly during teaching and learning, as well as application of the four stages model. For example, when learning how to apply Tan-chord theorem, one can learn to identify a tangent and a chord during the process unknowingly. Thus, learners were following what they had learnt from the classroom in the post-intervention test. Work from Learner 3, both classwork and post-intervention test, were used to elucidate that.

CLASSWORK (learner 3)





RTP: $F\hat{B}C = \hat{D}$ ✓ $\hat{C} + \hat{A} =$

(ii) $A\hat{B}E = \hat{C}$ ✓

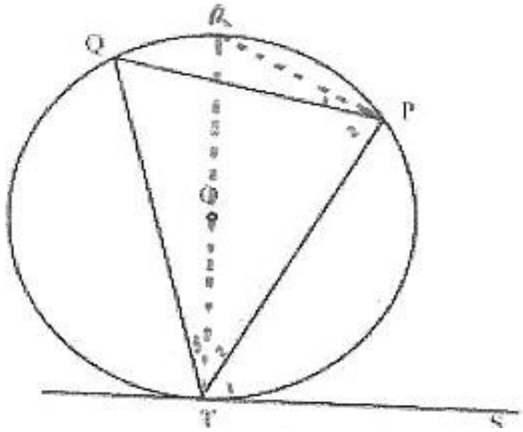
Proof: $\hat{B}_1 + \hat{B}_2 = 90^\circ$ (radius ⊥ tangent) ✓

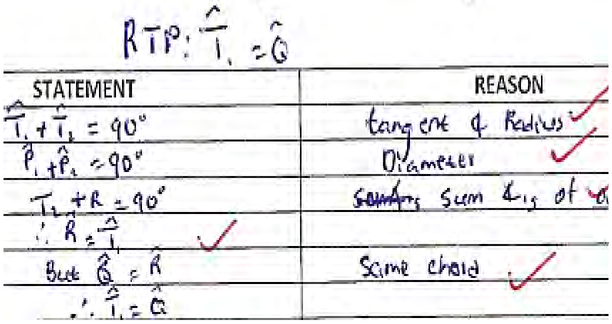
$A\hat{E}B = 90^\circ$ (∠s of Δ ✓

∴ $B_1 = \hat{C}$ ✓

But $\hat{D} = \hat{G}$ (subtended by BE) ✓

POST-INTERVENTION TEST (learner 3)





RTP: $\hat{T}_1 = \hat{Q}$

STATEMENT	REASON
$\hat{T}_1 + \hat{T}_2 = 90^\circ$	tangent ⊥ Radius ✓
$\hat{P}_1 + \hat{P}_2 = 90^\circ$	Diameter ✓
$T_2 + R = 90^\circ$	~~sum~~ sum ∠s of Δ
∴ $\hat{R} = \hat{T}_1$ ✓	
But $\hat{Q} = \hat{R}$	Same chord ✓
∴ $\hat{T}_1 = \hat{Q}$	

Figure 28: Recognition of different parts and lines in a circle or diagram

There was still a challenge with some learners who still had misconceptions with parts of a circle like a distinction between a tangent and a chord that protrudes to the outside of a circle or a cyclic quadrilateral and just a quadrilateral in a circle. Even though such misconceptions were not common towards the end of the topic, some learners could display them in their workings at some points.

Learners' recognition of different angles skills

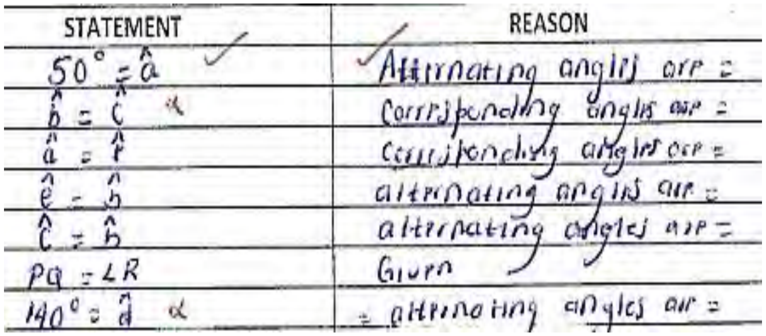
Some scholars Muhembo (2017) refer this skill as homomorphic visualisation. In this type of visualisation, some of the elements have certain mutual relation that imitates well the relationship between the abstract objects and provides with support to guide imaginations, while proving mathematical processes. The skill could be displayed through comparing angles in problem-solving, marking angles in diagrams, naming angles during problem-solving and giving angles values. The first stage of the model required one to highlight angles given in the statement. In stage 2 of the model, one is expected to assign angles with values whenever is necessary and in stage 3 one is expected to name and determine values of angles using application of theorems and through calculations. From Figure 23, the regularity of use of the skill by learners was as follows: Learner 2 employed the skill 23 times correctly and 8 times incorrectly in the post-intervention test, while in the pre-intervention test had applied the skill recognition of different angles once correct and 4 times incorrect. Learner 11 applied the skill of recognition of different parts and lines in a circle or diagram 14 times correctly, twice incorrectly and once inappropriately in the post-intervention test; yet in the pre-intervention test used twice correctly. Learner 12 applied the skill 6 times correctly and 6 times incorrectly in the post-intervention test but in the pre-intervention test, applied once inappropriately. Learner 3 used the skill of recognition of different angles correctly 18 times in the post-intervention test, yet in the pre-intervention test had applied the skill once inappropriately and incorrectly 3 times. Learner 7 applied the skill of recognition of different angles 22 times correctly in the post-intervention test yet in the pre-intervention test applied the skill thrice correctly, 4 times incorrectly and once inappropriately. Learner 17 applied the skill 12 times correctly in the post-intervention test while in the pre-intervention test used the skill of recognition of different angles once correct, once inappropriately and thrice incorrectly. Work below from Learner 2 and 7 shows how the skill has generally been applied (see Figure 29).

The information from Figure 29 revealed that there was a huge positive change in the application of the skill from the pre-intervention test to the post-intervention test across all the categories. From document analysis, the number of inappropriate and incorrect applications became less as the number of correct applications progressively upsurges. However, some underperformers were still having challenges with application of theorems to make judgements between two angles, even though they were doing right in the class activities. I realised that sometimes some correct statements were found to be supported with wrong reasons as well.

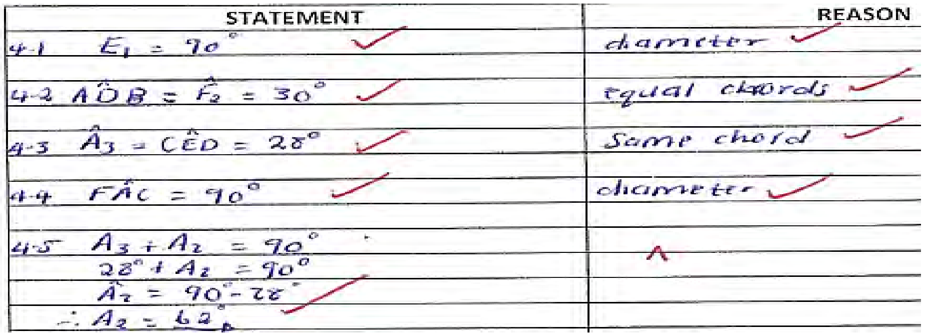
PRE-INTERVENTION TEST WORK

POST-INTERVENTION TEST WORK

Learner 07

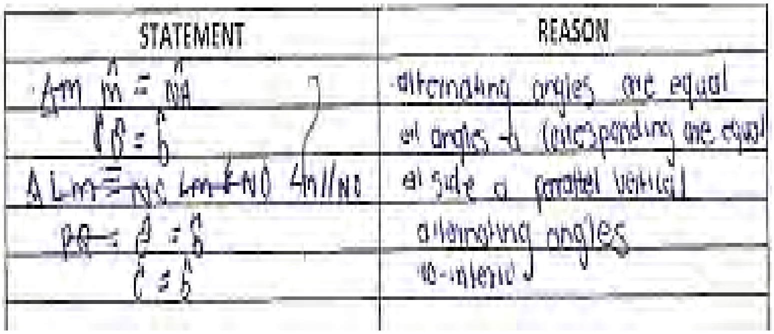


STATEMENT	REASON
$50^\circ = \hat{a}$ ✓	✓ Alternating angles are =
$\hat{b} = \hat{c}$	Corresponding angles are =
$\hat{a} = \hat{e}$	Corresponding angles are =
$\hat{e} = \hat{b}$	alternating angles are =
$\hat{c} = \hat{b}$	alternating angles are =
$PQ = \angle R$	Given
$140^\circ = \hat{d}$	= alternating angles are =

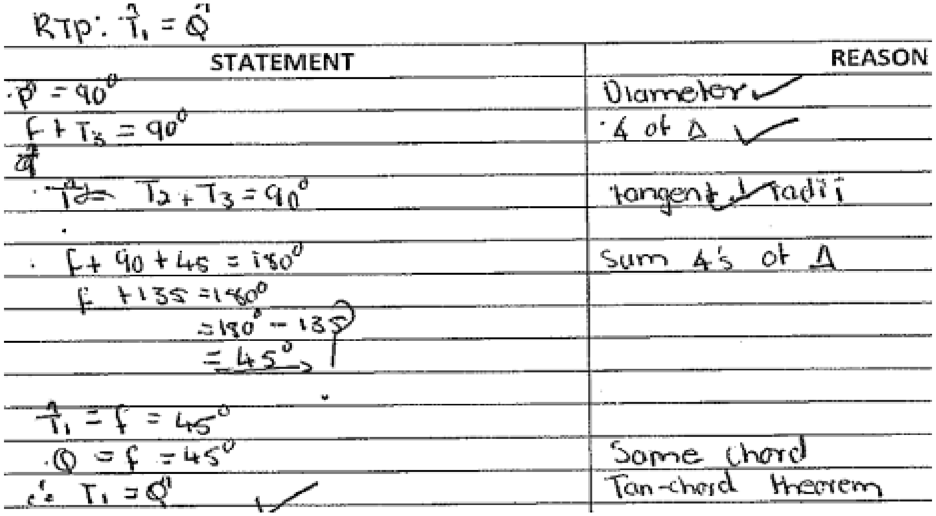


STATEMENT	REASON
4.1 $E_1 = 90^\circ$ ✓	diameter ✓
4.2 $A\hat{O}B = \hat{F}_2 = 30^\circ$ ✓	equal chords ✓
4.3 $\hat{A}_3 = C\hat{E}D = 28^\circ$ ✓	Same chord ✓
4.4 $F\hat{A}C = 90^\circ$ ✓	diameter ✓
4.5 $A_3 + A_2 = 90^\circ$	^
$28^\circ + A_2 = 90^\circ$	
$\hat{A}_2 = 90^\circ - 28^\circ$ ✓	
$\therefore A_2 = 62^\circ$	

Learner 02



STATEMENT	REASON
$\angle m$ $\hat{m} = \hat{n}A$	alternating angles are equal
$\hat{B} = \hat{b}$	alt angles + corresponding are equal
$\Delta Lm = NO$ ~~Lm~~ $\neq NO$ $\angle n // NO$	alt side a parallel vertical
~~PQ~~ = $\hat{\beta} = \hat{b}$	alternating angles
$\hat{c} = \hat{b}$	co-interior



RTP: $\hat{T}_1 = \hat{Q}$

STATEMENT	REASON
$\hat{P} = 90^\circ$	Diameter ✓
$f + T_3 = 90^\circ$	∠ of Δ ✓
~~T2=~~ $T_2 + T_3 = 90^\circ$	tangent ⊥ radii
$f + 90 + 45 = 180^\circ$	Sum ∠'s of Δ
$f + 135 = 180^\circ$	
$= 180^\circ - 135$	
$= 45^\circ$	
$\hat{T}_1 = f = 45^\circ$	
$Q = f = 45^\circ$	Some chord
$\therefore T_1 = \hat{Q}$ ✓	Tan-chord theorem

Figure 29: Recognition of different angles analysis

Verbalisation Skills

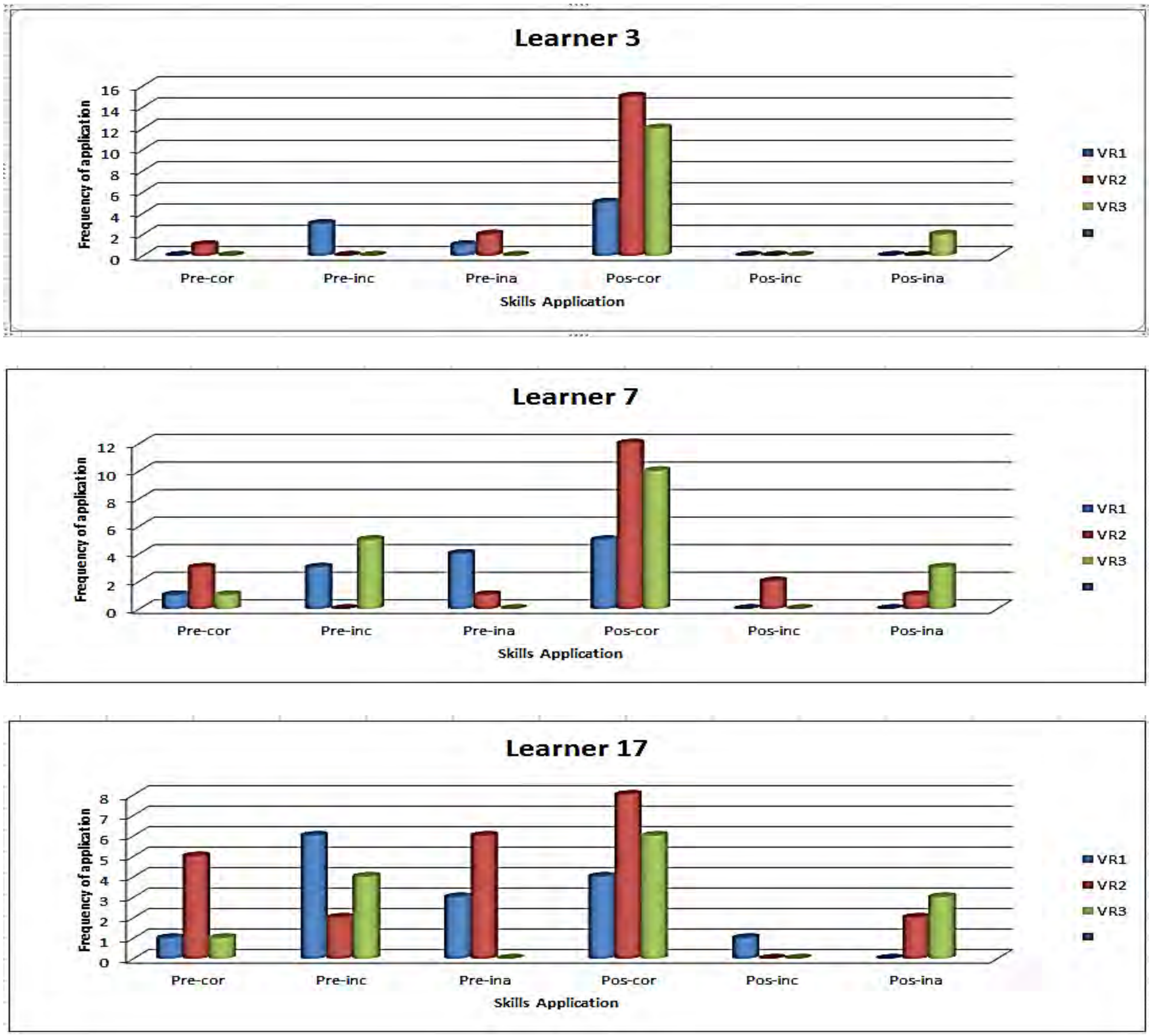


Figure 30: Verbal skills analysis

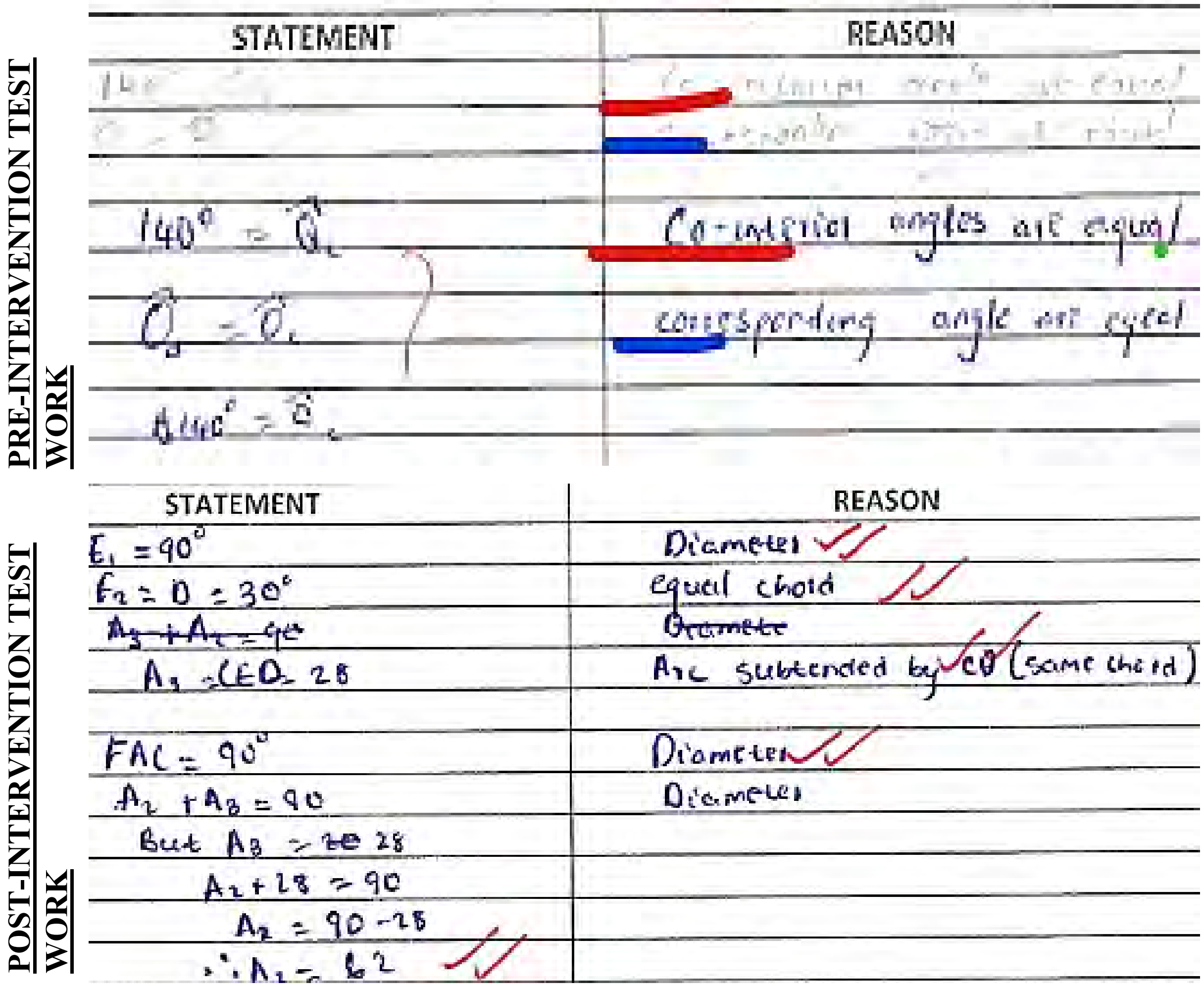
Verbalisation is one of the themes of this study focusing on the verbal component of problem-solving. Ahlum-Heath and Di Vesta (1986) concluded that verbalisation was most helpful during the initial flexible stages of learning to solve problems. Thus, in this research study, verbalisation was taught first as the foundation of the four stages model.

Learners' verbal descriptions skills

Verbal descriptions formed the basis for teaching the four stages model thus were taught initially and utilised throughout the journey of geometric teaching and learning. This skill of verbal descriptions is one of the skills that make up the theme of verbalisation skill. The verbal description skill was applied in question 3, 4 and 5 of the post-intervention test. The comparison between the rates of use of the skill in the post-intervention test compared with pre-intervention test (see Figure 30) was as follows: Learner 3 employed the skill 5 times correctly in the post-intervention test yet in the pre-intervention test applied the skill of verbal descriptions thrice inappropriately and once incorrectly. Learner 7, learner 11 and learner 12 utilised the skill 5 times correctly in the post-intervention test, yet in the pre-intervention test Learner 7 applied the skill once correctly, thrice incorrectly and 4 times inappropriately. Learner 17 applied the skill of verbal descriptions 4 times correctly and once incorrectly in the post-intervention test, yet in the pre-intervention test had applied the skill of verbal descriptions thrice inappropriately and 6 times incorrectly. Learner 2 applied the skill of verbal descriptions 10 times correctly, 8 times incorrectly and 4 times inappropriately in the post-intervention test, while in the pre-intervention test used the skill 4 times inappropriately and once incorrectly. Learner 11 in the pre-intervention test applied the skill of verbal descriptions once correctly and 4 times inappropriately. In the pre-intervention test Learner 12 applied the skill 8 times incorrectly. This is further explained by the work from Learner 3 and 11 (see Figure 31).

As shown in Figure 31, it was problematic to make geometric arguments in the pre-intervention test even though the learners were showing to have an idea of how it was supposed to be done. From observation of class activities, learners started to show to have an organised and clear way of making informed geometric arguments after being introduced to the four stages model and Grade 11 Geometry content. This skill was taught during the learning of stage 1, stage 2 and stage 3 of Polya's model, and was very useful in addressing questions that needs one to state a theorem, proving examinable theorems, solving riders and making logical proofs. In other words, the skill was a need in all the sections of geometric learning. Across the board, the skill was applied better in the post-intervention test unlike in the pre-intervention test.

LEARNER 3



PRE-INTERVENTION TEST WORK

STATEMENT	REASON
[illegible]	[illegible]
[illegible]	[illegible]
$140^\circ = \hat{Q}_2$	Co-interior angles are equal
$\hat{Q}_3 = \hat{Q}_1$	corresponding angle are equal
$140^\circ = \hat{Q}_1$	

POST-INTERVENTION TEST WORK

STATEMENT	REASON
$E_1 = 90^\circ$	Diameter ✓✓
$F_2 = D = 30^\circ$	equal chord ✓✓
~~$A_3 + A_2 = 90$~~	~~Diameter~~
$A_3 = CED = 28$	Arc subtended by ✓ CD ✓ (same chord)
$FAC = 90^\circ$	Diameter ✓✓
$A_2 + A_3 = 90$	Diameter
But $A_3 =$ ~~90~~ 28	
$A_2 + 28 = 90$	
$A_2 = 90 - 28$	
$\therefore A_2 = 62$ ✓✓	

Figure 31: Verbal descriptions analysis

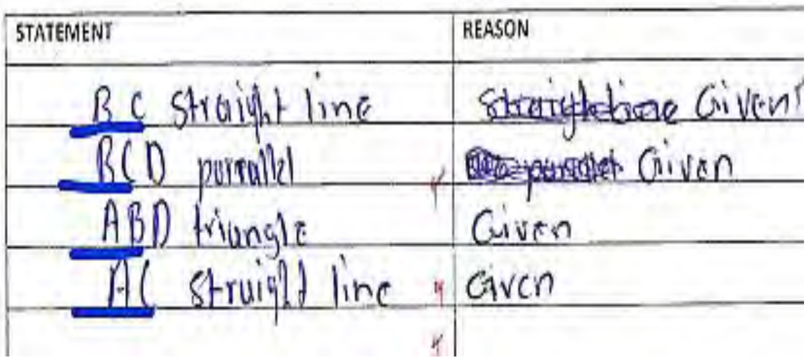
Learners' skill of naming of shapes, objects, and parts

The skill of naming shapes, objects and parts in Geometry is one of the verbalisation skills needed in geometric problem-solving. In order to do this, learners must be able to identify the characteristics and relationships of objects (comparing geometric figures to see how they are alike and how they are different and being able to distinguish between sets and subsets of quadrilaterals).Thus, the skill of naming of shapes, objects, and parts could be applied in question 1, 3, 4 or 5. From Figure 30, the occurrences of the use of the skill in the post-intervention test in comparison to its application in the pre-intervention test is as follows: Learner 3 used the skill of naming of shapes, objects, and parts 15 times correctly in the post-intervention test; yet in the pre-intervention test used it once correctly and twice inappropriately. Learner 7 applied the skill 12 times correctly, twice incorrectly and once inappropriately in the post-intervention test, whereas in the pre-intervention test applied thrice correctly and once inappropriately. Learner 17 applied the skill of naming of shapes, objects, and parts 8 times correctly and twice inappropriately in the post-intervention test; while in the pre-intervention test used 5 times correctly, twice incorrectly and 6 times inappropriately. Learner 2 applied the skill of naming of shapes, objects, and parts 8 times correctly, 3 times incorrectly and 3 times inappropriately in the post-intervention test; while in the pre-intervention test twice correctly and 6 times inappropriately. Learner 11 applied the skill of naming of shapes, objects, and parts 16 times correctly, twice incorrectly, and once inappropriately in the post-intervention test; while in the pre-intervention test had employed the skill 5 times correctly and twice inappropriately. Learner 12 applied the skill of naming of shapes, objects, and parts 8 times correctly and 7 times incorrectly in the post-intervention test, whereas in the pre-intervention test applied once correctly, twice incorrectly and 4 times inappropriately. As shown in Figure 32, I used Learner 7 and 12 work below to demonstrate the above statement.

Kotzé (2007) argued that learners must first attach some form of meaning to objects (identification of geometric figures), which are consequently ordered together in groups that make sense to the learner (analysing the attributes of geometric figures, such as mastering properties of angles, sides, and diagonals). As displayed in Figure 32, learners in the pre-intervention test were able to name shapes, parts, and objects, but without answering questions at hand. The only difference with the post-intervention test is that in the post-intervention test, the learners were now able to give correct names in answering questions during problem-solving except with some learners, especially in the underperformance category, who, at times, still could make same mistakes as in the pre-intervention test.

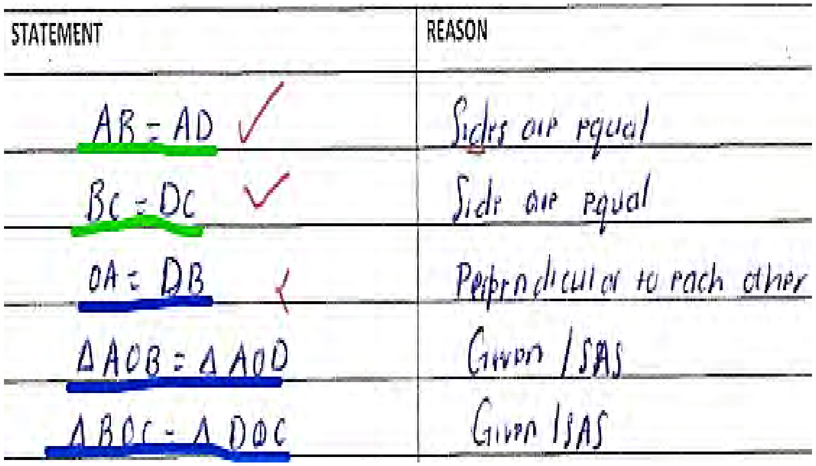
PRE-INTERVENTION TEST WORK

Learner 12



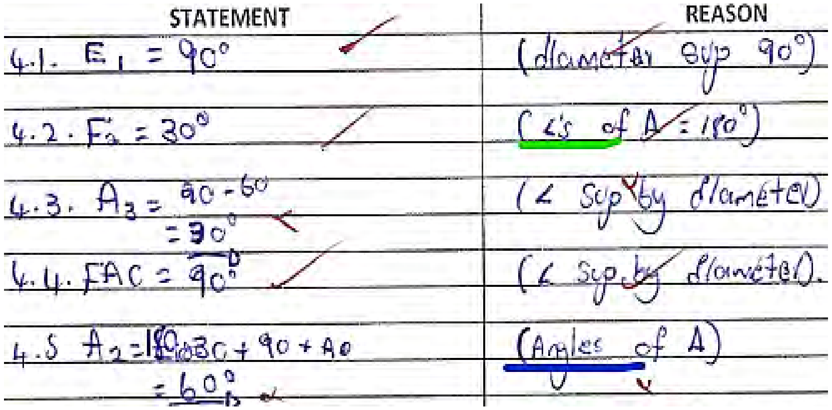
STATEMENT	REASON
BC straight line	~~straight line~~ Given
BCD parallel	~~parallel~~ Given
ABD triangle	Given
AC straight line	Given

Learner 07

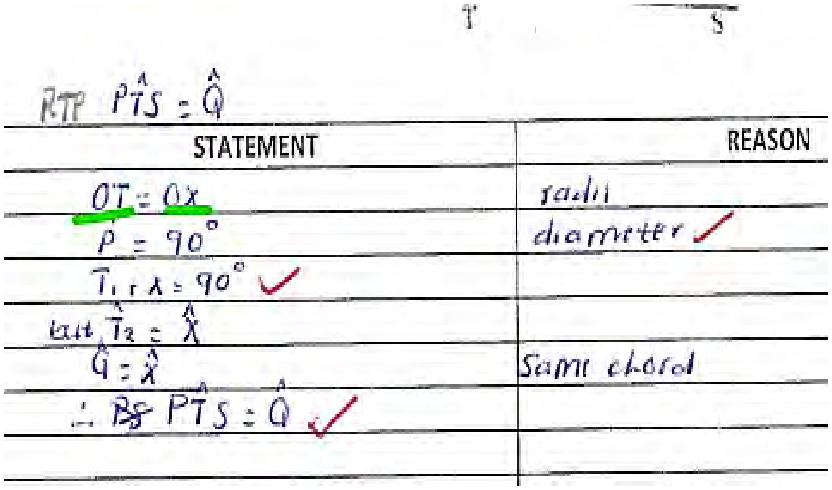


STATEMENT	REASON
AB = AD ✓	Sides are equal
BC = DC ✓	Side are equal
OA = DB	Perpendicular to each other
ΔAOB = ΔAOD	Given / SAS
ΔBOC = ΔDOC	Given /SAS

POST-INTERVENTION TEST WORK



STATEMENT	REASON
4.1. $E_1 = 90°$ ✓	(diameter sup 90°)
4.2. $F_2 = 30°$ ✓	(∠'s of Δ = 180°)
4.3. $A_3 = 90 - 60$ $= 30°$	(∠ sup by diameter)
4.4. FAC = 90° ✓	(∠ sup by diameter).
4.5 $A_2 = 180\ 30 + 90 + 60$ $= 60°$	(Angles of Δ)



RTP $\hat{PTS} = \hat{Q}$

STATEMENT	REASON
OT = OX	radii
$P = 90°$	diameter ✓
$T_1 + X = 90°$ ✓	
but $\hat{T}_2 = \hat{X}$	
$\hat{Q} = \hat{X}$	Same chord
∴ ~~BS~~ $P\hat{T}S = \hat{Q}$ ✓	

Figure 32: Naming of shapes, objects, and parts analysis

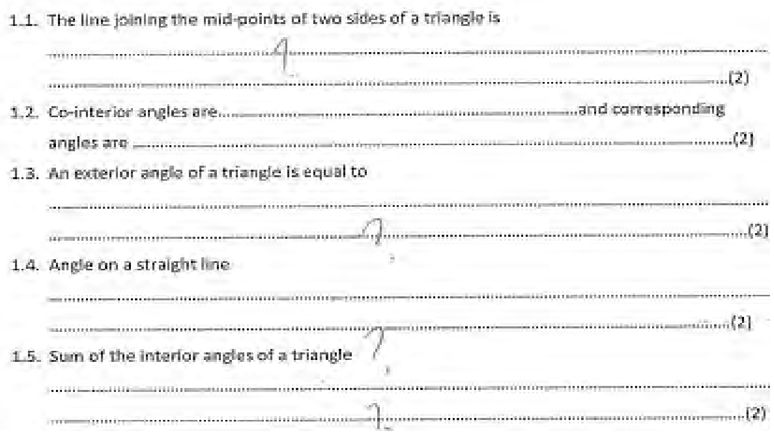
Learners' skill of Definitions and Stating in own words

The skill of defining and stating in own words is used in Geometry to state geometric theorems in most cases. This is part of verbalization, where one is expected to define and write down a geometric theorem of axiom. This skill was most developed during the introduction of Geometry and during learning of the first stage two stages of the four stages model. For one to be able to utilise the model effectively, defining and stating theorems is a requirement. Question 1, 3, 4 and 5 were demanding for the application of this skill. The incidence of the use of this skill by learners in the post-intervention test vs pre-intervention test (see Figure 30) is as follows: Learner 3 used the skill of definitions and stating in own words 12 times correctly and twice inappropriately in the post-intervention test while in the pre-intervention test never applied the skill. Learner 7 applied the skill correctly 10 times and inappropriately 3 times in the post-intervention test, whereas in the pre-intervention test applied it correctly once and 5 times incorrectly. Learner 17 utilised the skill of definitions and stating in own words 6 times correctly and 3 times inappropriately in the post-intervention test, while in the pre-intervention test, applied the skill once correctly and 4 times incorrectly. Learner 2 employed the skill 17 times correctly in the post-intervention test, yet in the pre-intervention test applied once correctly and thrice incorrectly. Learner 11 applied the skill of definitions and stating in own words twenty 3 times correctly and twice inappropriately in the post-intervention test, while in the pre-intervention test, applied the skill thrice correctly and thrice incorrectly. Learner 12 applied the skill of definitions and stating in own words 3 times correctly and 14 times incorrectly during the post-intervention test, yet in the pre-intervention test, employed the skill once correctly and 5 times incorrectly.

Learner 3's and 17's work was used below to illustrate how the skill has been applied (see Figure 33). The information below revealed there was then a big difference in the quantity of occurrences on the use of this skill in the post-intervention test compared to pre-intervention test. From the observations done in the classroom and on exercise books, most learners developed the skill during the planning stage of the model. Even the number of correct applications in the post-intervention test is far more than that of the pre-intervention test. However, some learners were failing to express themselves well when writing these theorems, with others confusing some of the theorems ending up writing the theorems with incomplete meanings. Generally, the application of this skill was better applied in the post-intervention test than it was in the pre-intervention test.

PRE-INTERVENTION TEST WORK **POST-INTERVENTION TEST WORK**

Learner 03



1.1. The line joining the mid-points of two sides of a triangle is

? ...(2)

1.2. Co-interior angles are ....and corresponding

angles are ....(2)

1.3. An exterior angle of a triangle is equal to

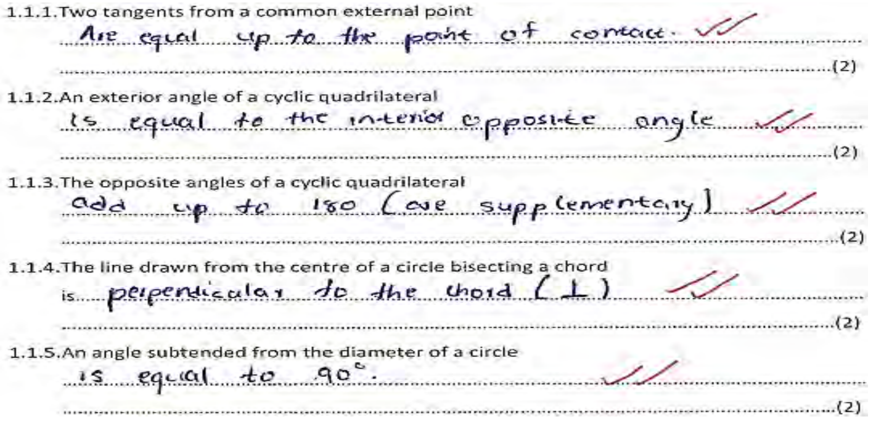
? ...(2)

1.4. Angle on a straight line

? ...(2)

1.5. Sum of the interior angles of a triangle

? ...(2)



1.1.1. Two tangents from a common external point

Are equal up to the point of contact. ✓✓ ...(2)

1.1.2. An exterior angle of a cyclic quadrilateral

is equal to the interior opposite angle ✓✓ ...(2)

1.1.3. The opposite angles of a cyclic quadrilateral

add up to 180 (are supplementary) ✓✓ ...(2)

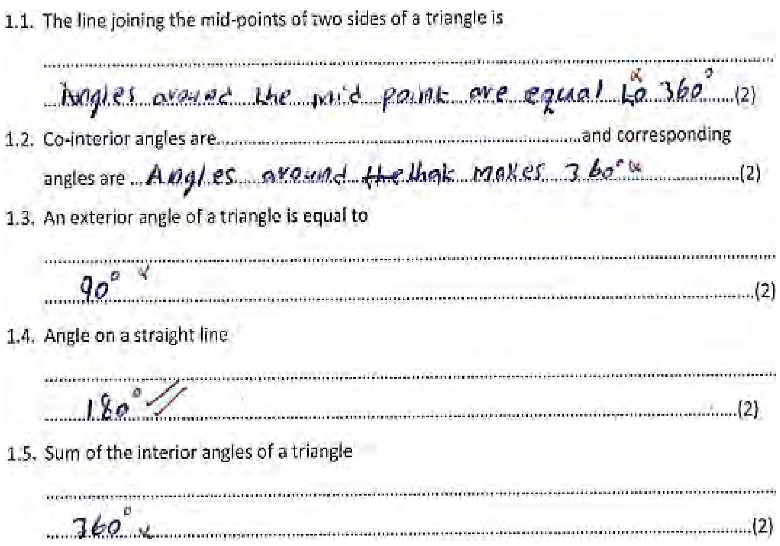
1.1.4. The line drawn from the centre of a circle bisecting a chord

is perpendicular to the chord (⊥) ✓✓ ...(2)

1.1.5. An angle subtended from the diameter of a circle

is equal to 90°. ✓✓ ...(2)

Learner 17



1.1. The line joining the mid-points of two sides of a triangle is

Angles around the mid point are equal to 360° ✗ ...(2)

1.2. Co-interior angles are ....and corresponding

angles are Angles around ~~the~~ that makes 360° ✗ ...(2)

1.3. An exterior angle of a triangle is equal to

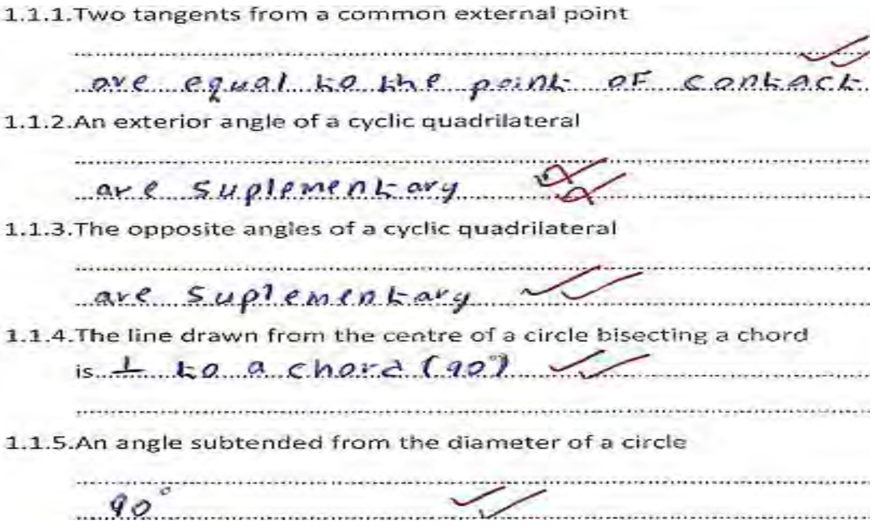
90° ✗ ...(2)

1.4. Angle on a straight line

180° ✓✓ ...(2)

1.5. Sum of the interior angles of a triangle

360° ✗ ...(2)



1.1.1. Two tangents from a common external point

are equal to the point of contact ✓✓

1.1.2. An exterior angle of a cyclic quadrilateral

are suplementary ✗✗

1.1.3. The opposite angles of a cyclic quadrilateral

are suplementary ✓✓

1.1.4. The line drawn from the centre of a circle bisecting a chord

is ⊥ to a chord (90°) ✓✓

1.1.5. An angle subtended from the diameter of a circle

90° ✓✓

Figure 33: Analysis of definitions and Stating in own words

Drawing Skills

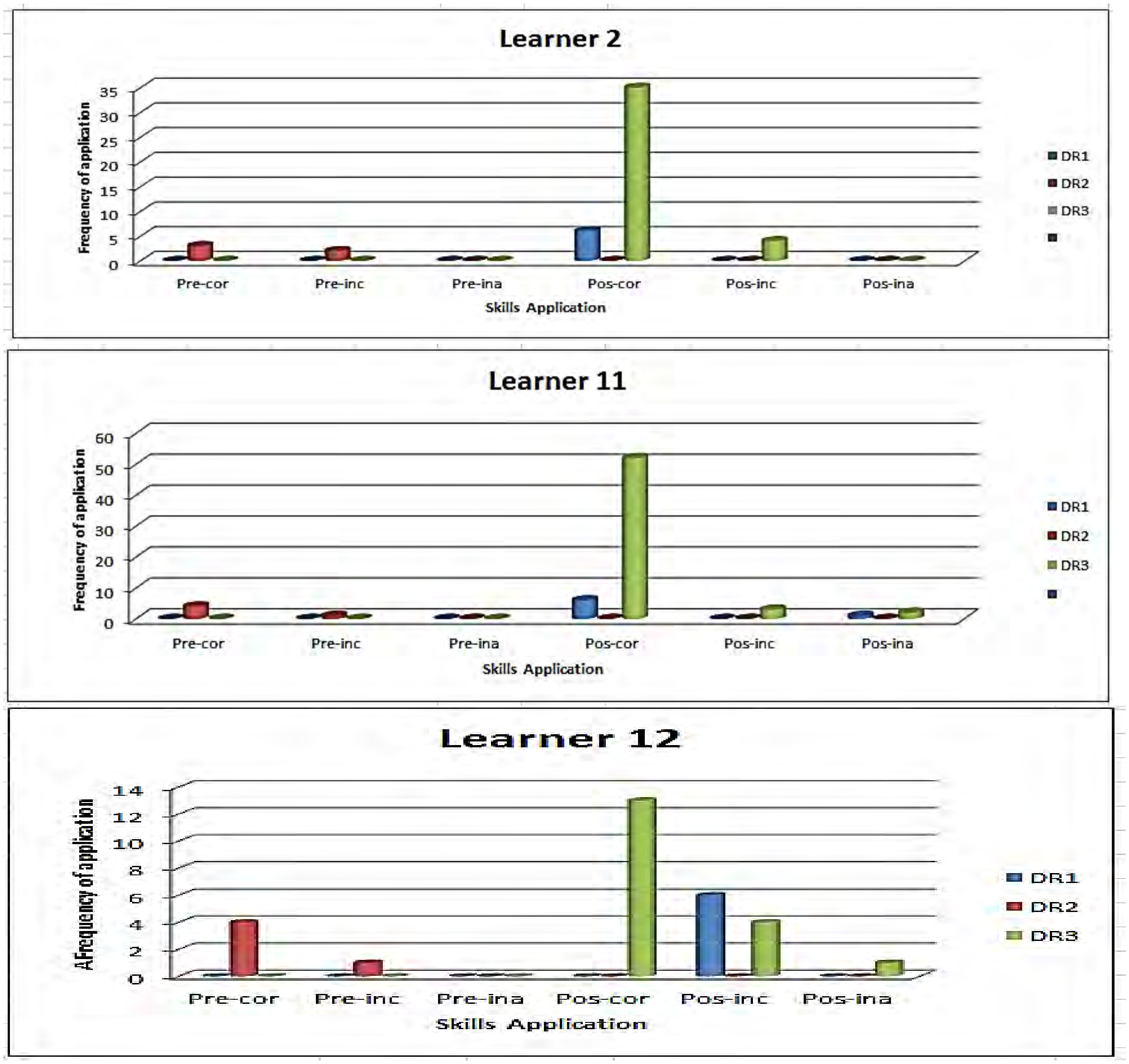


Figure 34: Drawing skills analysis

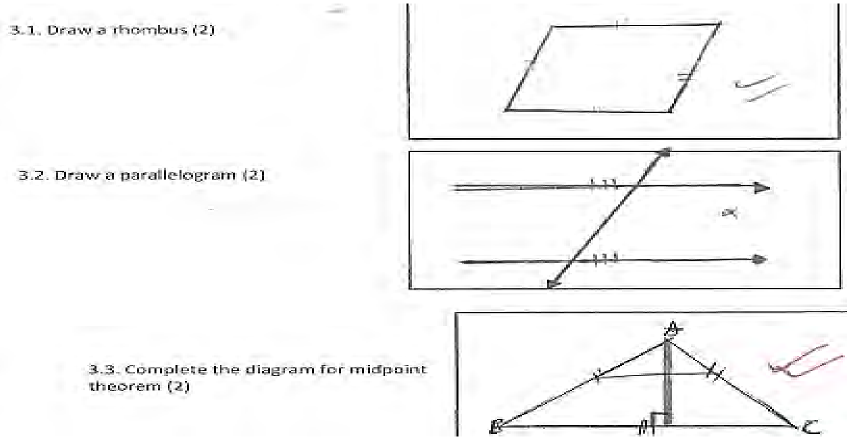
Learners' skill of constructing of diagrams based on its properties.

The skill of construction of diagrams based on their properties was employed by the six learners in question 3 of the pre-intervention test and all the learners failed to apply the skill in the post-intervention test except Learner 12. The reason might be that at FET there is no part where learners are tested on constructing diagrams based on their properties. During classroom teaching and learning as well as utilization of the four stages model, the teacher did not include much use of this skill as there was nowhere it could be tested for. The frequency of use of the skill in the pre-intervention test compared to post–intervention test (see Figure 34) was as follows: Learner 3 applied the skill of constructing diagrams based on its properties only once correctly and four times incorrectly. Learner 11 applied the skill four times correctly and once incorrectly on the same question. Learner 12 applied the skill of constructing diagrams based on its properties correctly in 4 situations and incorrect in 1 situation in the pre-intervention test whereas in the post-intervention test applied the skill once correctly. Learner 2 applied the skill of constructing diagrams based on its properties 5 times and was correct in 3 cases. The learner was incorrect when applying the skill in 2 scenarios. The skill of constructing diagrams based on its properties was applied in the same manner by Learners 7, 12 and 17. Learner 7 applied the skill correctly in 4 scenarios and incorrectly in 1 scenario. Learner 17 also made use of the skill of constructing diagrams based on its properties in 5 cases, where in one case it was correct application but in 4 cases it was incorrect. Figure 35 shows extracts of the work by Learner 11 and Learner 12 to show how the skill has been applied by the learners.

As shown in the Figure 35, there was a small section where learners were supposed to make constructions of diagrams based on their properties and the diagrams were most targeting shapes that relate to theorems such as circle, triangles, cyclic quadrilaterals, and others. One learner who tried to make use of the skill did that during application of the re-planning emerging skill. This was after proving a shape to be a cyclic quadrilateral then had to come back on the diagram and make a sketch of the circle that could make up the cyclic quadrilateral so that could further use it to find the other solutions. Even from classwork observations, the skill of construction of diagrams based on their properties was indirectly applied, and some learners at some points could show that they had forgotten some of the properties of some shapes. All in all, the skill was poorly demonstrated in the post-intervention test compared to the pre-intervention test.

PRE-INTERVENTION TEST WORK

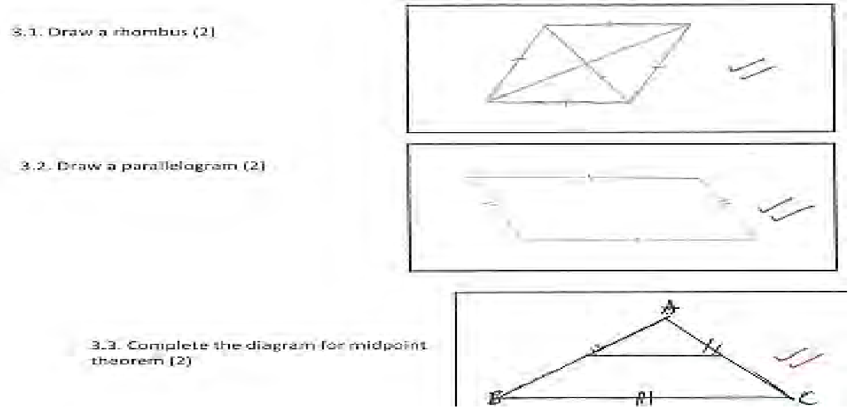
Learner 12


3.1. Draw a rhombus (2)

3.2. Draw a parallelogram (2)

3.3. Complete the diagram for midpoint theorem (2)

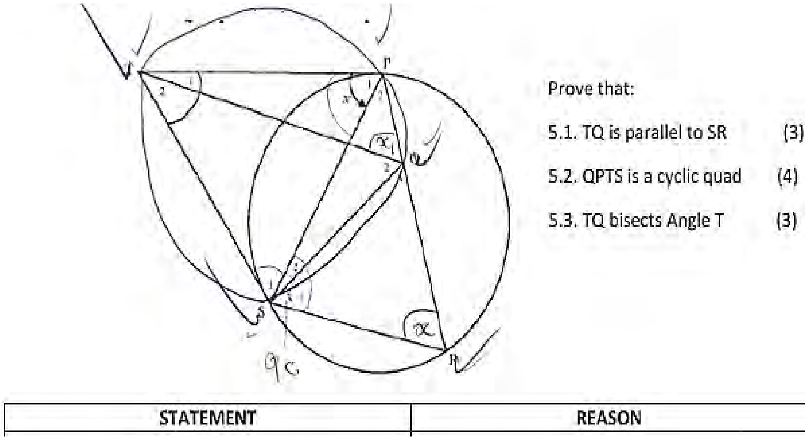
Learner 11


3.1. Draw a rhombus (2)

3.2. Draw a parallelogram (2)

3.3. Complete the diagram for midpoint theorem (2)

POST-INTERVENTION TEST WORK



Prove that:

5.1. TQ is parallel to SR (3)

5.2. QPTS is a cyclic quad (4)

5.3. TQ bisects Angle T (3)

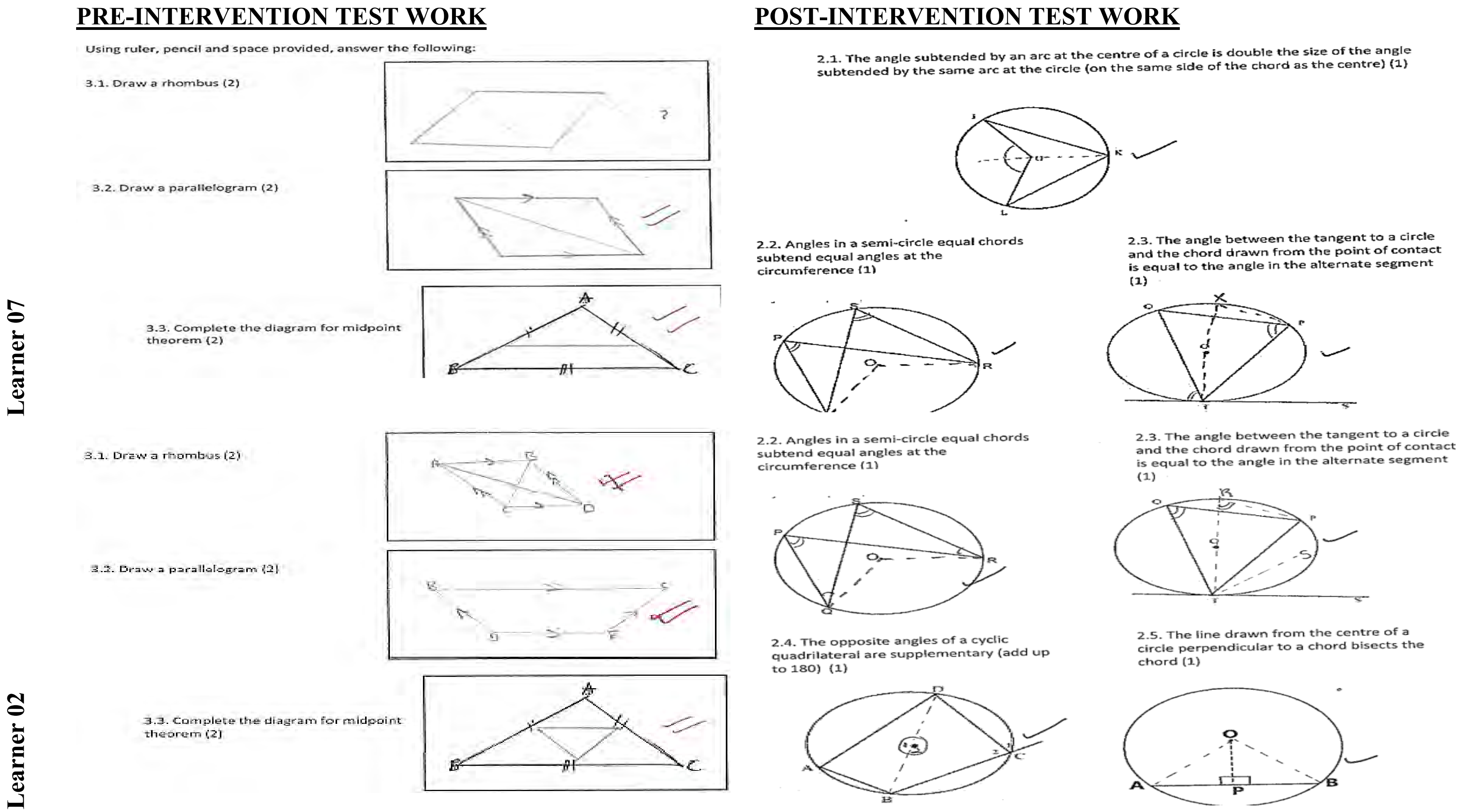
STATEMENT	REASON

Figure 35: Analysis of Learners' skill of constructing of diagrams based on its properties

Learners' skill of Sketching diagrams related to theorems and/or axioms.

Sketching diagrams is believed to be a very useful part of problem-solving in Geometry. According to Nunokawa (2006), the contribution of drawings in generating new information is to make an unintended combination or configuration of known elements which leads to an emergent pattern. Nunokawa (2006), further argued that even when the solver can find an appropriate emergent pattern, the information the solver will obtain is not necessarily an idea that will directly imply solutions, but additional information that can deepen the solver's understanding only a little further. The skill of sketching diagrams related to theorems and/or axioms was learnt during stage 2 (planning stage) of the four stages model. In this case, one was expected to make constructions to be able to make deductive proofs or sketches that will be used to dissemble a diagram into different theorems for problem-solving purposes. The skill of Sketching diagrams related to theorems and/or axioms was more applicable to Question 2, 3 and 5 of the post-intervention test. All the learners failed to apply this skill in the pre-intervention test. Learners were found applying the skill of sketching diagrams related to theorems and/or axioms in the post-intervention test as follows as shown in Figure 34. Learner 2 employed the skill of Sketching diagrams related to theorems and/or axioms 6 times correctly. Learner 3 also applied 6 times correctly as well as Learner 7. Learner 11 used the same skill 6 times correctly and once inappropriately. Learner 12 applied the skill of sketching diagrams related to theorems and/or axioms 6 times incorrectly and Learner 17 applied the skill of sketching diagrams related to theorems and/or axioms 5 times correctly. This is demonstrated using work from learner 2 and 7 (see Figure 36).

As displayed in Figure 36, the learners started to utilise the skill in the classroom during the use of the four stages model. The learners in class activities were using this skill to be able to deal with a problem at hand. As was noted by Todd (2021) that most given geometric diagrams can be in fact a parametrized family of diagrams, it means one's ability to break down the diagram into the family member diagrams can be a break through. Besides, in answering questions, I was also expecting learners to display this skill in planning for problem-solving, but I could not find any in the post-intervention test who did that. During class activities learners were encouraged to incorporate some drawings as some of the planning technique. Some could apply it at some point even though there were very few. I suspected that most learners found it unnecessary to make drawings if one believes he understands the theorem itself.



PRE-INTERVENTION TEST WORK

Using ruler, pencil and space provided, answer the following:

3.1. Draw a rhombus (2)

3.2. Draw a parallelogram (2)

3.3. Complete the diagram for midpoint theorem (2)

3.1. Draw a rhombus (2)

3.2. Draw a parallelogram (2)

3.3. Complete the diagram for midpoint theorem (2)

POST-INTERVENTION TEST WORK

2.1. The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre) (1)

2.2. Angles in a semi-circle equal chords subtend equal angles at the circumference (1)

2.3. The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment (1)

2.2. Angles in a semi-circle equal chords subtend equal angles at the circumference (1)

2.3. The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment (1)

2.4. The opposite angles of a cyclic quadrilateral are supplementary (add up to 180) (1)

2.5. The line drawn from the centre of a circle perpendicular to a chord bisects the chord (1)

Figure 36: Sketching diagrams related to theorems and/or axioms analysis

Learners' skill of adding useful elements to a diagram

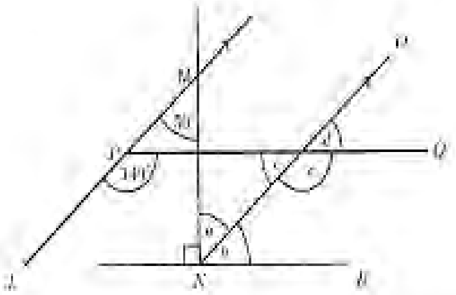
The skill of adding useful elements to a diagram was never applied by any of the learners in the pre-intervention test. The skill was only found applied in the post-intervention test. Schoenfeld (1985) cited by Jones (2013) argued that more explicit teaching that focuses on the uses of diagrams (for example, what to read from diagrams) might be helpful for learners of Mathematics. This might apply in Geometry if the learners can make use of the diagrams enough to solve geometric problems. In this case, the skill was well applicable during the teaching and learning of the utilisation of stage 2 of the four stages model which is the planning stage. After listing the theorems related to the key words, one is expected to go on the diagram and add all applicable information deduced from the theorems stated and any other relevant information. In question 2, 3, 4 and 5, this skill of adding useful elements to a diagram could be applied in one way or another. As is displayed in Figure 34, the frequency of the application of the skill of adding useful elements to a diagram as was applied in the post-test is as follows: Learner 2 employed the skill 35 times correctly and 4 times incorrectly in the post-intervention test, with Learner 3 applying the skill of adding useful elements to a diagram 41 times correctly. Learner 7 applied the skill of adding useful elements to a diagram 24 times correctly in the post-intervention test, with Learner 11 applying it 52 times correctly, 3 times incorrectly and twice inappropriately in the post-intervention test. Learner 12 employed the skill of adding useful elements to a diagram 13 times correctly, 4 times incorrectly and once inappropriately in the post-intervention test; while Learner 17 applied the skill of adding useful elements to a diagram 21 times correctly. To shed more light on the application of the skill, work from Learner 11 and 17 was used as displayed in Figure 4.34.

Figure 37 shows that learners started to apply the skill in the classroom activities. The same skill was seen being applied by every learner in the post-intervention test in one way or another. The skill was taught in proving examinable theorems, solving riders, as well as making deductive proofs. When doing document analysis during class activities, I realised that this skill was one of the widely applied skills since it was a requirement when using the four stages model to do part of the planning by adding some information on the diagram. Of course, no marks were awarded for this skill as it acted to an end. This is a powerful skill when combined with other skills, because if one could apply the skill right but fail to present the arguments for solutions, one would not get any marks.

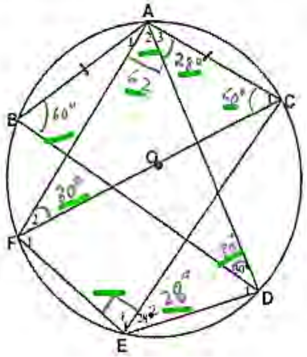
PRE-INTERVENTION TEST WORK | **POST-INTERVENTION TEST WORK**

Learner 11

5. Find each of the unknown angles marked in the figure below. Give a reason for your answers (2X5=10)



In the diagram, O is the centre of the circle. Chords AB = AC. Angle CED = 28° and ADB = 30°



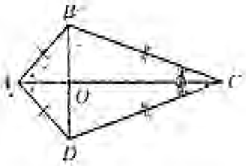
Calculate, with reasons, the sizes of the following angles:

4.1. E_1 (2)
4.2. F_2 (2)
4.3. A_3 (2)
4.4. FAC (2)
4.5. A_2 (2)

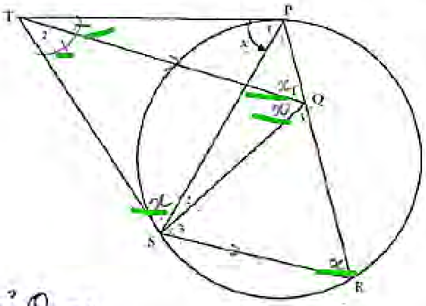
Learner 17

Question Four

Use the sketch of quadrilateral ABCD to prove that the diagonals of a kite are perpendicular to each other (10)



In the figure below, TP and TS are tangents to the given circle. R is a point on the circumference. Q is a point on PR such that angle $Q_1 = P_1$ and SQ is drawn. Let angle P_1 be x



Prove that:

5.1. TQ is parallel to SR (3)
5.2. QPTS is a cyclic quad (4)
5.3. TQ bisects Angle T (3)

RTP: $Q_2 = S_1$

Figure 37: Adding useful elements to a diagram analysis

Logical Skills

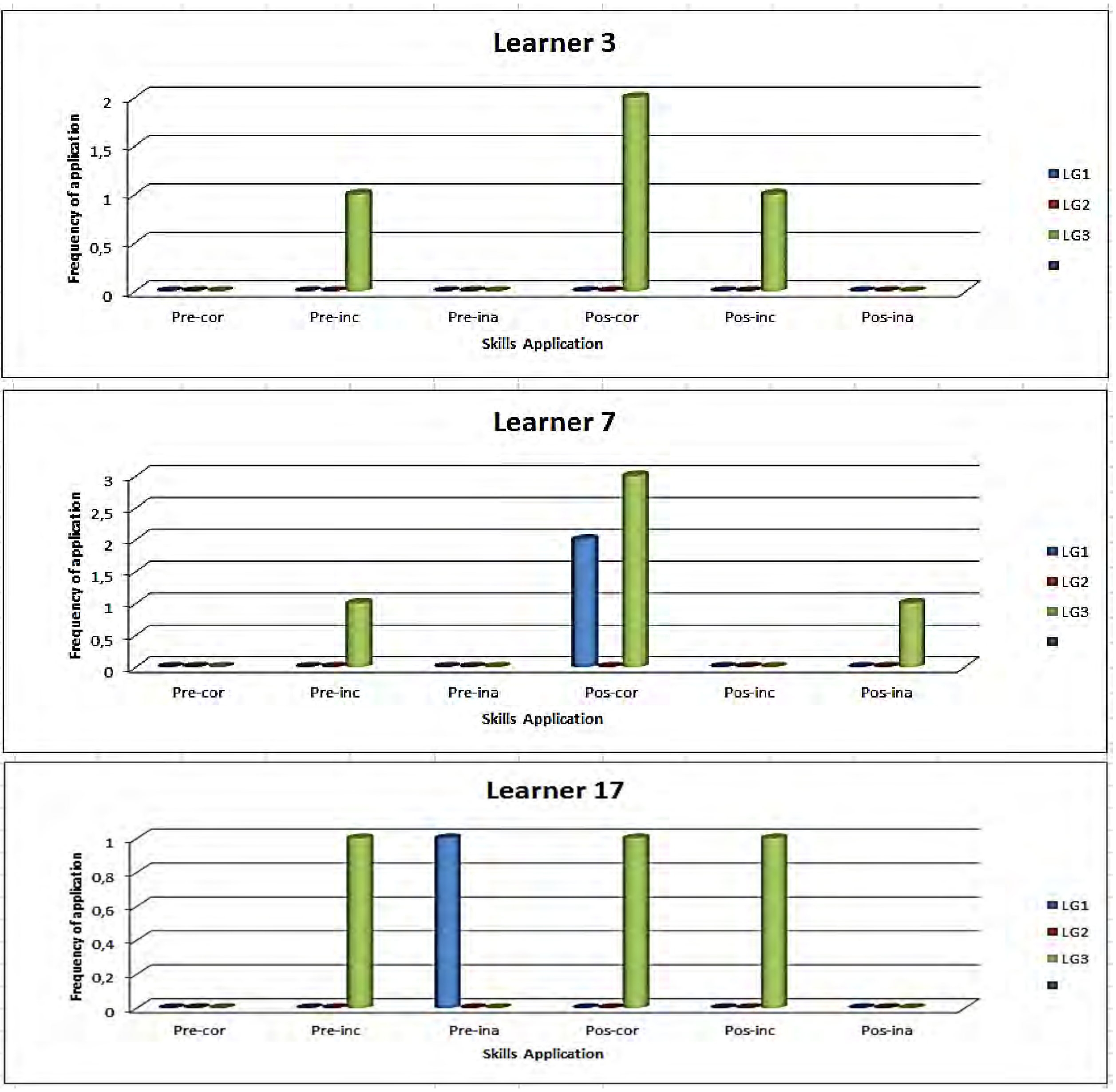


Figure 38: Logical skills analysis

Learners' development of logical proofs skills

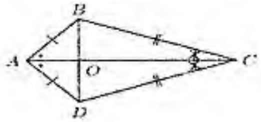
The skills of recognition of differences and similarities between geometric objects and that of classification of geometric objects by their properties were only applied by few learners in the pre-intervention test. There was no evidence of the application of these two skills in the post-intervention test. The only skill that had evidence of its application in the post-intervention test was the skill of development of logical proofs. Cheng & Lin (2009) states that geometric proof development is a process of constructing a sequence of arguments from X to Y with supportive reasons, and hence, is also called deductive reasoning and proving. The skill to make logical proofs was applicable to question 3 and 5. The regularity of its use in the post-intervention test vs in the pre-intervention test (see Figure 38) goes like this: Learner 2 applied the skill twice correctly, once times incorrectly, and once incompletely in the post-intervention test; while in the pre-intervention test, they managed to employ the skill once incorrectly. Learner 3 applied the skill twice correctly and once incorrectly in the post-intervention test, yet in the pre-intervention test, had used the skill once incorrectly. Learner 7 utilised the skill in the post-intervention test 3 times correctly and once incompletely, while in the pre-intervention test, they applied the same skill once incorrectly. Learner 11 applied the skill 3 times correctly and once incorrectly in the post-intervention test, while in the pre-intervention test, they applied the skill once incorrectly as well. Learner 12 applied the skill in the post-intervention test 4 times incorrectly, yet in the pre-intervention test applied the skill once incorrectly and once inappropriately. Learner 17 also applied the skill in the post-intervention test once correctly and once incorrectly, while in the pre-intervention test applied the skill once incorrectly. For example, Learner 2's and 7's work below can elucidate how the skill was applied in general (see Figure 39).

As shown in Figure 39, this skill was scarcely applied in the pre-intervention test and was one of the most targeted areas during classroom lessons and utilisation of the four stages model. I started teaching logical proofs when approaching the concept of solving examinable theorem proofs. Most learners, including those in the under-performance category, could do these proofs correctly. I suppose some learners were just cramming the steps involved in proving these theorems since they had failed to prove the very same theorems in the post-test, yet got the proofs correctly in the class activities. The concept developed when approaching proofs of tangents, diameter, parallel lines, isosceles triangle, angle bisection, cyclic quadrilateral and so forth. Such proofs required much reasoning and problem-solving skills including the skill of finding an RTP.

PRE-INTERVENTION TEST WORK

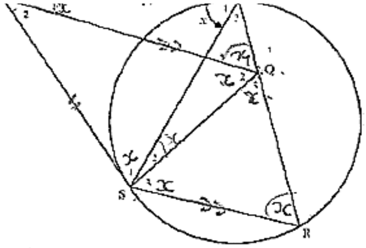
POST-INTERVENTION TEST WORK

Learner 07



STATEMENT	REASON
AB = AD ✓	Sides are equal
BC = DC ✓	Side are equal
OA = DB	Perpendicular to each other
ΔAOB = ΔAOD	Given / SAS
ΔBOC = ΔDOC	Given / SAS

STATEMENT	REASON
5.1 RTP: $\hat{R} = \hat{Q}_1$	
$\hat{P}_1 = R = x$	?
$\hat{Q}_1 = \hat{P}_1 = x$	given ✓
∴ $\hat{Q}_1 = \hat{R}$ ✓	
∴ TQ is parallel to SR	
5.2 RTP: $\hat{S}_1 = \hat{Q}_1$	
$\hat{P}_1 = \hat{Q}_2 = x$	same chord?
$\hat{S}_1 = \hat{P}_1 = x$	?
$\hat{Q}_1 = \hat{P}_1 = x$	given ✓
∴ $\hat{S}_1 = \hat{Q}_1$	
∴ QPTS is a cyclic quard ✓	
5.3 RTP: $\hat{T}_1 = \hat{T}_2$	
$\hat{P}_1 = \hat{S}_2 = x$	alternate
∴ $\hat{T}_1 = \hat{S}_2 = x$	Same chord ✓
$\hat{P}_1 = \hat{T}_2 = x$	same chord



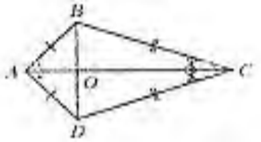
Prove that:

5.1. TQ is parallel to SR (3)

5.2. QPTS is a cyclic quad (4)

5.3. TQ bisects Angle T (3)

Learner 02



STATEMENT	REASON
ΔBCO = ΔDCO	all ~~side are equal~~ SAS Sides are equal
ΔΔABC ≡ ΔΔADC	the angles Δ are perpend parallel
line AC // Δ BDC	line is ⊥ to the surface
~~AB = A~~ AB = AD	Sides are equal
Δ ~~ABD~~ ΔABD = ΔBDC	SAS

RTP: ~~$R = Q_1$ Corresponding Angles~~ $P_1 = S_1 = x$ opp ∠ of Δ

STATEMENT	REASON
$P_1 = Q_1 = x$	Given ✓
$R = S_1 = x$	tan-chord.
$Q_1 = P_1 = x$	Same chord
$Q_2 = S_3 = x$	~~alternat corr~~ alternating ∠ are =
∴ $R = Q_1 = x$ ✓	~~corr~~ Converse (corresponding angles)
RTP: $P_1 = S_1 = x$ opp angles of Δ	
statement	Reasons
$P_1 = Q_1 = x$	✓ corresponding ∠s are =
$Q_1 = T_1 + T_2 = x$	exterior ∠ = interior opp ∠
$P_1 = S_2 = x$	alternating ∠'s

tangent
- tangent ⊥r
- 2 tangent
- tan-chord theorem

Parallel
- alternating
- correspond
- co-interior

Figure 39: Development of logical proofs skills analysis

Application Skills

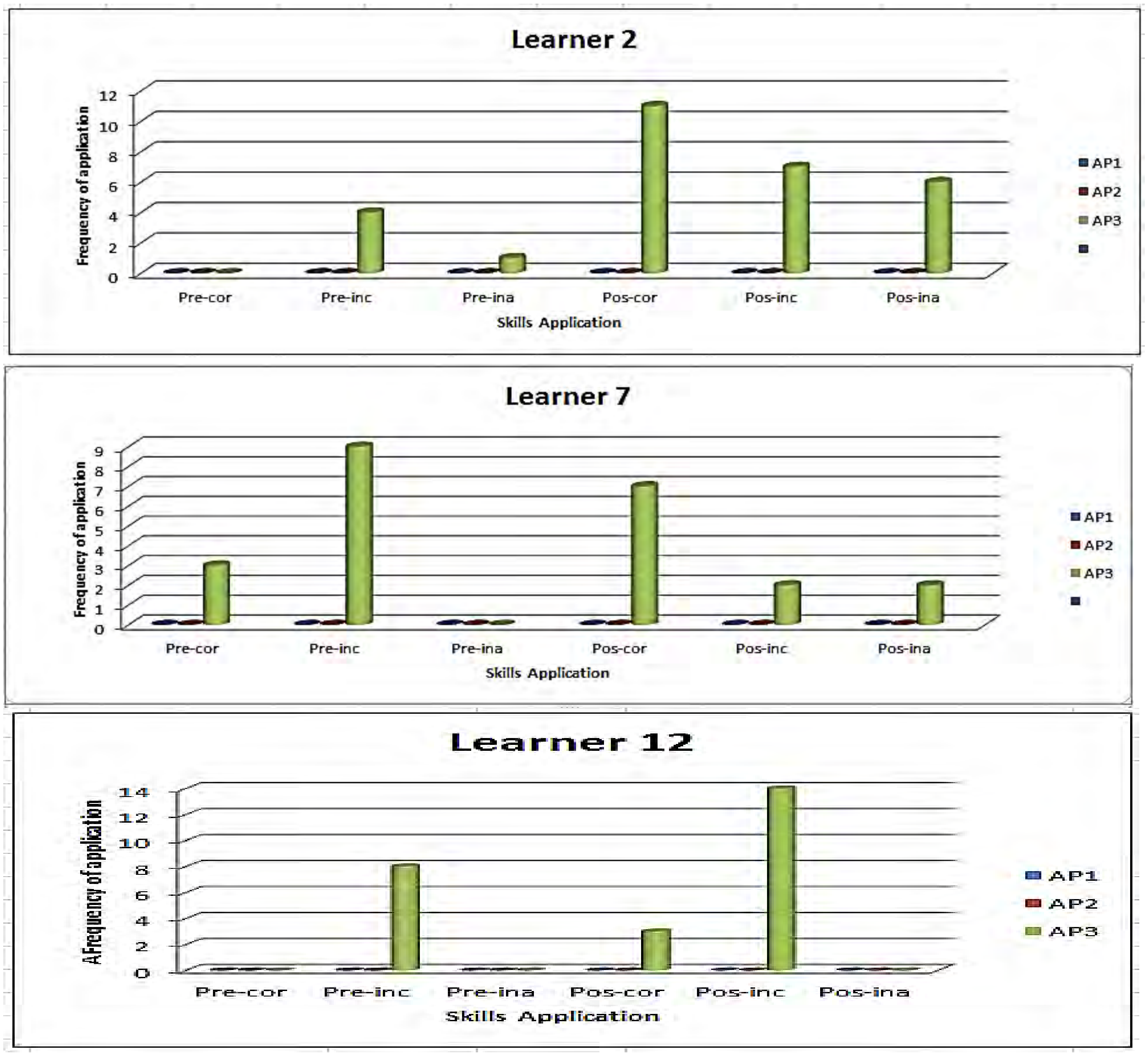


Figure 40: Application skills analysis

Application of theorems and axioms in problem-solving skills by learners

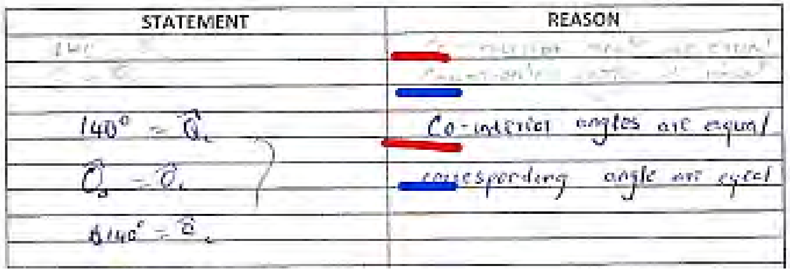
Learners employed the skill of application of theorems and axioms in problem-solving in question 3, 4 and 5. This skill is one of the extensively used one by learners in the post-intervention test. From Figure 40, the frequency of use of the skill by learners in the post-intervention test compared to pre-intervention test is like this: Learner 2 applied the skill 11 times correctly, 7 times incorrectly and 6 inappropriately in the post-intervention test, yet in the pre-intervention test had applied once inappropriately and 4 times incorrectly. Learner 3 utilised the skill 12 times correctly and twice inappropriately in the post-intervention test, while in the pre-intervention test had applied it incorrectly 5 times. Learner 7 also applied the skill 7 times correctly, twice incorrectly and twice inappropriately in the post-intervention test, whereas in the pre-intervention test had applied the skill thrice correctly and 9 times incorrectly. Learner 11 employed the skill in the post-intervention test 15 times correctly and once inappropriately, whilst in the pre-intervention test applied the same skill once correctly and 4 times incorrectly. Learner 12 used the skill 3 times correctly and 14 times incorrectly in the post-intervention test, while in the pre-intervention test applied the skill 8 times incorrectly. Learner 17 employed the skill in the post-intervention test 6 times correctly, twice incorrectly and thrice inappropriately, yet in the pre-intervention test applied twice inappropriately and 7 times incorrectly. I gave an example using learner 3 and 17 (see Figure 41).

The skill of application of theorems and axioms in problem-solving was expected to be done mostly in supporting geometric statements. As displayed in Figure 41, there was a wide movement in the use of the skill from the pre-intervention test to the post-intervention test. All learners from all the categories showed a noticeable shift in the quantity of the use of the skill. There was a rise in the correct use of the skill compared to the number of incorrect and inaccurate applications of the skill. This is a sign of strength because this is one of the areas where marks are awarded for in Geometry. In other words, correct values or statements with wrong reasons are not awarded marks in the examination marking centers. Therefore, I gave emphasis on how to make meaningful geometric arguments. This was mostly learnt during the implementation stage of the four stages model. In this case, it is the stage when the learner starts to get marks in his/her problem-solving process. However, most learners in the underperforming category had challenges with the skill of application of theorems and axioms in problem-solving in making logical geometric proofs.

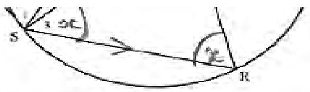
PRE-INTERVENTION TEST WORK

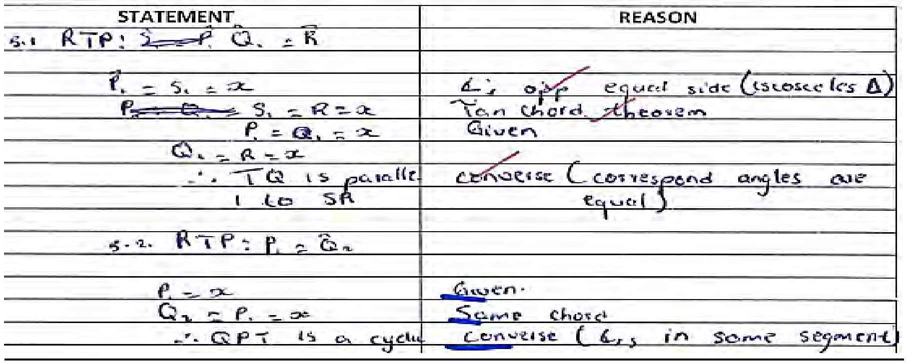
POST-INTERVENTION TEST WORK

Learner 03



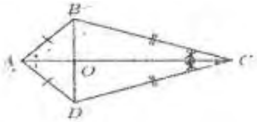
STATEMENT	REASON
[illegible]	[illegible]
[illegible]	[illegible]
140° = $\hat{Q}_1$	Co-interior angles are equal
$\hat{Q}_2$ = $\hat{Q}_1$	corresponding angle are eqeal
140° = $\hat{Q}_2$	

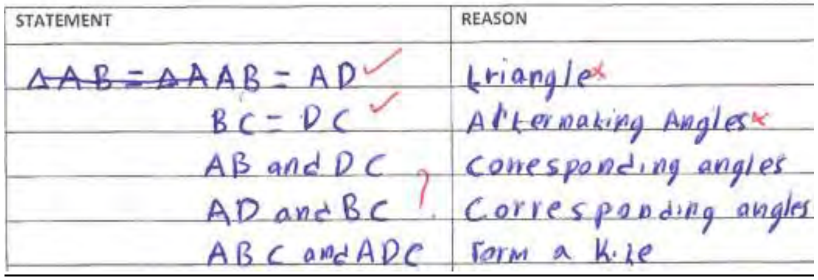




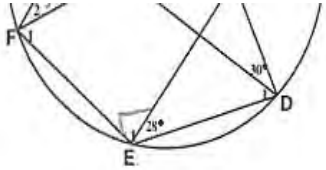
STATEMENT	REASON
5.1 RTP: ~~$\hat{S}$~~ $\hat{P}$ $\hat{Q}_1$ = $\hat{R}$	
$\hat{P}_1$ = $\hat{S}_1$ = x	∠s opp equal side (isosceles Δ)
~~P = Q~~ S_1 = R = x	Tan chord theorem
P_1 = Q_1 = x	Given
Q_1 = R = x	
∴ TQ is parallel to SR	converse (correspond angles are equal)
5.2. RTP: P_1 = $\hat{Q}_2$	
P_1 = x	Given.
Q_2 = P_1 = x	Same chord
∴ QPT is a cyclic	Converse (∠s in same segment)

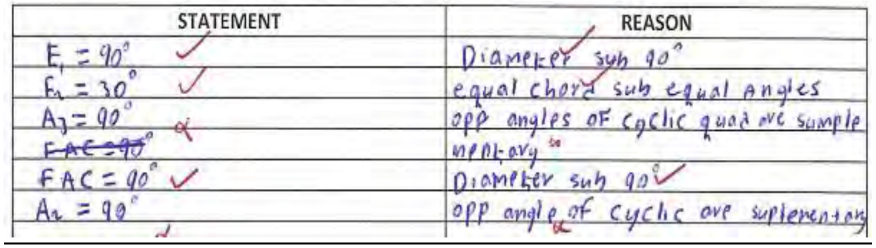
Learner 17





STATEMENT	REASON
~~ΔAB = ΔA~~ AB = AD ✓	triangle ✗
BC = DC ✓	Alternating Angles ✗
AB and DC	Corresponding angles
AD and BC ?	Corresponding angles
ABC and ADC	form a kite





STATEMENT	REASON
E_1 = 90° ✓	Diameter ✓ sub 90°
F_1 = 30° ✓	equal chord ✓ sub equal angles
A_3 = 90° ✗	opp angles of cyclic quad are supplementary
~~FAC = 90°~~	
FAC = 90° ✓	Diameter sub 90° ✓
A_2 = 90°	opp angle ✗ of cyclic are suplementary

Figure 41: Application of theorems and axioms in problem-solving analysis

Emerging Themes

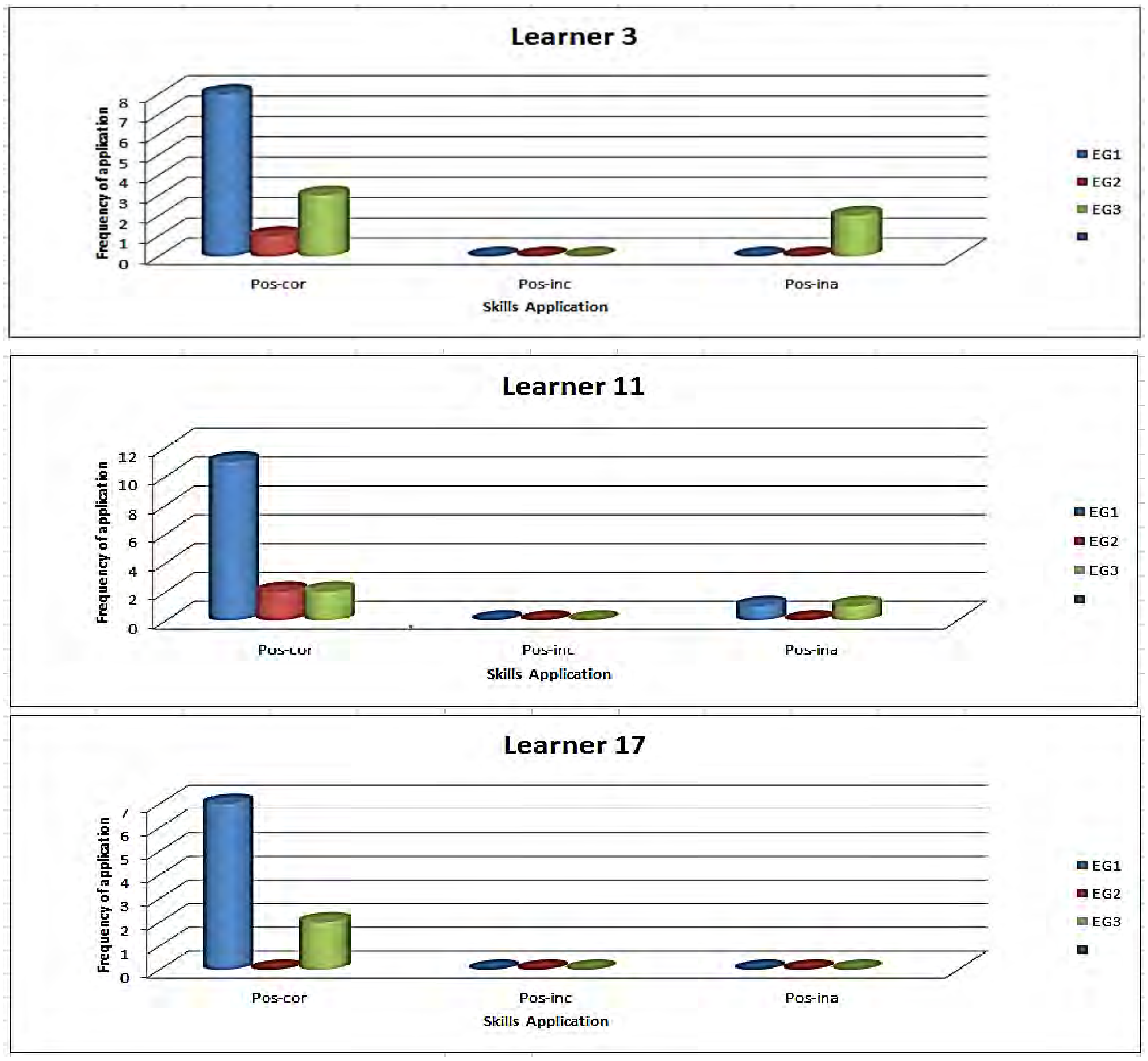


Figure 42: Emerging skills analysis per learner

Emerging themes comprised all the skills that were realised during teaching and learning not originating from literature. These skills include re-planning skill, computational skill and skill of finding RTP.

Learners’ *re-planning skill.*

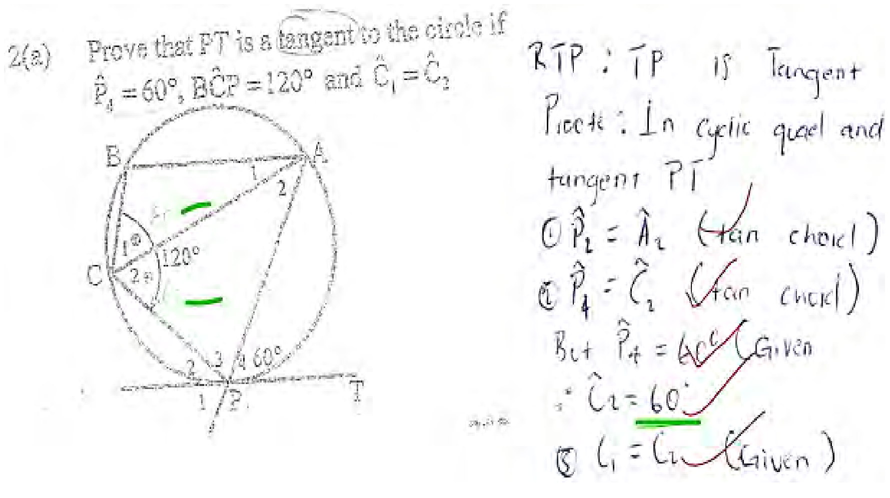
Re-planning skill is the ability to take answers got and put back in the situation to help find other answers. This skill was discovered as a solution to one of the weaknesses of Polya’s model. The four stages model is linear in nature, making one stage followed by another but in reality, I understood that the stages interlink. For example, re-planning needs one to reflect (stage 4) on the answer, and if it is correct take it back to the diagram to do a further analysis (stage 2) and find more solutions. The skill was expected to be applied to question 3, 4 and 5 of the post-intervention test paper whereby any answer got could be taken back in the diagram to assist finding of other solutions. During the application of the Polya model, the skill of re-planning was found very useful and commonly used by learners in problem-solving. The re-planning skill was also applied by several learners in the post-intervention test. The rate of use of this skill in the post-intervention test is as follows: Learner 2 applied the re-planning skill 10 times correctly and twice inappropriately. Learner 3 applied the same skill 8 times correctly in the post-intervention test. Learner 7 applied the re-planning skill 8 times correctly, whereas Learner 11 applied that same skill 11 times correctly and once inappropriately. Learner 12 applied the re-planning skill 5 times correctly with Learner 17 applying the same skill 7 times correctly. I used learner 3's, 11's and 17's their work to shed more light on how they were generally applying the skill.

As illustrated in Figure 4.43 below, the learners were taught to find answers and take them back to the diagram to use them to find the other angles. Learners did this in the classroom as well as in the post-test, although some few learners failed to follow this when writing the post-intervention test. This skill helps learners to get the actual sense of problem-solving which, according to Atsara, Sukoriyanto and Muksar (2021) refers to one's ability to thoroughly understand the question, analyse the provided information, and check the correct strategy for synthesizing the solution to the problem. Thus, by taking other solutions to find other solutions, learners developed an understanding that problem solving is a process and requires one to take different parts and synthesize them to come up with a solution.

CLASSROOM WORK

POST-INTERVENTION TEST WORK

Learner 12



2(a) Prove that PT is a tangent to the circle if $\hat{P}_4 = 60°$, $B\hat{C}P = 120°$ and $\hat{C}_1 = \hat{C}_2$

RTP : TP is Tangent

Proof : In cyclic quad and tangent PT

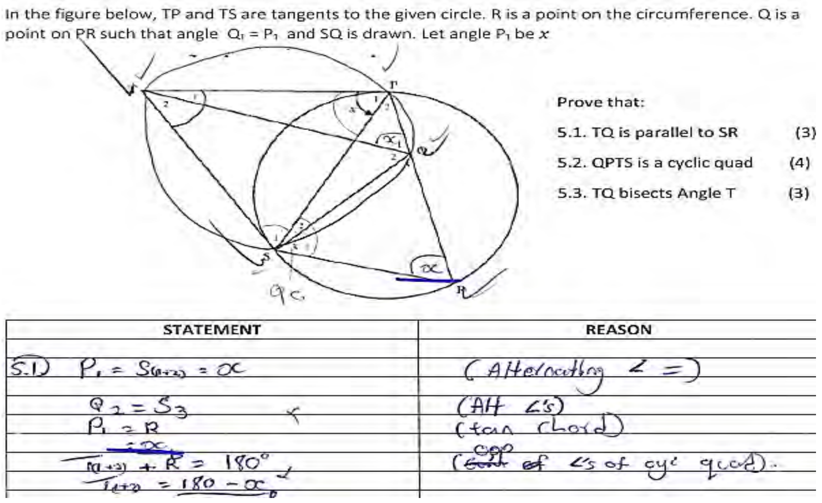
① $\hat{P}_2 = \hat{A}_2$ (tan chord)

② $\hat{P}_4 = \hat{C}_2$ (tan chord)

But $\hat{P}_4 = 60°$ (Given)

$\therefore \hat{C}_2 = 60°$

③ $C_1 = C_2$ (Given)



In the figure below, TP and TS are tangents to the given circle. R is a point on the circumference. Q is a point on PR such that angle $Q_1 = P_1$ and SQ is drawn. Let angle P_1 be x

Prove that:

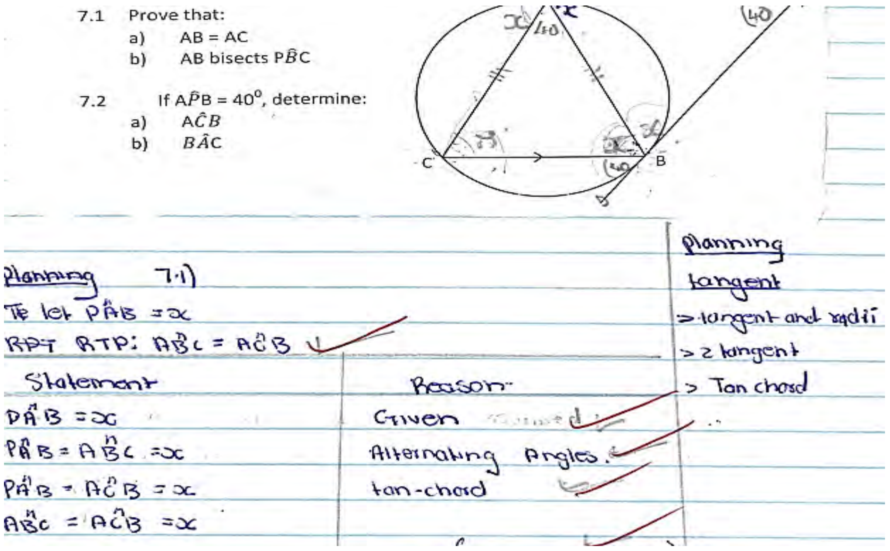
5.1. TQ is parallel to SR (3)

5.2. QPTS is a cyclic quad (4)

5.3. TQ bisects Angle T (3)

STATEMENT	REASON
5.1) $P_1 = S_{(1+2)} = x$	(Alternating ∠ =)
$Q_2 = S_3$	(Alt ∠'s)
$P_1 = R$	(tan chord)
$= x$	opp
$T_{(1+2)} + R = 180°$	(~~Sum~~ ~~of~~ ∠'s of cyc quad)
$T_{(1+2)} = 180 - x$	

Learner 02



7.1 Prove that:

a) AB = AC

b) AB bisects $P\hat{B}C$

7.2 If $A\hat{P}B = 40°$, determine:

a) $A\hat{C}B$

b) $B\hat{A}C$

Planning 7.1)

~~TP~~ let $P\hat{A}B = x$

~~RPT~~ RTP: $A\hat{B}C = A\hat{C}B$

Statement	Reason
$P\hat{A}B = x$	Given
$P\hat{A}B = A\hat{B}C = x$	Alternating Angles.
$P\hat{A}B = A\hat{C}B = x$	tan-chord
$A\hat{B}C = A\hat{C}B = x$	

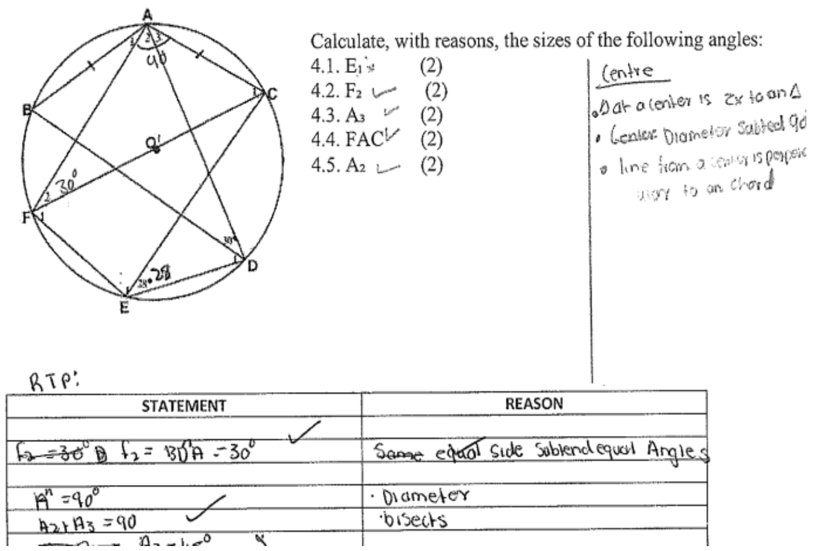
Planning

tangent

> tangent and radii

> 2 tangent

> Tan chord



Calculate, with reasons, the sizes of the following angles:

4.1. E_1 (2)

4.2. F_2 (2)

4.3. A_3 (2)

4.4. FAC (2)

4.5. A_2 (2)

Centre

- ∆ at a center is 2x to an ∆
- Center Diameter Subted 90
- line from a center is perpendicular to an chord

RTP:

STATEMENT	REASON
~~$F_2 = 30°$~~ $F_2 = B\hat{D}A = 30°$	~~Same~~ ~~equal~~ Side Subtend equal Angles
$A^n = 90°$	· Diameter
$A_2 + A_3 = 90$	· bisects

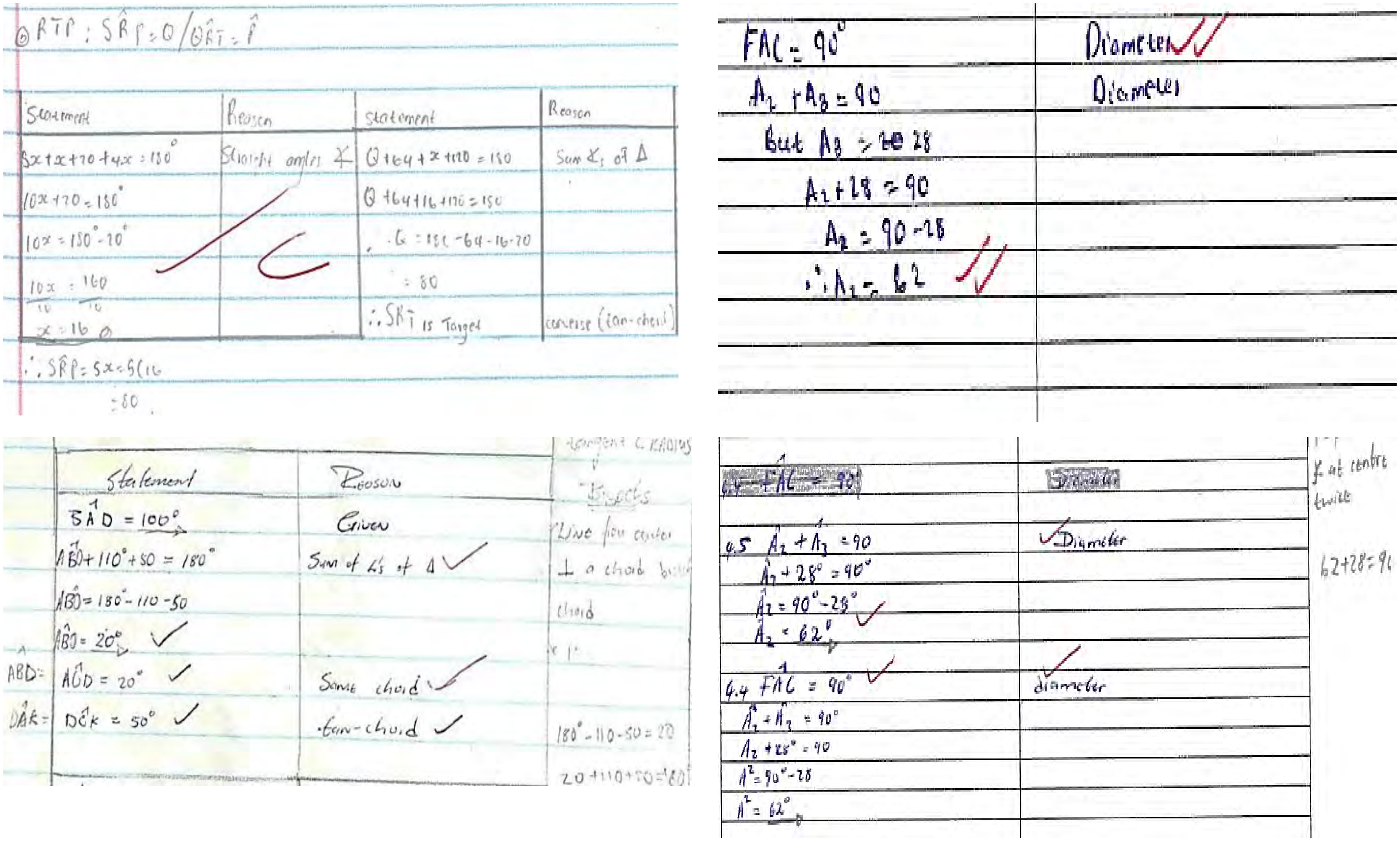
Figure 43: Re-planning skills analysis

Learners' computational skills.

Math computational skills comprise the basic arithmetic of addition, subtraction, multiplication, and division. These skills were found very critical in Geometry. During classroom teaching and learning applying the four stages model, I comprehended that, besides statement and reason format of arguing in geometric problem-solving, there are cases where one must apply the logic of computation to be able to break through. One can be very good at using theorems but if he/she is weak on applying computational skills, s/he will get stuck along the way. This skill involved calculations in the process of problem solving in Geometry besides supporting statements with reasons. The skill of computation was realised as a very critical skill that demanded one's calculative abilities manually and using a calculator, following a logical procedure. In the classroom as well as in the post-intervention test, the skill of computation was seen to be regularly used in problem-solving.

The incidence of application of this skill in the post-intervention test is like this: Learner 2 applied the computational skill twice inappropriately in the post test and Learner 3 used the same skill once correctly. Learner 7 employed it once correctly. Learner 11 applied twice correctly, and Learner 12 utilised the computational skill twice incorrectly, while Learner 17 never applied the skill in the post-intervention test. Work by Learner 2, 7 and 12 was used for illustration in this case, showing how learners were applying computational skills (see Figure 44). For example, Learner 3 sometimes had challenges in the classroom, getting answers wrong and writing corrections, but through more practice and pair work, the learner made it in the post-intervention test on questions that demanded application of computational skills. Learner 17 failed to apply the skill in the whole paper even though in the class activities he was doing it right in most cases. From this, I learnt that getting correct answers in the class activities gives one a higher chance to get it correct in the test, even though it is not guaranteed.

CLASSROOM WORK | **POST-INTERVENTION TEST WORK**



Learner 03

① RTP : $S\hat{R}P = Q / Q\hat{R}T = \hat{P}$

Statement	Reason	Statement	Reason
$5x + x + 70 + 4x = 180°$	Straight angles ∡	$Q + 64 + x + 120 = 180$	Sum ∡s of Δ
$10x + 70 = 180°$		$Q + 64 + 16 + 120 = 180$	
$10x = 180° - 70°$		$\therefore Q = 180 - 64 - 16 - 70$	
$\frac{10x}{10} = \frac{160}{10}$		$= 80$	
$x = 16$		$\therefore S\hat{R}T$ is Tanget	converse (tan-chord)

$\therefore S\hat{R}P = 5x = 5(16$

$= 80$

$FAC = 90°$	Diameter ✓✓
$A_2 + A_3 = 90$	Diameter
But $A_3 = ~~20~~\ 28$	
$A_2 + 28 = 90$	
$A_2 = 90 - 28$	
$\therefore A_2 = 62$ ✓✓	

Learner 11

	Statement	Reason	tangent ⊥ radius
	$B\hat{A}D = 100°$	Given	Bisects
	$A\hat{B}D + 110° + 50 = 180°$	Sum of ∠'s of Δ ✓	Line from center ⊥ a chord bisects chord
	$A\hat{B}D = 180° - 110 - 50$		
$\hat{ABD} =$	$A\hat{B}D = 20°$ ✓		
	$A\hat{C}D = 20°$ ✓	Same chord ✓	
$D\hat{A}K =$	$D\hat{C}K = 50°$ ✓	tan-chord ✓	$180° - 110 - 50 = 20$
			$20 + 110 + 50 = 180$

~~4.4 $F\hat{A}C = 90°$~~	~~Diameter~~	∡ at centre twice
4.5 $\hat{A}_2 + \hat{A}_3 = 90$	✓ Diameter	$62 + 28 = 90$
$\hat{A}_2 + 28° = 90°$		
$\hat{A}_2 = 90° - 28°$ ✓		
$\hat{A}_2 = 62°$		
4.4 $F\hat{A}C = 90°$ ✓	✓ diameter	
$\hat{A}_2 + \hat{A}_3 = 90°$		
$A_2 + 28° = 90$		
$A^2 = 90° - 28$		
$A^2 = 62°$		

Figure 44: Computational skills analysis

Learners' required To Prove (RTP) finding skill.

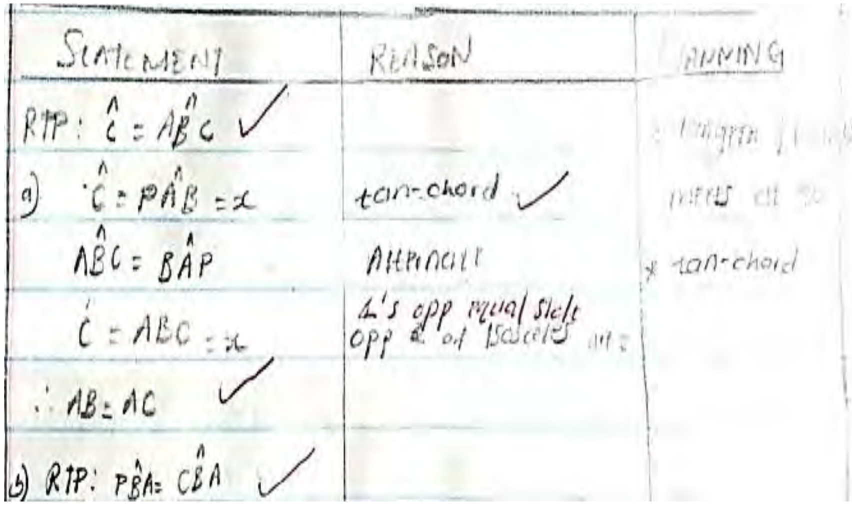
Required to prove (RTP) is that which needs to be shown to imply a proof. For example, if a question says prove a chord to be a diameter; the RTP will be to show that an angle subtended from that chord is right angle or to show that the chord concerned is perpendicular to a tangent at point of contact. Thus, when a question demands that someone makes a logical proof, one must know the required to prove (RTP) for him/her to know what is that which needs to be achieved. This proved to be a very critical skill in logical geometric proofs and the skill was developed during classroom learning when applying the four stages model and was also found displayed by several learners in problem-solving in the post-intervention test. The frequency of application of this skill by learners in the test was as follows: Learner 2 employed the skill of required To Prove (RTP) finding thrice correctly and once incorrectly. Learner 3 used the same skill correctly 3 times and twice inappropriately. Learner 7 applied the skill of Required to prove (RTP) finding correctly 4 times. Learner 11 utilised it correctly two times and once inappropriately. Learner 12 never applied the RTP finding skill. Learner 17 applied the same skill twice correctly. I demonstrated using an extract from work by Learner 3, 11 and 17 on how the skill had been applied from different performance categories.

Just like Makhubele (2015), I discovered that learners had challenges with proof questions. From classroom observations, I understood that most learners still lacked the ability to come up with the RTP so that they can develop a logical, coherent argument. Some moderate performers and some under-performers showed their incapacity in the classroom activities and on the post-intervention test. Just like drawing and visualisation skills, the RTP finding skill is used to an answer and no marks are awarded for it, yet it helps one to get the direction leading to the answer. Even though I emphasised that they should take time looking for the RTP before implementation, some learners were still caught offside in class activities and in the post test. They were found already solving the problem before finding the RTP. Such an act is normally a good recipe for failure in geometric proofs.

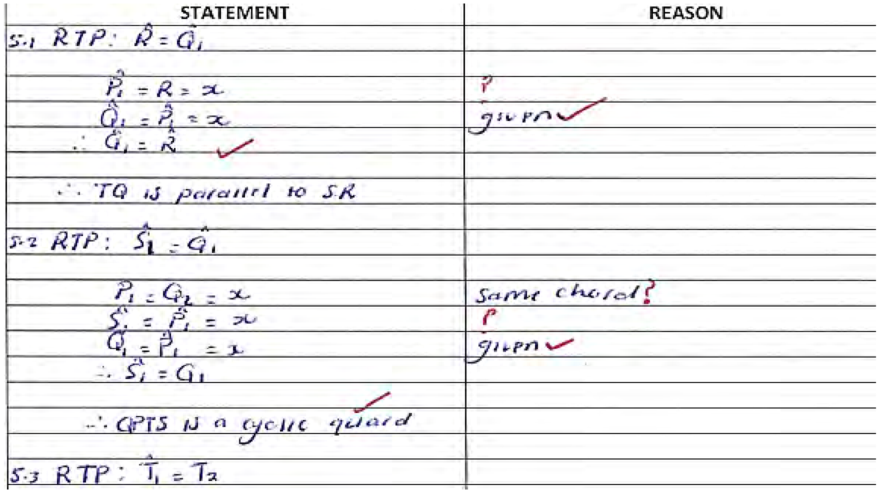
On the same note, it was painful that some learners came up with a correct RTP but failed to make logical deductions to come up with a correct proof.

CLASSROOM WORK | **POST-INTERVENTION TEST WORK**

Learner 07

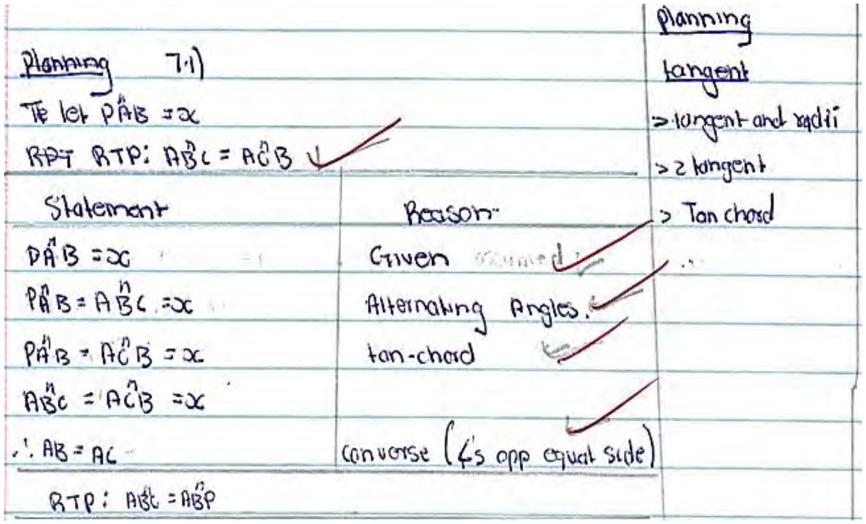


SIATEMENT	REASON	ANNING
RTP: $\hat{C} = A\hat{B}C$ ✓		[illegible]
a) $\hat{C} = P\hat{A}B = x$	tan-chord ✓	[illegible]
$A\hat{B}C = B\hat{A}P$	Alternate	✗ tan-chord
$\hat{C} = ABC = x$	∠'s opp equal side opp ∠ of Isosceles	
∴ AB = AC ✓		
b) RTP: $P\hat{B}A = C\hat{B}A$ ✓		



STATEMENT	REASON
5.1 RTP: $\hat{R} = \hat{Q}_1$	
$\hat{P}_1 = R = x$	?
$\hat{Q}_1 = \hat{P}_1 = x$	given ✓
∴ $\hat{G}_1 = \hat{R}$ ✓	
∴ TQ is parallel to SR	
5.2 RTP: $\hat{S}_1 = \hat{Q}_1$	
$P_1 = Q_2 = x$	same chord?
$\hat{S}_1 = \hat{P}_1 = x$	?
$\hat{Q}_1 = \hat{P}_1 = x$	given ✓
∴ $\hat{S}_1 = G_1$	
∴ QPTS is a cyclic quard ✓	
5.3 RTP: $\hat{T}_1 = T_2$	

Learner 02



Planning 7.1)

~~T~~ let $P\hat{A}B = x$

~~RPT~~ RTP: $A\hat{B}C = A\hat{C}B$ ✓

Statement	Reason
$P\hat{A}B = x$	Given ✓
$P\hat{A}B = A\hat{B}C = x$	Alternating Angles. ✓
$P\hat{A}B = A\hat{C}B = x$	tan-chord ✓
$A\hat{B}C = A\hat{C}B = x$	✓
∴ AB = AC	converse (∠'s opp equal side)

RTP: $A\hat{B}C = A\hat{B}P$

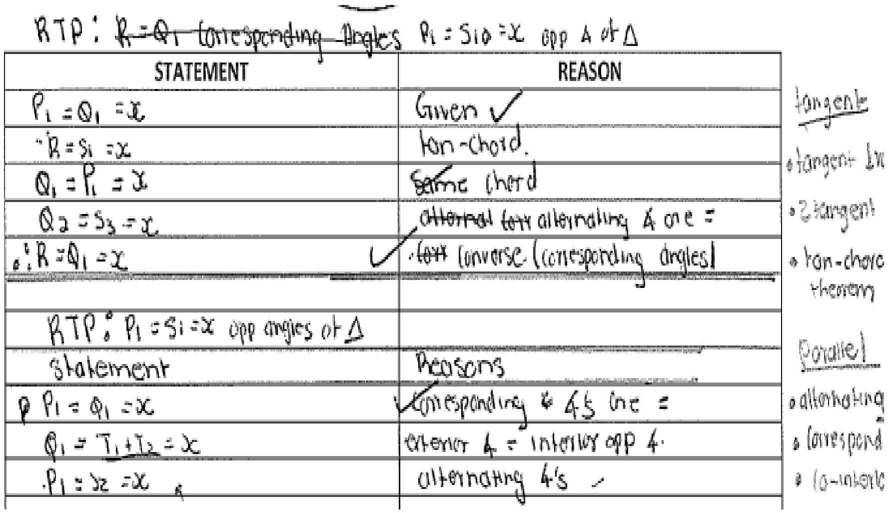
Planning

tangent

> tangent and radii

> 2 tangent

> Tan chord



RTP: ~~R = Q₁ corresponding Angles~~ $P_1 = S_1 = x$ opp ∠ of Δ

STATEMENT	REASON
$P_1 = Q_1 = x$	Given ✓
∴ $R = S_1 = x$	tan-chord.
$Q_1 = P_1 = x$	~~Same~~ chord
$Q_2 = S_3 = x$	~~alternal~~ ~~text~~ alternating ∠ are =
∴ $R = Q_1 = x$ ✓	~~text~~ converse (corresponding angles)
RTP: $P_1 = S_1 = x$ opp angles of Δ	
Statement	Reasons
$P_1 = Q_1 = x$	✓ corresponding ∠s are =
$Q_1 = T_1 + T_2 = x$	exterior ∠ = interior opp ∠.
$P_1 = S_2 = x$	alternating ∠'s

tangent

- tangent [illegible]
- 2 tangent
- tan-chord theorem

Parallel

- alternating
- correspond
- co-interi

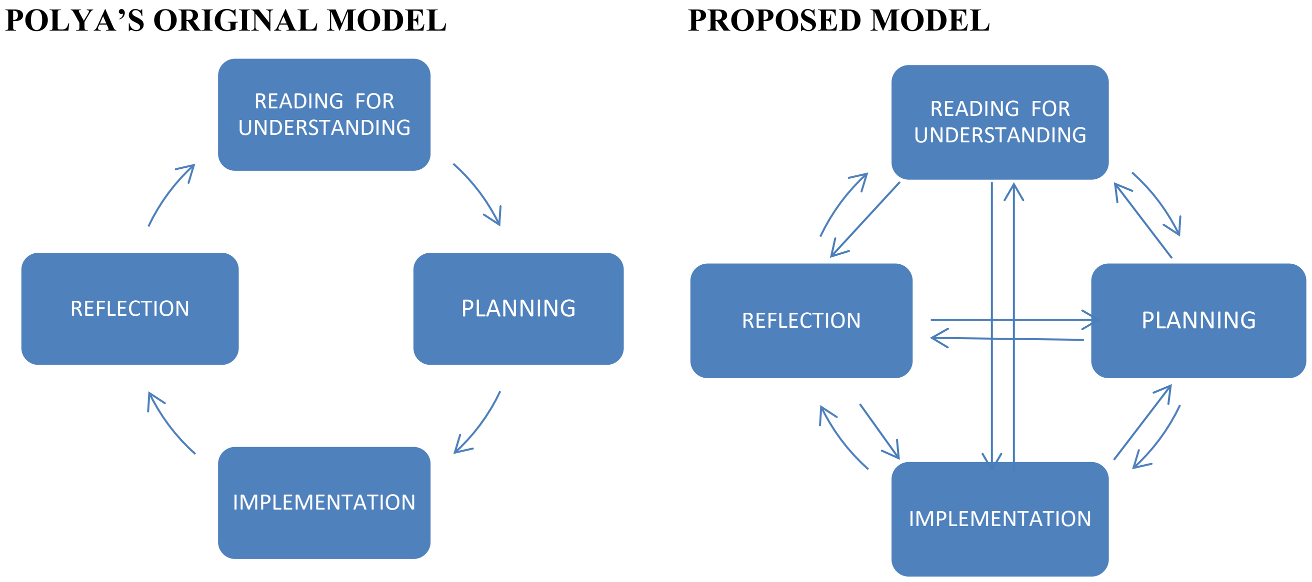
Figure 45: Finding of Required to Prove (RTP) skills analysis

Summary of the use of the four stages model and post test results

On account of the results of the research, conclusions can be made in terms of the implementation of the four stages model in geometric problem-solving as follows: 1) reading for understanding of geometric problems can generally be done by all learners both in the class activities and in the test except some under-performers; 2) planning stage in problem-solving is done by learners, but some of them do not do this step when writing tests; 3) implementation stage of problem-solving was done by the learners in the test, but since some of them could not understand the problems well, they could not solve the geometric problems correctly; and 4) on look back stage, although learners understand that this aspect is important, in practice, most of the learners did not do it even in class activities, worse in tests. This could be because they thought they were doing it right or that it was time wasting.

On the application of the themes generally, I realised a positive shift in some themes, but no major change in the other themes, after comparing the frequency of application of these themes. I also noted that, from the skills that constitute different themes, most of the learners showed a significant positive change in the general frequency of application, and in the number of correct applications. I also realised that there was a noticeable negative change from the pre-intervention test to the post-intervention test in the number of incorrect and inappropriate applications of the skills. I also discovered that there was evident positive overall difference on the general performance of learners between the pre-intervention test and the post intervention test in terms of marks for the whole test and per theme according to the structure of the two tests.

Besides application of existing themes and use of the four stages model, I also discovered some emerging themes. These themes emerged during the utilisation of the four stages model. Even though these themes were discovered in the process of model utilisation, I noted that some learners still could not apply them effectively. Out of the three discovered themes, some learners were failing to apply some or all of them in the process of problem-solving. One of the themes (re-planning skill) was discovered after seeing that the four stages by Polya were not practically linear. Figure 46 shows the original and proposed model. The proposed model was developed from the themes that emerged during the study. I realised that, during problem-solving, some answers could be derived direct from the statement and be put straight in the diagram and reflected upon before any proper planning. During implementation, one could go back and re-read the statement to gain more understanding if things are not working out. If one has been implementing from planning and found out things were not working out, one could go back to planning stage as well and find a way out.


POLYA'S ORIGINAL MODEL

PROPOSED MODEL

Figure 46: Proposed practical model developed from Polya's model

If an answer could be derived direct from the statement, it was very common for one to take the answer from the implementation and use it in planning stage to find ways to proceed. If the planning stage seemed not to be working out, one could always go back, read again the statement, and proceed with planning. All these dynamics showed that, in practical problem-solving, the process does not follow a linear sequence of steps.

4.4. GEOMETRY LEARNING CHALLENGE EXPERIENCED BY SELECTED GRADE 11 LEARNERS WHEN SOLVING PROBLEMS IN GEOMETRY

Data collected from the interviews, observations and document analysis are presented in this section to answer the third research question: ***What Geometry learning challenges are experienced by selected Grade 11 learners when solving problems in Geometry?*** There were some challenges encountered by both learners and I during the use of the four stages model. Data concerning the challenges was collected from the interviews and through observation during class lessons as learners were using the four stages in geometric problem-solving and from learner's exercise books.

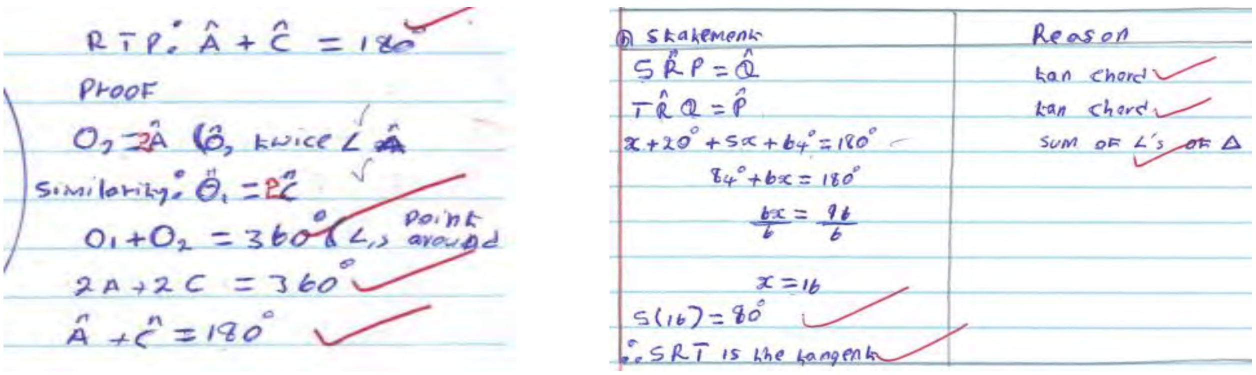
4.4.1. *Practice related challenges*

Facchinetti, (2019) believes the adage 'practice makes perfect' is proving particularly relevant in the field of neuroscience, where studies show that exposure to repeated experiences of a topic are more likely to build lasting neurological pathways. The same applies to Geometry in this case. From the interviews held, it shows that lack of practice was one of the challenges faced by some learners in conceptualising Geometry and the four stages model as a problem-solving tool. Some learners indicated that problem-solving in Geometry was a big problem during the first days of the use of the four stages model. They further indicated that it was only after some time of persistent practice that they started to understand the problematic concept. For example, Learner 7 said that when it comes to drawings and constructions it was hard for her at first, but after practice with the help of others, she now mastered the constructions for all examinable theorems and can now even help others get a better understanding on the concept. She mentioned that:

> *When making constructions I won't lie I found it challenging until I asked my classmates to help me up to the point when I feel I now understand it. I now don't have any problems with constructions….*

Learner 11's argument showed that he did not have any noticeable challenges in the application of logical skills. His explanation pointed again to the use of the four stages model, and how he could now simply apply it to deal with such questions. He mentioned that it became easier as he continued practicing. Learner 7 said:

> *With the four stages now, I am clear with the diagrams and everything, I find it becoming easier and easier especially if you know your Required to Prove (RTP), at first it was hard for me to find the RTP but now it's easy and once you identify the RTP, it's very easy to continue with the process.*



EARLY DAYS

RTP: $\hat{A} + \hat{C} = 180$

Proof

$O_2 = 2\hat{A}$ ($\hat{O}_2$ twice ∠ $\hat{A}$

Similarity: $\hat{O}_1 = 2\hat{C}$

$O_1 + O_2 = 360°$ (∠'s around point

$2A + 2C = 360°$

$\hat{A} + \hat{C} = 180°$

LATER DAYS

(b) Statement	Reason
$S\hat{R}P = \hat{Q}$	tan chord
$T\hat{R}Q = \hat{P}$	tan chord
$x + 20° + 5x + 64° = 180°$	sum of ∠'s of Δ
$84° + 6x = 180°$	
$\frac{6x}{6} = \frac{96}{6}$	
$x = 16$	
$5(16) = 80°$	
∴ SRT is the tangent	

Figure 47: Comparison of work presentation with time

Learner 3 also included the issue of practice in his statement and noted that he had issues with proofs until after some practice. He said that:

> *Cyclic quad is hard but parallel lines its easy because it's grade 10. Theorems such as alternating, corresponding, which are grade 10, 9,8 theorems so they are easy but when it comes to cyclic quad since we were not learning about it before it's a bit challenging. Now because we have learnt about it in grade 11, I enjoy proving it and I understand it better now. If it comes in a question paper, that's the area I get more marks…*

The challenge of practice can also be revealed from class activities. For example, some learners' presentations of their solutions got better through practice from the first day onwards.

All in all, lack of practice could have been one of the reasons for poor performance in different aspects of geometric problem-solving and four stages model utilisation. This is because learners required more time to practice Geometry as a concept and the model itself, together with its application. The challenge of practice might also be a result of the negative attitude towards the subject Mathematics in general. According to Gafoor and Kurukkan (2015), the most relevant reason observed by teachers for learners' difficulties in learning Mathematics is lack of sufficient effort by learners, and lack of awareness of the role of learners' self-efficacy for learning Mathematics. This was one of the challenges faced by the learners to lack practice and not get the intended results.

4.4.2. *Challenges specific to certain learners*

From the interviews it showed that learners can have different challenges even though they are from the same class. From analysis of learner responses, something that might be easy to some learners might be difficult to others. Learner 17 cited that the drawings and constructions done in Geometry were not equally easy, even though he did not have problems with the drawings and constructions. He affirmed that:

> *On constructions there are challenges here and there because on diagrams they give you different diagrams some of which I can do construction some I fail. I now don't have a problem with constructions, and it depends on the diagram….*

Learner 3 pointed out some specific theorems that gave him hard times. He said:

> *If they want drawings, sometimes it's hard especially the theorem which says an angle at the center is twice on the circumference is the one hard for me but the other ones like Tan-Chord theorem are easy to do the construction. I can quickly do such constructions*

> *and even on implementation its easy because you would have already done planning in the diagram and answers would already been there the just take it from the diagram to implementation…*

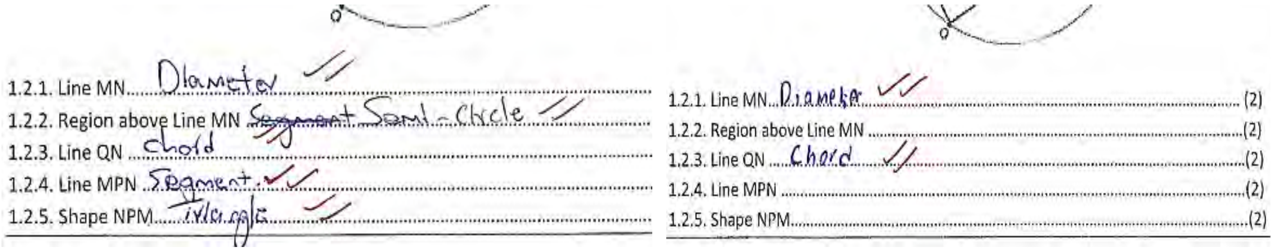
The same applied to Learner 17 who said:

> *I enjoy proving things like cyclic quadrilaterals because now I understand it, but the problem is when I am proving parallel lines, I get confused. I don't know why but I fail to prove parallel lines. I know the theorems that have to do with parallel lines, but I fail to prove lives being parallel. Not all of them of course but some of them depending on the diagram….*

All this shows that by virtue of being naturally different, learners will encounter different challenges. This was a challenge even to me as a teacher, in the classroom, every learner was unique, and their needs needed to be met uniquely as well. This forced me to be dynamic and flexible in my teaching methodology. This was also noticed in the post-intervention test whereby a question easy to one learner could prove hard to another. Figure 4.45 shows a question that proved easy to one of the underperformers (learner 12) but was a bit of a challenge to one of the moderate performers (learner 17).

LEARNER12

LEARNER17



LEARNER12

1.2.1. Line MN Diameter ✓✓
1.2.2. Region above Line MN ~~Segment~~ Semi-Circle ✓✓
1.2.3. Line QN Chord ✓✓
1.2.4. Line MPN Segment ✓✓
1.2.5. Shape NPM triangle ✓✓

LEARNER17

1.2.1. Line MN Diameter ✓✓ (2)
1.2.2. Region above Line MN (2)
1.2.3. Line QN Chord ✓✓ (2)
1.2.4. Line MPN (2)
1.2.5. Shape NPM (2)

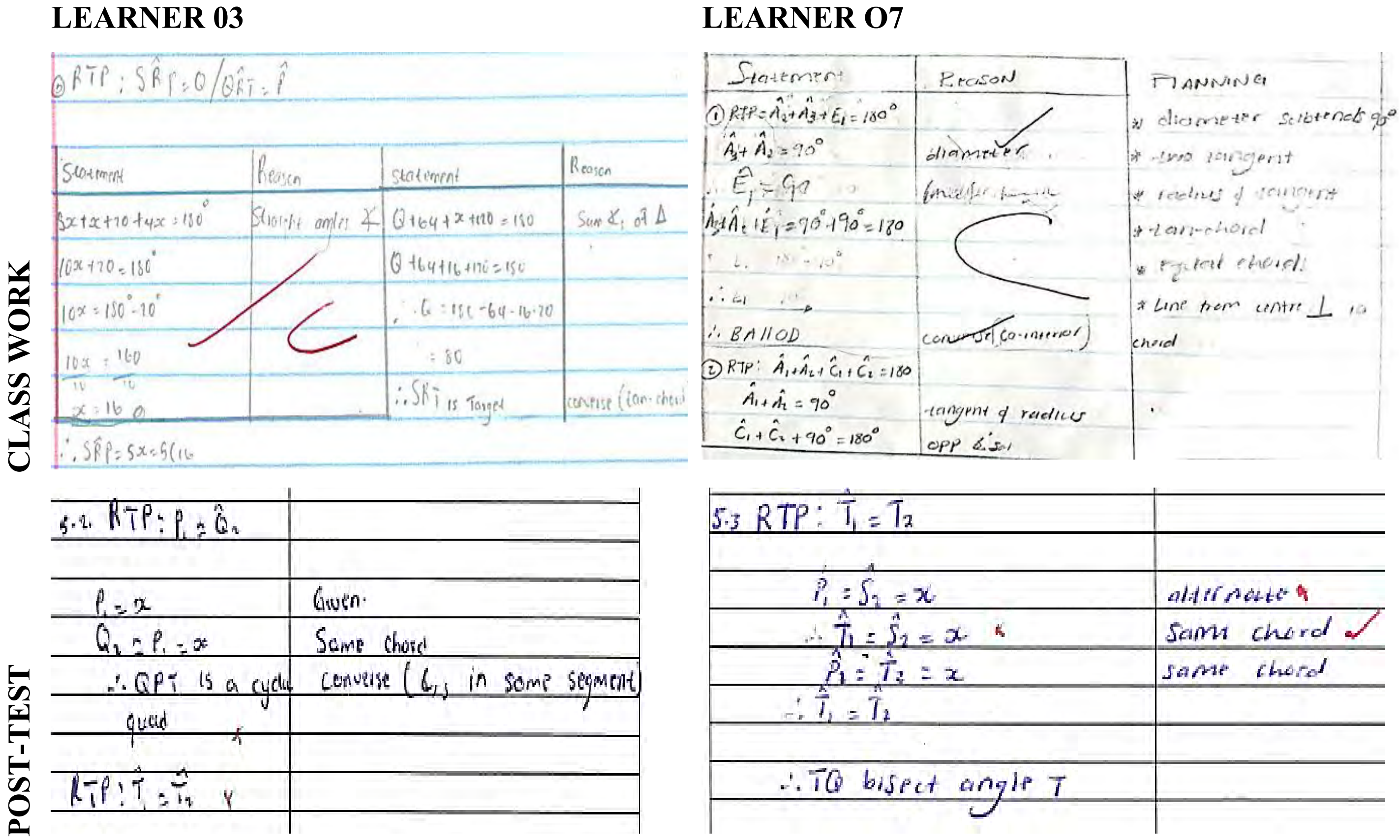
Figure 48: Comparison of learner challenges

The learners were from different categories and were all in one class. I feel this makes life difficult for some learners since, in most cases, I would use a blanket approach in teaching the learners. I realised that some learners failed to understand some of the concepts when marking class works, homework and short tests. The major reason for this challenge was that most learners were passive in the Mathematics classroom, even though they pretended to be attentive. Research has proven that the majority of teachers think that fear and shame are the main factors for causing learners' low participation and reluctance to ask (Nuri & Marsigit, 2019).

4.4.3. *Concept related challenges*

Some of the challenges mentioned in the interviews had to do with the concept of Geometry itself. For example, Learner12 mentioned that some theorems sounded similar, and this led to confusion in some learners, which led to learners giving inappropriate answers whereby one could give a correct statement but supported with an incorrect reason. Learner 12 mentioned something about theorems that sounded the same. He said:

> *".... Some theorems sound similar, and that thing confuses me. The other challenge is that sometimes I write a correct statement only to find the reason is the one which is not correct..."*



① RTP ; $S\hat{R}P = Q / Q\hat{R}T = \hat{P}$

Statement	Reason	Statement	Reason
$5x+x+70+4x=180°$	Straight angles ∡	$Q+64+x+120=180$	Sum ∡s of Δ
$10x+70=180°$		$Q+64+16+120=180$	
$10x=180°-70°$		$\therefore Q = 180-64-16-20$	
$\frac{10x}{10} = \frac{160}{10}$		$= 80$	
$x=16$		$\therefore S\hat{R}T$ is Tangent	converse (tan-chord

$\therefore S\hat{R}P = 5x = 5(16$

Statement	Reason	Planning
① RTP: $\hat{A}_2+\hat{A}_3+\hat{E}_1=180°$		* diameter subtends 90°
$\hat{A}_3+\hat{A}_2=90°$	diameter	* two tangent
$\therefore \hat{E}_1 = 90$	[illegible]	* radius ⊥ tangent
$\hat{A}_3+\hat{A}_2+\hat{E}_1 = 90°+90° = 180$		* tan-chord
[illegible]		* equal chords
∴ [illegible]		* Line from centre ⊥ to chord
$\therefore BA \parallel OD$	converse (co-interior)	
② RTP: $\hat{A}_1+\hat{A}_2+\hat{C}_1+\hat{C}_2=180$		
$\hat{A}_1+\hat{A}_2=90°$	tangent ⊥ radius	
$\hat{C}_1+\hat{C}_2+90°=180°$	opp ∠s of	

5.2. RTP: $\hat{P}_1 = \hat{Q}_2$

$P_1 = x$	Given
$Q_2 = P_1 = x$	Same chord
∴ QPT is a cyclic quad	converse (∠s in same segment)

RTP: $\hat{T}_1 = \hat{T}_4$

5.3 RTP: $\hat{T}_1 = T_2$

$\hat{P}_1 = \hat{S}_2 = x$	alternate
$\therefore \hat{T}_1 = \hat{S}_2 = x$	same chord
$\hat{P}_1 = \hat{T}_2 = x$	same chord
$\therefore \hat{T}_1 = \hat{T}_2$	

∴ TQ bisect angle T

Figure 49: Learners' work on geometric logical proofs

There were some challenges I faced in teaching the concept of logical proofs. Learners had challenges in understanding how to apply the converses of theorems in proving things like parallel lines, tangents, bisections, diameters, cyclic quadrilaterals and so forth. This was taught at the same time as the technique of finding the required to prove (RTP) was taught. It took a while for learners to internalise the concept, and others were still having confusion even up to the time the post-test was written. I realised when I was marking the post-test that most of the learners still lacked understanding of the concept as they underperformed on the question on the logical deduction of proofs. This is evident in the class activities as well as post-intervention test, where some sections like logical proofs were poorly attempted by learners. One reason that made the concept of Geometry a challenge could have been the negative attitude by learners towards the concept itself. Research continues to reveal that Geometry is still quite absent from the classrooms, especially in the early years; that it is disliked by most learners and misunderstood; and its notation is totally ignored (Sousa Melo & do Carmo Martins, 2015). This might have been the same scenario learners were facing in the classroom. Figure 49 shows some work from a moderate performance and high-performance category for both class activity and post-intervention test on logical proofs. The information shows that learners had challenges in the concept of logical proofs more than in other concepts. In this case, it is evident that some learners were getting it right when revising the class activities and homework, yet even during classroom activities others were finding it hard to answer this section as it required much reasoning.

4.4.4. *Curriculum- related challenges*

As a classroom practitioner, I observed some challenges during the application of the four stages model. I understood that learning the model itself needed time as it is a concept on its own. In the Mathematics curriculum, every topic is given a period to be covered and the expected time to be spent per lesson per week. This made the intervention require more time as there were two concepts to be covered simultaneously. Thus, I ended up acquiring extra time to be able to carry out the research. The stages involved in the utilisation of the four stages model, such as the planning stage are time consuming and demand more practice time so that the learners can be quick on their problem-solving. In other words, the model needs some time to be taught in isolation and then applied. Its application demands its own time as well to teach it. However, through practice, the time consumed in the application of the model would be compensated by the high probability of getting correct answers at the end. Learner 3 said:

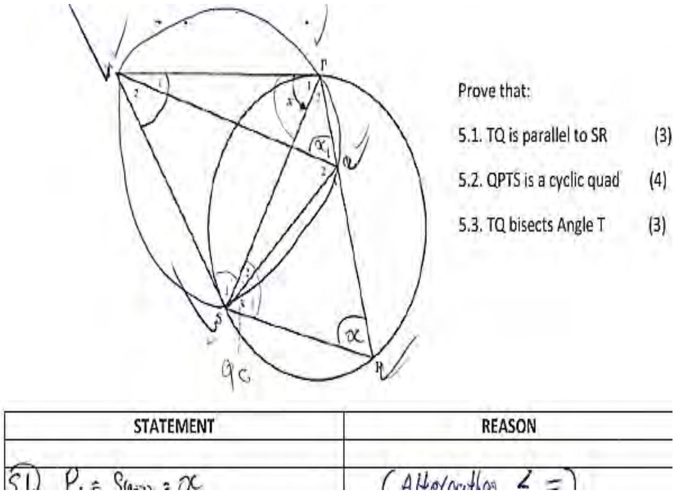
> *……. I can quickly do such constructions and even on implementation it is easy because you would have already done planning in the diagram and answers would already been there the just take it from the diagram to implementation….*

Just like Gafoor and Kurukkan (2015) note, in the current structure of the curriculum, teachers have been reporting that it is hard to allow more time for renewing previous knowledge especially in topics like Geometry. Such a loophole needed me to revisit prior knowledge to avoid more challenges when applying the four stages model. This needed more time to be added again to the stipulated time for the Geometry chapter in Mathematics. Some thinking moves related to mathematical problem-solving are usually neglected and not nurtured in the performance-based South African Mathematics education curriculum (Mahlaba, 2020). He further argued that the exercise of finding multiple solutions to mathematical problems is one of these thinking moves. This is a challenge to the learners as the curriculum would not give enough time for such activities, ending up making the teacher teach what is within the confinement of the curriculum only.

4.4.5. *Model application challenge*

I realised that most of the learners could apply the model very well during class activities and homework but could not apply some of the stages of the model in their problem solving in tests. Even though some of their solutions were correct, the application of the model was not physically seen in their workings. This might be because learners were still having their old mentality of not writing on and utilizing spaces on question papers and diagrams. Since some learners were not including some of the problem-solving steps learnt in the classroom, it was hard to detect whether some of their failures in problem-solving were a result of that or of other factors. Figure 50 illustrates how some learners, one from the underperformers category and another from high performers category, were eliminating some of the stages learnt for problem-solving in the post-intervention test.

LEARNER 12



Prove that:

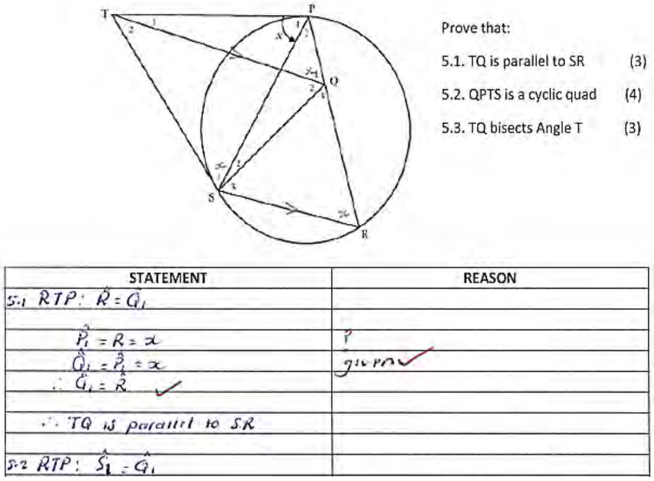
5.1. TQ is parallel to SR (3)

5.2. QPTS is a cyclic quad (4)

5.3. TQ bisects Angle T (3)

STATEMENT	REASON
5.1 P₁ = S = x	(Alternating ∠ =)

LEARNER 07



Prove that:

5.1. TQ is parallel to SR (3)

5.2. QPTS is a cyclic quad (4)

5.3. TQ bisects Angle T (3)

STATEMENT	REASON
5.1 RTP: $\hat{R} = \hat{Q}_1$	
$\hat{P}_1 = R = x$	?
$\hat{Q}_1 = \hat{P}_1 = x$	given ✓
∴ $\hat{Q}_1 = \hat{R}$ ✓	
∴ TQ is parallel to SR	
5.2 RTP: $\hat{S}_1 = \hat{Q}_1$	

Figure 50: Learners eliminating some steps in problem-solving in the post-intervention test

From Figure 50, Learner 12 did not apply the first stage of the model in answering a question in the test, even though he showed understanding of how to apply the stage during classwork and homework. This challenge was very common in the post-intervention test. Learner 7 did not do any planning on this question and was found not doing it correctly. In this case one cannot tell whether these acts were the cause of these learners not performing in these parts of the test. The other challenge observed was the application of the fourth stage of the model (reflection). From classwork, homework, and tests, learners were not applying the reflection stage properly. Some of the questions did not demand the application of the reflection stage, but even those questions which needed its application, most of the learners were not applying it. In the post-intervention test, the learners did not apply the reflection stage in their workings. From my own analysis, I realised that the reflection stage was not consistent in its application which ended up confusing some of the learners.

4.4.6. *Context related factors*

The study was done during Covid-19 era where social distancing was a protocol to be observed. It prohibited the use of group work. I normally used pair-work, where learners could discuss in pairs from a distance even though I felt it did not yield the actual intended results. Peer coaching and collaboration, class discussions and flipped classrooms were the most common methods used. I believe the learners learn more if they work collaboratively but did not manage to apply the strategy well during teaching and learning. Thus, I feel this was a challenge to my methodology during teaching and learning.

I had some challenges during the first days before learners got used to having a camera in the classroom. Even though the learners were aware that cameras would be part of the class during teaching and learning, I realised that some learners were being disrupted, especially during the first days. I could always remind learners to keep concentrating since the camera was only focusing on their work and not on them as individuals.

Another contextual challenge was the background of the learners. Learners in South African schools come from extremely diverse social, political, economic and cultural backgrounds, and this can be linked to the concern about the Mathematics education of South African learners (Kotze & Strauss, 2006). This might have affected the learners in one way or another. The school where the research was conducted is a school in deep rural areas of Kwa-Zulu Natal in South Africa. The school is still developing in terms of infrastructure and education of learners with no good pass history in the subject of Mathematics. This background was a challenge to me as a teacher and the learners directly and indirectly.

4.4.7. *Conclusion*

Carrying out this research was a challenge as I and the learners encountered several challenges. However, the research study was successfully done with an understanding that any research carried out will, in one way or another, have its own challenges. I therefore conclude by indicating that it was not easy to weigh and see which factor had most influence on the challenges faced by learners during teaching and learning for geometric problem-solving skills development.

CHAPTER FIVE 5

CONCLUSION AND RECOMMENDATIONS

5.1.INTRODUCTION

The purpose of this study was to explore and develop the mathematical problem-solving and geometric skills of Grade 11 learners in Euclidean Geometry. The research questions concentrated on the geometric skills learners could bring to class before being introduced to the four stages problem-solving strategy; how the learners could utilise the four stages in a social cognitive classroom; as well as the challenges learners encountered and their perceptions on the use of the strategy. In this study, I was concerned about developing learners' geometric understanding by making use of Polya's model. In this chapter therefore, I present a summary of the major findings of the study followed by a discussion on the significance of the study, as well as the limitations. I then present some recommendations originating from the study findings, including my own reflections of the research and suggestions for further research.

5.2.SUMMARY OF THE MAIN FINDINGS

Major findings of the study were presented according to the three research questions of this study. Learners were initially given a pre-intervention test designed to answer the first research question and to give an overview of which geometric skills learners brought to class before being introduced to Polya's model of problem-solving. During the intervention period, lesson observations and document analysis were carried out to answer the second research question. Document analysis was in the form of exercise books for six selected learners, which were further observed to see whether the four stages were being employed in geometric problem-solving. Several themes that were pre-determined and those that emerged in relation to the geometric problem-solving skills were discussed in chapter four. Semi-structured interviews on learners' challenges and perceptions were conducted after the lesson observations, to provide answers to the third research question of this study.

Research question 1

- *What geometric skills are demonstrated by selected learners prior to learning Grade 11 Geometry content?*

Pre-intervention test

The pre-intervention test was given, and the results revealed that most learners could move from previous grades to the next grades without prerequisite geometric skills attained. Very few learners managed to get at least a pass mark in the pre-intervention test. Better performance was realised in the application of drawing and visualisation skills. Verbalisation, application, and logical skills were poorly displayed despite these being the skills that are highly demanded for one to be awarded marks in an examination setting. All in all, the pre-intervention test results showed that there was need for intervention in the development of the learners' geometric skills. This necessitated the introduction of the Polya four stages model of geometric problem-solving as an intervention strategy.

The learners managed to display the following skills in the pre-intervention test that formed the themes for thematic analysis: recognition of a theorem or axiom; recognition of properties of shapes; recognition of different parts and lines in a circle or diagram; recognition of different angles, verbal descriptions and naming of shapes, objects and parts; definitions and stating in own words; sketching of diagrams related to theorems and/or axioms; constructing diagrams based on its properties; adding useful elements to a diagram; classification by their properties; development of logical proofs; as well as application of theorems and axioms in problem-solving. In trying to apply these skills, I noted that there were many cases of incorrect applications compared to inappropriate and correct applications. This was then a sign that there was a need for intensive development of geometric problem-solving skills of the learners.

Research Question 2

How can the use of Polya's model enable or constrain the development of problem solving and geometric skills in a selected Grade 11 Mathematics class?

Lesson observations and document analysis

During the classroom observations after the four-stage strategy had been introduced, learners were found applying the four stages effectively. All the learners could show reading for understanding by highlighting and/or underlining the key words. That was effectively applied from the beginning even though some learners could, at times, highlight the wrong words and fail to underline some key words. From document analysis, I realised that all the six learners whose exercise books were monitored could effectively apply the first stage (Reading for Understanding). Most learners preferred jotting down theorems (in short) that had to do with the key words highlighted in the second stage (planning). A few learners were able to include drawings in their planning. However, I came across some learners who wrongly stated the theorems or stated insufficient number of theorems for certain key words. From the observations and exercise book monitoring, I discovered that none of the learners left the part of planning blank. In the planning stage, learners could further interpret the diagram in relation to the theorems stated to see the applicability of the theorems, hence able to put some information on the diagram. The putting of information and filling up of diagrams was regarded as part of planning stage. Nevertheless, not all learners could take the already got answers back onto the diagram, but most could use them from their minds or workings, and others could not even use them at all.

Implementation stage (stage 3) was applied in the form of statement and reason. From lesson observations and exercise book monitoring, it was easy to start and get correct answers if the first and second stages were well done. Some learners were getting incorrect answers due to different reasons such as incorrect computations and application of wrong theorems, but what all of them could do right was to present the solutions in the right format of statement and reason. On the same note, having done stage 1 and 2 could make learners find where to start unlike before, where they could leave questions not attempted. As a part of problem-solving, some learners could cancel and try another way after discovering that one method was not working, which I regarded as part of the implementation stage. Even though learners were presenting their solutions in the right format, some learners gave wrong reasons for correct statements or the other way round.

The last stage of the four stages (look back) was a bit tricky for most learners to use and was the least applied of all the 4 stages. From lesson observations and exercise book monitoring, I realised that most learners could rarely check the correctness of their answers. Some problems demanded direct answers that did not need reflection, but for those that needed reflection, only a few learners could reflect. Several learners could do part of reflection by taking back the answers onto the diagrams for further use.

During lesson observations and exercise book monitoring, I discovered that there were other skills that emerged as learners were trying to develop their problem-solving skills using the four stages model. The skill of re-planning, computational skill, and that of coming up with the RTP were witnessed during this period. The skill of re-planning was whereby a learner could take an answer got back into the diagram and further use it in determining other missing values in solving of riders especially. On the same note, computational skills involved use of mathematical calculations and a device (calculator) in the process of geometric problem-solving. The skill of finding RTP was required most in dealing with proofs, as one was supposed to be able to determine the appropriate theory together with the correct set of angles that will make it possible to work out a deductive proof.

Post-intervention test

A post-intervention test was given to see if the four stages model had brought some changes to the learners' geometric understanding. The test was analysed individually and later as a comparison to the pre-intervention test. In the same line as Gopinath & Lertlit (2017) who witnessed improvement in performance of learners after being taught the use of Polya's model, I observed a good class performance in the post-intervention test. The majority of learners managed to get a pass mark in the post-intervention test. Most learners did better in the application of verbalisation and visualisation skills even though several learners had some challenges with the application of logical and application skills.

The pre-intervention test and the post-intervention test were compared to see if there was any noticeable change in the performance of learners overall, and geometric skills. This analysis shaped the quantitative part of the data analysis. The outcomes of the comparison showed that there was some improvement in the overall performance of the learners, and in all the different geometric skills except drawing skills, where the performance of the learners dropped. Thus, the post-intervention test revealed a development of the learners' geometric skills proving the usefulness of the four stages Polya's model of geometric problem solving.

From the post-intervention test, the learners displayed several geometric skills including those that emerged during the use of the four stages model. Since the frequency of application of the skills was always checked against its correctness, inappropriateness and incorrectness, the post-intervention test results showed that there was generally a higher frequency of correct and incomplete/inappropriate applications of the skills than incorrect applications. When doing a comparative analysis, I realised that the opposite was true with the pre-intervention test, where the highest frequency of occurrences with the application of the skills was witnessed on incorrect applications more than on correct and inappropriate applications.

Research Question 3

What Geometry learning challenges are experienced by selected Grade 11 learners when solving problems in Geometry?

This research question was answered through post-intervention interviews with some selected learners from the class, follow-up interviews, as well as during observations in the classroom. Participating learners in this study mentioned the various experiences they went through and how their perceptions on the use of the four stages were influenced after the introduction of Polya's model. Initially, all the learners claimed that they now liked solving geometric problems as compared to before and acknowledged that the model was indeed a useful tool that aided so much in developing their geometric skills.

The other research questions were meant to check learners' perceptions and challenges faced with geometric skills. On visualisation, most learners acknowledged that once someone does the first stage of the model correctly, it is very easy to interpret diagrams. One learner mentioned that he took it more like a game. None of the learners claimed to be having challenges in verbalisation as a skill. All the learners interviewed claimed that they did not have any difficulties in verbalisation as they already internalised all the theorems and could state all the theorems correctly except one learner who claimed to be confused sometimes with those theorems that share some similarities. The same applies for drawing and constructions. All the interviewees claimed that drawing skills were not an issue to them, especially those constructions required for all the examinable proofs in the exam. They mentioned that they could now prove all the theorems except one learner who said he was still troubled by one theorem.

From the responses from the interviews, learners noted that they now enjoyed even those questions that demanded logical and application skills. These are the questions that most of the learners initially left blank spaces not knowing where to start. Another learner mentioned that, if one could do stage 1 and stage 2 properly, life could easy when attempting such questions, especially when one could identify the RTP.

Learner responses were used to come up with the themes that were used for analysis of interview results. These themes were related to the challenges faced by learners during the use of the four stages model. The themes included: practice related challenges, challenges specific to certain learners, concept related challenges, curriculum- related challenges, model application challenge and context related factors. These challenges had effects here and there during the success of the four stages model in geometric skills development, but the major goals of the research study were achieved. In summary, even though learners were still having challenges in some sections, most of them exhibited a very big difference in interest in the concept and development in geometric skills.

5.3.SIGNIFICANCE OF THE STUDY

The poor performance of learners in Mathematics is a major concern, especially in Geometry. Many factors could be contributing to this including the challenges the educators encounter in the subject too. This study, therefore, informs and enriches my own classroom teaching practices as a high school Mathematics teacher. The study also endeavoured to provide recommendations and guidelines to fellow Grade 11 Mathematics teachers on the best ways to develop problem solving and geometric skills during teaching and learning of Euclidean Geometry, and how associated challenges faced by learners can be addressed.

5.4.CHALLENGES AND LIMITATIONS

The pandemic (Covid-19) affected the data collection process. The learners and the teacher were expected to observe social distancing, which resulted in group working not being possible. The learners could only sit in twos and not more than that.

The study had limitations resulting from the quantity of data collected. This study was a case study and only considered lessons and activities in Euclidean Geometry. The results could not be generalised. There is room for further research in this area involving larger scale action research to determine the effectiveness of using the four stages strategy in Mathematics problem-solving at large. Research involving other Mathematics topics should be explored as well. In this study, 20 learners were meant to take part but only seventeen did initially, and the other three joined later. This study was limited to a particular geographical area and data was collected from a small sample, implying the result of its findings cannot be generalised. Further researchers may also include the lower grades and explore the use of the four stages in problem solving.

5.5.RECOMMENDATIONS

As has been alluded in this research that Euclidean Geometry poses several challenges to learners' problem-solving ability, adopting an alternative teaching strategy for problem-solving in Euclidean Geometry will imply that many educators would shift from their present pedagogical methods of presenting geometric theorems as a finished product. From this study, the following recommendations are put forward to enhance learners' problem-solving ability in Euclidean Geometry.

Prior Geometric knowledge

This study suggests that Mathematics teachers attend more to prior knowledge before introducing the content of that concept for that grade. It was not easy to point out the actual reasons behind learners' inabilities to display skills related to prior knowledge, but the results of the research study revealed that learners lack prior knowledge of concepts learnt in the previous grades, especially in Geometry. Pre-tests and any form of diagnostic informal assessments are recommendable for analysis of learners' prior knowledge before teaching the current content.

Teaching of geometric theorems

This research study proposes that Mathematics teachers can make use of the four stages when teaching Euclidian Geometry. When using this method, it is advisable for teachers to create a mathematical learning environment conducive for learners to participate and for the teacher to guide the learners towards the understanding of geometric concepts through problem-solving. Social cognitive classroom environment can be created during problem-solving, as learners learn well through peer collaboration and observations. In this case, geometric skills will be the point of focus during problem-solving, and these skills include visualisation, verbalisation, drawing, application, and logic.

To begin with, verbalisation acts as a corner stone in geometric problem-solving since Geometry is all about theorems and one's ability to correctly make supported arguments. Thus, we need to re-look at the way definitions of geometric concepts are taught to learners. I support mathematicians like De Villiers (2008) for a "reconstructive" approach instead of the more regular direct axiomatic-deductive approach to teaching of definitions in Geometry. With reconstructive approach, we indicate that content is not directly introduced to learners as finished products of mathematical activity but is rather reconstructed during teaching in a typical mathematical manner. I noted that the pedagogical advantage of employing the reconstructive approach is that it allows the learners to become actively engaged in the construction of geometric definitions and theorems, hence internalising the meaning of the theorems instead of the wording. The learners will not memorise the theorems but rather know the theorems in such a way that they will be able to write them in their own words, as well as complete theorems regardless of how the examiner puts them forward.

Similarly, if verbalisation is done well, it makes visualisation and drawing easy. From this study, I recommend teachers to take some time relating verbalisation, drawing and visualisation before doing the actual problem-solving. Drawing will enable learners to make drawings from given theorems as well as separate diagrams from the bigger picture. Learners may not be on the same cognitive level therefore, it is advisable for learners to provide definitions of concepts, which align with their cognitive level. Learners' poor performance in Geometry can be attributed to, amongst other factors, the pedagogical strategy of the teacher, outdated curriculum, outdated textbooks that present ready-to use content, which learners are expected to memorise and accept without questioning. If learners performance in Geometry is to improve, then the educator needs to be the catalyst for change in this process (Mamona-Downs & Downs, 2005). Thus, to achieve this goal, we need to encourage learners to become actively engaged in the construction of knowledge in teaching and learning of Euclidean Geometry.

Teaching Geometric problem-solving

In a traditional classroom, learners are given predetermined statements to make proofs, and as a result learners assume that the statement must be true. Learners need to know how mathematicians use proof for the learners to appreciate the value of proof writing and to engage as mathematicians do. Learners need to be mindful that proof is an aspect found in the process of learning and discovering new Mathematics. Mathematicians first make conjectures based on observations, test these observations before formally publish his/her findings and allows their conjectures to be critiqued by fellow mathematicians before their conjecture is accepted as true. From this study, I recommend that as educators, we need to afford learners the opportunities to assess the thinking processes of their colleagues and their own. As was suggested by Groth (2018), I realised that it is advisable that the teacher's instruction aid learners to gradually progress from the lower levels of thinking before engaging the learners in the rigors of proof-oriented study of Geometry. When teaching riders and geometric proofs, I noted that most textbooks present solutions to geometric riders in a linear manner, implying that learners would be able to make sense of such a solution. However, such an approach may be less effective when dealing with more complex problems. I found a backward reasoning approach to be an approach that allows the learner to search for possible solutions to a given rider, using the given information, and making valid inferences from such information. This approach was enhanced during the use of the four stages model of geometric problem-solving where one must read for understanding (key geometric terms), do some planning (theorems related to key words, filling in the diagrams and add any relevant information), implementation (geometric statements and supporting arguments) as well as reflection. This encouraged the learner to be practically engaged during problem-solving, whereby one was supposed to make markings of key geometric terms in the statement, note down the geometric theorems related to the marked key words, check the applicability of the theorems by filling in the diagram, as well as adding any other relevant information. After that, the learner was then expected to present the solutions and reflect on them.

Learning environment

Since the method involves much writing as part of the problem-solving strategy, the use of the four stages in geometric problem-solving will make life interesting and easy for learners to tackle geometric problems especially when initiating problem-solving because most learners fail Geometry problem-solving from the very start as was noted by South African Mathematics internal moderators (2019) that most learners leave geometric questions unattended (Buthelezi & Sithole, 2019). I also understood that when doing problem-solving, especially with Riders, learners end up enjoying finding different angles using theorems and treat it as a game.

I would recommend that teachers create classroom environments that make learning of Geometry interesting. In this case, learners should also be viewed as intelligent and creative individuals whose questions are valued and teachers should afford the learners more time for discussions as this would give the learners effective decision-making powers in problem-solving. Thus, working in pairs or groups will allow for collaborative learning.

Teachers need to be encouraged to have positive attitudes towards the use of other strategies than the old routine ineffective ways. This is important because new ways of doing things allow for other associated advantages to be enjoyed by the learners. The four stages strategy should be used to encourage more the process of verbalisation and visualisation in the teaching and learning of Mathematics, especially core concepts of Geometry. This four stages strategy will help the learners to see and realise that Mathematics can be taught and learned in different ways that are easier to follow than the traditional methods.

5.6.SUGGESTIONS FOR FURTHER RESEARCH

Since this study was carried out in Euclidean Geometry, it will be interesting if this research is extended to other mathematical disciplines. I suggest a detailed study about the use of the four stages in problem solving with other GET and FET phase grades. This study was an interpretive study located in Nquthu at a farm school. It would be more interesting to see how the results would differ if the study was done at a school in town or in the location.

Further research on the use of the four stages in problem solving could be conducted to train Mathematics geometric teachers on how to incorporate it in their teaching of Euclidean Geometry. Further studies should also be conducted to find ways in which learners' geometric skills can be developed fully to prevent and improve the poor performance of Mathematics in South African schools. Research on the use of the four stages strategy as a tool in Mathematics in all grades could be undertaken in the future.

5.7.PERSONAL REFLECTIONS

This research study mission was quite a worthwhile experience for me in the research arena. The lens of this interpretive study was Social Cognitive Learning Theory. The concept of problem-solving challenged me to do an exploration and development of the mathematical problem-solving and geometric skills of my learners. The importance of problem-solving and geometric skills development in this study made me realise the importance of making use of the four stages strategy as the lack of problem-solving skills is claimed to be the most cause of poor performance in Mathematics. The research experience I gained in this study inspired me to want to undertake more research studies in the future. Finally, my grammatical, academic reading and writing were all improved through the writing of this research study.

5.8.CONCLUSION

This research study was centralised within the interpretive paradigm and reinforced by Social Cognitivism as the theoretical lens to investigate and explore Grade 11 learners' problem-solving skills in Euclidean Geometry. Data collection for this study was done using pre-intervention test, lesson observations, exercise books monitoring, after lesson follow-up interviews, post-intervention test and semi-structured interviews. The findings of this research study revealed that the four stages problem solving strategy has a high potential to improve problem solving and geometric skills of learners in Mathematics. The study further exposed that the use of the four stages in Mathematics problem solving excites independent thoughts in learners and makes Mathematics lessons very exciting.

5.9.REFLECTIONS ON MY RESEARCH JOURNEY

The road to get to this destination was not always smooth. I had to contend with challenges at home and at work. Despite the hard conditions, I was able to come out on the other end academically enriched and satisfied. The greater goal was to enrich myself and assist my colleagues who have trouble when teaching Euclidean Geometry, to teach it in a manner which allows for greater learner participation in the discipline. The dilemmas regarding Euclidean Geometry are not restricted to us only, it is a universal phenomenon experienced at various levels by different countries all over the world. Furthermore, this research study has advanced my research capabilities and taught me about learning and teaching Euclidean Geometry. My own teaching experience as a high school Mathematics teacher and when I joined the department of Education South Africa, I found myself working with learners in the FET phase which led me to combine my interest in Mathematics teaching and learning. It is this experience that has given me not only enormous insight, but also huge respect for learners. I look forward to sharing my experiences and knowledge gained from this research process with any interested entities.

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APPENDICES

Appendix A: Ethical Clearance

RHODES UNIVERSITY
Where leaders learn

Rhodes University, Education Faculty
Research Ethics Committee
PO Box 94, Makhanda, 6140, South Africa
Tel: +27 (0) 46 603 8393
Fax: +27 (0) 46 603 8028
email: e.rosenberg@ru.ac.za

https://www.ru.ac.za/researchgateway/ethics/

6 December 2022

Mr ratham hlupeni

g20h2444@campus.ru.ac.za

Dear Mr Ratham Hlupeni and Prof Marc Schafer, Dr Clemence Chikiwa

Re: Exploring Grade 11 learners' mathematical Problem-solving skills using Poyla's model during the learning of Euclidean geometry

APPLICATION NUMBER: 2022-4910-7322

This letter confirms that your research ethics renewal application has been reviewed and **APPROVED** by the Education Faculty Research Ethics Committee (EF-REC).

Approval is granted for 1 year. An annual progress report is required in order to renew approval for an additional period. You will receive an email notifying you when the progress report is due.

Should any substantive change(s) be made during the research process, that may have ethical implications, you should notify the Education Faculty REC Chair via email. This includes changes in investigators. The REC Chair will advise as to whether a new application is necessary.

Do keep this clearance letter secure and accessible throughout your study and after its completion. It will be needed when a thesis is examined and when publications are submitted to journals.

Please also submit a brief report to the REC Chair on the completion of the research. This can be done via email. The purpose of this report is to indicate whether the research was conducted successfully and whether any ethics-related matters arose that the committee should be aware of, in order to guide future studies.

Sincerely,

ERosenberg

Prof Eureta Rosenberg

Chair: Education Faculty Research Ethics Committee

Page 1 of 1

RHODES UNIVERSITY
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Rhodes University, Education Faculty
Research Ethics Committee
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https://www.ru.ac.za/researchgateway/ethics/

13/06/2021

Ratham Hlupeni

Education Department

g20h2444@campus.ru.ac.za

Dear MR RATHAM HLUPENI

Re: Exploring Grade 11 learners' mathematical Problem-solving skills using Polya's model during the learning of Euclidean geometry

APPLICATION NUMBER: 2021-4910-6041

This letter confirms that your research ethics application has been reviewed and **APPROVED** by the Education Faculty Research Ethics Committee (EF-REC). Your permission letter(s) where applicable have been received and you are free to proceed with your study.

Approval is granted for 1 year. An annual progress report is required in order to renew approval for an additional period. You will receive an email notifying you when the progress report is due.

Should any substantive change(s) be made during the research process, that may have ethical implications, you should notify the Education Faculty REC Chair via email. This includes changes in investigators. The REC Chair will advise as to whether a new application is necessary.

Do keep this clearance letter secure and accessible throughout your study and after its completion. It will be needed when a thesis is examined and when publications are submitted to journals.

Please also submit a brief report to the REC Chair on the completion of the research. This can be done via email. The purpose of this report is to indicate whether the research was conducted successfully and whether any ethics-related matters arose that the committee should be aware of, in order to guide future studies.

Sincerely,

ERosenberg

Prof Eureta Rosenberg

Chair: Education Faculty Research Ethics Committee

Page 1 of 1

Appendix B: Approval letter from the Department of education

KWAZULU-NATAL PROVINCE
EDUCATION
REPUBLIC OF SOUTH AFRICA

OFFICE OF THE HEAD OF DEPARTMENT

Private Bag X9137, PIETERMARITZBURG, 3200
Anton Lembede Building, 247 Burger Street, Pietermaritzburg, 3201
Tel: 033 3921062 / 033-3921051
Email: Phindile.duma@kzndoe.gov.za
Buyi.ntuli@kzndoe.gov.za

Enquiries: Phindile Duma/Buyi Ntuli **Ref.**:2/4/8/7111

Mr Ratham Hlupeni
P.O. Box 1461
VRYHEID
3100

Dear Mr Hlupeni

PERMISSION TO CONDUCT RESEARCH IN THE KZN DoE INSTITUTIONS

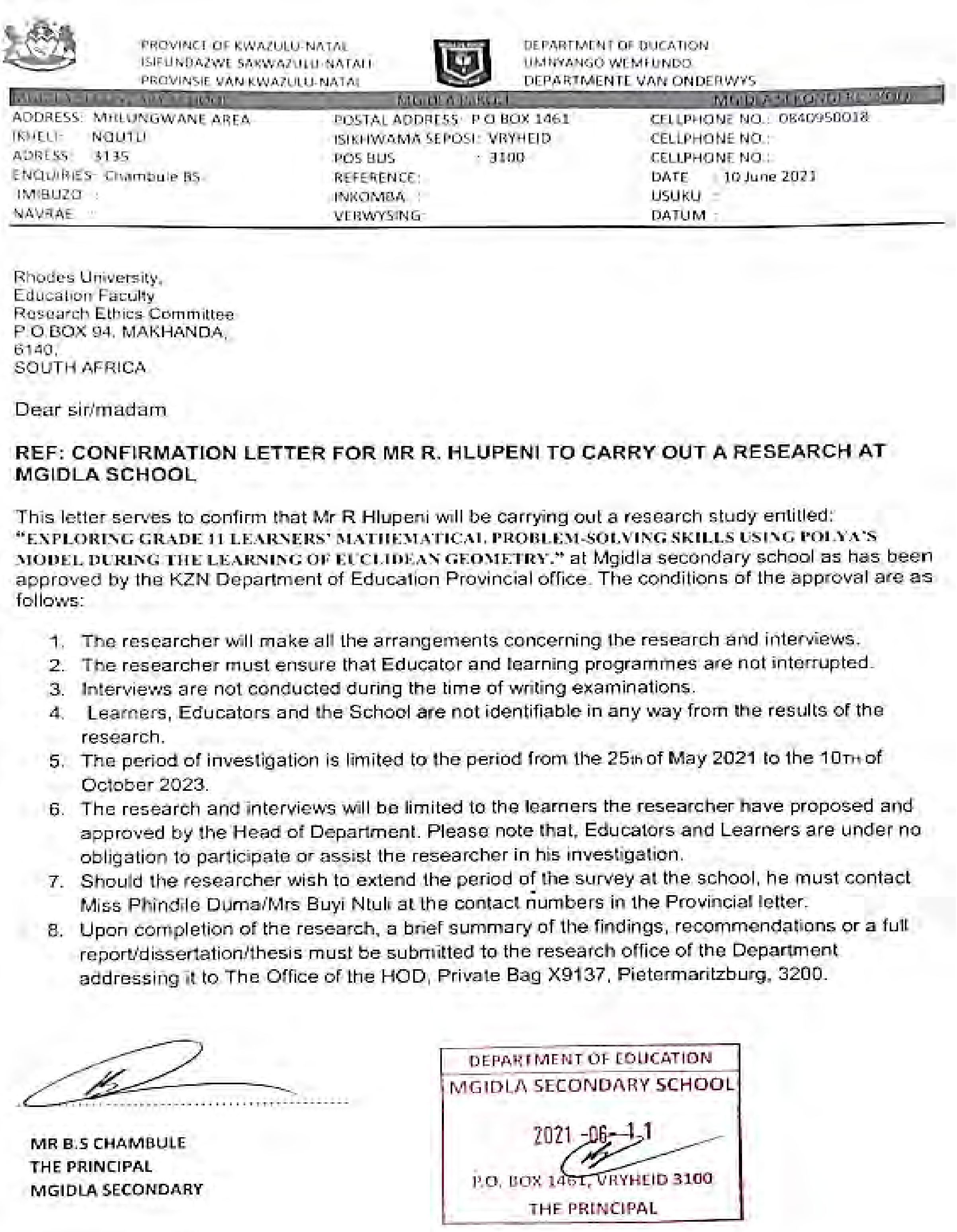
Your application to conduct research entitled: **"EXPLORING GRADE 11 LEARNERS' MATHEMATICAL PROBLEM-SOLVING SKILLS USING POLYA'S MODEL DURING THE LEARNING OF EUCLIDEAN GEOMETRY."**: in the KwaZulu-Natal Department of Education Institutions has been approved. The conditions of the approval are as follows:

1. The researcher will make all the arrangements concerning the research and interviews.
2. The researcher must ensure that Educator and learning programmes are not interrupted.
3. Interviews are not conducted during the time of writing examinations in schools.
4. Learners, Educators, Schools and Institutions are not identifiable in any way from the results of the research.
5. A copy of this letter is submitted to District Managers, Principals and Heads of Institutions where the Intended research and interviews are to be conducted.
6. The period of investigation is limited to the period from the 25th of May 2021 to the 10TH of October 2023.
7. Your research and interviews will be limited to the schools you have proposed and approved by the Head of Department. Please note that Principals, Educators, Departmental Officials and Learners are under no obligation to participate or assist you in your investigation.
8. Should you wish to extend the period of your survey at the school(s), please contact Miss Phindile Duma/Mrs Buyi Ntuli at the contact numbers above.
9. Upon completion of the research, a brief summary of the findings, recommendations or a full report/dissertation/thesis must be submitted to the research office of the Department. Please address it to The Office of the HOD, Private Bag X9137, Pietermaritzburg, 3200.
10. Please note that your research and interviews will be limited to schools and institutions in KwaZulu-Natal Department of Education.

Dr. EV Nzama
Head of Department: Education
Date: 25th May 2021

GROWING KWAZULU-NATAL TOGETHER

Appendix C: Approval letter from school principal



PROVINCE OF KWAZULU-NATAL
ISIFUNDAZWE SAKWAZULU-NATALI
PROVINSIE VAN KWAZULU-NATAL

DEPARTMENT OF DUCATION
UMNYANGO WEMFUNDO
DEPARTMENTE VAN ONDERWYS

[illegible]

ADDRESS: MHLUNGWANE AREA	POSTAL ADDRESS: P O BOX 1461	CELLPHONE NO.: 0840950018
IKHELI: NQUTU	ISIKHWAMA SEPOSI: VRYHEID	CELLPHONE NO.:
ADRESS: 3135	POS BUS : 3100	CELLPHONE NO.:
ENQUIRIES: Chambule BS	REFERENCE:	DATE : 10 June 2021
IMIBUZO :	INKOMBA :	USUKU :
NAVRAE :	VERWYSING:	DATUM :

Rhodes University,
Education Faculty
Research Ethics Committee
P.O.BOX 94, MAKHANDA,
6140,
SOUTH AFRICA

Dear sir/madam

REF: CONFIRMATION LETTER FOR MR R. HLUPENI TO CARRY OUT A RESEARCH AT MGIDLA SCHOOL

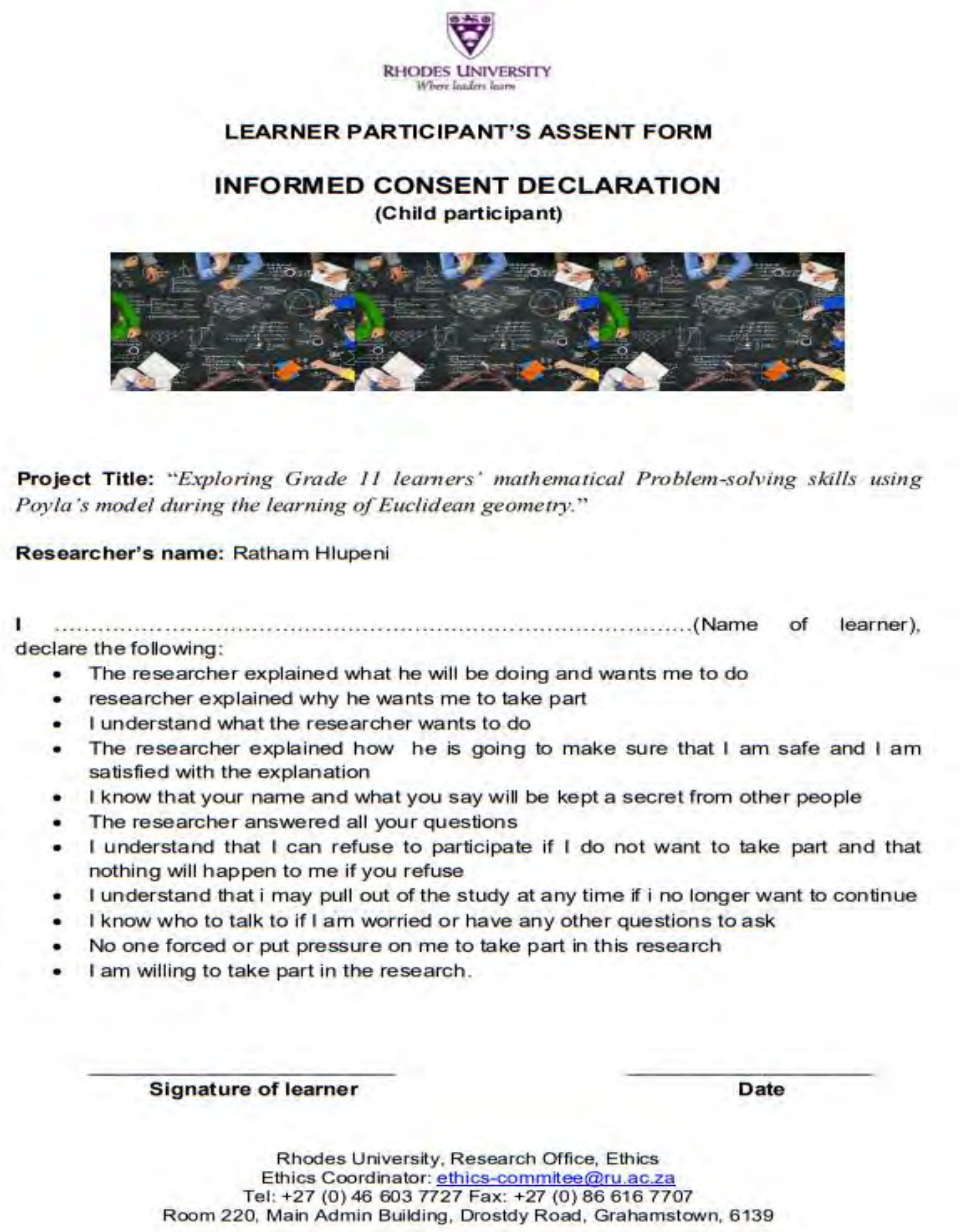
This letter serves to confirm that Mr R Hlupeni will be carrying out a research study entitled: **"EXPLORING GRADE 11 LEARNERS' MATHEMATICAL PROBLEM-SOLVING SKILLS USING POLYA'S MODEL DURING THE LEARNING OF EUCLIDEAN GEOMETRY."** at Mgidla secondary school as has been approved by the KZN Department of Education Provincial office. The conditions of the approval are as follows:

1. The researcher will make all the arrangements concerning the research and interviews.
2. The researcher must ensure that Educator and learning programmes are not interrupted.
3. Interviews are not conducted during the time of writing examinations.
4. Learners, Educators and the School are not identifiable in any way from the results of the research.
5. The period of investigation is limited to the period from the 25th of May 2021 to the 10TH of October 2023.
6. The research and interviews will be limited to the learners the researcher have proposed and approved by the Head of Department. Please note that, Educators and Learners are under no obligation to participate or assist the researcher in his investigation.
7. Should the researcher wish to extend the period of the survey at the school, he must contact Miss Phindile Duma/Mrs Buyi Ntuli at the contact numbers in the Provincial letter.
8. Upon completion of the research, a brief summary of the findings, recommendations or a full report/dissertation/thesis must be submitted to the research office of the Department addressing it to The Office of the HOD, Private Bag X9137, Pietermaritzburg, 3200.

MR B.S CHAMBULE
THE PRINCIPAL
MGIDLA SECONDARY

DEPARTMENT OF EDUCATION
MGIDLA SECONDARY SCHOOL
2021 -06- 11
P.O. BOX 1461, VRYHEID 3100
THE PRINCIPAL

Appendix D: Learner Participant consent form



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LEARNER PARTICIPANT'S ASSENT FORM

INFORMED CONSENT DECLARATION

(Child participant)

Project Title: *"Exploring Grade 11 learners' mathematical Problem-solving skills using Poyla's model during the learning of Euclidean geometry."*

Researcher's name: Ratham Hlupeni

I ………………………………………………………………………………(Name of learner), declare the following:

- The researcher explained what he will be doing and wants me to do
- researcher explained why he wants me to take part
- I understand what the researcher wants to do
- The researcher explained how he is going to make sure that I am safe and I am satisfied with the explanation
- I know that your name and what you say will be kept a secret from other people
- The researcher answered all your questions
- I understand that I can refuse to participate if I do not want to take part and that nothing will happen to me if you refuse
- I understand that i may pull out of the study at any time if i no longer want to continue
- I know who to talk to if I am worried or have any other questions to ask
- No one forced or put pressure on me to take part in this research
- I am willing to take part in the research.

Signature of learner **Date**

Rhodes University, Research Office, Ethics
Ethics Coordinator: ethics-commitee@ru.ac.za
Tel: +27 (0) 46 603 7727 Fax: +27 (0) 86 616 7707
Room 220, Main Admin Building, Drostdy Road, Grahamstown, 6139

Appendix E: Parent Consent Form

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PARENT/GUARDIAN INFORMED CONSENT

INFORMED CONSENT DECLARATION
(Parent or Guardian)

REQUEST FOR PERMISSION FOR YOUR CHILD TO TAKE PART IN RESEARCH STUDY

I, Ratham Hlupeni, a Master's degree student at Rhodes University (RU) in Grahamstown, South Africa would like to conduct a research for my Master's full thesis in Mathematics Education at the school where your child is attending. The study is entitled: "***Exploring Grade 11 learners' mathematical Problem-solving skills using Poyla's model during the learning of Euclidean geometry***". This research requires me to engage, observe and interview all Grade 11 Mathematics learners, for a period of at least three weeks. So your child is requested to take part since the study will be done during teaching and learning of Euclidean geometry.

Data collection will take place between May 2021 and October 2021. This research study will be conducted under the supervision of Dr Clemence Chikiwa from the Education faculty of Rhodes University. Learners will be observed while learning, they will also be interviewed during the course of this study. The study will involve video camera recordings of observations and interviews. Learners' names and their identities will be treated with confidentiality. It is within my ethical assurance that the name of the school and learner participants will remain anonymous and that all the information I collect as part of the research procedure will be accessible only to myself and my supervisors. Rhodes University Ethical Committee has approved this study. Please find an attached copy of the clearance letter from Rhodes University.

Should you require any further information, please do not hesitate to contact me or my supervisor. Our contact details are as follows:

Ratham Hlupeni: +27784138329 ***Clemence Chikiwa (PhD): +27466037210***
rathamhlupeni@gmail.com *C.Chikiwa@ru.ac.za*

Your permission to conduct this study will be greatly appreciated.

Yours sincerely,

Ratham Hlupeni
Master's student
Mathematics teacher (Mgidla SS, Nquthu Circuit).

Parent
I,……………………………,the parent/guardian to…………………………(Name of the learner),
Accept that my child be part of the research study.

Signature:……………………………………………… Date: ……………………………

Rhodes University, Research Office, Ethics
Ethics Coordinator: ethics-commitee@ru.ac.za
Tel: +27 (0) 46 603 7727 Fax: +27 (0) 86 616 7707
Room 220, Main Admin Building, Drostdy Road, Grahamstown, 6139

RHODES UNIVERSITY
Where leaders learn

IMVUME YOMZALI/YOMNAKEKELI

ISIMEMO SOKUVUMELANA

(Mzali)

ISICELO SEMVUME NGENGANE YAKHO UKUBA IBE YINGXENYE YONCWANINGO

Mina, Ratham Hlupeni, Umfundi weziqu ze-Masters esikhungwini semfundo ephakeme –i-Rhodes University (RU) ese- Grahamstown, eNingizimi Africa. Ngifisa ukwenza ucwaningo lwami lwe-Masters esifindweni sezibalo (Mathematics) kwezemfundo eskoleni lapho ingane yakho ifunda khona. Lolu cwaningo ludinga ngizinikele(engage), ngithathele kokwenzekayo, ngiphinde ngibuze imibuzo bonke abafundi abenza ibanga le-11 isifundo sezibalo. Isikhathi esinganga masonto amathathu kuyaphezulu, Ingane yakho kumele ibe yingxenye ngokuba ucwaningo luzobe lwenzeka ngesikhathi sokufunda nokufundiswa kwe- Euclidean Geometry. Ukuqoqwa kwemininingwane kuzoba phakathi kuka –Nhlaba ngonyaka wezi-2021 kuya kuMbasa ngonyaka wezi-2022. Lolu cwaningo luzokwenziwa ngaphanzi kobeluleki buka Dr Clemence Chikiwa osuka phansi kophiko lwemfundo e.Rhodes University luphinde lufake nokuqoshwa ngobuchwepheshe (camera) nenhlolokhona (interview).

Ikhomidi le Rhodes university likwemukele ucelakala ukuba uthole incwadi ye-clearence enanyathiselwe esuka ERhodes University. Ngiyaqinisekisa ukuba igama lesikole kanye nengane ezobe iyingxenye ngeke kwaziwe umuntu ngoba yimina nomeluleki kuphela onemvume yokuthinta imininingwane.

Uma ufuna ukwazi kabanzi ungangabazi ungathinta.mina noma maluleki wami.
Ratham Hlupeni: +27784138329 ***Clemence Chikiwa (PhD): +27466037210***
rathamhlupeni@gmail.com ***C.Chikiwa@ru.ac.za***

Imvume yakho ukuba ngenze lolu cwaningo iyoncomeka kakhulu.

Ozithobayo
Ratham Hlupeni
Master's student
Mathematics teacher (Mgidla SS, Nquthu Circuit).

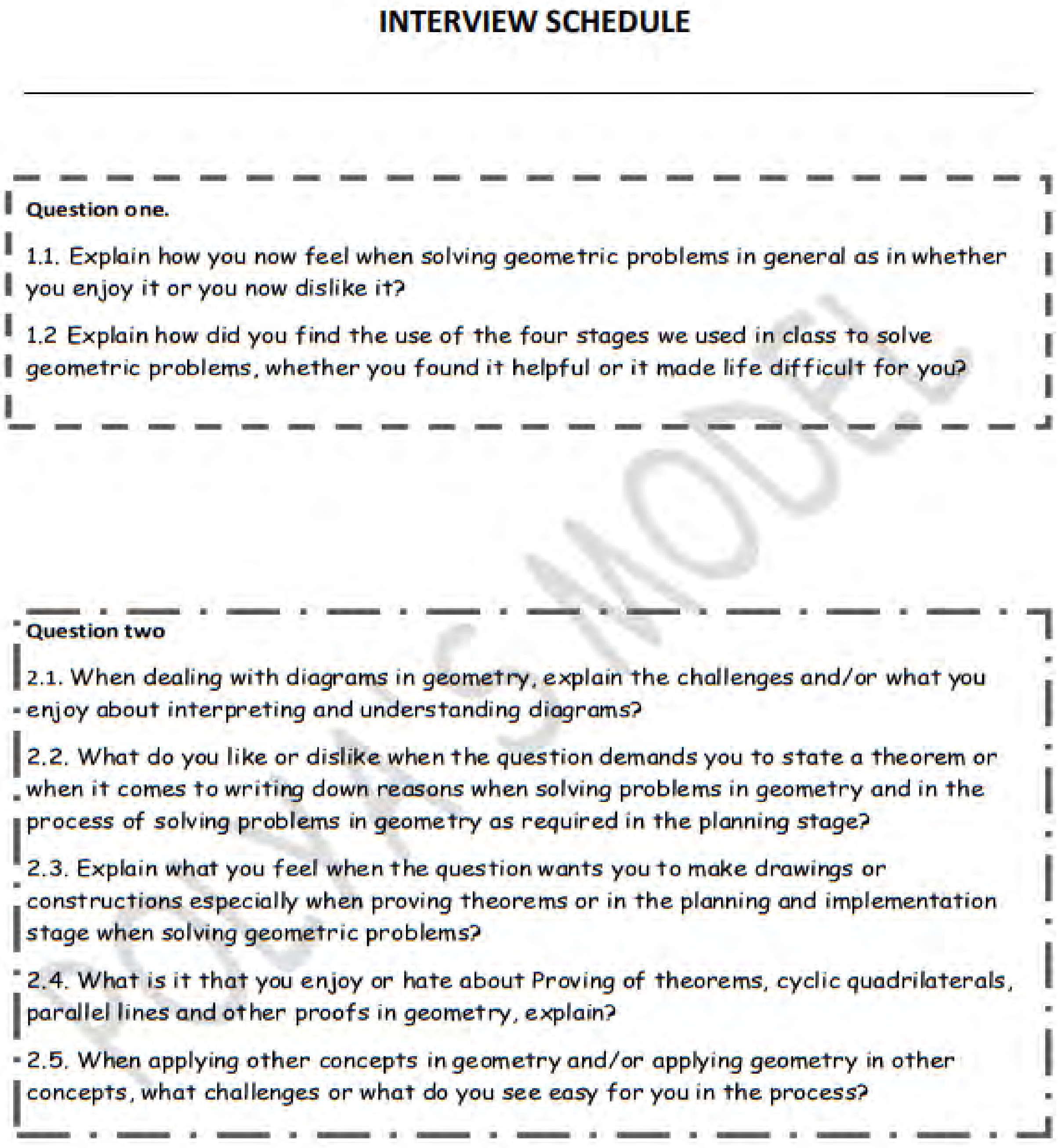
Parent

Mina…. ……………………….,mzali/mnakekeli ku…………………………(Igama lengane) ,uyavuma kuba ingane ibe yingxenye yocwaningo.

Signature:……………………………………………… Date: ……………………………….

Rhodes University, Research Office, Ethics
Ethics Coordinator: ethics-commitee@ru.ac.za
Tel: +27 (0) 46 603 7727 Fax: +27 (0) 86 616 7707
Room 220, Main Admin Building, Drostdy Road, Grahamstown, 6139

Appendix F: Interview Schedule



INTERVIEW SCHEDULE

Question one.

1.1. Explain how you now feel when solving geometric problems in general as in whether you enjoy it or you now dislike it?

1.2 Explain how did you find the use of the four stages we used in class to solve geometric problems, whether you found it helpful or it made life difficult for you?

Question two

2.1. When dealing with diagrams in geometry, explain the challenges and/or what you enjoy about interpreting and understanding diagrams?

2.2. What do you like or dislike when the question demands you to state a theorem or when it comes to writing down reasons when solving problems in geometry and in the process of solving problems in geometry as required in the planning stage?

2.3. Explain what you feel when the question wants you to make drawings or constructions especially when proving theorems or in the planning and implementation stage when solving geometric problems?

2.4. What is it that you enjoy or hate about Proving of theorems, cyclic quadrilaterals, parallel lines and other proofs in geometry, explain?

2.5. When applying other concepts in geometry and/or applying geometry in other concepts, what challenges or what do you see easy for you in the process?

Appendix G: Analytical Tool

Tool for analyzing geometric skills of learners

SKILL	DEFINITION	CODE	DESCRIPTION OF THE SKILL
1.Visual	Ability to envision a diagram	**VS**	**VS1**: Recognition of a theorem or axiom, **VS2**: Recognition of properties of shapes. **VS3**: Recognition of different parts and lines in a circle or diagram **VS4**: Recognition of different angles
2.Verbal	Ability to name, describe, state, explain or define	**VR**	**VR1**: Verbal descriptions **VR2**: Naming of shapes, objects and parts **VR3**: Definitions and Stating in own words
3.Drawing	Ability to draw, sketch, construct and put labels	**DR**	**DR1**: Sketching of diagrams related to theorems and/or axioms **DR2**: Constructing diagrams based on its properties **DR3**: Adding useful elements to a diagram
4.Logical	Ability to differentiate, classify and prove	**LG**	**LG1**: Recognition of differences and similarities between geometric objects **LG2** Classification by their properties **LG3**: Development of logical proofs
5.Applied	Ability to apply what is known in another situation	**AP**	**AP1**: Recognition of the physical model of the Geometry object, **AP2**: Application of the properties of the Geometry model to the physical object. **AP3**: Application of theorems and axioms in problem-solving.
6.Emerging	Any skill that emerges during the use of the model	**EG**	**EG1:** Re-planning skill **EG2**: Computational skill **EG3:** Finding of RTP

Appendix H: Analytical Tools

Tool for analysing presence of Polya's stages in learners' work

Polya's Stage	Code	Definition	Observable Indicators
1. Reading for understanding	**RU**	Understanding the problem has to do with understanding the language of the problem statement, knowing what has been asked to be found or shown	**RU1**: Underlining, highlighting, circling or any markings on important terms in the statement.
			RU2: Marking of important information on the diagrams
			RU3: Noting down of important terms
			RU4:restating the problem in own words
2. Planning	**PL**	Involves a skill in choosing the appropriate strategy to solve a problem.	**PL1:** Use a variable and choose helpful names for variables or unknowns
			PL2: Drawing diagrams/pictures
			PL3: Solve a simpler version of the problem
			PL4: list related theorems and/or axioms
			PL5: Look for a pattern or patterns
			PL6: Make a list
3. Implementation	**IP**	Carrying out the plan by selecting a strategy and if it is not working out one has to realise this and change it for another	**IP1:**Trials of implementation of any or some of the plans
			IP2: retrying another method if the first fails.
4. Looking Back	**LB**	Critical examination of the solution obtained to ascertain if the result is correct or whether the plan can be used to solve another problem	**LB1:** Workings to check the correctness of an answer got in stage3
			LB2: Taking back the answer from stage3 back in the diagram

Appendix I: Pre & Post Intervention Tests

QUESTION ONE.

PRE-INTERVENTION TEST [1Hr]

Complete the following geometric theorems or Axioms:

1.1. The line joining the mid-points of two sides of a triangle is

………………………………………………………………………………………………

………………………………………………………………………………………………

…………………………………………………………………(2)

1.2. Co-interior angles are……………………………………………………………………and corresponding angles are

………………………………………………………………………………………………

……………(2)

1.3. An exterior angle of a triangle is equal to

………………………………………………………………………………………………

………………………………………………………………………………………………

……………………………………………………………………..…(2)

1.4. Angle on a straight line

………………………………………………………………………………………………

………………………………………………………………………………………………

…………………………………………………………………(2)

1.5. Sum of the interior angles of a triangle

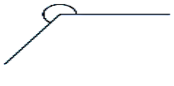
………………………………………………………………………………………………

………………………………………………………………………………………………

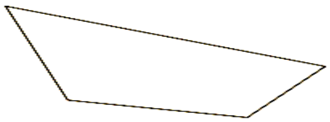
……………………………………………………………………(2)

QUESTION TWO

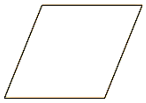
Name the Following angles and shapes (2X5=10)

SHAPE/ ANGLE	NAME
2.1 	
2.2 	

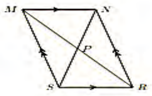
2.3



2.4



2.5



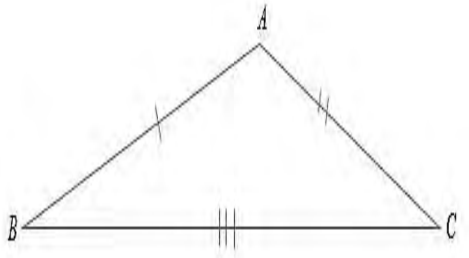
QUESTION THREE

Using ruler, pencil and space provided, answer the following:

3.1. Draw a rhombus (2)

3.2. Draw a parallelogram (2)

3.3. Complete the diagram for midpoint theorem (2)

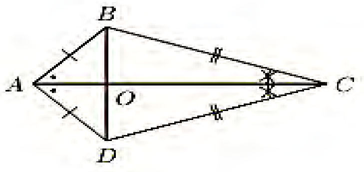


3.4. Complete this diagram to show alternate, co-interior and corresponding angles (2)

3.5. Draw a right angled isosceles triangle (2)

Question Four

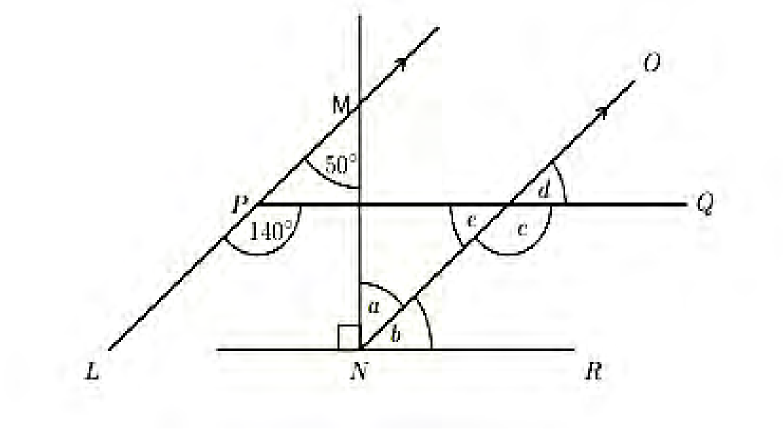
Use the sketch of quadrilateral ABCD to prove that the diagonals of a kite are perpendicular to each other (10)



STATEMENT **REASON**

QUESTION FIVE

5. Find each of the unknown angles marked in the figure below. Give a reason for your answers (2X5=10)



STATEMENT **REASON**

Post-Intervention Test

Question one.

POST-INTERVENTION TEST [1 Hr]

Complete the following geometric theorems:

1.1. Two tangents from a common external point

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………(2)

1.2. A line drawn parallel to one side of a triangle

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………..(2)

1.3. The opposite angles of a cyclic quadrilateral

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………..(2)

1.4. The line drawn from the centre of a circle bisecting a chord

is………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………….(2)

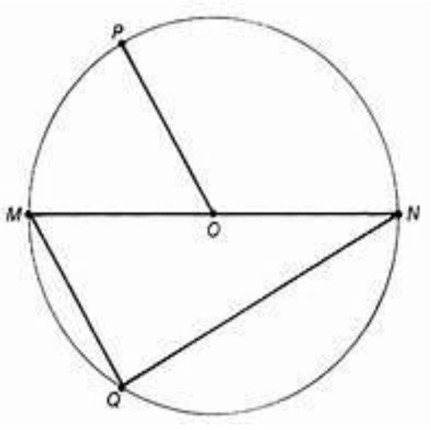
1.5. An angle subtended from the diameter of a circle

…………………………………………………………………………………………………

…………………………………………………………………………………………………

…………………………………………………………………….(2)

QUESTION TWO



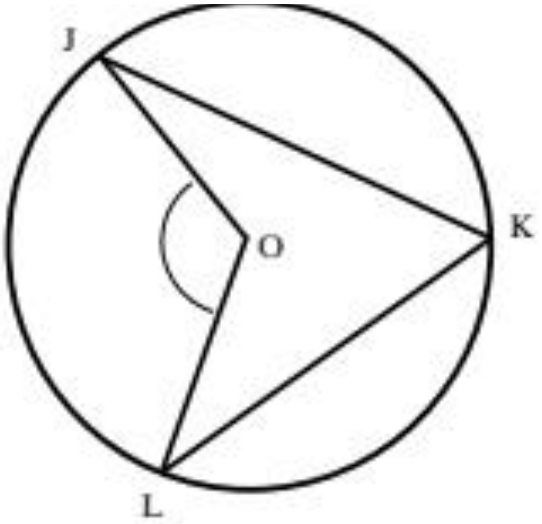
2. Name the following parts of a circle from the circle below.

2.1. Regions opposite line NQ……………………………………………………………… (2)
2.2. Line MN ……………………………………………………………………………(2)
2.3. Line QN … ……………………………………………………………………(2)
2.4. Line MPN ………………… ………………………………………………………(2)
2.5. Shape NPM……………………………………………………………………….. (2)

QUESTION THREE

Using ruler and pencil, draw a construction on the diagrams below to prove the respective geometric theorems.

3.1. The angle subtended by an arc at the center of a circle is double the size of the angle subtended by the same arc on the circumference (2)



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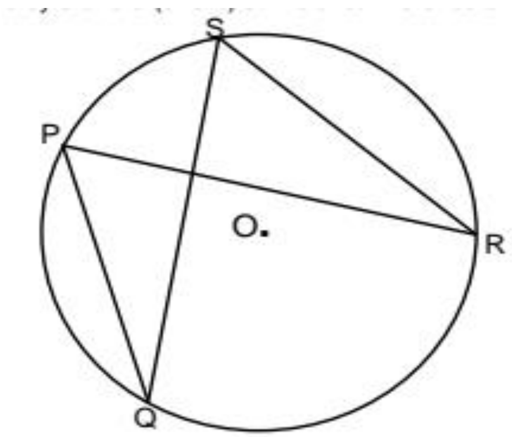
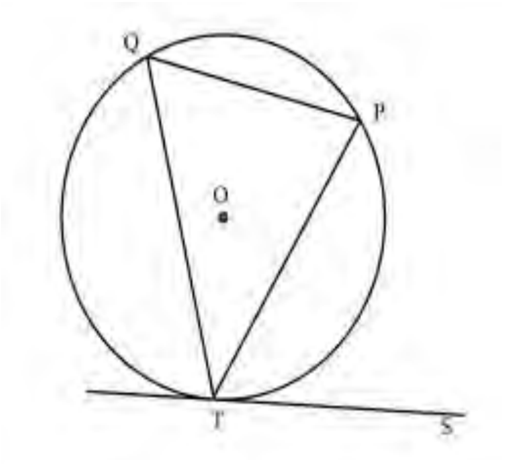
3.2. Equal chords subtend equal angles at the circumference (2)

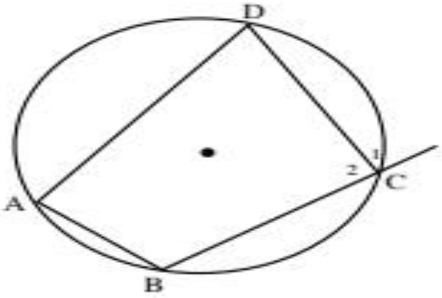
3.3. The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle subtended by the chord in the alternate segment (2)

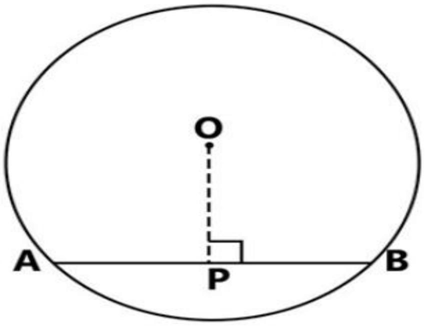




3.4. The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle (2)

3.5. The line drawn from the center of a circle perpendicular to a chord bisects the chord (2)





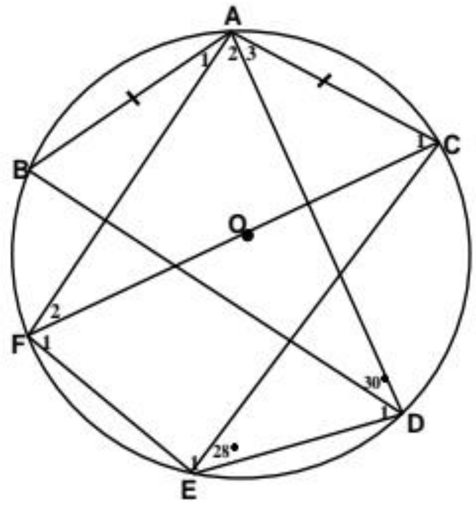
QUESTION FOUR

Use the construction on 3.3 to prove the theorem that states that: the angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment (Tan-chord theorem) (5)

STATEMENT	**REASON**

QUESTION FIVE

In the diagram, O is the center of the circle. Chords AB = AC. Angle CED = 28º and ADB = 30º



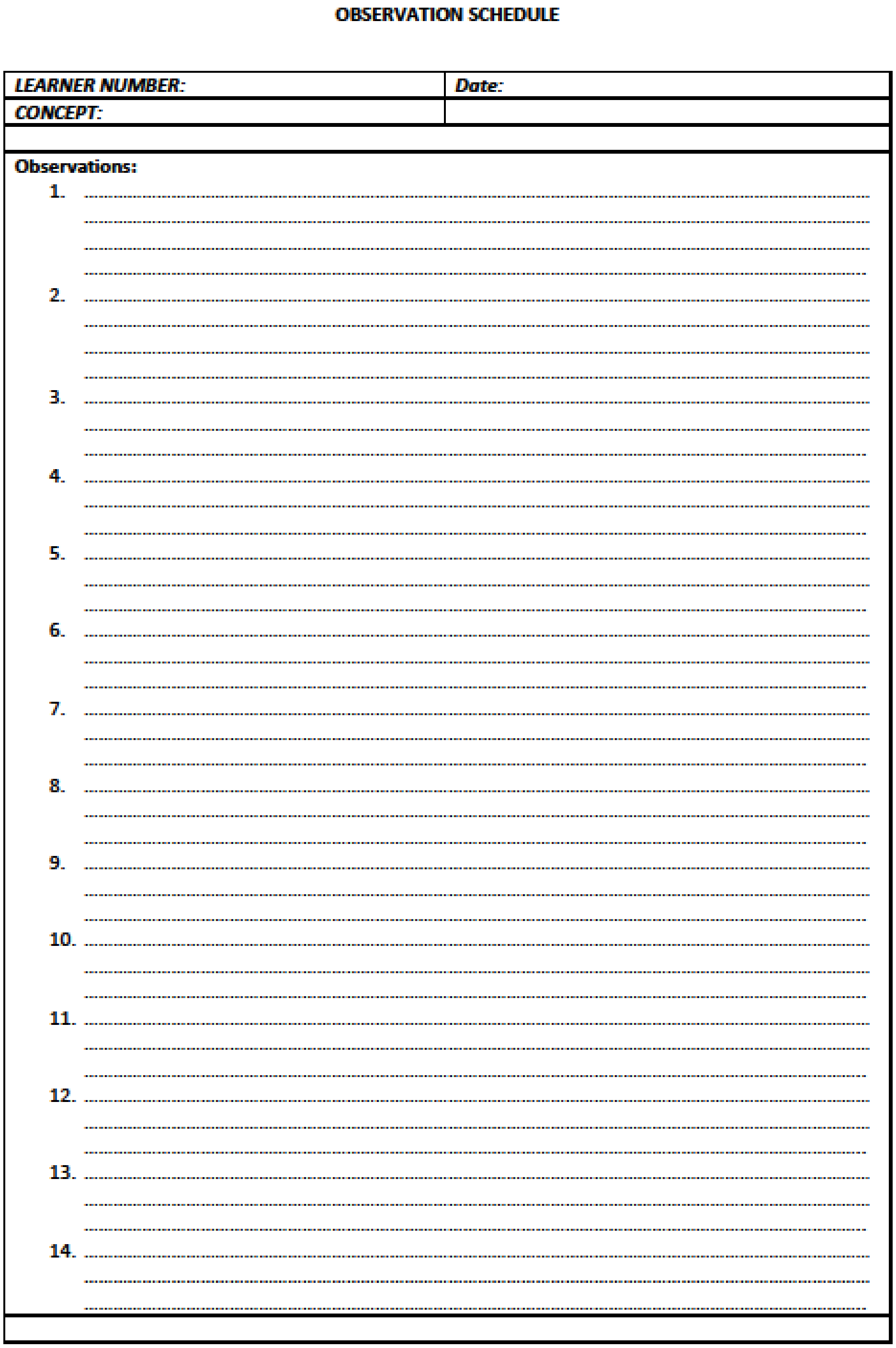
Calculate, with reasons, the sizes of the following angles:

5.1 FEC (2)
5.2. AFC (2)
5.3. CAD (2)
5.4. FAC (2)
5.5. FAD (2)

STATEMENT	**REASON**	**STATEMENT**	**REASON**

[50 marks

Appendix J: Observation schedule



OBSERVATION SCHEDULE

LEARNER NUMBER:	***Date:***
CONCEPT:	

Observations:

1. ....
2. ....
3. ....
4. ....
5. ....
6. ....
7. ....
8. ....
9. ....
10. ....
11. ....
12. ....
13. ....
14. ....

Appendix K: Certificate of editing

SOL PLAATJE
UNIVERSITY

Dr Jabulani Sibanda
Senior Lecturer: English Education
School of Education
Tel: (053) 491-0142
Email:Jabulani.Sibanda@spu.ac.za
Alternate e-mail: jabusbnd@gmail.com
Website: www.spu.ac.za
Cell: 0845282087
19 May 2023

RE: CERTIFICATE OF LANGUAGE EDITING

To whom it may concern

I hereby confirm that I have proofread and edited the following **Dissertation** using Windows 'Tracking' System to reflect my comments and suggested corrections for the author(s) to action:

Exploring Grade 11 Learners' Mathematical Problem-Solving Skills Using Polya's Model During the Learning of Euclidean Geometry.

REFERENCE

- **Author(s):** Ratham Hlupeni
- **Student Number:** G20H2444
- **Affiliation:** Rhodes University

Although the greatest care was taken in the editing of this document, the final responsibility for the product rests with the author(s).

Sincerely

19.05.2023

SIGNATURE

This certificate confirms the language editing I have done in my personal capacity and not on behalf of SPU