CONCEPTUAL DIFFICULTIES IN CHILDREN'S UNDERSTANDING OF SECONDARY SCHOOL ALGEBRA IN STANDARDS 6, 7 and 8. (THE APPLICATION OF THE CSMS ALGEBRA TEST TO A SAMPLE OF SOUTH AFRICAN SCHOOL CHILDREN).

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C O N T E N T S

PART I

Literature Survey

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PART I

LITERATURE SURVEY.

1. INTRODUCTION

1.1 THE NATURE OF MATHEMATICS

Whitehead (1948, p.291) maintains that mathematics is "... concerned with the logical deduction of consequences from the general premises of all reasoning." He also believes (op.cit., p.117) that "mathematics is the most powerful technique for the understanding of pattern, and for the analysis of the relationships of pattern." This idea is echoed by Sawyer (1955, p.12) who defines mathematics as "the classification and study of all patterns." Here 'pattern' is used in its wider sense to cover any kind of regularity that can be recognised by the mind. Einstein is reputed to have said that mathematics is a free invention of the human intellect. No longer need it "be bound either by empirical considerations or by compliance with the ordinances of some imaginary ideal world of mathematics." (Wilder, 1972, p.40)

It would appear from the above that the abstract and aesthetic qualities of mathematics are stressed. Indeed, Lamon (1972, p.6) says that the architecture of mathematics is founded on a hierarchy of abstractions, while Dieudonné (1972, p.100) considers that the essence of mathematics lies in the power to create abstractions and then reason with them. He believes that the greatest steps forward in mathematics have been linked to progress in the capacity to raise oneself higher in the realm of abstraction. The axiomatic method and emphasis on operational rules are therefore essential to the learning of mathematics.

One gets the impression from what has been said about the abstract qualities of mathematics that it is a body of knowledge which is pursued for its own sake - for its fascination and challenge, and that it can only be justified for its innate qualities. It would seem that the problems of mathematics are suggested by its own subject matter without any concern for the real world. It should, however, be remembered that historically, mathematics had its roots in the problems of the physical world, and only after that, did abstraction and generalization lead mathematics to evolve independently of these problems. The trend is constantly towards abstraction and generalisation but problems suggested by the outside world continually make new demands on mathematics. (Inc. Ass. of Assistant Masters, 1957, p.4). In this respect Skemp (1971, p.13) states that

> Mathematics is the most abstract, and so the most powerful of all theoretical systems. It is, therefore, potentially the most useful; and scientists in particular, but also economists and navigators, businessmen and communications engineers, find it an indispensable 'tool' ... for their work.

Wilder (1972, p.47) echoes this idea when he predicts that in its further development "Mathematics seems destined, under the influence of environmental and internal hereditary stresses, to become even more abstract and at the same time even more powerful in its capacity for dealing with reality."

To the pupil and the layman however, the value of mathematics will be seen mainly in its social usefulness and in its capacity for dealing with the problems of their environment. Much of the aesthetic and abstract nature of mathematics will remain hidden from them, for this can only come about once enough mathematics is known to appreciate the logical organisation of the subject and its systematic structure. Gagné (1972, p.169) feels that social usefulness is an essential criterion in deciding what mathematics should be taught at school. He feels that mathematics is too important to serve only the goal of aesthetic enjoyment. To conclude this section on the nature of mathematics, the words of Courant and Robbins seem appropriate. They state that "For scholars and laymen alike it is not philosophy but active experience in mathematics itself that alone can answer the question: What is mathematics?" (Courant and Robbins, 1941, p.5)

1.2 THE NATURE OF THE PROBLEM

The application of mathematical ways of thinking to nearly all the major fields of knowledge in recent times has increased the value of mathematics in society. (See Krutetskii, 1976, p.6; Watson, 1976, pp. 121 - 3; Fey, 1983, pp. 1167 - 8, etc.). Schools, therefore, are increasingly confronted with the task of developing to the maximum the mathematical abilities and interests of all its pupils with the view of meeting the needs of society. This includes an appreciation and understanding of the aesthetic and abstract qualities of mathematics as well as a knowledge of its utilitarian value in society. The question arises whether schools are achieving this?

With regard to mathematics education Dienes (1971, p.1) states that

> "There can hardly be a single member of the teaching profession concerned with teaching mathematics ..... who can honestly say to himself that all is well with the teaching of mathematics. There are far too many children who dislike mathematics, ... and many who find great difficulty with what is very simple. ... the majority of children never succeed in understanding the real meaning of mathematical concepts."

Reporting on his research project on factors affecting the learning of mathematics among Scottish school children Giles had this to say:

> "Despite the recent changes in the content of school mathematics, and despite the national importance of having a 'mathematically numerate' population, ... the situation is, still, that a large proportion of pupils at secondary schools do not like mathematics, and that very little is known about the factors affecting these attitudes."
>
> (Giles, 1981, p.7)

Regarding the level of attainment reached by children he makes this comment:

> "It seemed to me that a considerable proportion of the pupils in their third year in the secondary school were unable to cope with basic mathematical notation and concepts despite the considerable time they had spent being taught the subject."
>
> (Op. cit. p.11)

Some of the findings in the 1982 Cockcroft Report regarding the teaching of mathematics in British schools, were:

(a) very little of the mathematics taught in schools was remembered or used by the pupil after leaving school.

(b) Far from leading to any aesthetic appreciation for the subject, mathematics is revealed to be an area which is generally disliked. To many it creates a sense of failure with resultant feelings of fear, helplessness and anxiety which are carried through to adult life.

An aspect of mathematics which gives great cause for concern is algebra. There are indications of both lack of basic knowledge and a general lack of understanding of key concepts. (Giles, 1981, p.143) The CSMS study (Hart, 1981, p.118) revealed that in the algebra test the majority of 13, 14 and 15 year olds were not able to cope consistently with items that could properly be called algebra at all.

The present comparative study suggests that a similar situation exists in our South African schools.

A study by Harper (1980, pp.237 - 243) indicated that a minority of pupils from two grammar schools in England had acquired a conceptual understanding of the algebraic variable. "... what passes for 'algebra' in many classrooms is, to many pupils, little more than a manipulative exercise using letters empty of cognitive content." (Op. cit., p. 242)

The problem regarding algebra is certainly not a new one. In 1919 Sir Percy Nunn, considering the boundary between arithmetic and algebra, came to the conclusion that the two topics could not be separated by appealing to the subject matter. They required different attitudes of mind - arithmetic dealt with methods of calculation, whereas algebra dealt with the processes involved in such calculations. (In Harper, 1980, p. 238). Bertrand Russell (1927, p. 89) states:

> "When it comes to algebra and we have to operate with x and y there is a natural desire to know what x and y really are. That, at least, was my feeling. I always thought the teacher knew what they were but wouldn't tell me."

Rosnick (1981, p.418), in his article on misconceptions concerning the concept of variable, has this to say:

> "Letters in equations were my initial mathematics downfall. ... It was my first sight of the α's and β's of abstract algebra that pushed me into a frenzied retreat from mathematics. At that time (ten years ago) I identified mathematics as being little more than obscurity by abstraction and dehumanisation by symbolization."

The difficulties regarding mathematics in general and algebra in particular can be seen, therefore, to be very real. There is a path of increasing abstraction from junior school to high school, and as mathematics becomes more abstract, the symbols become more obscure. Unfamiliarity with mathematical symbols and their underlying concepts leads many a child to a loathing of mathematics and to dropping it at the earliest opportunity. To help pupils to understand algebra, then, ways and means must be found of helping them to learn a new numerical language.

1.3 OUTLINE OF THE STUDY

With the above in mind, it was decided to base this study on the child's conceptions (or misconceptions) of the algebraic variable using the algebra test and mark scheme developed by the CSMS (Concepts in Secondary Mathematics and Science) research team. During the years 1974 - 9 the CSMS research programme, based at Chelsea College, University of London, investigated children's understanding of mathematics from the ages 11 to 16 years. The mathematics component of this research, CSMS (M), began with the notion of attempting to trace the development of 'levels' of understanding in different topic areas of secondary school mathematics. (Hart, 1980, p.151). The intention was to develop a hierarchy of understanding in mathematics which would provide information to teachers and curriculum developers. By using the CSMS (M) algebra test and mark scheme, it would be possible in this study, to observe aspects of algebraic conceptualisation at the early algebraic stage in a sample of South African school children of equivalent age to those tested by the

CSMS team, and to compare these findings with those of the CSMS research.

By its nature this piece of research is very restricted with regard to time and resources, preventing in-depth study. It was never-the less hoped that the knowledge gained would be useful to teaching, especially in the area of the child's first contact with letters being used as mathematical symbols.

It might be appropriate at this stage to discuss the different paradigms used in cognitive development research. Carpenter (1980, pp. 146 - 8) distinguishes between two different conceptions of cognitive development. One is based on the organismic model and is represented by those who work within the Piagetian or neo-Piagetian paradigm. The other is the mechanistic model which is concerned with acquiring knowledge by learning sets of skills and co-ordinating these by chain-like associations to yield new combinations. Lovell describes those working within the latter model as skill-integrationists (eg. Bruner and Gagné) (Lovell, 1980, p.2).
The organismic model places far more emphasis on internal psychological mechanisms needed to explain intelligent behaviour than does the mechanistic model. Primarily the organismic model focuses on how the child processes or operates on information in a variety of problem situations, while the mechanistic model is more concerned with the product or structure. It should be clear, then, that one important aspect of the organismic model is the clinical interview which puts emphasis on the thinking processes of children.

It can also uncover misconceptions or give a wealth of information on incorrect responses. Objective instrumentation (eg. written tests) tends to focus on the end product - an overall test score for example, which cannot provide as much information on the thought processes of the child as can the interview.

The Piagetian type of research was evident in the CSMS study which focussed on "pupils' thinking as shown by their explanations and justifications" and gave "recognition to the active construction of knowledge of pupils." (Hart, 1980, p.1). These aspects imply that a "hierarchy of understanding cannot be identified simply by examining the logical structure of mathematics or by analysing a given concept as a network of prerequisite concepts or skills." (Op. cit. p.1) This procedure would probably be followed by proponents of the mechanistic model. The results of the CSMS study were based on written tests administered to approximately 10 000 schoolchildren. However the tests were designed to probe understanding and to assess this understanding in the sense of "what a child can do?" rather than "what does the child know?" (Hart, 1980, p.6) i.e. the focus was on the process and not the product. This was achieved through interviews which were "an integral part of the methodology, being used to inform and interpret the results of the written tests besides being used as a method of assessing the suitability of the test items." (Op. cit., p.6)

Being so closely related to the CSMS research, this piece of research follows the same paradigm. It was decided that interviews should form part of the research, but for a slightly different reason. The CSMS algebra test and mark scheme was used in this study, so there was no need to use the interviews to assess the suitability of the test items or to interpret the results of the tests. This had already been done by the CSMS team. The reason for interviewing was rather to discover whether the methods used, and errors made, by the children in this study were similar to those codified in the CSMS mark scheme. They could also bring to light other methods or errors not discovered by the CSMS study. Another reason for conducting the interviews was to gain experience in this aspect of research.

2. CONCEPT DEVELOPMENT, COGNITIVE GROWTH AND ACQUISITION OF KNOWLEDGE

2.1 CONCEPTS, SCHEMAS AND UNDERSTANDING

2.1.1 CONCEPTS.

The term "concept" is very much referred to in education, but is seldom defined. Novak (1978, p.3) describes concepts as inventions of man used to describe observed regularities in events. Pikas feels that a concept has been formed when a common response is given to dissimilar stimuli. (In Sowder, 1980, p.244). Gagné defines a concept in mathematics as an abstract idea which enables people to classify objects or events and to specify whether the objects or events are examples or non-examples of the abstract idea. (In Bell, 1978, p.108). Skemp (1971, p.22) states that a number of experiences which have something in common is required for the formation of a concept. It can be seen then that the ability to abstract certain unifying features from sets of objects, events or experiences and to classify these according to some specific criteria is necessary for concept formation.

If the objects, events or experiences are in the outside world any concepts formed are called primary concepts (Skemp, 1971, p.25; 1972, p.3) and are relatively easily formed. It is important, however, to realise that a concept <u>is</u> the <u>abstraction</u> and <u>not</u> the name or symbol associated with it. Concepts which are abstracted from other concepts themselves are called secondary concepts (Skemp, 1971, p.25). Novak (1978, p.3) calls these higher order concepts <u>propositions</u> while Sowder (1980, p.246) calls them <u>principles</u>. Skemp (1972, p.32) states that, in general, "Concepts of a higher order than those which a person already has cannot be communicated to him by a definition, but only by arranging for him to encounter a suitable collection of examples." From the learning point of view, then, the teacher should introduce more advanced topics by arranging for children to have sufficient mental experiences from which they can abstract their own higher order concepts. Once having formed the concept, a definition would give greater meaning and generality to the concept. In mathematics most mental experiences do not involve examples from the outside world (relations between objects), but involve, rather, relations between other concepts. It should therefore be ensured that these concepts are already firmly established in the mind of the learner. (Skemp, 1971, p.32)

Whatever method a teacher might employ for concept learning, accurate recall of examples or non-examples of a concept, or the recitation of a definition does not constitute sufficient evidence that a concept has been learned. Rather, positive feedback can only be obtained if pupils can respond correctly to examples not used in the learning situation. "Concept learning, then, is manifested by the ability to generalise beyond the instances used in the learning situation and beyond the physical similarities present in some instances."

(Farrell and Farmer, 1980., pp. 134 - 5)

2.1.2 SCHEMAS.

A concept which is formed in the mind of the learner is of necessity embedded into a structure of other concepts. This conceptual or mental structure is called a schema. A schema constantly changes as new concepts are assimilated, while any change in a schema is called accommodation. (Skemp, 1972, p.9) Assimilation and accommodation will be discussed in greater detail in section 2.2.1 on Piaget. It should, however, be noted at this stage, that Skemp's idea of accommodation is not exactly like that of Piaget. (For a detailed discussion on this see Skemp (1972).

2.1.3 UNDERSTANDING.

Concept assimilation implies the understanding of the new experiences - in other words "to understand something means to assimilate it into an appropriate schema." (Skemp, 1971, p.46). Skemp (1972, pp.12 - 17) distinguishes between two types of understanding.

(1) relational and (2) instrumental understanding.

He defines relational understanding as "... understanding both what you do and why you do it." (Skemp, 1972, p.12). He disparagingly refers to instrumental understanding as "rules without reasons." (op. cit. p.12) After a series of articles by Byers and Herscovics (1977) and Backhouse (1978), Skemp (1979) accepts that there is a third kind of understanding, viz. logical understanding. Agreeing with Byers' and Herscovics' formulations he now states that:

(1) INSTRUMENTAL UNDERSTANDING is evidenced by the ability to apply an appropriate rule to the solution of a problem without knowing why the rule works.

(2) RELATIONAL UNDERSTANDING is evidenced by the ability to deduce specific rules or procedures from more general mathematical relationships.

(3) LOGICAL UNDERSTANDING is evidenced by the ability to connect mathematical ideas and to combine these into chains of logical reasoning.

(Skemp, 1979, p.45)

Some of these ideas will be discussed in greater detail in Section 4 with particular reference to algebra.

It follows from the above that the appropriate schema into which a concept will be assimilated will depend on how it is understood. One way of deciding whether a schema is appropriate or not is to consider the goal that is to be achieved. Skemp (1979, p.45) puts it this way: "Although the subject matter ... may be the same, for various reasons and on various occasions the goals of learning may well be different, with the likelihood that different kinds of schema will be appropriate." For example, if the immediate goal of a child is to pass a test, he may simply rely on instrumental understanding and the schema formed will be short term.

2.2 COGNITIVE DEVELOPMENT

The difficulties that children experience in learning and understanding mathematics could be attributed to difficulties in building up a conceptual structure or schema. All too many children try to learn mathematics according to rote-memorised unconnected rules (instrumental learning) which are much harder to remember than an integrated conceptual structure. (Skemp, 1971, p.31) But how are conceptual structures formed and what are the processes involved in acquiring knowledge? A number of theories are briefly described below.

2.2.1 PIAGET

The major theory concerned with conceptual learning is that of Piaget. His contention is that there are essential differences in the modes of reasoning which children use as they grow older. These modes can be divided into three broad stages from infancy to maturity. The stages form an unvarying sequential order, but the rate at which individuals advance from one stage to the next is not necessarily constant.

The first, or pre-operational stage, can be divided into two parts. The first part is referred to as the sensori-motor period and occurs from birth to approximately two years of age. During this time intelligent behaviour is acquired without symbolization or language. The child learns by direct contact with his environment through his senses and actions. The remainder of the pre-operational stage extends from roughly two years to seven years of age. During this period children learn to represent objects or events by symbols - particularly in the form of language. However, this symbolization also takes the form of imitation, symbolic games and drawing. The ability to form mental images also appears. Children at this stage are essentially egocentric; that is, they have implicit belief in their own ideas and view everything in relation to themselves.

From about seven or eight years of age the child's thinking becomes operational in that his actions can now be internalized in thought. (Piaget, 1972, p.122) These operations are essentially reversible enabling the child to conserve - to recognize the invariance of quantities under certain transformations which change their appearance. Reversibility, according to Piaget (Op. cit, p.123) is present in two forms: (1) negation (or inversion) and (2) reciprocity (or compensation). This, however, will be taken up in more detail later when the Piagetian stages are considered with particular reference to mathematics. The ability to conserve is task orientated. For example, conservation of number begins at about age seven. This gradually extends so that weight can be conserved at about nine years, while the ability to conserve volume only occurs at roughly eleven or twelve years of age. This phenomenon of the same operation applying to different tasks at different ages is called horizontal décalage and remains a problem in Piagetian stage theory. (Lovell, 1972, p.3; 1978, p.99)

The ability to classify, to match two sets by one-to-one correspondence and to make a series of objects by ordering them is also characteristic of this stage. Thus, in Piaget's words "The concrete operational thinker collects results, classifies and orders them, and establishes correspondences." (In Bell, Costello and Küchemann, 1983, p.53). His reasoning at this stage, however, is limited by his experience with material reality. This second major stage of development is therefore referred to as the concrete operational stage.

At the age of eleven or twelve the child begins to enter the final stage of development - the formal operational stage. At this stage he is freed from the constraint of material reality and is able to think in terms of the possible. His thinking is hypothetico-deductive - that is, he is able to hypothesize in terms of available data and to test this logically. His reasoning would, therefore, follow the pattern: 'If this were true, then ...'. The formal thinker also has the potential for combinatorial analysis. That is, when combinations of variables are involved, he is able to consider all possible combinations of these in a systematic manner by varying one factor at a time while holding the others constant. The concrete thinker, on the other hand, is usually unable to separate the variables in a systematic way, and may alter two or more at a time. (cf. Piaget's pendulum experiment for example.) Formal operations are also characterised by propositional thinking. This type of thinking is called second degree thinking by Piaget and is characterised by the ability to form relations between relations. Children at the concrete level can only cope with relations between objects. In summary, the following passage from Farrell and Farmer outlines the manner of formal operational thinking:

> Presented with a new situation, the adolescent begins by classifying and ordering the concrete elements of the situation. The results of these concrete operations are divested of their intimate ties with reality and become simply propositions that the adolescent may combine in various ways. Using combinatorial analysis, the student regards the totality of combinations as hypotheses that need to be verified and rejected or accepted.
>
> (Farrell and Farmer, 1980, p.63)

Other necessary conditions for advanced thinking are the acceptance of lack of closure (ALC) and the ability to cope with multiple interacting systems (MIS). (Collis, 1978, pp.223 - 227; Lunzer, 1979, pp.212 - 216). These will be discussed later with particular reference to mathematics.

The transition from one stage to another does not occur overnight nor does advancement to a later stage mean that the earlier intellectual abilities are eradicated. The earlier stages are subsumed under the later ones with the child retaining the ability to operate at a lower level. Indeed, an adolescent, or indeed, an adult, at the formal operational stage often resorts to concrete reasoning. Piagetian theory holds that cognitive development takes place through the process of adaptation which has two components (i) assimilation and (ii) accommodation. Through assimilation new experiences are incorporated into an existing schema which must then be restructured to accommodate the new input. The interplay between assimilation and accommodation is such that there is always equilibrium in the child's schema or mental structure. The process whereby equilibrium is reached is called equilibration. The constant restructuring of the schema through accommodation results, in time, in a qualitative change in the schema, with corresponding shift in thought patterns. Assimilation and accommodation thus take on new forms, representing new levels of equilibration which are the developmental stages (pre-operational etc.) described above. (Di Vesta, 1982, p.286; Novak, 1978, p.10; Bell, 1978, pp. 100 - 101). Other factors which have some influence to bear on the child's advancement through the developmental stages are maturation, physical experience, logico-mathematical experience and social interaction. (Bell, 1978, p.100)

2.2.1.1 QUESTIONING PIAGET'S THEORY

Although the Piagetian approach to cognitive development has played a major role in research in mathematics education, it has not been free from criticism. For example, the age and sequence patterns put forward by Piaget have been challenged by some researchers. It is maintained that his method of comparing mean ages of development for different samples of children is inadequate for verifying the existence of sequences. Rather, a longitudinal study with repeated measures on the same subjects is necessary to confirm the existence of invarient developmental sequences. (Fey, 1982, p.1173; Carpenter, 1980, p.156). There also appears to be some doubt as to the developmental order of certain sequences. For example, whereas Piaget and Inhelder initially proposed that conservation and transitivity developed synchronously, many studies have found that conservation develops before transitivity, while Brainerd's study revealed that the order was reversed. (Carpenter, 1980, p.157). It appears that the type of tasks used in testing have some bearing on the results.

It is also believed by some investigators that the ages Piaget lays down for his developmental stages are too rigid, and that acquisition of higher stages can be accelerated with suitable training. They feel that his equilibration model does not account entirely for the acquisition of logical structures. Yet others believe that cognitive growth is not stage dependent, but a continuous process, dependent on the acquisition of specific concepts pertaining to specific learning experiences. Ausubel, for example, follows this line of thought as will be seen later. Whilst Bruner believes in different structural levels, he feels that Piaget's model has no inherent dynamic to cause the child to move from the equilibrium of one stage to that of the next higher stage. (Shiu, 1978, p.14).

From the educational point of view, Piagetian stage theory does sometimes have some negative consequences. For example, there are some who interpret the stages simplistically by assuming that up to the age of eleven years or so, concrete methods of teaching must be used and thereafter children are able to reason formally and manipulate abstract symbols. From previous discussion it can be seen that this interpretation is over simplistic in that, even if there is certainty regarding stage theory (as opposed to continuous progress), the transition from one stage to the next is not as abrupt as implied, nor is there agreement as to the exact ages for the onset of formal operations. Some feel that the formal operational stage occurs, on average, much later than eleven years. (Brown, 1979, p.361). Another negative consequence of the stage theory is that it can be interpreted as offering explanations why children seem unable to learn, and that it provides teachers with excuses for not teaching various ideas. (Fey, 1982, p.1173).

In an article, rather critical of Piaget's stage theory, Brown and Desforges (1977) conclude that Piaget's search for generalities in cognitive development has not been successful. They list several reasons for arriving at this conclusion, among them being:

(i) that Piaget formed various hypotheses based on his experiments with children, but failed to test these or possible alternative ones - implying that his interpretations MAY NOT necessarily be the correct ones.

(ii) the extent to which the stages can be meaningfully defined and the problem of horizontal décalage. They query "... how widespread this décalage can be without it seriously eroding the utility of the stage model." (p.12) They describe a number of studies over the age range of two years to adulthood to substantiate their claim that Piaget's description of stages is less than adequate.

(iii) the nature of the developmental processes. Among the criticisms in this section is the lack of evidence that the sequential development has been confirmed across different cultures. (p.13)

Lunzer (1979, p.221) feels that the groups of transformations of propositional statements labelled the INRC group by Piaget, is not as crucial to all formal thinking as Piaget claims and argues that MIS (Multiple interacting systems) is far more general. To back up his claim, Lunzer mentions some of Piaget's experiments which involve MIS but do not feature in the INRC group.

To conclude this section on the inadequacies of Piaget's stage theory the following passage seems appropriate:

> The Piagetian theory may be incomplete, indeed it may even be mistaken; however it does enjoy some predictive success, and it is at present undoubtedly the best theory we have available to help us understand the difficulties children experience in learning mathematical concepts.
>
> (Brown, 1979, p.371)

2.2.2 SOVIET STUDIES

Whereas Piaget and others regard cognitive development as taking place independently of the school curriculum, the Soviets assume that cognitive development and school learning are fundametally linked. Carpenter (1980, p.178) quotes El 'Konin and Davydov as stating that a pupil's mental development is determined by the content of what he is learning. Developmental stages are therefore not absolute, and it is believed that they can be significantly changed by altering the school curriculum. (Carpenter, 1980, p.178; Fey, 1982, p. 1173; Krutetskii, 1976, p. 331)

Another approach by the Soviets which provides an alternative to Piagetian theory involves the distinction between spontaneous and scientific concepts. Scientific concepts are the product of direct instruction from adults while spontaneous concepts result from concrete experience and the child's own mental effort. An interplay between the two is regarded as essential for development, with the formal structure of the scientific concepts helping to organise a child's spontaneous concepts into a coherent system. (Fey, 1982, pp. 1173 - 4; Carpenter, 1980, p. 180).

2.2.3 AUSUBEL

Ausubel is concerned with the distinction between rote learning and meaningful learning. To him learning is meaningful only when any new knowledge is non-arbitrarily and substantively related to relevant existing concepts or propositions in the learner's cognitive structure, thus providing an 'anchor' for the new knowledge. Rote learning, on the other hand, results in arbitrary incorporation of new knowledge into the cognitive structure, with no relevant concepts to anchor the new knowledge. Further, meaningful learning does not result in new knowledge being added to existing concepts like layers in a layer cake. Rather, new knowledge is assimilated into existing relevant concepts by interacting with these concepts. The form of both the new knowledge and the anchoring concept is thus altered. Ausubel describes the above process as subsumption, while he labels the anchoring concept a subsumer.

An integral part of Ausubel's assimilation theory is progressive differentiation. As new knowledge is subsumed, the whole cognitive structure is modified, becoming more elaborated with new linkages being formed between the concepts. A concept is thus not 'acquired', but is in the process of being differentiated (Novak, 1978, pp.5 - 6), Ausubel's process of subsumption differs from Piaget's concept of assimilation in that, to Ausubel (1) new knowledge is linked to specifically relevant concepts or propositions and (2) the process is continuous and major changes in meaningful learning occur,not as a result of general stages of cognitive development, (as Piaget holds) but rather as a result of growing differentiation and integration of specifically relevant concepts in cognitive structure. (Op. cit., p. 5). Older children in general, therefore, are capable of more abstract thinking, not because of some unique cognitive capability associated with maturation, but rather because they possess a far more elaborated relative cognitive framework.

Another aspect to realise is that, according to Ausubel, young children can acquire a highly formal reasoning ability in fields for which they have had appropriate preparation (Novak, 1978, p. 25). This would seem to imply that advancement through Piaget's developmental stages could be accelerated with adequate training. Many studies by researchers have been concerned with this aspect, but there is little empirical evidence to show that attempts to accelerate development lead to any desirable outcomes. (Carpenter, 1980, p. 161; di Vesta, 1982, p. 294)

2.2.4 BRUNER

Bruner distinguishes between a theory of learning and a theory of instruction. A theory of learning concerns intellectual development and is descriptive of the mental activities of pupils at different stages of cognitive growth, whereas a theory of instruction is prescriptive - it prescribes how to teach pupils at the different stages of mental development. Much of Bruner's work has been concerned with instructional theories, and these have taken cognisance of the cognitive growth of children. In this respect he identifies three stages of intellectual development which correspond with his three modes of representation of knowledge. These are:

(1) The ENACTIVE stage. At this stage, any knowledge received cannot be used unless it is translated into physical action. It is characterised by 'knowing how' rather than 'knowing why'.

(2) The ICONIC stage. Here knowledge can be translated into images of concepts or principles contained in the knowledge, and involves reflective functioning rather than just physical action.

(3) The SYMBOLIC stage. Around adolescence language becomes increasingly important as the medium of thought, but does not entirely replace actions or images. The ability to cope with symbolic and logical propositions together with rules for transforming them develops and there is also an increasing ability to handle several variables simultaneously.

(Bell, 1978, pp. 138 - 145; Blunt, 1977 pp. 8 - 9)

There is some degree of correspondence between Bruner's three stages and Piaget's pre-operational, concrete and formal operational stages. Like Piaget, Bruner also believes that cognitive growth is not a constant, gradual process. Rather, it is discontinuous with new levels being reached only when certain cognitive capacities develop. (In this respect, both differ from Ausubel's ideas on cognitive development). Unlike Piaget, Bruner feels that the stages are not clearly linked with age. He also feels that cognitive growth results chiefly from external influences - the child's experiences with his environment. Piaget feels that advancement from one stage to the next results spontaneously with maturation, influenced only in a general way be experience.(Novak, 1978, p. 4). From the point of view of learning theory, Bruner feels that guided discovery learning leads to meaningful learning. Ausubel regards this method as inefficient, and maintains that meaningful learning can result from expository teaching if approached in the correct manner - advance organiser, etc. (Bell, 1978, p. 130; Novak, 1978, p. 2). Both Piaget and Bruner through his three modes of representation, would agree that in conceptual learning, experience with concrete or pictorial representations should precede symbolic work. (Sowder, 1980, p. 252)

2.2.5 INFORMATION PROCESSING.

Cognitive development studies following the Piagetian approach have concentrated on the structural component of cognitive growth. A contrasting approach, inspired by the advent of the computer, is that of information processing which stresses the importance of analysing the actual thinking processes of humans. Davis, Jockusch and McKnight (1978, p. 14) state that "computers have led to what is virtually a new discipline, so-called artificial intelligence - the study of information-handling processes as they are carried out by sophisticated computer programmes, and the search for parallels between computer information-handling and human information-handling." The usefulness of the computer metaphor for analysing human mathematical thought is described in detail in an article by Davis, Jockusch and McKnight (1978) and there is a more concise account of these ideas in an article by Davis and McKnight (1979). Some of these will be discussed in Section 3 of this study.

There have been attempts to relate Piaget's stage theory to certain aspects of information processing theory viz. the processing capacity of children. Halford (1978, pp. 212 - 213) proposed that Piaget's four stages of cognitive development, the sensori-motor , pre-operational, concrete operational and formal operational, could be characterised by four levels of reasoning. These levels differ in the complexity of the relationships between symbol systems and environment elements. The child's ability to function at the respective levels corresponds with his short-term memory capacity. At level 1, which corresponds with Piaget's pre-operational stage, the child must possess a short-term memory span of two chunks, while the concrete and formal operational stages (levels 2 and 3) require 4 and 6 chunks respectively. A child at the sensori-motor stage (level 0) has a memory span of 0 chunks.

Pascual-Leone (In Collis, 1975, p. 141; Halford, 1978, pp. 213 - 215; Carpenter, 1980, p. 182; Bell, Costello and Küchemann, 1983, p. 29) puts emphasis on the child's processing capacity. He maintains that the number of chunks of information a child can integrate simultaneously grows with age. From the early pre-operational stage, the central information processing capacity (M-power) of the child increases by one chunk every two years, so by the late formal operational stage (15 - 16 years) his processing capacity will have been increased by seven chunks.

There has been a certain amount of empirical evidence to suppport the theories on memory span described above. For example, in a mathematical context, Collis made a study of the elementary arithmetical operations and their relationships to memory span. He found evidence to suggest that there is reasonable correspondence between his results and those predicted by Pascual-Leone's formula for processing capacity (M-power). (Collis, 1975, p. 141). Collis' work will be discussed in more detail in Section 3.

3. OPERATIONAL THINKING IN MATHEMATICS

3.1 CONCRETE AND FORMAL OPERATIONS

Piaget's theory, described in the previous section, is at present the major theory of conceptual learning. However many of his concepts concern the field of physical phenomena and little is known about what kinds of behaviour his conceptualisations refer to in the field of school mathematics - especially with respect to algebraic symbolism. Collis (1969, 1974, 1975, 1978) has done important research in this connection and his experimental work has attempted to make up for some of the deficiencies in Piaget's theory, particularly with respect to the differences between concrete and formal operations in elementary mathematics. Halford (In Collis 1978 and Halford, 1978), Lunzer (1973) and Collis (1974, 1975, 1978) have each written on closure as a means of showing the fundamental differences between concrete and formal operational reasoning. Collis' work, however, pertains particularly to the mathematical field and hence this study. His ideas will therefore be discussed in more detail below.

3.1.1 ACCEPTANCE OF LACK OF CLOSURE (ALC)

Collis proposes that as a child moves up the continuum of cognitive functioning, from the concrete operational level to the formal operational level, he develops a greater degree of tolerance for unclosed operations. In this respect, he distinguishes between four different levels of closure with respect to the elementary arithmetical operations (1974, pp. 5 - 6, 1978, pp. 223 - 4). The lowest level is apparant from about 7 plus years of age. Here a child is able to operate with only two elements at a time, as long as immediate closure is possible to guarantee the uniqueness of the result. That is, the two elements connected by the operation must actually be replaced by a third element belonging to the same set. The four arithmetic operations can only be meaningful at this level if used singly with small numbers which the child can relate to the physical world with which he is familiar. For example, he would be able to decide that 3 + 5 = 8 by imagining sets of three and five objects put together and counted. He would also consider the statement 3 + 5 = 8 to be meaningful.

From about 10 years, the child is able to regard the results of performing an operation as unique without having to make a replacement to guarantee this. Expressions involving numbers outside his empirically verifiable range would thus be meaningful to him (e.g. 536 + 347). He would also be able to use expressions involving more than one operation, as long as each step in the sequence could be closed before proceeding to the next (e.g. 3 + 7 + 2).

The third level begins at about 12 plus years. The ability to refrain from actual closure, as long as a unique result is guaranteed at any required time, becomes more general. Collis refers to this stage as the period of <u>concrete generalisations</u>, because a few specific instances would be sufficient to satisfy a child at this stage of the reliability of a rule, provided that the rule could be applied only in cases where the operations involved give rise to a unique result. He would thus be able to determine that $\frac{382 \times 743}{382} = \frac{672 \times 743}{672}$, but would not necessarily be able to understand and use meaningfully the generalization $\frac{m.a}{m} = \frac{n.a}{n}$.

Finally, from about 15 years, the adolescent is able to consider closure in the formal sense in that he is now able to work with operations as such without having to relate them or the elements to physical reality. For example, he is able to understand that $a * b = b * a$ where the operation $*$ is suitably defined. A younger child tends to regard the operation $*$ as necessarily being one of the known operations +, -, x or ÷. The adolescent is now able to understand the true meaning of a relationship as he no longer needs to rely on immediate closure to guarantee a unique result. For example, in the formula $V = L \times B \times H$, besides being able to obtain unique results by substituting numbers for letters, he would be able to predict what would happen to V if, for example, L were halved, B doubled while H remained constant. Further, at this stage, one or two specific demonstrations would not satisfy an adolescent of the generality of a rule.

3.1.1.1 SOME CRITICISMS OF ALC AS AN INDEX OF FORMAL REASONING

The notion that ALC is an index of advanced reasoning has not been accepted without criticism. Following the Lunzer - Pocklington study, suggestive that the ability to avoid drawing premature closure is a necessary condition for most forms of advanced reasoning (Lunzer, 1979, pp. 213 - 214), a study by Wollman, Eylon and Lawson (1979) does not appear to bear out this conclusion. The results of their study show that ALC was observed at the 7 - 8 year-old level - an age at which Lunzer reported no ALC. On a simpler task they observed spontaneous ALC in a majority of 5 - 6 year-olds (p.663). They suggest that any improvement of ALC with age might not be so much a cognitive phenomenon suggestive of advancing stages in operational thinking, as a result of improved working memory abilities.

As a result of her work in the SESM (Strategies and Errors in Secondary Mathematics) project, Booth (1983) finds reason to suggest that ALC may not necessarily be a function of formal operational thinking. The outcome of a teaching programme designed to improve a child's notion of ALC showed that notable gains were made in all the classes tested, regardless of age or level of mathematical ability. The apparant effectiveness of the teaching programme suggests that the notions involved in ALC were not beyond the conceptual grasp of the children tested (ages 12 - 14 years). The report also suggests that the notion of ALC is not at the same level of conceptual difficulty as that of the letter in algebra as a generalised number. This latter notion is linked to the attainment of formal operationality.

It would seem, then, that although ALC may have an important part to play in the development of reasoning, it is not the sole condition for advanced reasoning. Lunzer (1979, p.215) suggests that another factor in the development of advanced thinking is the ability to handle multiple interacting systems. He feels that although ALC is a key feature of most tasks requiring formal reasoning, its role is more of an enabling one, determining the complexity of the system within which a child or adolescent can work meaningfully.

3.1.2 MULTIPLE INTERACTING SYSTEMS (MIS)

Lunzer distinguishes between simple and complex systems. In a simple system solutions to tasks can only be effected by assimilation to one set of covarients. (Lunzer 1979, p. 215, Collis 1978, p. 227). Complex tasks on the other hand involve more than one system of covariation. Any successful solution to such tasks depends upon the interaction between two or more systems. A situation of this nature is termed a complex system by Lunzer and he refers to complex tasks as tasks involving Multiple Interacting Systems (MIS). In Inhelder's and Piaget's pendulum experiment, for example, where a child must show that the period is dependent on its length and is independent of other factors such as mass, the relevance of MIS is clear. Here the factors of length and mass each represent a system, and the effect on the period could be a result of either system taken separately or an interaction between the two. It is the task of the child to disentangle the actions of the two systems concerned and to single out length as being the relevant factor.

Collis (1978, p. 227) illustrates the idea of MIS with an example taken from school mathematics. He found evidence to show that children at the later concrete operational level can work meaningfully with formulae such as $A = L \times B$, the area of a rectangle. That is, they are able to recognise that the area of any rectangle can be found if the necessary units for L and B are known. The system here is simple as it involves only a single system of covariation - the area (L x B) changes as the size of the rectangle changes. However, when changes in one or more of the variables A, L and B must be related to changes in one or more of the others, the system becomes complex since it involves an interaction between two systems. For example, if A were to remain constant while B varied in some way, there would be two variations involved viz: B is varied and L must be varied in a compensatory way to keep the product L x B, constant. Only adolescents at the formal operational level are able to handle problems of this nature.

3.1.3 NEGATION AND RECIPROCITY

In section 2.2.1 it was mentioned that Piaget's notion of reversibility is present in two forms (1) Negation (or inversion) and (2) reciprocity (or compensation). In Piagetian terms, negation is the cancelling of an action by the inverse transformation. For example, if a child moves an object from A to B, this change can be negated by his moving the object back to A. On the other hand the child could reproduce the original situation between himself and the object by leaving the object at B and moving his own body to B. In the latter case the moving of the object has not been cancelled by the child, but simply compensated for by the reciprocal strategy of moving himself to B. (Piaget, 1972, p. 123). Piaget's beam balance experiment offers another clear illustration of inverse and reciprocal strategies. For example, adding a weight to one pan can be negated by removing it. However, adding the same weight to the other pan to maintain balance would be a reciprocal or compensatory strategy.

Piaget points out that negation (inverse) and reciprocity constitute two essential kinds of reversability which do not reach synthesis in a single system until the formal operational level. Generally the inverse strategy is used at the lower operational levels, while the reciprocal, as well as the inverse strategies are available at the higher (formal) level.

In the elementary mathematical context Collis demonstrates the link between ALC and the child's notion of an inverse by using a simple equation such as $x + 3 = 8$ as an example. (Collis, 1974, pp. 8 - 9, 1975, pp. 38 - 40, 1978, pp. 225 - 227). He identifies four levels at which children operate to find the value of x in the equation.

At the lowest level (8 - 9 years) the problem is viewed as a counting task where the child merely has to count on from 3 until 8 is reached and record the number of units counted, or he could use addition tables, e.g. $1 + 3 = 4$, $2 + 3 = 5$, etc. The notion of an inverse operation in order to isolate x has not yet appeared and the operational sign + is only required as the necessary stimulus to set the child responding in terms of counting.

Collis regards the second and third levels (10 - 15 years) as very similar, differing only subtly in the child s notion of the negating mechanism. At both levels children regard both sides of the equation as representing a unique number. $x + 3$ must therefore be regarded as closed, thus focussing attention on the operation uniting the two elements. At the 2nd level the value of x can be found by regarding it as something unique to which 3 has been added. The effect of adding 3 can be "destroyed" by taking 3 away. Since

$x + 3$ is equal to 8, x must be 5. At the 3rd level the inverse process becomes an "undoing" of an operation rather than one which "destroys" the effect of the operation. Thus x and $x + 3$ are both regarded as unique but unknown numbers and the value of x can be found by applying the inverse operation - that is, adding 3 can be "undone" by subtracting 3.

At the final stage (16 plus years) the adolescent is able to focus on the operation itself. Further, he does not need to regard either side of the equation as unique - x could be variable or constant. In order to isolate x, then, his task would be to find the operation which would operate on the given operation so as to negate it. This operation is then applied to both sides of the equation so as to maintain the original relationship or balance. Thus $x + 3 = 8$ becomes $x + 3 - 3 = 8 - 3$, then $x + (3 - 3) = 5 + (3 - 3)$ from which $x = 5$. Here x is isolated by the use of the reciprocal strategy. This leaves the original operation of addition untouched while its effect is neutralized by the compensatory operation of subtraction on both sides.

The relevance of MIS to the difference between the inverse and reciprocal strategies should also be readily discerned. The inverse strategy involves operating immediately and directly on the latest operation performed without necessarily considering the effect on any interacting operation. This could be likened to Lunzer's simple system. On the other hand the reciprocal strategy does take account of the possible interactions between variables and would thus be analagous to a complex system. In this case negation need not be applied immediately and directly to the latest operation, but could be applied to another part of the system. Collis (1974, p. 10)

gives an example to illustrate that in mathematics the reciprocal strategy is sometimes essential rather than optional, and the use of the inverse strategy in such cases could give rise to serious errors. For instance in the equation $\frac{x}{2} + 3 = 7 - 2$, the fraction is eliminated by multiplying throughout by 2 to give $x + 6 = 14 - 4$. A child, with only the inverse strategy available, might operate directly and immediately on the fraction in order to clear it, as follows $\frac{x}{2} \times \frac{2}{1} + 3 = 7 - 2$, thus destroying the original relationship. A child operating at this level has difficulty in deciding when such a procedure is incorrect (as above), and when it would be correct to operate on one term only, e.g. $\frac{\frac{1}{2}x}{\frac{1}{3}} + x$ is equal to $\frac{\frac{1}{2}x}{\frac{1}{3}} \times \frac{6}{6} + x$ because $\frac{6}{6}$ is simply a replacement of the identity element. In Collis' words then "The availability of the reciprocal transformation implies the ability to handle the total relationship whereas the negative strategy tends to apply to one operation alone."
(Collis, 1974, p. 10).

3.1.4 THE EFFECT OF OPERATIONS AND ELEMENTS ON THE COGNITIVE LEVEL OF MATHEMATICAL PROBLEMS

Collis (1978, pp. 228 - 235) clearly indicates to what extent the elements involved in a mathematical problem and the operation (s) performed on them influence the level of difficulty of the problem. At the early stages of a child's development, the elements with which a child works must be "real" things. As he grows older there is a gradual development of the ability to work with elements more remote from his "real" world, culminating eventually in the ability to use completely abstract elements.

At the early concrete level, then, the elements would be small integers, while at a higher level the child would be able to work with large numbers outside his verifiable range. Finally, at the highest level, the elements would be mathematical symbols (algebraic variables). Regarding the operations on these elements, the lowest level would involve actual physical manipulation of objects. The ability to handle single operations, then multiple operations which can be closed sequentially develops, until eventually the adolescent reaches a stage where he is able to work with operations on operations. Figure (i) below indicates the dimensions upon which the items Collis used in his study varied. (Collis, 1978, p. 229)

		OPERATIONAL STRUCTURE DIMENSION Concrete level	 Formal level
ELEMENT DIMENSION	Concrete level	$8 \times 3 = 3 \times \triangle$ $8 + 4 - 4 = \triangle$ (a)	$7 - 4 = \triangle - 7$ $4 + 3 = (4+2) + (3-\triangle)$ (b)
	Formal level	$a \times b = b \times \triangle$ $4283 + 517 - 517 = \triangle$ (c)	$576+495 = (576+382)+(495-\triangle)$ $a \div b = 2a \div \triangle$ (d)

Fig. (i) Analysis of item types by dimension and level.

Items in cell (a) above, generally involve only one operation which can readily be closed. In cases where an operation is followed by an inverse, this is a simple negation which neutralises the operation by "undoing" it. In all the items only small numbers are used so that there can be no doubt as to the uniqueness of the result. Although in cell (b) some items still only contain one operation, the level of difficulty has been raised by the introduction of a reversal. For instance, the item

$7 - 4 = \Delta - 7$ suggests an unclosed operation and in order to find Δ a child has to find a number which gives a result of 3 when 7 has been subtracted from it. The difficulty is that whereas the difference of 3 is suggested by subtraction in the first place on the left side, and where subtraction is again suggested on the right hand side, the child must, in fact, perform a reversal and add the 3 to the 7. (It is interesting to note that in a test submitted to the writer's standard 6 class of 52 pupils, 16% of them did not recognise the non-commutativity of subtraction and gave an answer of $7 - 4 = 4 - 7$ to the above item. This fact is also reported by Brown (1979, p. 366). A possible explanation for this error is that children perceive the "-" sign as indicating a difference, and the order of recording this in symbolic language is unimportant to them).

Where more than one operation is involved in cell (b) items, this usually involves negation in the form of compensation. Although in an item such as $4 + 3 = (4 + 2) + (3 - \Delta)$, Δ can be found by sequentially closing each binary operation, this tactic is rendered more difficult or impossible in cell (d) items because of the higher level of abstraction of the elements. In this case the compensatory tactic is necessary, thus focussing attention on the operations themselves. Cell (c) items are of similar structure to cell (a) items, but once again attention is focussed on the operation by making the elements more abstract.

Collis predicted that the structural (operation) dimension of mathematical problems ought to be more critical than the element dimension and this was borne out by the results of his study. He found that items with concrete content (elements) and concrete structure (cell (a) type) could be solved by most 9 year olds most of the time, while items with formal content and concrete structure (cell (c) type) were achieved well at the 10 or 11 year old level. Children from about 12 to 15 years were able to achieve success with items with concrete content and formal structure (cell (b) type), while the final set of items, those with both dimensions at the formal level (cell (d) type), were only solved successfully by the 15 to 17 year old group. (Collis, 1978, p.233).

It is clear then, that operations are the main cause of difficulty, but these cannot be divorced entirely from the elements involved. There is an interaction between the two. However, when the operational level is kept constant, the difficulty of a problem will depend on the abstractness of the elements, while difficulties in the operations are more readily overcome when simple elements are used.

3.1.5 CORRESPONDENCE BETWEEN COLLIS' LEVELS AND PIAGET'S STAGES

In the description of Collis' work above it was seen that he identified four levels of cognation with respect to ALC, the child's notion of negation and his ability to cope with the interaction between the elements and operations of mathematical problems. Collis states that these four stages correspond to the four stages within Piaget's concrete and formal operational levels viz :

STAGE 2A - Early concrete operational (about 7 - 9 years)

STAGE 2B - Late concrete operational (about 10 -12 years)

STAGE 3A - Early formal operational (about 13 - 15 years)

STAGE 3B - Late formal operational (15 plus years)

Collis prefers to regard Piaget's stage 3A as still being concrete operational, albeit of a more advanced kind, because at this level the child's notion of ALC is still rooted in empirical reality. As was mentioned earlier, Collis refers to this stage as the concrete generalization period. In his words he states that

> The child remains convinced of the correctness of his deductions, not in terms of the structure itself remaining consistent at all times, but in terms of, 'it has worked for a number of specific examples and one can always return to an empirical ikon to justify a particular procedure.'
>
> (Collis 1978, p. 244)

4. PROBLEMS IN LEARNING ALGEBRA

4.1 WHAT IS ALGEBRA?

The algebra referred to in this study is school algebra - that is, the algebra of number or as it is also termed "generalised arithmetic." By this is meant that letters or other symbols are used for numbers and for the writing of simple general statements representing arithmetic rules and structures. For children who have been drilled from an early age in working with numbers per se, the transition to generalised arithmetic can cause particular conceptual problems. The problems may not be caused by the letters or symbols themselves, but by the new meanings that must be attached to them. Mathematics is not essentially the learning of symbols, but rather the development of ideas that need to be symbolised in order to be communicated. One of the difficulties for the initiate into the symbolic world of algebra is that different meanings can be attached to the same symbol or letter. Morgan (1972, p.6) makes this point clear when he says this of a mathematician:

> When he writes 'x' he may be using that mark to represent a number, a group of related numbers, or something to which a number can be meaningfully attached such as a length or a likelihood. But his 'x' is just as likely to be standing for a logical proposition, a pattern, an operation, or even a concept which cannot be translated into words, let alone numbers.

To add to the confusion, a particular letter or symbol may be referred to by a variety of names by different people or books and in different contexts. For example, in the equation $y + 2 = 7$ the 'y' can be referred to as a placeholder, an unknown or a variable. The last mentioned, however, would be incorrect.

Conceptual difficulties experienced by children in generalised arithmetic could be attributed to the ways in which they interpret or misinterpret the symbols or letters at different stages of their development. This fact will be considered below.

4.2 MEANINGS CHILDREN GIVE TO LETTERS (THE CSMS STUDY)

In the CSMS (M) study (Hart, 1980, 1981), Küchemann identified six different ways in which children interpret letters in algebra. (Küchemann 1980 a; 1980 b, pp.28 - 29; 1981, pp. 104 - 112) These are:

1. Letter evaluated.
2. Letter ignored (not used).
3. Letter used as an object.
4. Letter used as a specific unknown.
5. Letter used as a generalised number.
6. Letter used as a variable.

Not all of the above categories are relevant to each item in the CSMS algebra test (see copy in appendix 1), and the interpretation used by children depends on the particular item and also on its complexity (level of difficulty). In general, categories 1, 2 and 3 above represent a lower level of understanding of the algebraic letter. They indicate ways in which children may interpret letters so as to avoid using them as unknown numbers. Categories 4, 5 and 6 represent different theoretical uses of the letter and Küchemann maintains that for a child to have any real understanding of the beginnings of algebra at all, he should at least be able to cope with items that require the use of a letter as a specific unknown. (Küchemann, 1981, p.105). In the CSMS study most of the children tested could not do this consistently. It will be seen from the present study that a similar situation exists in the sample of South African children tested.

A description of the six categories listed above follows. Most of the items referred to in the discussion are from the CSMS algebra test, a copy of which appears in appendix 1.

4.2.1 LETTER EVALUATED

In this category children assign a numerical value to the letter at the outset so as to avoid having to operate on an unknown. For example in item 5 (iii) ($e + f = 8$, $e + f + g = \ldots$), children may write 12 instead of $8 + g$. Here they are unable to tolerate the unclosed answer $8 + g$, and resolve the difficulty be evaluating g, e.g. ($4 + 4 + 4 = 12$).

Another way in which the letter could be evaluated is to assign to it the number of its position in the alphabet. This would give an answer of 15 in the above item. (g is the 7th letter in the alphabet). Letter evaluated also applies to questions such as 6(i) 'find a in $a + 5 = 8$' where children are asked to find a numerical value. Shiu (1978, pp. 128 - 143) shows that where there is a choice between evaluating and ignoring a literal term children at a lower level prefer the evaluation strategy. e.g. in the item

$$E + 17 = 36$$

$$\text{so} \quad E + 12 = \square$$

younger children (11 - 12 years) tend to evaluate E from the first equation and then substitute this value into the second. She calls this strategy 'sequential closure'. In the other approach the child would match the literal terms and then use a patterning technique with the numerical terms. His reasoning would be that since 12 is 5 less than 17, the number in the box must be 5 less than 36. Shiu calls this a 'displacement strategy'. The latter strategy demands a higher degree of tolerance of Collis' ALC and Shiu found that it was only adopted by mathematically competent subjects of 16 - 17 years although many still preferred sequential closure. (op. cit. p.143). (See also Bell, Costello and Küchemann, 1983, p.136).

4.2.2 LETTER NOT USED.

Here the letter is either ignored or given no meaning if acknowledged. For example in item 5 (i) 'if a + b = 43, then a + b + 2 =...', although there are two unknowns, nothing needs to be done to them. By using a matching procedure they can effectively be ignored. This focuses attention on the + 2 and the child would reason that since the left-hand side is 2 more than a + b the answer must be 2 more than 43. This, in fact, is what Shiu calls the 'displacement strategy'. Shiu (1978, pp. 128 - 129) states that this approach is inadequate for subtraction if used uncritically. For example, in the item

$$F - 11 = 18$$
$$\text{so} \quad F - 9 = \square$$

the reasoning that since 9 is 2 less than 11 so the answer is 2 less than 18 is incorrect. A more successful strategy would be to argue that in the case of F - 9, 2 less is being subtracted, therefore the answer is 2 more than 18. Here, of course, a knowledge and experience of working with directed numbers simplifies the tasks involved in using the displacement strategy. A similar example 5 (iii) was used in the CSMS test viz: "If n - 246 = 762 then n - 247 = ... ". Here the larger numbers tend to lead to arithmetical errors if the letter is first evaluated.

The incorrect answer, 7n, to item 4 (ii) 'add 4 onto 3n' illustrates the case where a letter is acknowledged but given no meaning. Here the elements which are meaningful (the numbers 3 and 4) are 'properly' combined while the letter is simply left as it is.

4.2.3 LETTER AS OBJECT.

Here the letter is used as shorthand for an object (eg 'a' stands for apple) or as an object in its own right (eg 2 a's and 5 a's make 7 a's).

Thinking of the letter in this way is a highly effective way of simplifying some algebraic expressions. For example 2a + 5b + a (item 13 (iv)) becomes 2 apples and 5 bananas and another apple giving 3 apples and 5 bananas or simply 2 a's and 5 b's and another a gives 3 a's and 5 b's altogether.

Examples which can be solved effectively and easily using the letter as an object often appear in geometric form. eg. item 9(i) " 

$p = ...$ ". Here the item can be solved by thinking of 'e' as a name or label for each of the sides (and not as the unknown length of the side) and the task is simply one of collecting the e's together to give p = 3e.

Although using the letter as object reduces the letter's meaning from something quite abstract to something within the child's concrete reality, there are many times when such usage is quite inappropriate. This occurs when items involve objects (eg students, professors etc. in the example below) but where the letters are defined as representing <u>numbers</u> of objects.

Küchemann (1982, p.47) and Rosnick (1981, p.418) give the following example to illustrate this point:

> Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this university". Use S for the number of students and P for the number of professors.

In a group of 150 second year engineering students less than two-thirds were able to give the correct answer, S = 6P, in any form. The most common error was 6S = P, which Rosnick refers to as the 'reversed equation' (p.419). Here S and P are simply being used as shorthand for 'student' and 'professor' rather than variables standing for the <u>number</u> of students and professors, and the statement is read as "there are six students for every professor."

In the above example, treating the letter as object leads to a patently wrong answer. However there are times when the distinction between letter as object and letter as number is not so clear cut. Küchemann (1982, p.47) illustrates this with the following example.

> Write an expression for the total cost of three bars of chocolate and a packet of crisps, when a bar of chocolate costs x pence and a packet of crisps costs y pence.

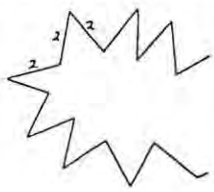
Here the correct answer, $3x + y$, can be arrived at incorrectly by treating the letter as an object eg. "3 bars and a packet of crisps" or correctly by thinking of this as "3 bars at x pence each and a packet of crisps at y pence."

The tendency to use letters as symbols for objects rather than for numbers of objects is a common and serious stumbling block in algebra.

4.2.4 LETTER AS SPECIFIC UNKNOWN

Here the letter is seen as a numerical entity in its own right - a particular but unknown number which can be operated upon directly despite any resulting lack of closure. eg. a child is willing to accept that $3n + 4$ is the answer to item 4(ii) "add 4 onto 3n". He accepts the lack of closure and realises that this is all that <u>can</u> be done to combine the elements since n is an unknown.

In item 9 (iv)



part of this figure is not drawn. There are n sides altogether, all of length 2

p =

Children are required to operate on n as a specific unknown giving the answer 2n. Here n is clearly defined as a number although its value is unknown. A sizeable proportion of children tend to assign a value to n (eg 16, as the figure shows 16 and a bit sides) and arrive at an answer of 32. It is clear that these children are not willing to accept n as a numerical entity in its own right.

4.2.5 LETTER AS GENERALISED NUMBER

In this case children must see the letter as being able to take on several values rather than representing a specific value. This is an extension of viewing the letter as a specific unknown. In item 16 of the CSMS test "What can you say about c if $c + d = 10$ and $c < d$" a common response was to list several values for c (eg. c = 1, 2, 3, 4) indicating that these children were aware of the concept of generalised number albeit in a rather limited sense. The most common response was to list only one value for the letter c viz. $c = 4$. These children apparently are still operating at the "specific unknown" level.
Another example which tests the concept of generalised number is item 18 (ii) "Is the following always, never or sometimes (when) true? $L + M + N = L + P + N$". In order to answer this correctly children must realise that M and P can take on many values and that sometimes these values may coincide. They must also be able to tolerate the ambiguous answer 'sometimes'. The most common answer is that the statement is never true. Children who give this answer are probably operating at the 'letter as object' level. They might argue that M and P are different letters, therefore the two sides can never be equal. (Mangoes cannot be pears).
Küchemann points out that in general, items which require letters to be seen as generalised numbers prove to be more difficult than those where the letter need only be seen as a specific unknown. He suggests that it might be the case that children can

handle specific unknowns before they conceive of generalised numbers. For a fuller discussion on this see Küchemann (1980a, p.58 or 1981, pp. 109-110)

In the case of letter as generalised number Küchemann points out that a distinction can be made between a letter taking on several values in turn or a letter representing a set of values simultaneously. (1980a, p.57; 1981 p. 109). He notes further that the notion of 'simultaneous values' leads to the concept of variable.

4.2.6 LETTER AS VARIABLE.

At this level the child sees the letter as representing a range of values as well as being aware that some kind of relationship could exist between two such sets of values. In other words, he would be able to describe the degree to which changes in one set of values are determined by changes in another. This means that the child is able to establish a second-order relationship between them. Consider for example, the answer to item 22 viz. $5b + 6r = 90$. The equation gives the relationship between b and r, the number of blue and red pencils. A child who gives a single pair of numbers as the solution eg. (6;10) is operating at the 'specific unknown' level, while a child who realises that the relationship could hold for several pairs of numbers eg. some or all of (6;10), (12;5), (0;15), (18;0) is able to interpret the letter as a generalised number. Although this latter view indicates that the child realises that b and r <u>can</u> change, it is not sufficient to show that he understands <u>how</u> they change with respect to each other. A child who can establish a further relationship between b and r such as "as b increases, r decreases" or even better "the increase in b is more than the (corresponding) decrease in r'' is aware that a second-order relationship exists between b and r. This puts him at a higher level than children who can only interpret the letter as a specific unknown or generalised number. The child at this level understands the letter as a variable. (Küchemann 1980a, p.60; 1981, pp.110-111). The following, then, is an operational definition of a variable:

> Letters are used as variables when a second (or higher)-order relationship is established between them.
>
> (Küchemann 1981, p.111)

Lastly, the point should be made that the common practice of using the term "variable" as a blanket term for letters in generalised arithmetic obscures the different meanings that children give to the letters and is so broad as to make the term redundant. (Küchemann 1980a, p.58)

4.3 LEVELS OF UNDERSTANDING IN ALGEBRA (CSMS ALGEBRA TEST)

As was mentioned in section 1.3, a major aim of the CSMS research was to provide a hierarchy of understanding in mathematics topics which would be of use to teachers and curriculum developers. This was to be based on what the children seemed to understand rather than on a logical analysis of mathematics. Although the original desire was to assess the demand of mathematics in terms of the Piagetian levels, this idea was later abandoned because of the formidable problems encountered. (Hart 1980, pp.75 - 76). The hierarchies and the notion of stages were thus arrived at by using a statistical analysis on the tests themselves. This involved the use of the homogeneity coefficient ϕ , clustering according to facility, and testing scalability of the groups of items using Guttman scalogram analysis. For a short description of this analysis see Hart (1981, pp. 6 - 8) and for a detailed discussion on the method used see the CSMS (M) monograph (Hart 1980, pp. 95 - 101).

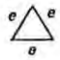
Using this analysis in the algebra test, items that correlated well with each other were classified into four 'levels of understanding.' Of the original 51 items, twenty-one were rejected either because they came from a question with a large number of similar items (eg. Question 13) or because their correlations were relatively low. This left thirty items that were assigned to the four groups.

In Section 8,2 it can be seen how the items were divided into the four levels and what interpretation of the letter was deemed adequate for a correct response to an item.

4.3.1 RELATIONSHIP BETWEEN LEVELS OF UNDERSTANDING AND WAYS OF INTERPRETING THE LETTER.

With regard to the interpretation of the letter, it should be noted that items at levels 1 and 2 can be solved by evaluating the letters, not using them, or treating them as objects. i.e. they can be solved without having to operate on letters as unknowns. Levels 3 and 4 require that letters be treated as specific unknowns, generalised numbers or variables.

The difference between levels 1 and 2 and levels 3 and 4 is essentially one of item complexity. For example, the level 1 item "find a if a + 5 = 8" can be solved by simply recalling a number bond, whereas the level 2 item "find u if u = v + 3 and v = 1" requires that a child be able to cope initially with an ambiguous statement.

Similarly in the level 1 item "  $p = \ldots\ldots$ " the child simply has to collect three similar objects together giving p = 3e, whereas in an equivalent level 2 item "  $p = \ldots\ldots$ " the objects differ, which means that the answer 4h + t cannot be closed. It can be seen then that to answer level 2 items there must be a willingness to accept answers which are to some extent ambiguous or unclosed. A similar distinction exists between level 3 and level 4 items. eg. at level 3 the item "add 4 onto 3n" basically involves a single operation, whereas in the level 4 item "multiply n + 5 by 4" two operations are involved which must be co-ordinated (Küchemann, 1980a, pp. 69-70)

4.3.2 THE CSMS ALGEBRA TEST LEVELS AND PIAGET'S STAGES.

There is a basic difficulty in trying to link the four levels established in the algebra test to the Piagetian stages in that most of Piaget's work relates to science rather than mathematics.

However Küchemann (1980a, pp.131 - 134) did try to make a comparison using Piaget's pendulum task on some of his subjects. The results do suggest that there is some degree of correspondence between his levels and Piaget's stages. Further support is given by considering the ways in which the children interpret the letters. (Küchemann, 1980a. p.134). For example, evaluating letters, ignoring them or treating them as objects (levels 1 and 2) suggest a reliance on directly intuitable reality (concrete operations), whereas operating on unknowns and being able to work with second-order relations (levels 3 and 4) are characteristics of formal thought as suggested by Collis (See section 3.1.4 and 3.1.5 of this study.)

Taking the above into consideration Küchemann suggests the correspondence between his algebraic levels and Piaget's substages is as follows:

LEVEL 1	Below late concrete	(2A to 2B)
LEVEL 2	Late concrete	2B
LEVEL 3	Early formal	3A
LEVEL 4	Late formal	3B

Seen in this light the levels do give a reasonable guide to the cognitive levels of children.

4.4 AN ALTERNATIVE FRAMEWORKS MODEL.

From what has been said above, it can be seen that difficulties experienced by children in algebra can be attributed, in part, to the way they interpret the letter. For example, their confusion over the distinction between the letter as representing the number of objects or as representing the object itself, or their difficulty in understanding the notion of letters as generalised numbers rather than standing for specific numbers. Booth (1983, pp. 2 - 4) suggests two other areas of difficulty viz:

(1) NOTATION AND CONVENTION: CONJOINING IN ALGEBRAIC ADDITION

An example of this is where children write the conjoined answer, ab, instead of a + b for algebraic addition. This probably occurs as a result of their unwillingness to accept the unclosed answer.

(2) FORMALIZATION AND SYMBOLIZATION OF METHOD. Difficulties in this area stem from the fact that children often do not understand the underlying process or structure of an arithmetical problem and can therefore not generalise in the algebraic case. This may be the result of children applying rote-memorised methods (cf. Skemp's instrumental understanding), or of using informal procedures not taught in the classroom (idiosyncratic methods). These informal methods may be effective in the solution of simple problems but are often not readily applicable to problems with

large numbers or in algebraic representation. Further, children consider mathematics to be a subject where the aim is to find numerical answers to problems. For example, they may not consider that using an appropriate method with correct symbolization is all that is required of them, i.e. "They do not consider that a 'method' statement can also be an answer". (Booth, 1983, p.4) This dual nature of mathematical notation is also reported on by Davis, Jockusch and McKnight (1978, p.100). They state that one of the hardest things for a child to accept is that an expression like x + 7 is "both a 'recipe' that tells us what to do, and also a name for the answer we get if we do it." (their emphasis). To a child the "+" sign usually implies that a standard numeral is required for the sum.

The conceptual difficulties with the algebraic letter identified above have so far been accounted for within a Piagetian framework (as interpreted by Collis and Küchemann), i.e. in terms of concrete as opposed to formal operational thinking. An interesting alternative model that could account for these difficulties is suggested by Booth (1983, pp. 4 - 5). She refers to it as an 'Alternative Frameworks' model. Essentially this suggests that prior to their introduction to elementary algebra children have been working within an arithmetic 'framework of knowledge' in which (1) 'closed' numerical answers are required, (2) obtaining the correct answer by whatever method makes sense is more important than understanding the mathematical structure of the problem. (3) conjoining does imply addition eg. 27 means 2 x 10 + 7 or $1\frac{1}{2}$ means $1 + \frac{1}{2}$. (4) Symbols representing quantities have always signified unique values and (5) letters often refer to an object or measure rather than its quantification eg. 6m = 6 metres. Booth notes that it is not surprising that children, on meeting algebra for the first time adhere to 'the rules of the game' already established within an arithmetic framework of knowledge with inevitable consequences as far as the understanding of algebra is concerned. (p.5)

Booth discovered that by changing the framework of reference of the children she used in her study, using a specially designed teaching programme, many of their conceptual difficulties with respect to accepting unclosed answers and formalization of method vanished. At the same time, however, they appeared to show resistance to the notion of letters as generalised numbers. She suggests that this is indicative of the involvement of maturation-linked cognitive factors as well as the establishment of a more appropriate framework of reference. (Booth, 1983, p.9). This seems to indicate that a child's conceptual understanding in certain areas of generalised arithmetic depends on his attainment of a certain cognitive level and on the 'framework of knowledge' within which he is working.

4.5 RANDOM ERRORS RELATED TO A CHILD'S MISCONCEPTIONS IN ALGEBRA.

Many difficulties and errors in elementary algebra arise from children's lack of understanding of the relationship between their own arithmetic concepts and the symbols and algorithms taught in school.

An example given by Davis and McKnight (1979, p. 99) involves an understanding of the 'zero product principle' viz: if $(x - 3)(x - 5) = 0$ then either $x - 3 = 0$ or $x - 5 = 0$. Here the specific numerical values 3 and 5 are irrelevant, but the specific numerical value 0 is not, and children fail to realise this. This regularly leads to the principle being misused by children who apply it in cases where the product is not zero, eg. if $(x - 3)(x - 7) = 8$ then $x - 3 = 8$ or $x - 7 = 8$.

This type of error can be explained in terms of Matz's "surface level" rules and "deeper level" rules. (See Davis and McKnight, 1979, pp. 98 - 99). However it could possibly be explained more simply in terms of Skemp's ideas of understanding. A child who makes this type of error has probably relied on instrumental understanding (rules without reasons) rather than on relational understanding (knowing which rule to use and why it works).

Other examples of typical errors or misconceptions which could be explained in terms of instrumental versus relational understanding are $\frac{2 + x}{x} = 2$ and $x^3 + x^2 = x^5$. Bester (1982, pp. 34 - 36) suggests that 'interference theory' could account for the occurrence of such errors. Briefly, this theory suggests that subject matter is recalled incorrectly when previous (or subsequent) learning of similar yet different subject matter in mathematics could confuse the logic of what is learnt. For example, in the examples above interference could be caused by $\frac{2 \times x}{x} = 2$ and $x^3 \times x^2 = x^5$ respectively, resulting in the observed errors. It is suggested that if true relational learning takes place, errors of this nature could be avoided. It is important then, that teachers should be explicit in pointing out the difference between mathematical concepts despite similarities that might exist.

The last type of error to be discussed in this section are errors which Davis and McKnight refer to as 'natural language' errors (1979, p. 109) These commonly occur at the early algebraic stage. For example, pupils who read the expression 2x - x as "two x take away x" nearly always write the result as 2x - x = 2. To them 'taking away x' means literally removing the 'x' from 2x leaving 2. (The writer has frequently noted this type of error in his standard 6 classes.)

In connection with language and mathematics, there is little doubt that children have problems in going from statements written in ordinary language to equivalent statements written in mathematical or symbolic language. Problems of this nature involve a too mechanical translation from one language to the other. For example Bell (1980, p. 161) notes that a statement such as "there is one more rabbit than there are hutches" is very often written as $r + 1 = h$ in symbolic terms. Other examples of this nature have already been discussed eg. writing $6S = P$ for the statement "there are six times as many students as professors." Errors of this nature, as noted previously, probably originate from misinterpreting the letter in algebra - using the letter as an object rather than as a symbol standing for the number of objects.

The examples referred to in this section perhaps serve to illustrate that children often proceed by manipulating meaningless symbols with no attempt to consider what the symbols mean. If they paused to consider that symbols represent numbers and checked their answers accordingly, many errors could possibly be avoided.

5. SUMMARY, IMPLICATIONS AND CONCLUSION

5.1 SUMMARY AND IMPLICATIONS.

In Section 2 of this study it was seen that the major theory of conceptual learning this century is that of Piaget. In spite of flaws and weaknesses in his theory (eg. horizontal décalage or the lack of a satisfactory criterion for defining formal operations) or alternative theories (eg. Ausubel's or information processing), there appears to be much supporting evidence that the stage concept theory cannot be abandoned. (See, for example, Halford, Lovell, Collis, etc.)

An important aspect that arises out of subsequent research is that the ages at which children progress from stage to stage cover a wider range than would be expected from Piaget's descriptions. Shayer has shown, for example, using class versions of some of Piaget's tasks on a representative sample of British school children, that the onset of formal operations occurs, on average, much later than 11 years. Further, he claims that the proportion of school children who reach the full formal operational stage is not more than about 20%. (In Brown, 1979, p.361 and Bell, Costello and Küchemann, 1983, p.72). This means that the majority of pupils at high school only reach the concrete operational level and would therefore be unable to reason effectively in an abstract or formal manner.

From the first part of this study about the nature of mathematics, it was seen that although the origins of mathematics were embedded in concrete reality, it eventually transcended this reality to become a process dealing with abstract relationships, some of which are far removed from the perceived reality. Accepting Shayer's claim that most children at secondary school have not yet reached the stage of formal operations, coupled with the fact that much of the mathematics taught at school requires the ability to deal with abstract relationships, it is not surprising that children find it a difficult subject - a subject which makes little sense and which is generally associated with a lack of success.

Another aspect arising from the Piagetian paradigm and which has some bearing on teaching is the interest Piaget shows in processes rather than structure (contrast with the skill-integrationist approach described in section 1.3). As was mentioned, this focuses on pupils' thinking as shown by their explanations and justifications. The advantage of this approach is that it enables the exploration of the particular framework of ideas held by children when they approach new topics. They derive this conceptual framework from past experience, and if it does not relate to a new situation with which they are faced, their view must either be modified or a new and separate framework must be built up.' (Lovell,1980, p.3) This links up with the alternative frameworks model described in section 4.4 where children approach secondary school mathematics with an arithmetic framework of knowledge. Karmiloff-Smith and Inhelder (In Lovell,1980, p.3) have established that children do not easily relinquish a theory which they have accepted and understood, and prefer an entirely new theory to modifying an existing one. The importance of this with respect to the arithmetic framework of ideas, firmly established in children's minds in junior school and the influence it bears on their understanding of generalised arithmetic in secondary school can be appreciated. Here the task of the teacher is rendered difficult in that he must encourage children to modify their existing frameworks to encompass the old and new ideas in a single and broader system.

Piagetian theory offers a general description of the characteristic modes of reasoning and type of task with which children are capable of dealing at different stages of their intellectual development. Most of these tasks relate to physical phenomena however, and it was seen in section 3 how Collis, in particular, attempted to relate mathematical tasks to Piaget's stage theory. His work showed that children operate at four levels with respect to the elements and structural complexity (operations) of mathematical items, their notion of inverse and their tolerance of unclosed answers. These levels correspond closely with Piaget's four stages, but it was seen how Collis differs slightly from Piaget in his interpretation of the third stage (stage 3a). Whereas Piaget regards this stage as early formal operational, Collis, regards it as still being concrete operational in that children still see the need to verify general rules empirically. He therefore refers to it as the period of concrete generalisations.

With regard to the elements in mathematical items, Collis made a distinction between small numbers, large numbers and letters and he argued that difficulties stemmed from the extent to which they lacked meaning for the child. In section 4.2 it was seen that Küchemann identified six different meanings that children give to letters. Their choice of meaning to a large extent depends on their cognitive level. He found that the majority of school children interpreted the letter in a way that avoided having to regard it as an unknown. This would seem to indicate that they are unable to cope with letters in a formal manner. This is consistent with Shayer's claim that only a small proportion of children at school reach the formal operational level.

5.2 CONCLUSION.

Two important points seem to emerge from what has been said up to now. The first is that mathematics is probably very much more difficult for children than either teachers or the general public realise. Some parts of it, however, may appear to be easy because they consist of symbol-manipulations which can be done using rote-memorised methods without an understanding of the underlying operations involved. The second is the slow pace at which children progress through the concrete operational stage - indeed few seem to reach beyond it while at school. As noted earlier, this second point to a large extent accounts for the first. In their teaching, then, teachers should use as many concrete references as possible to illustrate the basic concepts upon which mathematics is built. It seems that the development of mathematical skills (symbol-manipulation) is an overriding objective in teaching to the neglect of developing an understanding of the underlying principles and structure of mathematics. This is possibly because it is far easier to achieve and leads to better results in the short term.

Finally, teachers should find what the 'knowledge base' of the children they teach is, and build on that. Too often they base their teaching on what they _feel_ the child should know, rather than on what the child actually knows.

PART II

EMPIRICAL WORK

6. INTRODUCTION

The aim of this research project was to:

(a) observe aspects of algebraic conceptualisation at the early algebraic stage in a sample of South African school children.

(b) compare these findings with those of the Algebra section of the C S M S (M) study. (Hart 1980, 1981; Küchemann 1980a, 1980b, 1981)

(c) observe whether there was any notable difference between boys and girls of equivalent ages with respect to their understanding of generalised arithmetic. This aspect, however, formed only a minor aspect of the project.

In order to implement these aims, permission was obtained from Dr. Kathleen Hart, leader of the C S M S mathematics team, to use the algebra test and marking scheme of the C S M S study. The test was adapted for South African conditions by changing the words 'pence' into 'cents' and 'pounds' into 'rands'. Further, it was translated into Afrikaans for the benefit of Afrikaans speaking children.

A number of children from the writer's own classes were chosen for interviewing. The purpose of this was twofold:

(i) to verify whether the responses (specific errors or wrong methods) were indeed the same as those indicated in the C S M S marking scheme and

(ii) to gain experience in this aspect of research.

6.1 RESEARCH DESIGN

The research carried out in this study can be described, basically, as ex-post facto. This involves studying an existing condition and searching back for plausable causal factors or reasons for the condition. In this study an attempt has been made to identify the conceptual difficulties experienced by school children at the early algebraic stage and to search for possible reasons or causes for these difficulties.

Typical of most ex-post facto research, lack of control precludes the investigator from establishing, with certainty, a causal relationship between phenomena. Even if an association has been found between variables, this may not be a cause-effect relationship but a result of some hitherto unthought of common factor or factors. Furthermore, a phenomenon may result not only from one or more causes, but also from one cause at one instance and another cause in another instance. The possibility of reverse causation also exists, i.e. when a relationship has been found between two variables, it may be difficult to decide which is the cause and which is the effect. For example, does violent behaviour result from watching violent films, or does a violent temperament promote the tendency to watch violent films?

In this study a child's conceptual difficulties in algebra have been ascribed to a number of causes, chief among them being the cognitive level of the child coupled with the abstract nature of mathematics. The possibility of reverse causation can be ruled out, but antecedents such as home background, teacher effectiveness, intelligence, sex, etc. may also play a part to a lesser or greater degree. It is likely, though, that some or all of these factors may have an influence to bear on the cognitive level of the child.

From what has been said above about lack of control, etc. it may be argued that ex-post facto research is of little value. However, in research involving the complexities of social and educational phenomena it is, in most instances, the only feasable way - the rigorous experimental method may simply not be possible. A researcher cannot, for example, exercise control by manipulating home background, intelligence or teacher personality, or cause one group to become failures or drop-outs, etc. Nevertheless, in ex-post facto research it is sometimes possible to introduce a certain measure of control by carefully selecting the samples used. For example, a number of possible factors could be eliminated by selecting a sample as homogeneous as possible on a given variable. In this study the samples were made as homogeneous as possible by choosing children only from typical academic high schools serving white middle class communities.

Although ex-post facto designs have their limitations they can be useful in providing pointers for subsequent research of a more rigorous nature. For example, the CSMS project was followed up by the SESM programme, the aim of which, was to investigate the causes underlying selected errors which the earlier CSMS project had shown to be widely prevalent. (See Booth, 1983.)

In section 8 of this study it can be seen that the bulk of the research has taken the form of a cross-sectional study. That is, children's progress from std. 6 through to std. 8 is assessed by comparing the performances of three sets of different children from each of the three standards. Advantages of this method are that there are no real difficulties in obtaining large samples in each of the year groups and results can be obtained relatively quickly. However a weakness lies in the fact that inferences concerning a child's progress from std. 6 to std. 8 are based on the performances of different children in each of the three standards. On the other hand a longitudinal study can trace the progress of particular children from std. 6 through to std. 8 and any inferences concerning their progress are therefore probably more reliable. Its disadvantage though, apart from the length of time it takes, is that of experimental mortality. There is bound to be a loss of pupils through dropout over the years. It can be seen in section 8.4, which describes a very small longitudinal study using the writer's own classes, that an original sample of 50 std. 6's resulted in a sample of only 20 children in std. 8.

6.2 THE C S M S ALGEBRA TEST

The C S M S algebra test used in this research project was designed to probe understanding of basic concepts across a broad range of algebraic tasks, rather than to test algorithms or teacher taught methods. These tasks include substitution, simplifying expressions and constructing, interpreting and solving equations. The items were chosen to span a wide range of difficulty and in writing the items words which could be termed 'technical' were avoided. In this respect, in-depth interviewing with children formed an integral part of the development of the items. These helped to (i) pinpoint terms and expressions in the questions which gave rise to difficulties and which could be replaced so that the questions could be more easily understood, (ii) find methods (leading to correct and incorrect answers) used by children. Answers which resulted from specific incorrect methods were noted and coded for use in marking the written tests of the large sample. (Hart 1981, pp 1 - 6)

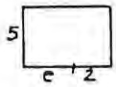
The final version of the written test comprised 51 items which altogether required under one hour to complete. The test can be seen in Appendix I. It must be pointed out that this version was the one used in the large scale survey of the C S M S study. Questions 10 and 23 have been omitted because of the extremely poor response in the initial trials. As described in section 4.2 and 4.3 of this study the test was subjected to a statistical analysis resulting in 21 of the items being rejected. The remaining 30 items were sorted out into four levels of understanding generalised arithmetic. (See Section 8.2)

6.3 THE MARKING SCHEME OF THE C S M S ALGEBRA TEST

The marking scheme developed alongside the test and it was only finalised after several hundred completed scrips were examined for unexpected answers. Where these answers made sense, an attempt was made to incorporate them into the marking scheme (Küchemann, 1980a, p.43) A copy of the marking scheme appears in the appendix. The answers to each item were coded 0 to 9. Correct answers were coded 1 and, where appropriate, code 2 was given to 'weak' correct answers. Code 3 was used for answers that were ambiguous or in some way inadequate without being explicitly wrong. (eg. the answer $c = 10 - d$ would be given code 3 for item 16 "what can you say about c if $c + d = 10$ and $c < d$?") Code 9 was used for miscellaneous wrong answers, while 0 was used when no answer was given at all. Code 4 was commonly used when incorrect punctuation (operational signs) was used (eg. $3n \times 4$ instead of $3n + 4$ for item 4(ii) "add 4 onto 3n"); code 5 when the numbers in an item were combined correctly while the letter was essentially ignored (eg. 7n instead of $3n + 4$); code 6 when the letter was ignored altogether (eg. 7 instead of $3n + 4$); code 7 when the letter was given a value according to its position in the alphabet, while code 8 was used essentially for other wrong numerical answers. Küchemann (1980a, p.44) illustrates this scheme using item 7 (iv) which conforms in most respects to the pattern described above. This is reproduced in figure 6.1 below.

Fig. 6.1 Marking scheme used for item 7(iv)

What is the area of this shape?



Code	Response	Meaning of Code
1	5(e+2) 5e + 10	Correct answer
2		(Where appropriate, Code 2 is given for "weak" correct answers.)
3	5e+2 e+2 x 5 5e2 MISUNDERSTOOD: PERI.	The answers are ambiguous due to inadequate punctuation: the child may have been trying correctly to express e AND 2 multiplied by 5.
4	5e x 2	Here the punctuation is not so much inadequate as definitely wrong.
5	e + 10 10e 7e	For e+10 the numbers have been combined correctly but the letter has essentially been ignored. The same thing seems to be happening to the letter in the other answers though they occurred less often.
6	10 7	The numbers have been combined but the letter has been ignored entirely.
7	35 50	e seems to have been given the value 5 from its position in the alphabet.
8	25 20 30	e seems to have been evaluated geometrically (the base of the rectangle looks to be about 5 units long).
9		All other wrong answers
0		Item omitted

Note: responses written small occurred less frequently but shared some of the characteristics of the other responses under the same code.

The marking scheme for some of the examples in the test were much simpler than that described above. For example the answers to questions 7(i), 7(ii), and 8 were marked correct (code 1), incorrect (codes 8 and 9 depending on whether the incorrect answers were numerical or not), or omitted (code 0). Here the purpose of the questions was simply to determine whether children could calculate areas or perimeters of specific rectangles or quadrilaterals. On the other hand certain questions required much more complicated mark schemes. Küchemann (1980a, p.45) illustrates such a scheme using item 17(i). This is reproduced in figure 6.2 below.

FIG. 6.2 Marking scheme used for item 17(i)

Mary's basic wage is R20 per week.
She is also paid another R2 for each hour of overtime that she works.

If h stands for the number of hours of overtime that she works, and
if W stands for her total wage (in Rands)
write down an equation connecting W and h ....

Code	Response	Meaning of Code
1	W = 20 + 2h w = 20 + 2 × h	Correct answer
2	TWO NUMERICAL PAIRS OF VALUES (CORRECT) W=28 h=4 W=30 h=5, etc OR 28,4 30,5 etc	Though this would seem to be a much lower level answer than that of code 1, it does express the important notion that a letter can take more than one value. (In the event less than 1% of all children gave a code 2 response.)
3	W + 2h = wages W+2h 20 + 2h EXPRESSIONS INVOLVING Π_w, etc (CORRECT). AMBIGUOUS PUNCTUATION: W=20+h2	The answers are ambiguous or inadequate (eg W is used for the basic wage) but the letters are at least being used as numbers (not objects as below).
4	W=20+h 20+h W = wages + h Total wage = W+h W= W+h h+w	An attempt is being made to establish a general relationship but there is a tendency to use the letters as objects: h seems to stand for the hours of overtime rather than the number of hours.
5	W+h. W∩h wh	The answer is again general, but the letters are being combined in the most primitive way (simply "juxtaposed" or "associated").
6	ONE NUMERICAL PAIR OF VALUES (CORRECT) W=28, h=4 etc OR 28,4 etc	The letters have been evaluated, and only one pair of (correct) values is given.
7	2h = 24 W 4h = 28 W etc. 24 = W+2h 28 = W+4h etc.	The letters have again been evaluated (correctly) but the letters are then used as objects: 2 hours overtime gives a total Wage of 24.
8	W=20 h=2 20W+2h W=10h 22 = W+h R22	In some way these answers all involve the given (ie most obvious) numbers, 2 and 20.
9		All other answers, including arithmetical errors.
0		No response.

As in the case of many items in the test, there is some degree of similarity in the meanings of codes 5, 6, 7 and 8 for the two items illustrated above (7(iv) and 17(i)). Code 6, 7 and 8 involve answers which are essentially numerical, while in code 5 the elements are combined in a very direct and primitive way.

Küchemann (1980a, p.46) regards question 3 as one of the most important items in the test. In this question children were asked to choose the larger of 2n or n + 2 and explain their answer. Here the mark scheme had to be modified to accommodate the different answers. The most common response was 2n without any explanation or with an inadequate explanation such as "because it's multiply". This was given code 4. The other definite answer given was n + 2. This was coded 8. Occasionally it was claimed that the two expressions were the same. This was also coded 8. The point of the question was to see whether children realised that the relative sizes of the expressions depended on the value of n, and that the critical value of n is 2 when the two expressions are equal. Answers were only classified under code 1 if children made this point clear in their explanation, eg. "2n is bigger if $n > 2$" or if they listed several values of n around the critical value to illustrate that the relative sizes of the expressions depended on the value of n. Code 2 was assigned to conditional answers where only one value of n was given at or below n = 2. If no values of n were listed, or if no explanation was given, conditional answers were given code 3. Figure 6.3 below illustrates the marking scheme for question 3. (Küchemann 1980a, p.47)

Fig. 6.3 Marking scheme used for question 3

Which is the larger, 2n or n + 2 ?
Explain: ....

Code 1	Code 2	Code 3	Code 4	Code 8	Code 9/0
"DEPENDS" + awareness of at least two of n = 2 n = 1 n = 0 n negative ('n>2' OK!)	"DEPENDS" + only ONE of n= 2 n = 1 n = 0 n negative	"DEPENDS" or "Usually 2n" but NONE of n = 2,1,0,neg	2n	n + 2 "SAME"	Others/ omitted

The frequency of the codes for each item in the C S M S test for the standard 6, 7, 8 and 9 samples used in this study is shown in appendix 3, as well as those of the 1976 2nd, 3rd and 4th year samples of the C S M S study. It should be pointed out that each item in the test is assigned a variable number as shown in the copy of the test in appendix 1. These variable numbers are used in the code frequency tables.

The facilities were obtained by combining codes 1 and 2 in most cases except as indicated in figure 6.4 below, or when code 2 was not used at all. This follows the procedure used in the C S M S study (Küchemann 1980a, p.47)

FIG. 6.4 Codes used to determine the facility of algebra items. Items whose facilities were determined other than by combining codes 1 and 2.

ITEM NO.	VARIABLE NO.	CODE USED TO DETERMINE FACILITY
14	38	1, 2, 3
19 (i)	46	1
19 (ii)	47	1
20	48	1, 2, 3

Recently (1985) the National Foundation of Educational Research (N F E R - Nelson) in England have published a simplified version of the marking scheme described above. This is now freely available.

6.4 LEVELS OF UNDERSTANDING IN THE C S M S ALGEBRA TEST

One of the aims of the C S M S study was to develop a hierarchy of understanding in different topics in school mathematics to facilitate teachers and others in making decisions about sequencing and content of teaching material. These hierarchies were, in practice, different clusters or groups of items. To form a group the items had to satisfy certain criteria (See Hart 1980, pp. 102 - 103; 1981, p. 7) Broadly they had to be homogeneous in content, at approximately the same level of difficulty (facility) and share a degree of consistency in relative difficulty. This gave rise to 30 of the items in the algebra test being selected into four levels of difficulty. Figure 6.5 shows how items were grouped into the four levels.

Fig. 6.5 LIST OF ITEMS COMPRISING EACH LEVEL IN THE ALGEBRA TEST

LEVEL	NO. OF ITEMS	
1	6	Item No. 5(i); 6(i); 7(ii); 8; 9(i); 13(i) Variable 10 13 16 19 20 29
2	7	Item No. 7(iii); 9(ii); 9(iii); 11(i); 11(ii); 13(iv); 15(i) Variable 17 21 22 26 27 32 39
3	8	Item No. 4(ii); 5(iii); 9(iv); 13(ii); 13(viii); 14; 15(ii); 16 Variable 7 12 23 30 36 38 40 41
4	9	Item No. 3; 4(iii); 7(iv); 13(v); 17(i); 18(ii); 20; 21; 22 Variable 5 8 18 33 42 45 48 49 50

The level of understanding of a pupil is assessed by determining the most difficult cluster of items on which he correctly answers at least two-thirds of the items. Children who pass a higher group of items without passing all easier groups are said to be 'error-types'. This in practice happened very infrequently. From figure 8.11 (p.98) it can be seen that there are distinct cut off points between the different levels of the algebra test for each of the year groups of the C S M S study. These levels were arrived at using a large representative sample of the English population. Jesson (1983), using two contrasting schools, found that the levels determined in the C S M S algebra test were robust enough to withstand marked departures from the national sample used for deriving them and that, in assessing the levels of understanding of pupils in algebra, considerable confidence could be placed on the consistency and stability of the levels so determined for any measure of ability. (Op. cit. 1983, p.135). In other words, the hierarchy was independent of the abilities of the children.

Figure 6.6 gives a description, in general terms, of each level in the algebra test (Hart 1980, p.111)

Figure 6.6 Levels of the Algebra hierarchy.

LEVEL	DESCRIPTION OF THE GROUP OF ITEMS
O	The criterion of 2/3 of level 1 items not satisfied.
1.	The significance of letters need not be realised; letters can be ignored, evaluated or seen as objects.
2.	Letters used as objects or evaluated but there is need to use mathematical syntax.
3.	Letters as specific unknowns. Lack of closure must be accepted.
4.	The manipulation of specific unknowns where this involves co-ordination of operations or where the letters represent numbers of objects or their cost etc. rather than the objects themselves. Letters as variables.

7. SAMPLE AND ADMINISTRATION OF TESTS

7.1 DETAILS OF SAMPLE

Altogether seven Government schools in the Border area of the Cape Province were used. These schools were typical homogeneous academic high schools serving white middle class families in both rural and urban areas. Five of the schools were co-educational, three of which consisted of both English and Afrikaans speaking pupils, while one was only English and the other only Afrikaans. Of the two remaining schools, one was an English girls' school and the other an English boys' school. The selection of the pupils used in each school was left to the principals or senior mathematics teachers of the respective schools, and in many cases an entire year group in the school was used.

Fig. 7.1 Breakdown of number of pupils from each school

	SCHOOL							
STANDARD	A	B	C	D	E	F	G	TOTAL
6	4	47	28	32	57	30	50	248
7	9	21	32	25	81		37	205
8		81			35		15	131
9							33	33
TOTAL	13	149	60	57	173	30	135	717

In actual fact the results of the 30 standard 6 pupils from school F were not used as it was felt from their results that this class had been primed by their teacher. The actual size of the standard 6 sample was therefore only 218.

It can be seen from the above table that the samples for standards 6 and 7 are far more representative than for standard 8 where only three schools were used, more than half being from school B. Moreover, whereas mathematics is a compulsory subject for all pupils in standards 6 and 7 this does not obtain in standard 8 or higher where pupils may choose to take mathematics. The standard 8 sample is, therefore, rather more select and is probably not representative of the population as a whole. This renders it unsatisfactory to compare the standard 8 results with those of the 4th year group in the C S M S study where ALL pupils were required to take mathematics. The original intention was not to use standard 9 pupils in the study. It was felt, however, to be of some merit to see how pupils who had been exposed to algebra for some years would do in the test. The sample, however, is not representative, coming from only one school - the writer's own class. The class was rather weak, 25 of the 33 pupils taking mathematics on the standard grade. The results do not form part of the study but are included purely out of interest.

7.2 ADMINISTRATION AND MARKING OF TESTS

In June of 1983, the writer administered the C S M S algebra test to his own 6, 7 and 8 mathematics classes as a small pilot study (although these do form part of the final sample described above.) These tests were marked according to the C S M S marking scheme during the midyear holidays and any difficulties encountered in using the marking scheme were carefully noted. The writer was very fortunate in being able to discuss these difficulties and other problems related to the coding and tabling of the results with Dr. Kathleen Hart during a five day conference held in Bloemfontein in July 1983. At this meeting Dr Hart suggested that a statistical analysis to determine a hierarchy of understanding would be out of the scope of a half thesis of this nature, and that the four levels determined in the C S M S study should be used in this study.

The rest of the testing was done during the third quarter of 1983, and the tests were administered by the mathematics teachers of the various schools used. Instructions for the administration of the tests were given to the teachers. In these instructions, the teachers were asked to explain to the pupils that the test was set with a view of obtaining a clearer understanding of children's basic grasp of school algebra. Consequently it would not be used to assess a particular child's ability or to make any judgement of any nature about a particular child. The children should therefore have no fear in writing the test but should relax and answer the questions to the best of their abilities. As the aim of the test was to get a clearer idea of the way in which children think in mathematics, it was explained to the teachers that the completion of the test was of more importance than strict adherence to the time limit of one hour. In practice, however, most children finished well within the hour. A fair proportion did not answer all the questions presumably because they were unable to do them.

The teachers were also asked to complete a questionaire. In this they were asked to assess the general mathematical ability (above average, average or below average) of the class or group completing the test. They were also asked to list any items or topics in the test that the pupils had not yet been taught in each year group at their particular school. This information showed that most teachers assessed their classes as having average ability - a very small proportion indicated that a particular class was above or below average. However, not much reliance can be placed on this subjective information except to assume that the children were fairly representative of the children in these schools. Further information indicated that items involving the formation of equations or expressions, the solving of equations, or the manipulation of expressions involving the co-ordination of more than one operation had not been covered very extensively in many of the standard 6 classes. This would probably account for the low facility for these particular items at the standard 6 level.

In January 1985 the writer submitted the test to his own standard 8 pupils, twenty of whom had previously written the test in June 1983, when they were in standard 6. This information provided the basis for a longitudinal study, albeit on a small sample. This will be described in section 8.4 (p. 105)

7.2.1 TABULATION OF CODES AND DETERMINATION OF LEVELS

The tests were marked and the codes tabulated as shown in appendix 4 These tables were used for entering the codes into a computer which then determined the frequency of each code in percent. (See code frequency tables in appendix 3) The tables were also used to determine the level of each child. This was done by counting the number of items children answered correctly in each cluster of items forming a level. Children were regarded as being at a certain algebraic level if they correctly answered at least two thirds of the items at that level and no higher level. (In fact the actual criterion was 4/6, 5/7, 5/8 and 6/9 correct for each of the four levels respectively).

Each child's score (number of items answered correctly from the 30 selected items forming the four levels) was also determined and recorded on the table of codes.

7.3 INTERVIEWS

After marking the written tests, six children from the writer's own classes were selected for interviewing. The aim of the interviews has already been described in Section 6.

The interviews, which were tape-recorded for future analysis, were all held in private. The following basic procedure was adopted:

(i) As in the case of the written tests, it was pointed out that the reason for interviewing was not to make any judgement about the child's ability. No 'right' answers were being sought, but rather a clearer understanding of the child's thinking. The child was therefore urged to speak his thoughts out aloud.

(ii) Items from the test were worked through one at a time. After having completed an item the child was asked to explain how he had done it. The answers and explanations given by the children indicated that they were uninhibited and co-operative. Occasionally it happened that a child was unable to proceed. In such cases a certain amount of 'leading on' was required. The following interview with Roger, an intelligent 13 year old in standard 6 illustrates this point:

QUESTION 17: (i) Mary's basic wage is R20 per week. She is also paid another R2 for each hour of overtime she works. If h stands for the number of hours overtime she works and if W stands for her total wage (in rands), write down an equation connecting W and h.

(ii) What would Mary's total wage be if she worked 4 hours of overtime?

Roger was not certain how to start, probably as he was not very familiar with equations and less familiar with translating word sentences into algebraic sentences.

I: If you wanted to work out how much she earned in a week how would you start?

R: You'd put her R20 for the week and then however many hours she works overtime.

I: Yes, well you know how many hours overtime she works.

R: Four hours

I: No, that four hours only comes in the second part. We are still busy with the first part. So how many hours overtime does she work?

R: h

I: So could you work out how much she earns in overtime if she works h hours?

R: No, because you don't know what h is.

The above example also serves to illustrate that Roger was unable to cope with h as a specific unknown or generalised number. He felt the necessity to evaluate h, first by giving it the value of 4 and finally by saying it was impossible to find the amount earned for overtime if the value of h was not known. That Roger was unable to cope with generalised numbers is further illustrated in the following example:

<u>QUESTION 18:</u> (ii) Is the following always, never or sometimes (when) true. $L + M + N = L + P + N$?

I: You've said it can never be true. Why did you say that?

R: 'Cos it's L and there isn't a M and there's a P and so it wouldn't be the same.

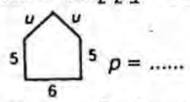
I: Why, because M is not the same as P?

R: Yes.

It could be argued that the interviewer begged the final response by his last question, but nevertheless it is clear from Roger's initial statement that he did not realise that M and P could have the same value.

Initially the writer made the mistake of trying to work through all the items in the test. This proved unsatisfactory in that the interviews became too long and the children too tired to concentrate satisfactorily. In the other interviews, which were limited to 30 minutes, items selected for discussion depended on the child's responses in the written test.

Nothing really new emerged from the interviews, but they did confirm that the responses of the children in this study were similar to those described in the C S M S study. It can therefore be fairly confidently assumed that the use of the C S M S marking scheme was not out of order in this study. Further examples follow to illustrate this point.

The first is from an interview with Jeanne, an average 14 year old in standard 7. She was not happy to accept the unclosed answer of 16 + 2u for item 9iii "  " and wrote down the following answer: 16 + 2u = 18u (Code 5 in the C S M S marking scheme)

I: How did you get that answer?

J: I added 5 and 5 and 6 and got 16 and those are both u's so that's 2u.

I: So how did you get 18u?

J: Well I just thought I must add up all the sides.

I: So do you think 18u is correct?

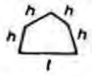
J: I'm timesing 18 by u. You are going to get a bigger number.

I: Are you not satisfied with that answer there? (points to 16 + 2u)

J: I don't know ... maybe I shouldn't have added them up 'cos they are not like terms.

I: Maybe you shouldn't have, so why did you?

J: Because it doesn't look good like that.

Although Jeanne was tempted to add the two terms together in the above example, she did not fall for this temptation in the previous item. She gave the unclosed answer of 4h + t for 9(ii) "  $p = \ldots\ldots$ ". Unfortunately this question was omitted from the interview but a probable reason for not closing is that the coefficient of t is an implied "1" and both terms contain letters so the temptation to combine the two terms was not so great.
From the above interview it is also clear that she is aware of the concept of unlike terms.
The temptation to evaluate a letter is strong, as the following interview shows. Carol is an average 15 year old in standard 7. She gave an answer of 12 for item 5(iii) "e + f = 8, e + f + g = ..." (code 8 in the C S M S marking scheme)

I: How did you get an answer of 12?

C: I took it for granted that e and f were equal.

I: In fact you took it for granted that e, f and g were all equal - equal to 4. Do they have to be?

C: It doesn't tell you what value e and f are so you must make up your own value for g as well.

I: Assuming that g is 4 you must get an answer of 12, but does g have to be 4?

C: No.

I: So if g does not have to be 4, what would you write down for your answer then?

C: Any number

I: What number?

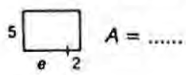
C: Any number higher than 8.

I: So it must be a number?

C: Yes - any number except imaginery.

It is clear that Carol does not accept g as an entity in its own right and therefore cannot cope adequately with letters as specific unknowns.

In most of the interviews it became clear that there was confusion with the operations of addition and multiplication. Two examples from the interview with Yolanda, a 15 year old standard 7 pupil are given below to illustrate this.

In item 17(iv) " 

" Yolanda wrote 10e for the area (as most did). (Code 5)

I: What is the length of that side? (points to the side with the e and the 2)

Y: Well the length of it is 2 and e. That will be 2e.

After an example with a number in the place of e it appeared that she realised that the length was 2 + e. However when it came to working out the area she argued thus:

Y: 2 plus e times 5 will be equal to 2e times 5 and that is 10e.

She wrote the following down as she spoke:

2 + e x 5

2e x 5 = 10e

I: But is 2 + e equal to 2e?

Y: Yes, 2 plus e is 2e to me - yes, they are the same.

In question 3 "which is bigger 2n or n + 2? Explain." Yolanda gave 2n as the answer with the explanation that "2 x n is bigger than n + 2."

I: How did you work out that 2n should be bigger?

Y: Well I don't know. I think 2n is bigger than n plus 2 because when you've got 2n like that without anything in between it means 2 times n. But I don't know, because 2 times n is 2n and n plus 2 will also be 2n.

It is clear from these two examples that the confusion is probably due, not so much as to when to multiply or when to add, but rather to the symbolic notation. She realises that 2n is 2 x n, but also wants to write 2n for n + 2 (or 2e for 2 + e). The ambiguity in writing the same expression for both operations does not seem to worry her. The conjoined answer "2e" for the length of the side of the rectangle does mean 2 + e to her in this context. This was verified in the interview when she was asked to replace e with a number. Taking e to be 4 she stated without hesitation that the length of the side was 6 and not 8. This example serves to illustrate that symbolism in generalised arithmetic is probably one of the big stumbling blocks for pupils. The writer did not pursue what Yolanda understood by the answer '10e' (10 x e or 10 + e?). The chances are that she did not give it a thought, as the answer is wrong using either interpretation. This seems to be a common failing with school children (as indeed with many adults!) They do not reflect on the logic of their answers. However when pointed out, they realise their error as the following conversation with Roger about his answer for question 22 shows. ("blue pencils and red pencils, etc.") He wrote down 90c = b + r. (Code 4)

I: You've got 90c = b + r. What does this mean?

R: That the blue pencils and red pencils added together equal to 90 cents.

I: But what do b and r stand for?

R: The number of blue and red pencils.

I: So here you've added two numbers together and got an answer of 90 cents. Do you think that can be right if you add b and r together?

R: No.

I: Why?

R: Bécause that will give you the total number of pencils.

I: So what's wrong with the statement you've written down?

R: There shouldn't be 90 cents there. etc.

With a number of leading questions Roger eventually arrived at the correct answer.

8. ANALYSIS AND DISCUSSION OF RESULTS

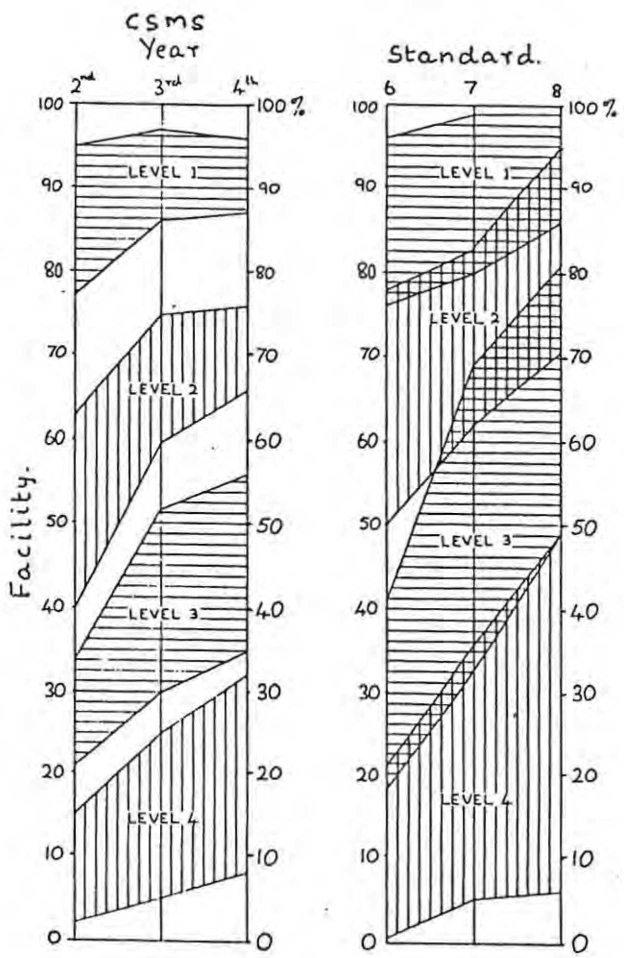
8.1 INTERPRETATION OF THE LETTERS

It is not proposed to discuss in detail how the children in this study interpreted the letters in each of the items comprising the four levels of the CSMS test. It has already been pointed out (section 7.3, p.77) that these interpretations are similar to those identified in the CSMS study - a fact which is borne out by comparing the percentages of children in this study and the CSMS study responding to each code in the marking scheme. (see code frequency tables in appendix 3). Children's interpretation of the letters has already been discussed in section 4.2 (pp. 41-48). A fuller discussion by Küchemann can be found in Hart (1981, pp 104-112). Reference to the different interpretations for each of the items will be made, however, in the next section on levels of understanding.

8.2 LEVELS OF UNDERSTANDING IN ALGEBRA

As described in section 6.4 (p.69) items which correlated well with each other in the CSMS study were classified into four levels of understanding. The same combination of items in each level was used in this study. Figure 8.1 below shows the facility bands for each level, for the 2nd, 3rd and 4th year groups of the CSMS study and the corresponding facility bands for the same year groups in this study.

FIG. 8.1 RANGE OF FACILITIES FOR QUESTIONS IN EACH LEVEL OF THE ALGEBRA TEST.

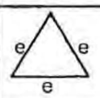
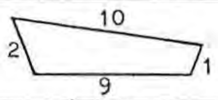
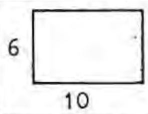


It can be seen that there are distinct cut off points between neighbouring pairs of levels for each age group in the CSMS study - a fact which does not obtain in this study, especially for the older age groups. However, a study by Jesson (1983, pp. 125-135) has shown that the levels so measured do have consistency and stability for any measure of ability of pupils and that the use of these levels can satisfactorily be applied to differing populations (op cit., p.135).

These levels are shown in figs. 8.2 to 8.5 below. (Küchemann, 1980a, pp. 64-69). In each of the tables items are shown in abbreviated form. The facilities given are for 3rd years in the CSMS study and the equivalent age group in this study (standard 7). Other common responses are also given together with the percentage frequency. The interpretation of the letter deemed adequate for a correct response is also listed.

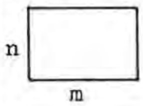
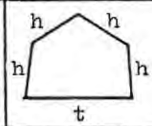
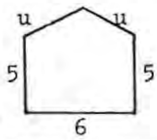
LEVEL 1. As can be seen from the facilities the items in this level are all very easy.

Fig. 8.2 Items assigned to level 1

Facility					%		
Std.7	CSMS	Item	Level 1 items	Other responses	Std.7	CSMS	Interpretation of letter deemed adequate for correct response
99	94	9i	 p =				Object
98	97	8	 p =				No letters involved
96	92	6i	What can you say about a if a + 5 = 8 ?				Evaluated
95	97	5i	If a + b = 43 a + b + 2 = ...				Not used
90	86	13i	2a + 5a =				Object
80	89	7ii	 A =	perimeter = 32	13		No letters involved

LEVEL 2.

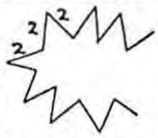
Fig. 8.3 Items assigned to level 2

Facility		Item	Level 2 items	Other responses	%		Interpretation of letter deemed adequate for correct response
Std.7	CSMS				Std.7	CSMS	
82	61	11i	What can you say about u if u = v + 3 and v = 1	u = 2	7	14	EVALUATED, but need to cope with ambiguity of u=v+3 (one unknown is 3 more than another unknown), and not reduce this ambiguity to 'u and v together equal 3' which leads to u=2.
81	68	7iii	 A =	perimeter 2m + 2n	6		OBJECT
77	60	13iv	2a + 5b + a =	8ab	11	20	OBJECT, but need to avoid temptation to close (8ab which is also ambiguous)
74	75	15i	Numerical item concerned with diagonals of polygon				No letters involved
68	68	9ii	 p =	4ht or hhhht	24	20	OBJECT, but need to avoid temptation to close.
66	62	11ii	What can you say about m if m = 3n + 1 and n = 4				EVALUATED, but need to cope with (temporary) ambiguity of m=3n+1.
62	64	9iii	 p =	2u16 or 2u.25.6 or uu556	17	16	OBJECT, but need to avoid temptation to close (2u16)

Children at level 2 are better able to cope with temporary ambiguities (11(i) and 11(ii)) and are more willing to accept unclosed answers. This, of course, is more fully realised at higher levels. For example 11% of all the standard 7's closed the answer for the simplification of 2a + 5b + a and wrote 8ab. Of these 77% were at level 1, 18% at level 2 and 5% at level 3. In standard 6 25% gave the answer 8ab of which 76% were at level 1. This is similar to the situation reported on in the CSMS study (Küchemann, 1981, pp. 113-114). Similarly, of the 10% of standard 7's who left item 11(i) out completely or gave the answer u = 2, 63% were at level 1.

LEVEL 3.

Fig. 8.4 Items assigned to level 3

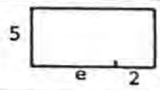
Facility Std.7	Facility CSMS	Item	Level 3 items	Other responses	% Std.7	% CSMS	Interpretation of letter deemed adequate for correct response
69	47	13 viii	3a - b + a =				SPECIFIC UNKNOWN. Use of letters as object breaks down. 3 apples take away a banana makes little immediate sense.
65	45	13ii	2a + 5b =	7ab	32	34	SPECIFIC UNKNOWN. Use of letters as objects does not help. Must avoid tendency to close (7ab)
54	52	15ii	A figure with k sides has ..... diagonals.				SPECIFIC UNKNOWN.
45	41	5iii	If e + f = 8 e + f + g = ...	12 9 15	23 11 0	26 6 2	SPECIFIC UNKNOWN. Though e and f can be ignored by matching (as in 5i) g can not; nor can g be evaluated (4+4+4=12, etc.)
45	41	14	What can you say about r if r=s+t and r + s + t=30	r=10	21	20	SPECIFIC UNKNOWN. r can not be evaluated directly (as in 10+10+10=30)
40	38	9iv	Part of this figure is not drawn...  n sides altogether. p=	p=32, 34, etc.	39	18	SPECIFIC UNKNOWN. n can not be used as an object(as in 9i,9ii, 9iii), nor can n be evaluated (by literally closing the figure:p=32 etc.)
34	36	4ii	4 added to n can be written as n + 4. Add 4 onto 3n	7n 7	50 6	31 16	SPECIFIC UNKNOWN. n has to be operated upon, not avoided (4+3n → 7n)or ignored entirely(4+3n → 7)
33	30	16	What can you say about c if c + d = 10 and c is less than d	only one value, usually c = 4	42	39	GENERALISED NUMBER

As can be seen from the above table, children must be able to cope with letters as specific unknowns or generalised numbers in simple item-structure in order to answer the questions successfully. They must also be able to regard unclosed answers such as 8 + g or 3n + 4 as meaningful, even though the letters represent numbers rather than objects.

The facilities for the items in both the CSMS study and this study are comparable (except for 13(ii) and 13(viii) about which more will be said later). Regarding the other responses, it appears that there is a greater tendency for children in this study to avoid the use of, or to evaluate the letter in items 4(ii) and 8(iv).

<u>LEVEL 4</u>.

Fig. 8.5 Items assigned to level 4

Facility		Item	Level 4 items	Other responses	%		Interpretation of letter deemed adequate for correct response
Std.7	CSMS				Std.7	CSMS	
36	22	20	Cakes cost c pence each and buns b pence each. If I buy 4 cakes and 3 buns, what does 4c + 3b stand for?	4 cakes and 3 buns	24	39	SPECIFIC UNKNOWN (or generalised number). The temptation to use the letter as objects is particularly strong since the item involves objects
35	23	13v	(a - b) + b =				SPECIFIC UNKNOWN. The use of letters as objects is no longer plausible (an apple take away a banana..)
26	17	4iii	Multiply n+5 by 4	4xn+5 n+20 20	11 49 7	19 31 15	SPECIFIC UNKNOWN. Here it is necessary to coordinate two operations, and to recognise the ambiguity of an answer like 4xn+5
16	25	18ii	Is the following always, never or sometimes (when) true? L + M + N = L + P + N	Never	70	51	GENERALISED NUMBER. M and P can represent a range of values, which may coincide
14	12	7iv	 A=	5xe+2 e+10 10	4 61 3	18 28 13	SPECIFIC UNKNOWN
10	12	21	If $(x+1)^3+x = 349$ is true when x=6, what value of x makes $(5x+1)^3+5x = 349$ true?				GENERALISED NUMBER or variable. x can be represented by 5x, which results in the transformation ÷5
8	11	22	b blue pencils (5 pence each), r red pencils (6 pence each), cost 90 pence altogether... (i.e. 5b + 6r = 90)	b+r=90 6b+10r=90 or 12b+5r=90	33 2	17 6	SPECIFIC UNKNOWN or generalised number. Not letter as object (blue and red pencils cost 90 pence, etc.)
7	5	17i	Basic wages R20. R2 per hour overtime. W stands for <u>total</u> wage and h for number of hours overtime. Write an equation connecting W and h.	W=20+h W+h 20W+2h	20 19 17	13 14 11	SPECIFIC UNKNOWN or generalised number. Not letter as object (W=20+<u>h</u>ours overtime)
5	6	3	Which is larger, 2n or n+2 ? Explain	2n (because it's multiply, etc.) n+2 or the same	66 24	71 16	VARIABLE (2nd order relationship). Intuitively it is reasonable to assume 2n > n+2 (e.g. for n=10, 20>12). But, as n changes the difference between 2n and n+2 changes, so for some (smaller) value of n, 2n may be less than 'n+2'

At this level children must be able to cope with letters as specific unknowns in a more complex structure e.g. in items 4(iii) and 7(iv) it is necessary to be able to co-ordinate two operations. In these two items the facilities for the std 7's are better than the CSMS 3rd year facilities, but there is a greater tendency for the children in this study not to use the letter. However more children in the CSMS study ignore the letter entirely and give a purely numerical answer. A similar situation exists for item 4(ii) in level 3. A noticeably larger percentage of the CSMS sample were happy with ambiguous answers for items 4(iii) and 7(iv) ($4 \times n + 5$ and $5 \times e + 2$ respectively).

As in all the other levels, the std 7's do far better than the 3rd years in item 13 - the simplification of algebraic expressions. However, when it comes to the more difficult items where letters should be used as generalised numbers or variables, it appears in most cases that the CSMS 3rd years have the edge on the std 7's. (items 18ii, 21, 22, 17i and 3.) Moreover, there is a far greater tendency for the std 7's in this study to simplify the situation (incorrectly) by regarding the letters as objects. This suggests the possibility that the children in this study have been drilled to a greater extent in simplifying and manipulating algebraic expressions to the neglect of developing understanding. It seems that more emphasis has been placed on symbol manipulation and the development of skills rather than on developing basic mathematical concepts. This could explain why they do not achieve as well as their CSMS counterparts when it comes to the more difficult level 4 items. This phenomenon is also noticeable at the std. 6/2nd year level and surprisingly (perhaps significantly) at the std. 8/4th year level, (see Fig. 8.6, page 90) where the std. 8's form a select group of of children who have chosen to take mathematics. The 4th year group in the CSMS study all have to take mathematics as a subject. The observation described above is also perhaps borne out by the results of items 9ii and 9iii (level 2) where the CSMS facilities are slightly better for the 2nd and 3rd year samples and very close for the 4th year sample, and where the children in the CSMS samples are less inclined to close their answers in all three age groups. (See Fig. 8.6). It seems then, that when it comes to accepting lack of closure and using letters in the true algebraic sence (as generalised numbers and variables) the children in the CSMS sample display a relatively greater ability, which might indicate that they have a better understanding of basic mathematical concepts.

Regarding question 16 (level 3, $c + d = 10$, $c < d$, $c = \ldots$) it can be seen (Fig 8.6) that the facilities for the first two year groups in both studies are in close agreement. (The probable reason for the difference in the std. 8/4th year facilities has already been given). In this item the children were required to see that c could have more than one value, but as can be seen, a surprisingly large proportion gave only one value (usually $c = 4$) in all the year groups in both studies (even the std. 9's, while more than half the std. 6's gave only one value). However, most of the children probably would have been willing to find other values if prompted, as the following conversation with Yolanda, a standard 7 pupil, illustrates:

She gave the answer $c = 4$ and $d = 6$

I: Are these the only possible values for c and d?

Y: No, it could have been 3 or 7.

I: So why did you only give 4 and 6?

Y: Because they were the first that came to my mind.

Fig. 8.6 Facilities of items at each level and percentages of children who give other responses.

Item and Level	Item No.	Facility Std. 6	CSMS Yr. 2	Std. 7	CSMS Yr. 3	Std. 8	CSMS Yr. 4	Std. 9	Other responses	% Std. 6	CSMS Yr. 2	Std. 7	CSMS Yr. 3	Std. 8	CSMS Yr. 4	Std. 9
LEVEL 1 10, 1, 9, 2 p=...	8	96	95	98	97	99	96	100								
e, e, e p=..	9(i)	96	91	99	94	99	93	91								
a if $a+5=8$	6(i)	94	86	96	92	99	93	100								
$a+b=43$ $a+b+2=$	5(i)	93	92	95	97	99	95	100								
6, 10 A=	7(ii)	77	79	80	89	86	90	88	perimeter = 32	17		13		5		
$2a+5a=$	13(i)	76	77	90	86	93	87	97								
LEVEL 2 $u=v+3$ $v=1 \therefore u=$	11(i)	78	49	82	61	95	70	97	$u=2$	6	15	7	14	1	8	3
n, m A=	7(iii)	74	54	81	68	86	76	88	perimeter $2m+2n$	14		6		4		
Numerical - diagonals of polygon.	15(i)	70	63	74	75	86	72	97								
$m=3n+1$ $n=4$, $m=$	11(ii)	60	44	66	62	92	67	100								
h, h, h, h, t p=	9(ii)	54	58	68	68	76	73	82	4ht or hhhht	34	22	24	20	15	13	6
u, u, 5, 5, 6 p=	9(iii)	51	54	62	64	71	67	82	2u 16 or 2u.25. 6.	27	20	17	16	11	12	6
$2a+5b+a$	13(iv)	50	40	77	60	85	66	91	8ab	25	26	11	20	7	17	0
LEVEL 3 $2a+5b=$	13(ii)	41	29	65	45	71	51	79	7ab	54	45	32	34	24	34	18
diagonals of k-sided polygon.	15(ii)	36	34	54	52	74	54	94								
$e+f=8$ $e+f+g=$	5(iii)	35	25	45	41	66	50	79	12 g is next letter, 9 g is 7th letter, 15	22 18 3	29 10 4	23 11 0	26 6 2	11 6 2	16 6 1	0 6 0
2, 2, 2, n sides p=	9(iv)	29	24	40	38	66	41	82	p = 32; 34 etc.	40	25	39	18	18	17	3
$3a-b+a$	13(viii)	27	27	69	47	81	56	67								
r if $r=s+t$ and $r+s+t=30$	14	24	30	45	41	60	39	82	$r=10$	20	24	21	20	10	21	0
$c+d=10$ $c<d$. $c=?$	16	22	21	33	30	49	35	70	only one value usually c = 4	53	43	42	39	24	35	12
4 added to 3n.	4(ii)	18	22	34	36	54	41	76	7n 7	61 6	41 17	50 6	31 16	33 4	30 12	15 3
LEVEL 4 cakes and buns. 4c+3b	20	21	14	36	22	49	30	64	4 cakes and 3 buns.	27	36	24	39	21	26	18
$(a-b)+b$	13(v)	14	15	35	23	47	32	70								
multiply n+5 by 4	4(iii)	9	8	26	17	49	25	73	$4 \times n+5$ $n+20$ 20	12 57 5	12 39 16	11 49 7	19 31 15	17 27 3	18 29 11	9 12 3
10, e, 2 A=	7(iv)	6	7	14	12	24	16	27	$5 \times e+2$ e+10 or 10e 10	11 57 6	15 27 10	4 61 3	18 28 13	8 44 6	19 31 9	3 46 12
$L+M+N=L+P+N$.	18(ii)	5	11	16	25	19	27	58	NEVER	83	56	70	51	79	50	36
Blue and red pencils $5b+6r=90$	22	2	2	8	11	21	13	70	$b+r=90$ $6b+10r=90$ or $12b+5r=90$	18 2	11 6	33 2	17 6	38 0	17 5	18 3
Basic wage and overtime $w=20+2h$	17(i)	2	2	7	5	9	8	30	$w=20+h$ $w+h$ $20w+2h$	10 28 18	6 13 14	20 19 17	13 14 11	28 13 15	19 12 11	15 24 6
$(x+1)^3+x=349$ etc.	21	1	4	10	12	12	16	42								
Which is larger 2n or n+2	3	1	4	5	6	6	10	12	2n n+2 or same	66 28	62 24	66 24	71 16	80 8	66 15	76 9

This is probably typical of most of those who gave only one answer - to accept without consideration of other possibilities that the first answer that comes to mind is the only and sufficient answer to the question. Teachers should encourage children to overcome this blinkered approach to mathematics.

It is interesting to note that all the year groups in this study obtained lower facilities than their CSMS counterparts for item 7ii, where children were required to find the area of a rectangle with sides 6 and 10 respectively. (The CSMS 3rd and 4th year groups even obtained higher facilities than the std. 9's.) A relatively large number of the children in this study gave the perimeter instead of the area (see 'other responses' in Fig. 8.6). Unfortunately the corresponding figures for the CSMS study were not given, but from the code frequency tables in the appendix it can be seen that the number could not have been as great. (eg. 13% of std. 7's gave the perimeter, but not more than 7% of 3rd years could have done so.) Could this be an indication that children in the CSMS study read the questions more carefully, or is there genuine confusion over the difference between area and perimeter on the part of the children in this study? - a case of knowing the formulae but not knowing when to apply them. (cf. Skemp's instrumental understanding.) However, when it comes to using letters instead of numbers for the area (item 7iii, level 2) children in this study seem to do far better than their CSMS counterparts (in fact the facilities dropped in the case of the CSMS study but remained fairly constant in this study). The number of pupils who gave the perimeter appears to have dropped but this is probably misleading, as many of the children who gave 32 for the answer for 7ii attempted to write down the perimeter for 7iii but failed because of the more abstract elements - a common answer for these children was $m^2 + n^2$.

It was mentioned in Section 7.1 that a small sample (n = 33) of std 9 pupils were also tested at the end of their std 9 year. The pupils in the sample were generally rather weak, but even so it can be seen from Fig. 8.6 that there is a considerable gain in facility in most of the items at each level. As would be expected the gain in facility is greater at the higher levels - particularly level 4, although the facilities for items 7iv, 17i and 3 are still very low. It would appear then that the extra year's experience in algebra makes a noticeable difference to the performance of the pupils. However even in std 9 many pupils still tend not to use the letter or to regard it as an object (see 'other responses' in Fig. 8.6 for level 4 items 20, 4iii, 7iv, 18ii, 20ii and 17i.) A number also have a low tolerance for ALC (see 'other responses' in Fig. 8.6 items 9ii, 9iii level 2 and 13ii, 4ii level 3.) It would seem then that a fair proportion of the std 9's have not yet reached the formal operational stage. However not much reliance should be placed on these results as the sample was probably not representative, coming from only one school and being very small.

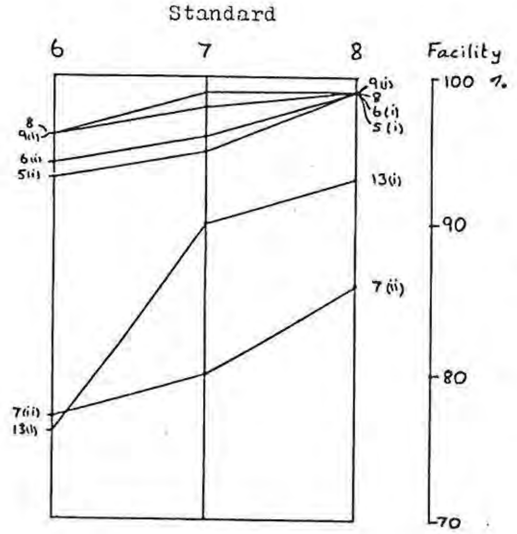
8.3 CHANGES IN PERFORMANCE WITH AGE

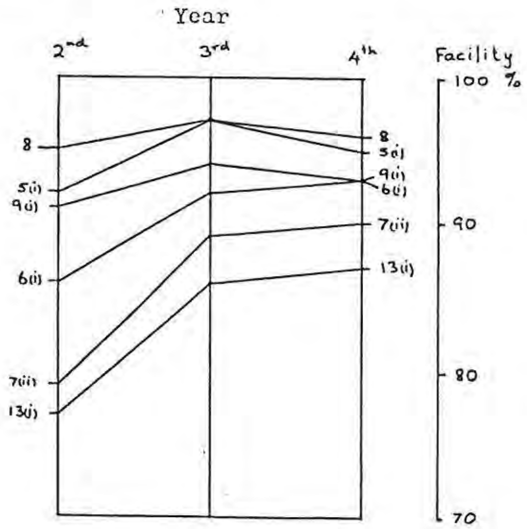
8.3.1 CROSS-SECTIONAL DATA. FACILITY OF ITEMS

The facilities for each of the 30 items selected to form the levels are shown graphically below (Fig. 8.7) for both the 1976 CSMS 2nd, 3rd and 4th year samples and the std. 6, 7 and 8 samples in this study.

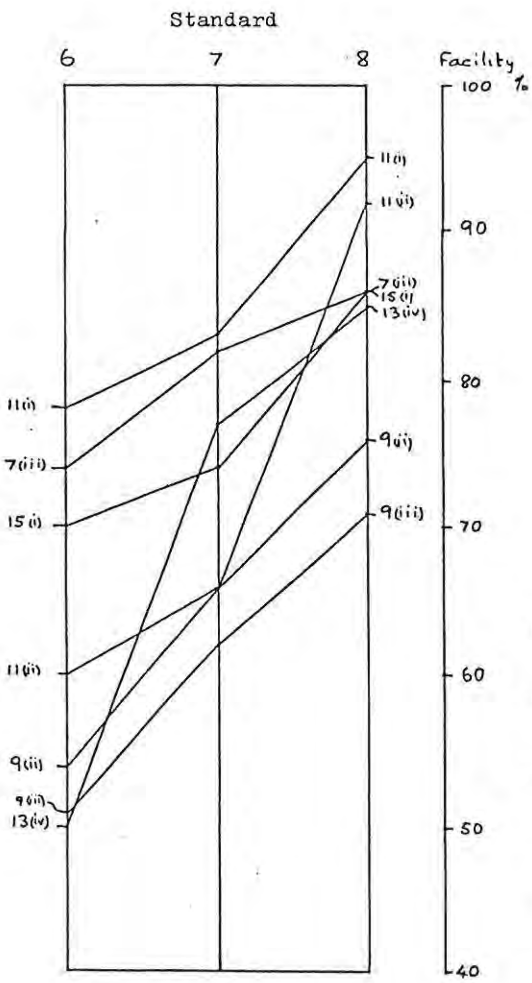
Fig. 8.7 Facilities of 30 selected algebra items for Stds. 6, 7, 8 and CSMS (1976) 2nd, 3rd, and 4th year samples.

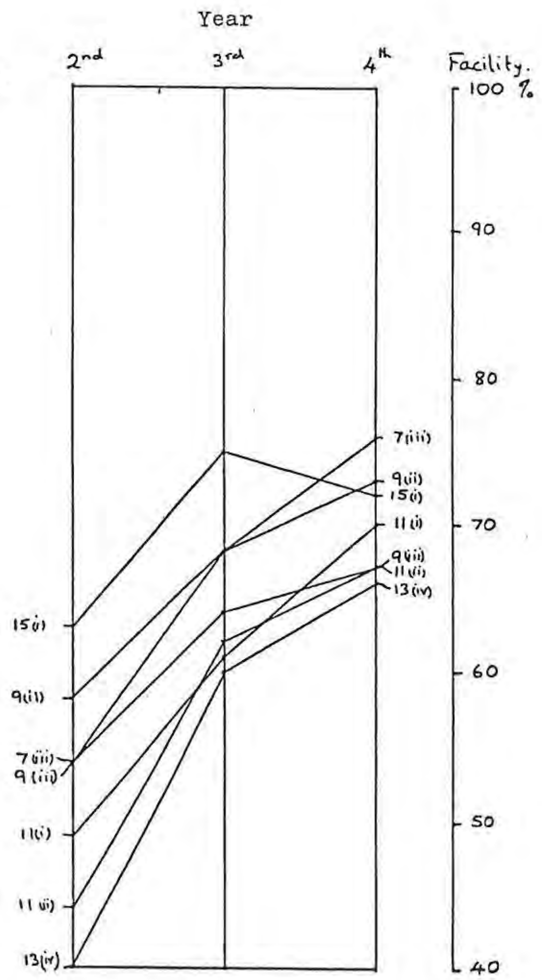
Level 1 items



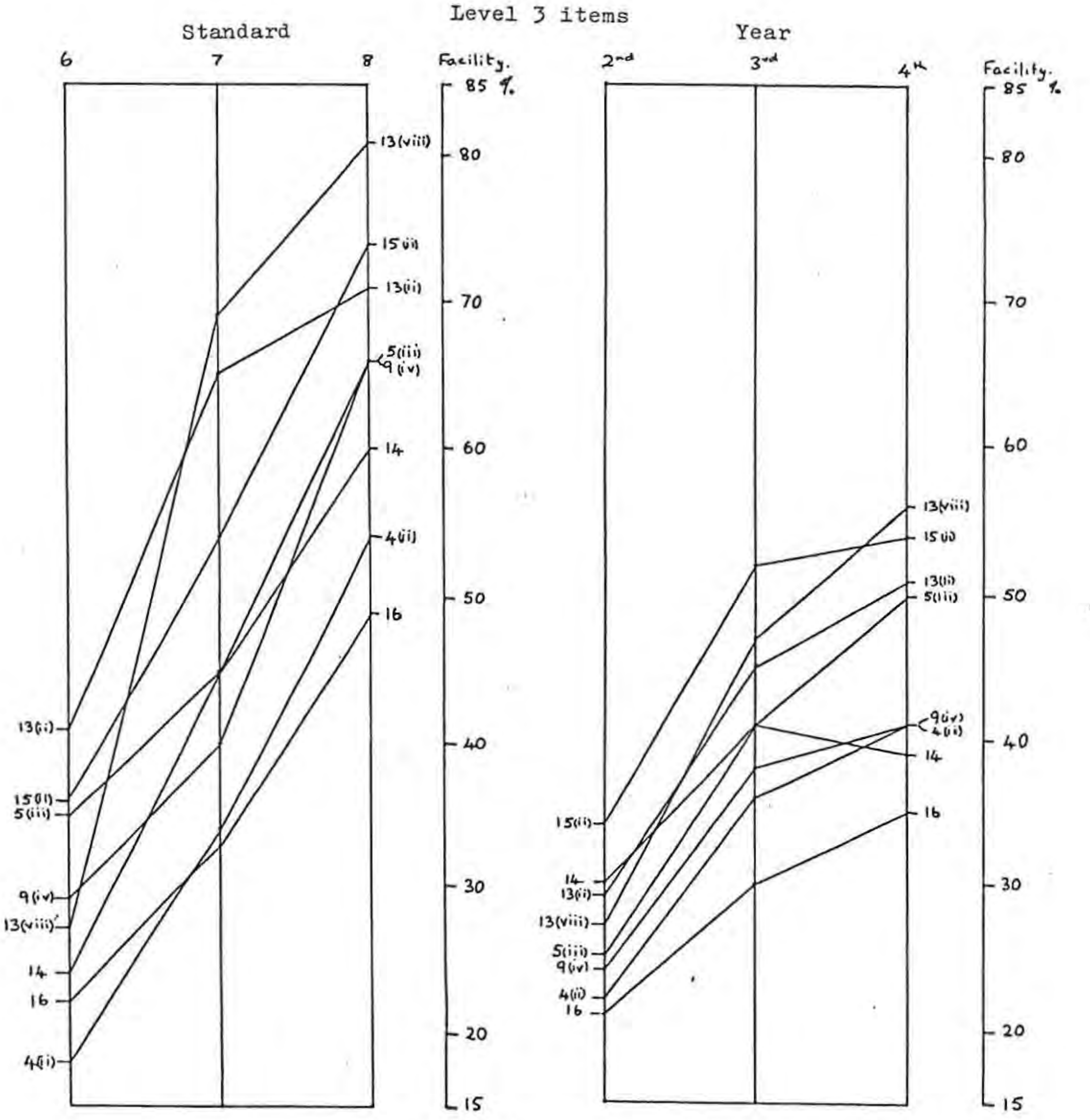


Level 2 items

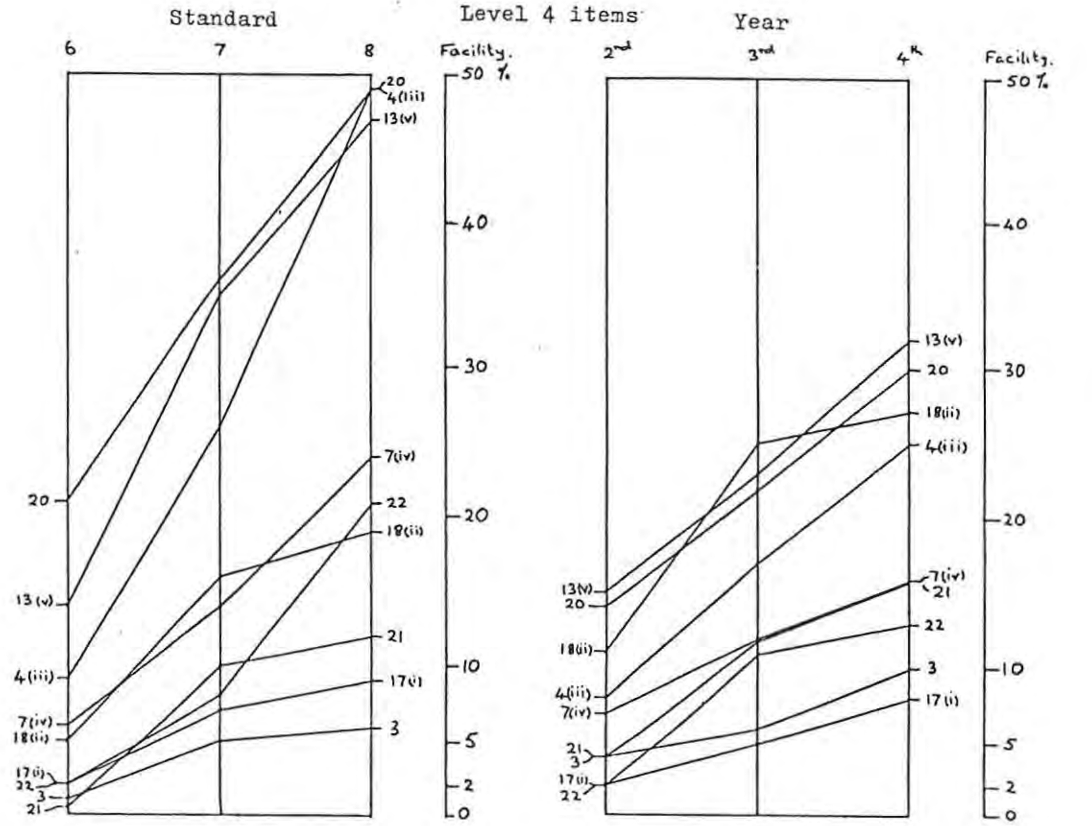




Level 3 items



Level 4 items



It can be seen that at each of the levels the increase in facility was generally greater from the 2nd to 3rd year samples than from the 3rd to 4th samples of the CSMS study (See also Fig 8.8 below). In some cases (particularly level 1) the performance actually declined from 3rd to 4th years. Küchemann (1980a, p.73) maintains that this is probably due to a ceiling effect, and to a growth in the proportion of children who reject mathematics, especially amongst those who find the subject difficult. (It should be remembered that all 4th years are required to take mathematics). The same cannot be said of this study. There is a steady increase in facility from std. 6 to 7 and from std. 7 to 8, although as Fig. 8.8 below shows, the increase from std. 6 to 7 is generally greater than from std. 7 to 8 (except for level 2)

Fig. 8.8 Average increase in facilities from Std. 6 to 7 to 8 and from 2nd to 3rd to 4th years for items at each level. (percentages to nearest whole number)

Item level	1	2	3	4		1	2	3	4
Std. 6 to Std. 7	4	11	26	11 %	2nd year to 3rd year	6	14	15	7 %
Std. 7 to Std. 8	3	12	19	9 %	3rd year to 4th year	0	5	5	5 %

Fig. 8.9 below gives the mean scores on 30 selected items for each year group. It also shows clearly that in the case of the CSMS study the improvement from the 2nd to 3rd year samples was greater than from the 3rd to 4th year samples, but in the case of this study the steady improvement was maintained from std. 6 to 7 to 8.

Fig. 8.9 Number of items correct (30 items)

	Mean	Increase		Mean	Increase
Std. 6	12,43		2nd year	11,6	
		3,59			3,15
Std. 7	16,02		3rd year	14,7	
		3,32			1,13
Std. 8	19,34		4th year	15,9	

The std. 8's in this study are a select group who have chosen to do mathematics as a subject and this explains their far superior performance over the 4th years in the CSMS study.

The relatively poor performance of the std. 6's in levels 3 and 4 particularly (See Fig. 8.10 below) can possibly be explained, in part, by their lack of familiarity with generalised arithmetic. Information from the std. 6 teachers of many of the classes who wrote the test indicated that items involving the formation of algebraic equations and expressions, the solving of equations or the manipulation of expressions involving the co-ordination of more than one operation had not been covered very extensively, if at all in some cases. Many of the level 3 and level 4 items involved these. However, cognitive ability must play a part to some extent. It would be expected that there would be more std. 7's and std. 8's entering the formal operational stage than std. 6's. Consequently their performance at the 3rd and especially the 4th levels would be better. However, it can be seen from the very low facilities of level 4 items that most children had not yet reached this stage even in std. 8.

It is clear from Fig. 8.7 that the rate of increase of facility from std. 6 to std. 7 is far greater for all the items in question 13 than for any other item in the test (This applies to all levels).

Question 13 concerns the simplification of algebraic expressions with or without brackets. This suggests that the std. 6's were handicapped by a severe lack of familiarity with this section of algebra. Once a minimum degree of familiarity is reached (in std. 7) progress would be more dependent on cognitive ability and would thus be expected to slow down.

This is verified by the lower rate of increase of facility from std. 7 to std. 8, even though the std. 8's form a select group.

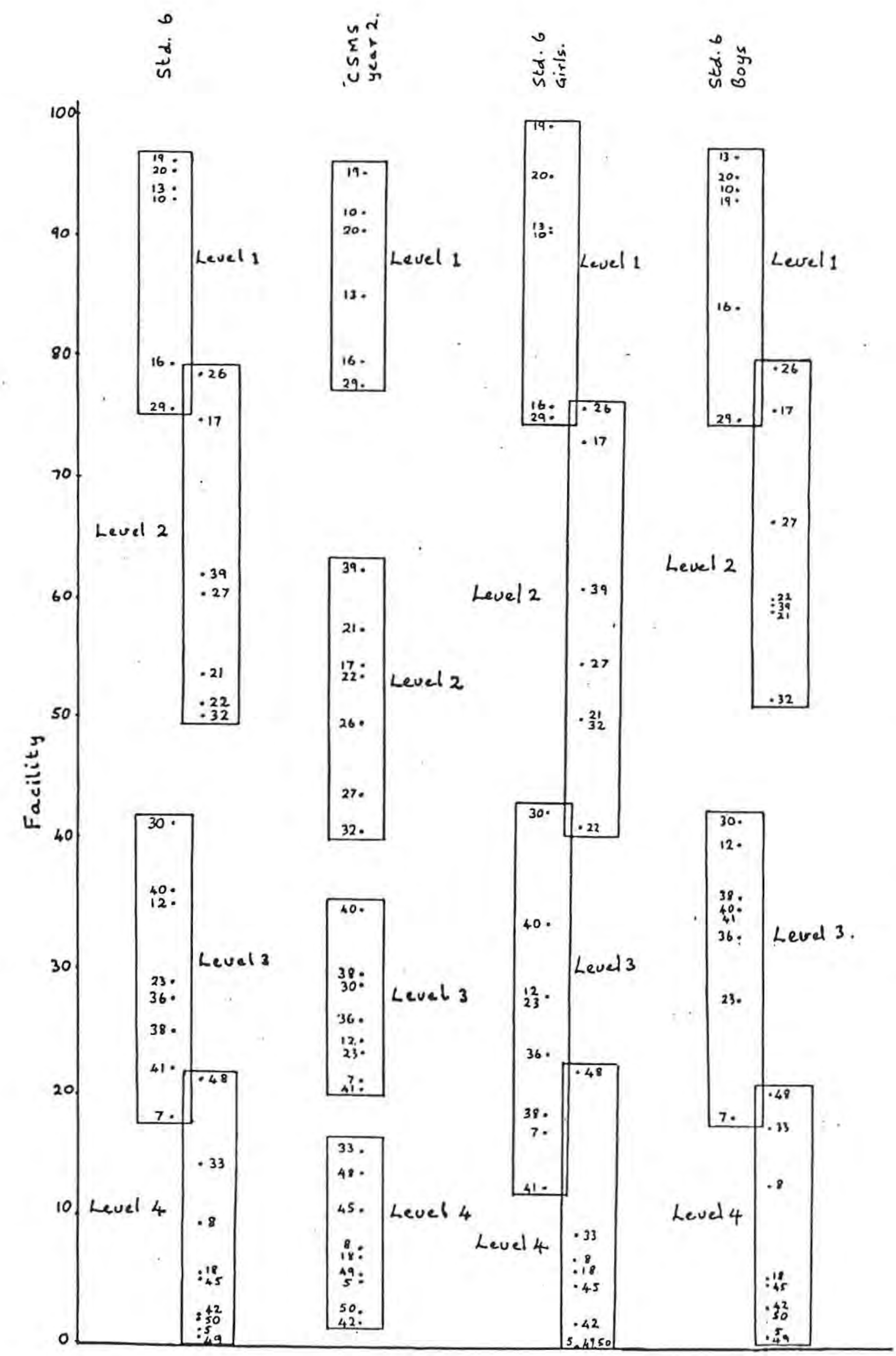
Fig. 8.10 Mean facilities of groups of items at each level.

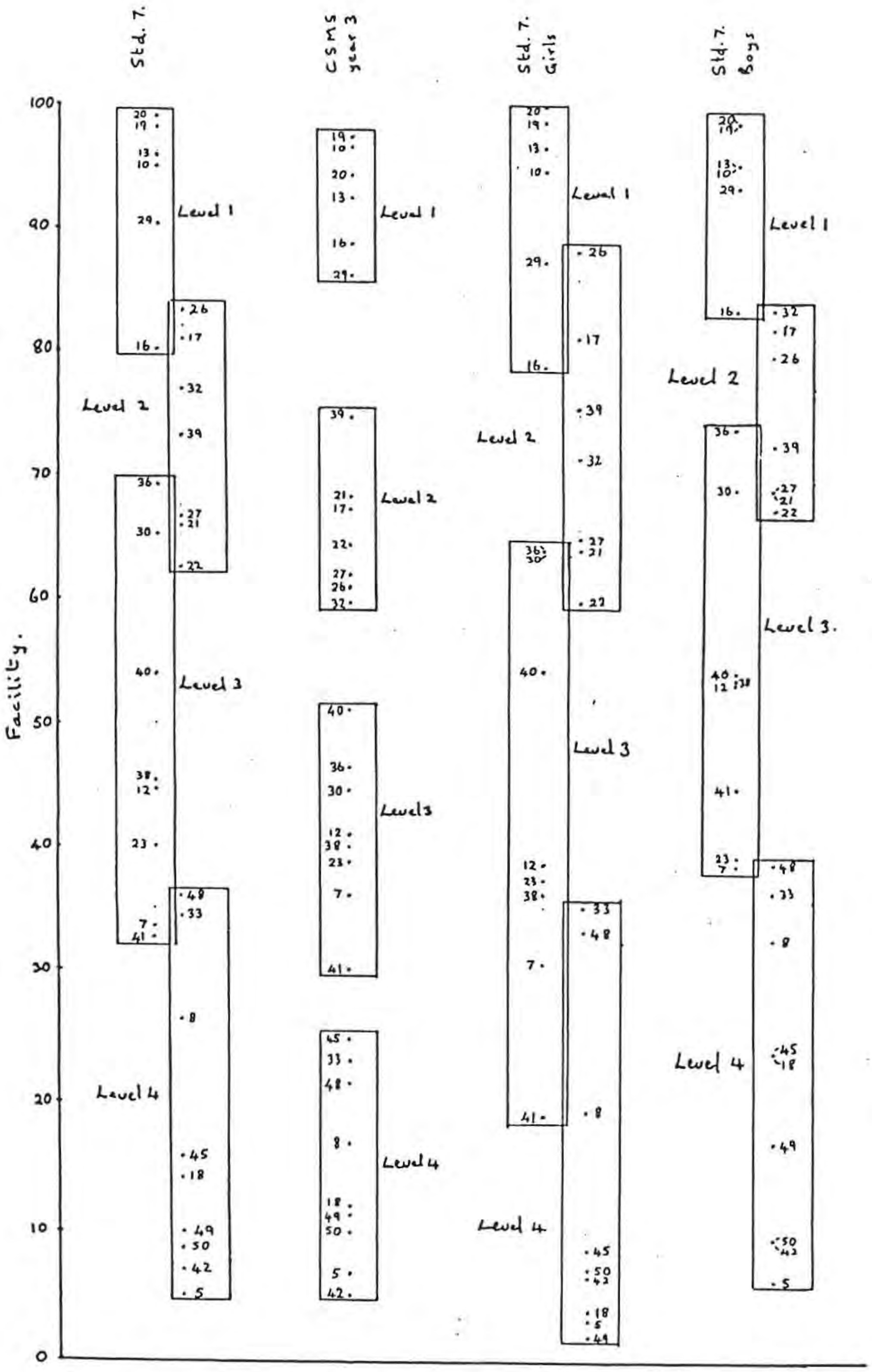
	Level 1			Level 2			Level 3			Level 4		
	13 yrs	14 yrs	15 yrs	13 yrs	14 yrs	15 yrs	13 yrs	14 yrs	15 yrs	13 yrs	14 yrs	15 yrs
This study	88,7	93,0	95,8	62,4	72,9	84,4	29,0	55,0	74,0	6,7	17,4	26,2
CSMS study	86,6	92,5	92,3	57,7	65,4	70,1	26,5	41,3	45,9	7,4	14,8	19,7
Difference	2,1	0,5	3,5	10,7	7,5	14,3	2,5	13,7	28,1	-0,7	2,6	6,5

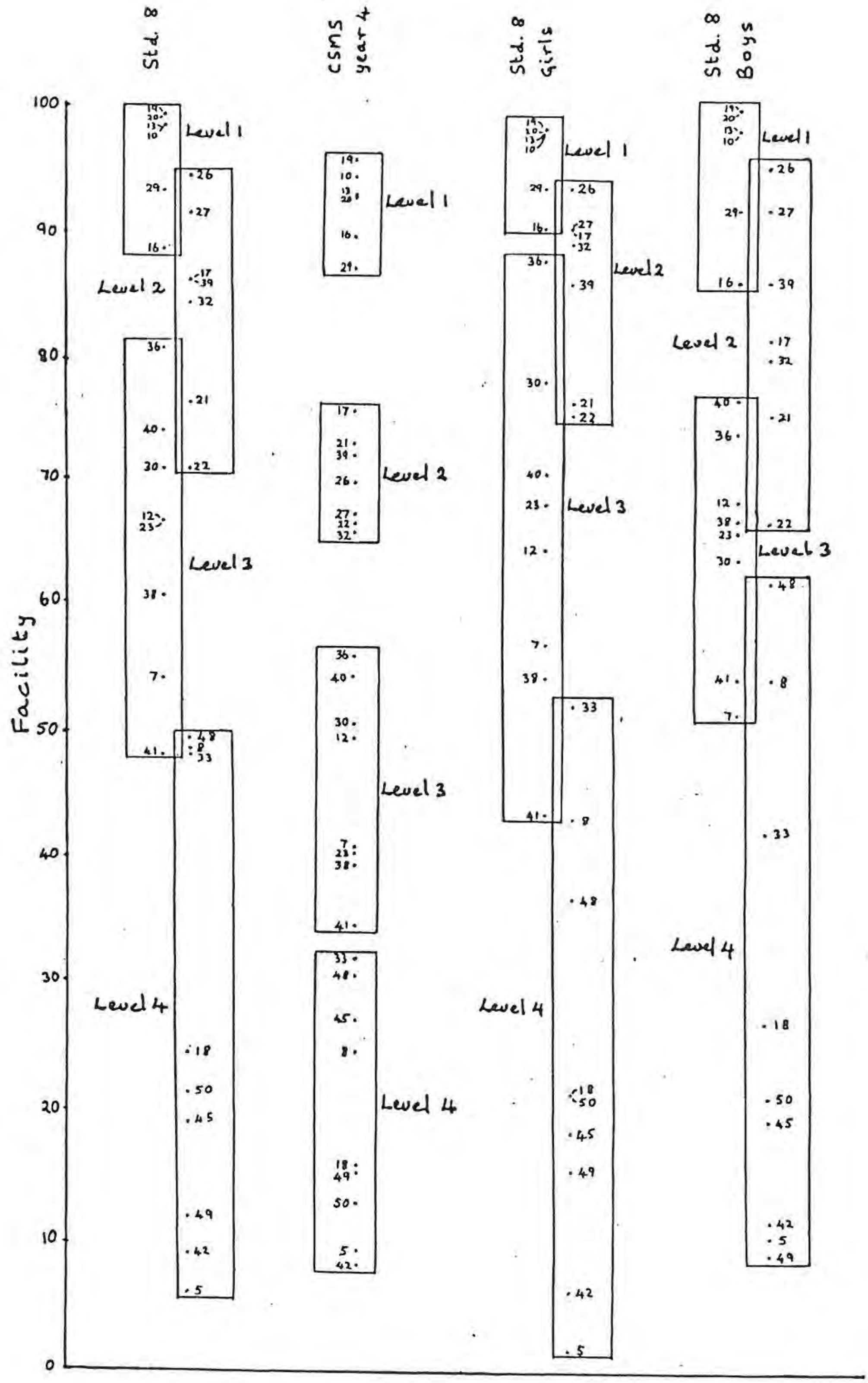
From the mean facilities in Fig. 8.10 above, it can be seen that the children in this study generally performed better than their CSMS counterparts at all levels (except std. 6/2nd year, level 4). Levels 2 and 3 discriminate between the facilities in the two studies to a greater extent than levels 1 and 4. This perhaps could be expected. On the easier level 1 items all children do well and on the difficult level 4 items all tend to do rather badly. The reasons for the better performances in this study are not clear (apart from the better performances of the std. 8's). Perhaps there is more drill and discipline in South African schools. In the short term this might improve scholastic achievement, but not necessarily an understanding of basic concepts, which could account for 2nd years performing slightly better than the std. 6's at level 4.

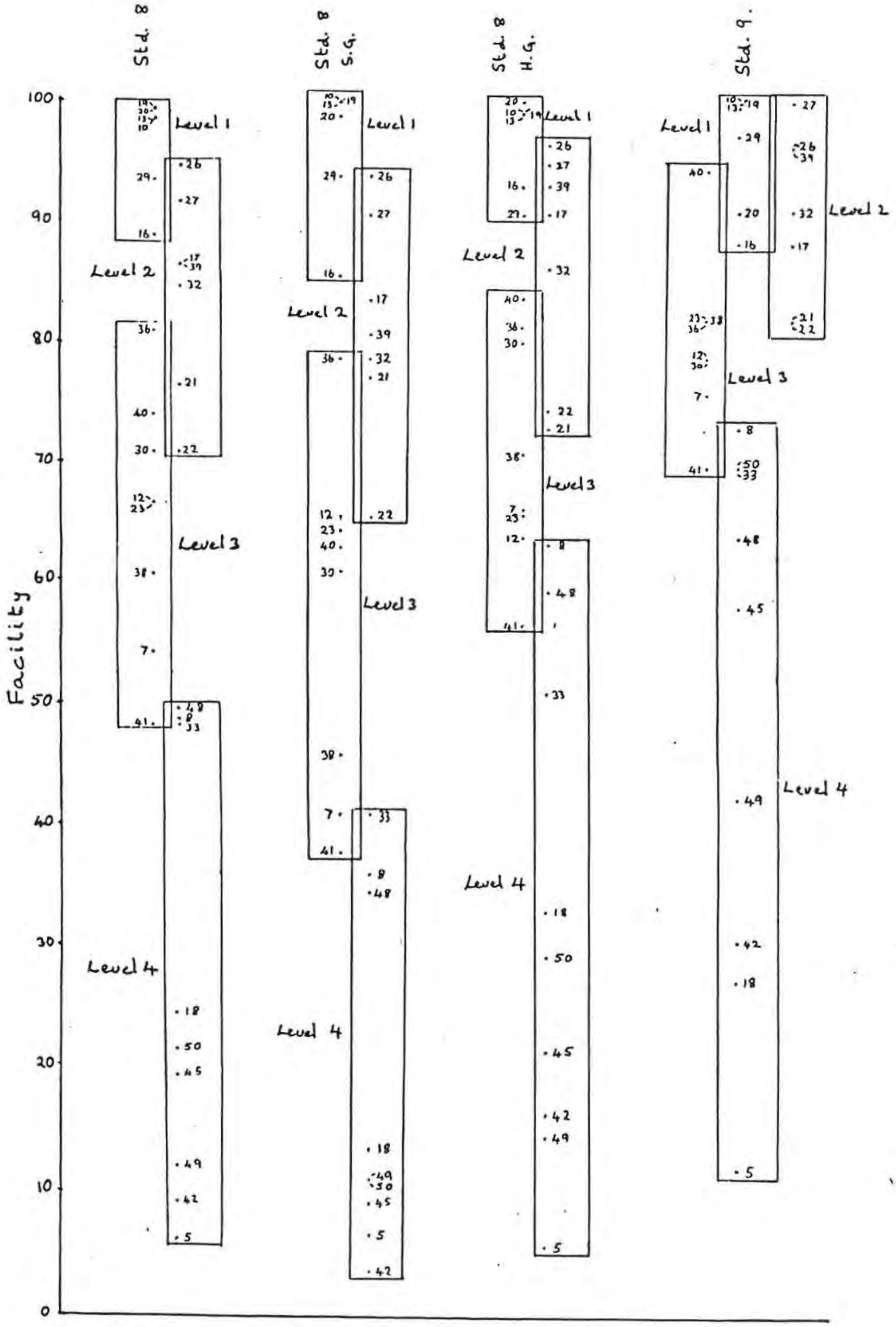
The facilities of the clusters of items forming the different levels is also shown graphically in Fig. 8.11 (pp. 98-101) Once again it is clear that the distribution of levels operates differently in the two studies, with the CSMS levels sharply differentiated while the levels in this study overlap to a marked extent in some cases. This is particularly noticeable for levels 2, 3 and 4 in the higher standards (8 and 9). What is also noticeable is that the range of facilities for the lower levels (1 and 2) tends to decrease, while there is a marked increase in the range for the higher levels (3 and 4) as the children get older (std. 6 through to

Fig. 8.11 Facilities and levels obtained from using the CSMS algebra test
Note: The numbers in the blocks refer to the variables in the original test.




Std. 7.
CSMS year 3
Std. 7. Girls
Std. 7. Boys
Facility.
100
90
80
70
60
50
40
30
20
10
0
Level 1
Level 2
Level 3
Level 4


Std. 8
CSMS year 4
Std. 8 Girls
Std. 8 Boys
Facility
Level 1
Level 2
Level 3
Level 4


Std. 8
Std. 8 S.G.
Std. 8 H.G.
Std. 9.
Facility
100
90
80
70
60
50
40
30
20
10
0
Level 1
Level 2
Level 3
Level 4

std. 9). This fact can be seen more clearly in Fig. 8.1. The same phenomenon occurs when the std. 8's are differentiated between higher grade and standard grade. The higher grade range for level 4 items (5% to 63%) is far greater than that for standard grade (4% to 40%). There are two factors which may contribute towards this: (1) The extra one or two years' experience in mathematics. (2) As children get older there is bound to be a greater percentage who would be able to reason in a formal manner (these would also be expected to take higher grade mathematics) and consequently better able to cope with some of the level 4 items.

Fig. 8.11 also differentiates between boys and girls in each standard, but more will be said about this in section 8.5.

8.3.2 CROSS-SECTIONAL DATA. DISTRIBUTION OF LEVELS.

It was mentioned in section 7.2 that children were assigned to one of the four algebra levels according to the highest level at which they could answer at least two-thirds of the items correctly. (Strictly 4/6, 5/7, 5/8, 6/9 for levels 1, 2, 3 and 4 items respectively). A child who could not answer at least two-thirds of level 1 items correctly was assigned to level 0. When determining the levels, 18 children were excluded altogether from each of the three samples because their performances did not scale, i.e. they reached criterion at one level, but not at all lower levels. In fact, all of them reached the criterion for level 3 but not for level 2. These children were called error types. Fig. 8.12 below shows the distribution of std. 6, 7 and 8 children at each level.

Fig. 8.12 Percent(given as a whole number) and cumulative percent of children at each level.

(i) Percentage of children at each level.

Child level	0	1	2	3	4
Std. 6	2	49	35	13	1
Std. 7	2	28	32	32	6
Std. 8	0	11	27	58	4

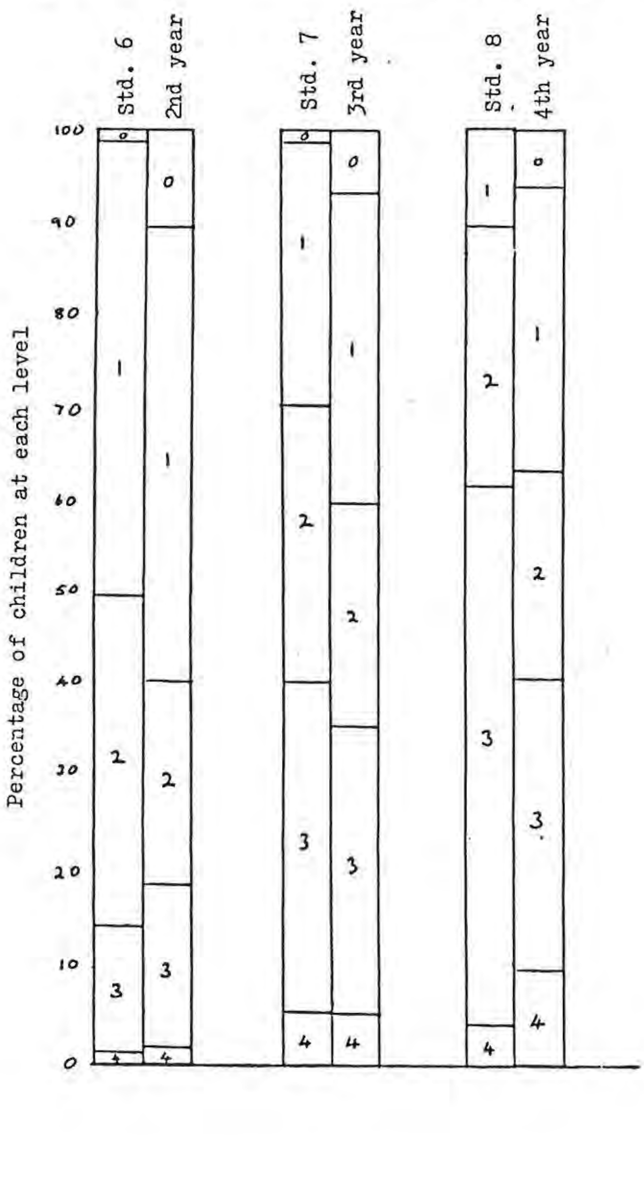
(ii) Cumulative percentage of children at each level.

Child level	0+	1+	2+	3+	4
Std. 6	100	98	49	14	1
Std. 7	100	98	70	38	6
Std. 8	100	100	89	62	4

It can be seen (especially from (ii)) that there is a steady improvement in performance from std. 6 to std. 7 to std. 8, although the comparatively greater improvement in performance between std 6 and std 7 than between std 7 and std. 8 is not as marked as it was using facilities. Surprisingly there is a drop in performance from std. 7 to std. 8 at level 4. This might be because the std. 8 sample was not as large and representative as the std. 7 sample. Nevertheless it is still surprising in the light of the std. 8's being a select group.

The relative proportions of children at each level are illustrated in Fig. 8.13 below, together with the corresponding proportions of children in the 1976 CSMS samples. (See data in Küchemann, 1981, p. 116)

Fig.8.13 Percentage of children in both studies at levels 0 to 4



Once again it can be seen that the performance of the children in this study is generally better than that in the CSMS study. Notable exceptions are: (1) The proportion of 2nd years at levels 3 and 4 is greater than the std 6's at the corresponding levels. (2) The proportion of children at level 4 is equal for the 3rd years and std. 7's. (3) There is a greater proportion of 4th year children at level 4 than std. 8 children. As noted above, this is surprising, as the std. 8's form a select group. Possible reasons for this have already been given. It seems then, that in assessing children in terms of levels, the CSMS samples have the edge on the corresponding samples in this study at level 4 (and level 3 in std. 6). This is in contrast to assessing them in terms of facilities where the std. 6's were the only group not to perform as well or better than their CSMS counterparts. (See section 8.3.1, fig. 8.10). However, assessment in terms of levels may present a truer picture - especially for the std. 8's at level 4. The mean facilities for the std 7's and 8's at level 4 are relatively high because three out of the nine items have very much higher facilities than the rest. (See fig. 8.7). At the std. 6 level this does not obtain to such a marked extent. The question arises why it is basically at level 4 that the CSMS results are equal or better than those in this study. Once again, it can only be conjectured that perhaps discipline in South African schools is greater and children more firmly drilled in algebraic manipulations and algorithms with not enough encouragement to think for themselves. Their performance on easier, more mechanical items, would then be better, but not in the case of the more abstract items (as in most of the level 4 items).

The Chi-square test was used to test for the significance of the differences between the distribution of levels between the standards and also between higher grade and standard grade in std. 8.

Fig. 8.14 Number of children at each level.

Std.	Level 0	1	2	3	4		
6	4	106	76	27	1	N=214	(4 error types)
7	3	56	64	64	12	N=199	(6 error types)
8	0	13	33	72	5	N=123	(8 error types)
8 HG	0	5	9	39	4	N=57	
8 SG	0	8	24	33	1	N=66	

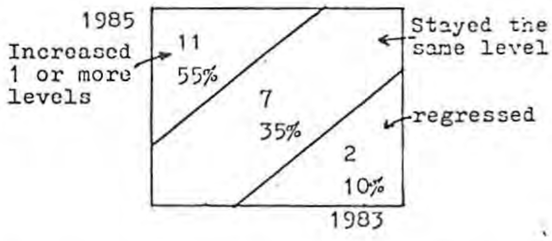
Levels 0 and 1 were combined for stds. 6, 7 and 8 as the expected values for level 0 were less than 5, while levels 3 and 4 were combined for the test between std. 8 higher grade and standard grade for the same reason. The results gave χ^2 = 40,28 for the difference between stds 6 and 7, χ^2 = 26, 17 for the difference between stds 7 and 8, and χ^2 = 7,95 for the difference between std 8 higher grade and standard grade. The first two differences were highly significant ($p < 0,01$), but the difference between std 8 HG and std 8 SG was only significant at the 5% level. However, the difference between the mean scores of the std 8 HG and SG pupils on the 30 selected items forming the four levels was highly significant. This difference (20,72 - 18,15 = 2,57) gave a t-value of 4,045 with $p < 0,01$. The t-test also gave highly significant differences between the mean scores of std's 6 and 7 (16,02 - 12,43 = 3,59) and std's 7 and 8 (19,34 - 16,02 = 3,32). The t-values were 7,32 and 5,85 respectively, both giving $p < 0,01$.

8.4 LONGITUDINAL DATA:

A group of 20 children from the writer's school who were tested in June 1983 when they were in std 6 were again tested in January 1985 - the beginning of their std 8 year. Unfortunately the sample was small, but this was the maximum number possible in the school. The same test was used in both cases. The children's performances were classified into one of the four levels for each test. The change in the number of children at each level from the first time of testing to the next is shown in fig. 8.15 below.

Fig. 8.15 Levels of 20 children in 1983 and 1985

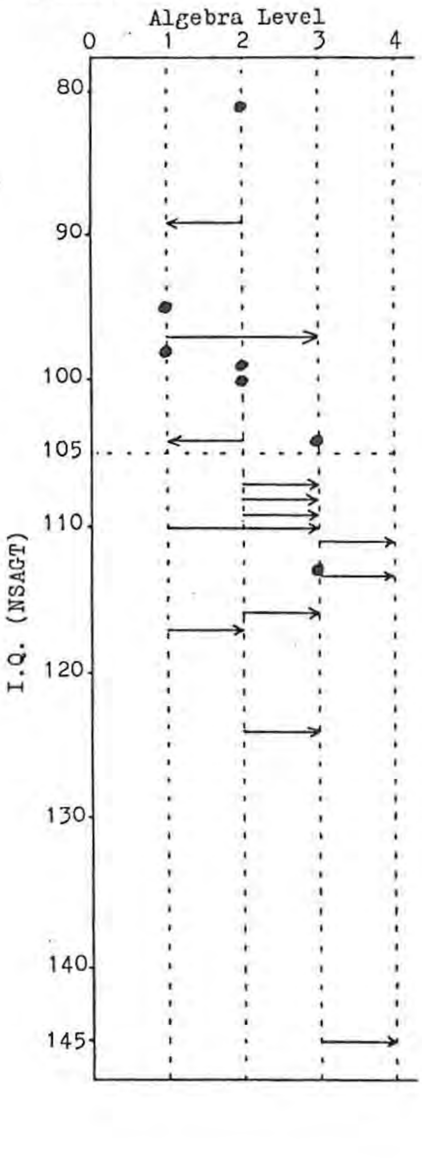
Level 1985 \ Level 1983	1	2	3	4	
4			3		3
3	2	5	2		9
2	1	3			4
1	2	2			4
	5	10	5		N=20



It can be seen that the proportion of children who regressed is small (10%), while most progressed by one or more levels. (55%).

The graph below (fig. 8.16) suggests that there is a strong relation between the change in level and IQ and also between the levels and IQ.

Fig. 8.16 Change in levels over 18 months against children's IQ.



It can be seen that only one child (out of nine) with IQ below 105 gained in level while two regressed and six stayed at the same level. However, in the group of eleven whose IQ's are above 105, all except one progressed by one or more levels. It can also be seen that all except two children in the lower IQ group (IQ < 105) were still at level 2 or below after 18 months - the higher levels of abstraction being beyond them. However, all except one child in the higher IQ group (IQ > 105) had reached level 3 or level 4 after 18 months (three having reached the highest level of abstraction). This is similar to the situation reported on in the CSMS study. (See Hart, 1981, pp. 185 - 186)

It may be argued that the 20 children in this longitudinal study formed a select group, as they were the ones who had chosen to do mathematics as a subject in std. 8. However the results do suggest that a number of the children should not have made this choice. A further argument against these results is that the children wrote the same test each time and many of the items may have been familiar to them at the second time of testing. The brighter ones especially may have remembered the questions. The writer never discussed any of the questions with his pupils and at no time did any of the pupils have access to the test in the interim period. Judging from the manner in which the children answered the questions at the second time of writing, it appeared that they had retained no memory of the test.

8.5 DIFFERENCE IN PERFORMANCE BETWEEN THE SEXES

One of the minor aims of the study was to see whether there was any difference between boys and girls of equivalent ages with respect to their understanding of generalised arithmetic. This was done by comparing their performances in the CSMS algebra test.

8.5.1 FACILITIES OF ITEMS

Separate printouts of the code frequences for boys and girls can be seen in the appendix. It should be noted that the number of boys and girls do not add up to the total in each sample in these tables, because a number of pupils did not give their name or sex on their test papers. These had to be excluded in this section of the work. The facilities and levels of the boys and girls are shown graphically in fig. 8.11 (p.98). From these it can be seen that the pattern for the boys and girls and for the whole group is similar. There does, however, appear to be a tendency for the lower ends of the rectangles surrounding each cluster of items defining a level to be lower for the girls than for the boys (except for std 8. levels 1 and 2). This means that the lower facilities for each level are less for the girls than for the boys, although these lower facilities are not always for the same items. This suggests that the performance of the boys may be slightly better. This observation is verified by comparing the mean facilities of boys and girls at each level. (Fig. 8.17 below)

Fig. 8.17 Mean facilities of clusters of items in each level

	Std. 6				Std. 7				Std. 8			
Level	1	2	3	4	1	2	3	4	1	2	3	4
Boy	89,5	65,4	33,4	8,2	93,5	74,7	53,6	21,8	95,3	82,9	65,3	28,8
Girl	87,7	58,1	21,1	5,9	92,5	72,3	42,9	13,1	96,8	86,1	65,6	24,2
Diff.	1,8	7,3	12,3	2,3	1,0	2,4	10,7	8,7	-1,5	-3,2	-0,3	4,6

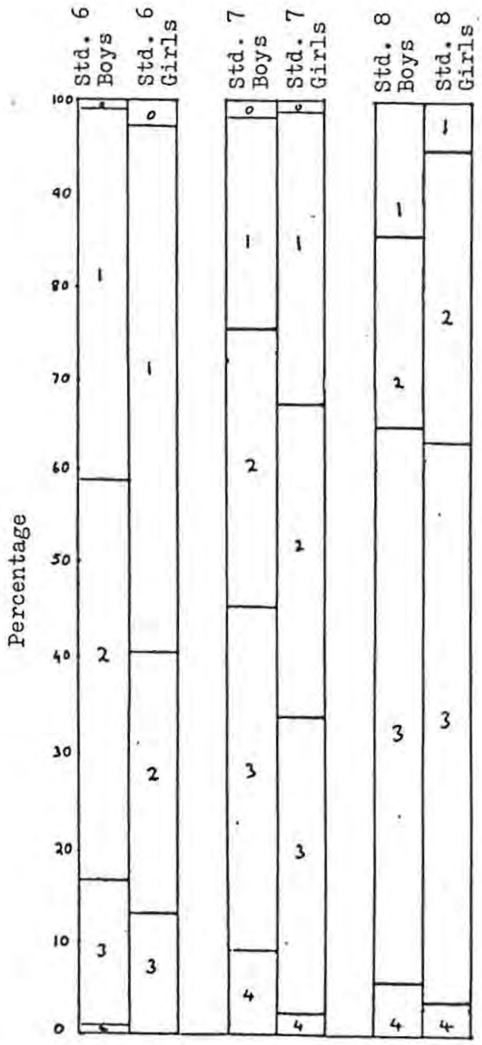
8.5.2 LEVELS OF CHILDREN

Fig. 8.18 below shows the cumulative percentages of boys and girls at each level. These are shown graphically in fig 8.19.

Fig. 8.18 Cumulative percentage of boys and girls at each level

Child level		0+	1+	2+	3+	4
Std. 6	Boys	100	98	57	16	1
	Girls	100	97	40	12	0
Std. 7	Boys	100	98	76	45	10
	Girls	100	99	67	33	2
Std. 8	Boys	100	100	85	64	5
	Girls	100	100	95	62	3

Fig. 8.19 Percentage of boys and girls at each level.



Here it appears that basically the performance of the boys is better than that of the girls for each standard (eg 57% of the std 6 boys are at level 2 or above while only 40% of the std 6 girls are at level 2 or above, etc.)

The Chi-square test was used to test for the significance of the observed differences in distribution. Fig 8.20 below gives the number of boys and girls at each level.

Fig. 8.20 Number of boys and girls at each level

		Level					Total	Error types
		0	1	2	3	4		
Std. 6	Boys	1	41	41	15	1	99	1
	Girls	3	58	28	12	0	101	3
Std. 7	Boys	2	22	31	35	10	100	5
	Girls	1	30	31	29	2	93	1
Std. 8	Boys	0	9	14	36	3	62	4
	Girls	0	3	21	35	2	61	4

Fig. 8.21 gives the values for χ^2 in each standard and the corresponding values for p.

Fig. 8.21 Chi-square values for the differences in levels between boys and girls

	χ^2	d.f.	p value
Std. 6	6,51	2	$0{,}05 > p > 0{,}01$
Std. 7	6,54	3	$p > 0{,}05$
Std. 8	4,44	2	$p > 0{,}05$

From the above table it appears that the observed differences in the distribution of the levels are not significant except, perhaps, in std 6 where there is a fairly significant difference.

8.5.3 MEAN SCORES ON 30 SELECTED ITEMS FROM THE TEST

The t-test was used to test for the significance of the difference between the means of the boys' and girls' scores on the 30 selected items comprising the four levels of the algebra test. Fig 8.22 gives the mean scores for each standard.

Fig. 8.22 Mean scores for boys and girls on the 30 items of the algebra test and t-values for the difference

Std.	Mean score Boy	Girl	Difference	t	p
6	13,19	11,78	1,41	2,285	$0{,}05 > p > 0{,}01$
7	17,05	15,12	1,93	2,425	$0{,}02 > p > 0{,}01$
8	19,38	19,47	-0,09	0,134	$p > 0{,}8$

In std 6 and std 7 the boys, as before, do better than the girls and both results are fairly significant. However the observed difference in the mean scores of the std 8 samples is not significant.

It appears, then, that in terms of facilities, levels and mean scores, the boys in std's 6 and 7 have a better understanding of generalised arithmetic then than the girls as measured by the CSMS algebra test. These differences in performances are not, however, highly significant (at the 1% level) and should be treated with some caution. On the other hand, it appears that in std 8 the girls are slightly better, but the difference is not significant and cannot therefore be accepted.

A possible explanation for the above observation is that in std's 6 and 7 mathematics is a compulsory subject. Boys, being more motivated than girls in terms of future careers etc. would tend to do better. On the other hand, in std 8 where children may choose to take mathematics as a subject, it seems reasonable to assume that boys and girls would be equally motivated, having chosen to do the subject. This argument suggests that more girls than boys might give up mathematics at the end of their std. 7 year. From fig. 8.20 above it seems that this is not the case - the numbers of boys and girls taking mathematics in std 8 are virtually equal. However, it is possible that the std. 8 sample used in this study is biased, coming from only three schools and being rather small.

9. DISCUSSION AND IMPLICATIONS FOR TEACHING

The CSMS algebra test was designed to test children's understanding of basic concepts, and not rote remembered rules or algorithms. Many of the individual items in the test, therefore, may not be in a form that is commonly found in text books or in examination papers. They do not, however, exceed the conceptual requirements of school algebra. The results of this research show that the conceptual demands of seemingly simple activities in school algebra are sometimes far greater than would have been expected. For example, a teacher might be forgiven for regarding the example "add 4 onto 3n" as being too simple to include in a mathematics examination. However only 18% of the std 6's, 34% of the std 7's and 54% of the std 8's could answer this question correctly. Even at the std 9 level where children had nearly completed their fourth year of algebra, 24% could not give the correct answer! The difficulty of this example lies in the fact that in primary school (and perhaps even later) children have been conditioned into believing that the aim in mathematics is to find numerical answers to problems. To the majority of them, then, the answer '3n + 4' is unsatisfactory because they feel they have not yet added. This observation is corroborated by the fact that 67% of the std 6's, 56% of the std 7's and 37% of the std 8's gave the answer '7n' or '7'.

At a much lower level it seems that a fair proportion of std 6, 7 and 8 pupils cannot even cope with simple numerical items involving areas of rectangles. 23% of the std 6's, 20% of the std 7's and 14% of the std 8's could not calculate the area of a rectangle with sides 6 and 10 respectively. Most of these gave the perimeter instead. This suggests that children do not read the questions properly or (more likely) that the concept of area is more difficult than would be expected. Children are told that the area is length times breadth and when they successfully complete an exercise of multiplications to get the areas of different sized rectangles, teachers (and the children themselves) may be misled into believing that the concept of area has been understood. Skemp refers to this as instrumental learning or 'rules without reasons'.

Perhaps there are two lessons to be learnt from these examples. The first is that teachers' judgements as to what children should be capable of are often based on intuition or assumption. They tend to overestimate a child's ability or underestimate the difficulty of mathematics. Secondly, the belief is false that understanding automatically follows the completion of a topic or (to put it in another way) that once children have been taught a topic (eg areas or fractions in primary school) they should 'know' it. How often does a teacher say the following or something similar to his class? "You were taught this in std 5 and so you should know it. There is no time to go over it again or we will never finish the syllabus." On the other hand, if a teacher does 'go over it' again he may erroneously approach it with the assumption that the children are in possession of certain basic facts (which perhaps are crucial) with the result that the whole exercise would be futile. Following from this, it should be clear that it is important to present content in such a way that a child can process and thus conceptualise it. It is therefore necessary to move away from telling or showing children what to do and to involve them more dynamically in the reasoning process. It is encouraging to see that the new syllabus (introduced in std 6 in 1984) and some text books based on it appear to be attempting just this. For instance in one text book a few basic conventions about algebraic symbolism or shorthand (eg 2 x n is written as 2n and x^3 means $x \times x \times x$ etc) are given and children thereafter are encouraged to discover for themselves short methods or rules for multiplication and division etc. through the exercises they do, and to test whether these methods always work. As the awareness of a rule develops at widely different stages in different pupils, pupils who have already discovered a rule and are convinced that it is correct; are urged to keep the knowledge to themselves. The emphasis is for the teacher not to verbalise rules or laws for the children eg. 'when you multiply, add the powers of like terms' etc. Unfortunately most of the text books do this. This encourages instrumental learning which, although makes for easier teaching and better results in the short term, eventually leads to the child being bogged down with an endless number of rote-memorised meaningless rules for symbol manipulation.

The results of the test show that the majority of std 6 and 7 pupils in this study were at level 2 or below (86% and 62% respectively). Even in std 8 where pupils take mathematics by choice 38% were below level 3. This indicates (in Piagetian terms) that their ideas are still based firmly on concrete reality and any demand for abstraction or formulation of strategy is beyond the majority of them. In terms of algebra this means that most children cannot cope with the letter as an unknown or generalised number and tend to minimise the conceptual difficulty by treating the letter as an object or (worse) by simply ignoring it.

Teachers should be aware of the different ways in which children, text-books or they themselves use letters in school algebra. A child's confusion regarding the algebraic letter could be a direct result of the teacher's own poor conceptualisation of what he teaches. With this in mind, a case could be made for the better qualified more experienced teachers teaching the junior mathematics classes where the seeds of basic algebraic concepts are sown, rather than the senior classes, which is usually the case.

Coming back to the letter as object, the temptation to use it in this way is strong. For example, in questions involving the simplification of algebraic expressions by collecting terms, this strategy proves most successful, and it would seem silly to think of the letter in any other way. Much of the work done in algebra at the std 6 level involves simplifications of this nature, as this type of manipulation is vital in the intermediate steps of the solutions of higher order problems. However the relative ease with which children can manipulate letters as objects can have serious consequences, in that this work is often seen as the end in itself and not as the means towards the end, where letters must be understood at the higher levels of interpretation (specific unknown, generalised number or variable) in order for a problem to have any meaning at all for a child. Consider, for example, the case where a teacher or text book (or both) has drummed it into the children that just as apples and bananas are different, so are a and b different and cannot therefore be added. (The writer has noticed that a recently published textbook uses this type of explanation for not being able to express a + b in a simpler way). It is perhaps not surprising then, that in the test 83% of the std 6's, 70% of the std 7's and (surprisingly) 79% of the std 8's stated that L+M+N = L+P+N could never be true. Being aware of the ways in which the letters can be interpreted however, does not clarify how the children should be taught. Should children be encouraged to use the letter as object in algebraic tasks until the time arises where treating the letter in this way is inadequate, or should teachers focus exclusively on tasks where the result of using the letter as object is definitely wrong, as in the example above or in the blue and red pencils problem in the test? This latter approach would preclude much of the necessary ground work in algebra. Küchemann sums up this dilemma in the following way:

> If we cannot ignore letter as object, nor simply state that it is wrong, perhaps the best we can do is continually to seek out demonstrations of its inadequacy, in the hope that children will eventually see that it should be abandoned.
>
> (1982 p.51)

Another fact that emerges from this study is that many of the erroneous concepts, algorithms and methods of symbolisation are fairly widespread. This is probably indicative of children's abilities at certain stages of their conceptual development. Perhaps they are unable to grasp the level of abstraction of the work being presented to them and erroneously substitute their own idiosyncratic or child methods to ease the cognitive burden. For example, by treating the letter as object, or worse by ignoring or evaluating it, when it should be treated as an unknown; by writing terms together (conjoining) because of low tolerance of ALC; by not performing all the operations when two or more operations have to be co-ordinated (this often follows from using incorrect syntax eg. $5 \times e + 2$ instead of $5(e + 2)$); by determining perimeter when area is asked for etc. If teachers are aware of the types of errors that are likely to occur at the various stages of development this knowledge could be used to effect in their lessons.

A mistake which teachers often make is to treat all errors in the same way and to feel that a demonstration of the correct method is sufficient for all the children to understand. Rather, children should be encouraged to recognise their errors by examining their work critically and reflecting on the logic of their answers. One way of achieving this, perhaps, is not only to give children the correct solution of a problem, but also incorrect ones and to point out (or preferably get the children to point out) where the errors are. In Skemp's terminology this could encourage, not only relational understanding, but also a logical understanding of the work, eg. when dealing with the problem concerning the area of a rectangle with sides 5 and e + 2 respectively, the teacher should give all the following: $5 \times e + 2$, $e + 2 \times 5$, $5(e + 2)$, $5e + 10$, $5e + 2$ and ask the children which they think are the correct answers and why. Or he could give something like the following as a solution to an equation:

$$x + 5 = 8$$
$$= 8 - 5$$
$$= 3$$

and ask the pupils if they can see what is wrong with it.

When comparing the performances of the children in this study with those in the CSMS study, it was seen (from the interviews and responses of the written items) that basically children from both studies tended to behave in the same way. The trend was for the facilities to be of a similar order and for the same types of errors to be made. There were, however, some noticeable differences. The std. 8's tended on the whole to do better than the corresponding 4th year group in the CSMS study. This is probably because the std. 8's form a select group consisting of children who chose to take mathematics as a subject. On the other hand <u>all</u> the children in the 4th year group of the CSMS study had to take mathematics. It was also noticed that in examples involving symbol manipulation or mechanical skills, the children in this study tended to do better than their CSMS counterparts. This is more noticeable at the std 7/3rd year level than at the std6/2nd year level where, presumably, a minimum level of familiarity with generalised arithmetic has not yet been reached. On the other hand when it comes to a higher degree of abstraction, where letters must be understood as generalised numbers or variables (as in the more difficult level 4 items) it appears that the children in the CSMS study tend to do better than their counterparts in this study, even at the std 8/4th year level. For example only 19% of the std 8's said that L + M + N = L + P + N could be true when M = P, while 27% of the 4th years got the correct answer. On the other hand 79% of the std 8's said that the two expressions could never be equal while only 50% of the 4th years did so (a similar situation existed in the other year groups). It has already been suggested that a possible reason for this is that there is a greater tendency for South African schools (at least as far as the schools in this study are concerned) to focus on the development of mathematical skills and symbol manipulation to the neglect of developing children's mathematical powers and understanding of underlying principles and structure than there is for the English schools to do so. However this is purely conjecture and is based on the feeling that drill and discipline is more rigid in South African schools than it is in English schools.

Concerning the difference in performance between boys and girls in this study, it would seem that the boys tend to perform slightly better than the girls at the std 6 and std 7 levels where mathematics is a compulsory subject for all pupils. It was suggested, as a reason, that boys are possibly more motivated than girls because of their future careers. At the std 8 level where pupils take mathematics by choice, there is virtually no difference between the performances of boys and girls in the CSMS algebra test. It would be reasonable to expect this as, presumably, having chosen to take mathematics, boys and girls would be equally motivated.

The first aim of the Cape Junior Secondary course in mathematics is to develop a love for, an interest in and a positive attitude towards mathematics by presenting the subject meaningfully. Knowledge gained from this study about (1) children's conceptual difficulties in connection with letters in algebra and (2) the wide spread errors made and wrong methods used by school children could possibly provide teachers with greater insight and clarity on how to achieve this aim. Further, it is clear that the majority of std 6's and std 7's are still at the concrete operational stage (levels 1 and 2) while much of the mathematics they are expected to learn is of an abstract nature. In order for mathematics to be meaningful, then, teaching should be firmly rooted in the stage of concrete operations.

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APPENDIX I

CSMS ALGEBRA TEST AND MARKING SCHEME

(HART, 1980, pp. 174 - 180)

Algebra 1

Name .... *School* .... *Class* ......

Date .... *Date of Birth* .... (*day* *month* *year*)

Boy or Girl ....

Trial Item 1

What number does a + 4 stand for if a = 2 ....

What number does 4a stand for if a = 2 ....

Trial Item 2

Fill in the gaps:
(work down the page)

$x \longrightarrow 3x$	$x \longrightarrow x+3$	$x \longrightarrow 7x$	$x \longrightarrow x+8$
$2 \longrightarrow 6$	$5 \longrightarrow 8$	$2 \longrightarrow .$	$3 \longrightarrow .$
$5 \longrightarrow .$	$4 \longrightarrow .$		
	$n \longrightarrow .$		

Now check your answers against the answers on the back page.

---

1. Fill in the gaps:

$x \longrightarrow x + 2$	$x \longrightarrow 4x$
$6 \longrightarrow$. Var.1	$3 \longrightarrow$. Var.3
$r \longrightarrow$. Var.2	

---

2. Write down the smallest and the largest of these: smallest largest

n + 1, n + 4, n - 3, n, n - 7. .... Var.4 ....

---

3. Which is the larger, 2n or n + 2 ? Var.5 ....

Explain: ....

---

4. <u>4 added to n</u> can be written as <u>n + 4</u>. Add 4 onto each of these:

8 n + 5 3n

...... Var.6 ...... Var.7 ......

<u>n multiplied by 4</u> can be written as <u>4n</u>. Multiply each of these by 4:

8 n + 5 3n

...... Var.8 ...... Var.9 ......

---

5. If a + b = 43 — a + b + 2 = Var.10 ......

If n - 246 = 762 — n - 247 = Var.11 ......

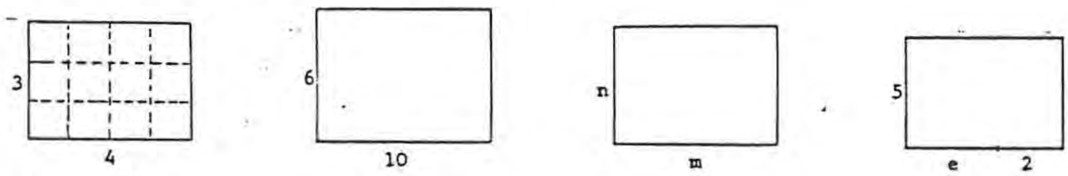
If e + f = 8 — e + f + g = Var.12 ......

6. What can you say about a if $a + 5 = 8$ ....Var.13..........

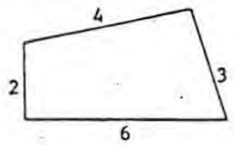
What can you say about b if $b + 2$ is equal to $2b$ ....Var.14..........

---

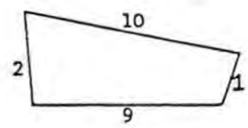
7. What are the areas of these shapes?



A = ...Var.15. A = ...Var.16.. A = ...Var.17.. A = .Var.18....

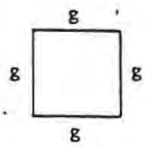


8. The perimeter of this shape is equal to $6 + 3 + 4 + 2$, which equals 15.



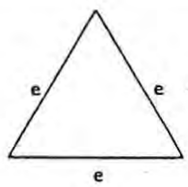
Work out the perimeter of this shape. p = ...Var.19......

9.

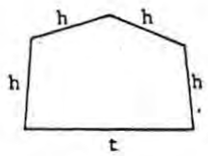


This square has sides of length g.
So, for its perimeter, we can write $p = 4g$.

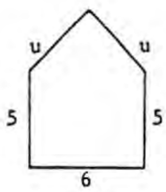
What can we write for the perimeter of each of these shapes?



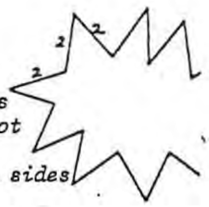
p = ...Var.20.....



p = ...Var.21.....



p = ...Var.22.....



Part of this figure is not drawn. There are n sides altogether, all of length 2.

p = ...Var.23.....

---

11. What can you say about u if $u = v + 3$ and $v = 1$ ....Var.26.......

What can you say about m if $m = 3n + 1$ and $n = 4$ ....Var.27............

12. If John has J marbles and Peter has P marbles, what could you write for the number of marbles they have altogether? ....Var.28......

---

13. a + 3a can be written more simply as 4a.

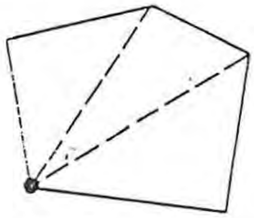
Write these more simply, where possible:

2a + 5a =	.Var.29........		
2a + 5b =	.Var.30........	3a − (b + a) =	..Var.34.......
(a + b) + a =	.Var.31........	a + 4 + a − 4 =	..Var.35.......
2a + 5b + a =	.Var.32........	3a − b + a =	..Var.36.......
(a − b) + b =	.Var.33........	(a + b) + (a − b) =	..Var.37.......

---

14. What can you say about r if r = s + t
and r + s + t = 30 ......Var.38......

15.



In a shape like this
you can work out the number of diagonals by
taking away 3 from the number of sides.

So, a shape with 5 sides has 2 diagonals;

a shape with 57 sides has ...Var.39.... diagonals;

a shape with k sides has ...Var.40.... diagonals.

---

16. What can you say about c if c + d = 10
and c is less than d ....Var.41.....

---

17. Mary's basic wage is R20 per week.
She is also paid another R2 for each hour of overtime that she works.

If h stands for the number of hours of overtime that she works, and
if W stands for her total wage (in Rands)
write down an equation connecting W and h: ....Var.42......

What would Mary's total wage be if she
worked 4 hours of overtime? ....Var.43..........

18. When are the following true -always, never, or sometimes?
Underline the correct answer:

$A + B + C = C + A + B$ Always. Never. Sometimes, when ...Var.44.......

$L + M + N = L + P + N$ Always. Never. Sometimes, when ...Var.45...

---

19. $a = b + 3$. What happens to a if b is increased by 2? ...Var.46...

$f = 3g + 1$. What happens to f if g is increased by 2? ...Var.47...

---

20. Cakes cost c cents each and buns cost b cents each.
If I buy 4 cakes and 3 buns,
what does $4c + 3b$ stand for? ...Var.48...

---

21. If this equation ⟶ $(x + 1)^3 + x = 349$
is true when $x = 6$,

then
what value of x
will make this equation ⟶ $(5x + 1)^3 + 5x = 349$
true?

Var.49

$x =$ ....

---

22. Blue pencils cost 5 cents each and red pencils cost 6 cents each.
I buy some blue and some red pencils and altogether it costs me 90 cents.

If b is the number of blue pencils bought, and
if r is the number of red pencils bought,
what can you write down about b and r? ...Var.50...

Trial Item
ANSWERS ⟶

What number does $a + 4$ stand for if $a = 2$ ...6...

What number does $4a$ stand for if $a = 2$ ...8...

$x \longrightarrow 3x$	$x \longrightarrow x+3$	$x \longrightarrow 7x$	$x \longrightarrow x+8$
$2 \longrightarrow 6$	$5 \longrightarrow 8$	$2 \longrightarrow 14$	$3 \longrightarrow 11$
$5 \longrightarrow 15$	$4 \longrightarrow 7$		
	$n \longrightarrow n+3$		

	1			2	3+	4+ CODES				5+			6		7+ AREA			
10	$x \to x+2$ 6→	r→	$x+4x$ 3→	SMALLEST/ LARGEST	WHICH IS LARGER? 2n OR n+2	ADD 4 ONTO n+5	3n	MULTIPLY BY 4 n+5	3n	a+b=43 a+b+2=	e+n=761 [illegible]	e+f=8 e+f+g=	a+5=8	b+2 IS EQUAL TO 2b	3□4	6□10	n□m	5□e+2
1 CORRECT	8	r+2	12	n+7, n+4	"DEPENDS" + AWARE OF AT LEAST TWO OF: n=2 n=1 n=0 n NEGATIVE [illegible]	n+9	3n+4	4(n+5) 4n+20	12n	45	761	8+g	3	2.	12	60	nm n×m	5(c+2) 5e+10
2 CORRECT WEAK					"DEPENDS" + ONLY ONE OF: n=2 n=1 n=0 n NEG												MISUNDERSTOOD BUT CONSISTENT WITH	
3 AMBIGUOUS [INADEQUATE PUNCTUATION, +, ×; NOT NECESSARILY WRONG]		r2 2r x+2		OTHER WAY ROUND n+4, n−7 [illegible]	"DEPENDS" OR "USUALLY 2n" BUT NONE OF: n=2, 1, 0, NEG	n+5+4 n+54 n9	3n+4 IFF "n+5+4" 3n4 n×3+4	4n+5 n+5×4 4n5 n54 n+54	3n×4 3n4 43n			8g		"b+2 = 2b"			n, m	5e+2 e+2×5 5e2 e25 MISUNDERSTOOD
4 AMBIGUOUS ASSOCIATION [WRONG PUNCTUATION, +, × BUT NOT WRONG OPERATION ON NUMBER]		r×2			2n	n×9 n+5×4	3n×4		12+n			8×g					n+m	5e×2
5 ASSOCIATION WITH LETTER [EXPLICIT WRONG OPERATION ON NO.]						n+20	7n 12n	n+20 n+9	7n									e+10 10e 7e
6 ASSOCIATION LETTER IGNORED		+2 2				9	7 12	20 9	7 12			9						10 7
7 SPECIFIC/ CONSISTENT [ALPHABET CODE]		t 20 [illegible] 21 22				23 22 24 19 w w+9 [illegible]	3r 3w	4s	3r			15 ONLY						35 (50)
8 SPECIFIC/ OBVIOUS (* ANY 'OTHER' [illegible])	12	*	7	*	n+2 "SAME"	*	*	*	*	*	763	12	*	b=1 b=0	*	*	12	25 20 30
9 OTHERS		INCLUDE 5		ONE OR MORE WRONG														
0 BLANK																		

MARKING SCHEME

8	9 PERIMETER: c, c, c	h, h, h, h, t	u, u, 5, 5, 6	n	10 CABBAGES AND TURNIPS 8c + 6t	No. of [illegible]	11		12 NUMBERS
22	3c e3	4h+t 4h+lt	2u+16 u^2+16 2^u+16	2n 2^n 2	EXPLICIT "COST OF C CABBAGES AND t TURNIPS"	c+t	4	13	J+P
	LONG: c+c+c	h+h+h+h+t	u+u+u+16 2u+2.5+6	n2 n^2 xy	LITERAL "8p×(C)abbages plus 6p×(t)turnips"				
	e.ee e,e,e (indexes)	hhhht 4ht	uu556 2u16	n, 2	IMPLICIT "(...) COST" "COST OF CABBAGES AND TURNIPS"	c,t ct	V= u−3	$\frac{n-1}{3}$	J,P JP J_r+P_r n_J+n_P JΛP
	cxexc	4hxt	2u×16	n+2	"THE NUMBER OF CABBAGES OR TURNIPS" (GENERAL)	c×t			J×P
		5ht	32u 18u 2u+150 2u+60	32n 32+n 17n 6n					
					1CAB, 1 TURNIP 8p+6p 14p	2			
	15 125			28 P	100 8CABS, 6 TURNIPS AND 100	100			26 27 25 Z
*	*	*	*	32+42 34 6 16 17	8CAB, 6TURNIPS (14 CABBAGES AND TURNIPS)	14	2	*	*
			ARITH ERROR	include 32+2n n×n nn	INCLUDE 14ct 14c+t 14c,t ct 8×c + 6×t				J=P

13

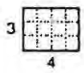
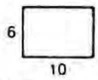
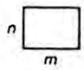
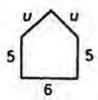
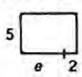
2a + 5a = 7a	
2a + 5b = 2a + 5b	3a − (b + a) = 2a − b
(a + b) + a = 2a + b	a + 4 + a − 4 = 2a
2a + 5b + a = 3a + 5b	3a − b + a = 4a − b
(a − b) + b = a	(a + b) + (a − b) = 2a

2a+5a	2a+5b	(a+b)+a	2a+5b+a	(a−b)+b	3a−(b+a)	a+4+a−4	3a−b+a	(a+b)+(a−b)
7a	2a+5b OR "IMPOSSIBLE" OR "BLANK" IF AT MOST ONE OTHER BLANK.	2a+b	3a+5b	a	2a−b	2a	4a−b 4^a-b	2a 2^a
								2a+b−b OR a+a
BRACKET EXPANDED: (EVEN IF WRONG) SIMPLIFIED BUT INADEQUATE NOTATION 'BLANK' IF ≥1 OTHER BLANK	a^2+ab a^2+b 2a+ab 2a,b 7ab a^2b	3a5b $2a^2+5b$	$ab-b^2$ ab−2b b^2a	$-3ab-3a^2$ $3a^2-b$ (2ab) 3ab−4a	a^2	4ab $3a^2-b$	a^2-b^2 a^2	
STRAIGHT FORWARD 2a×b		WRONGS	(OR × a−2b a+b	INSTEAD 4a−b	OF +) 2a+8 0 a	2a−b 3a−b	a 2a+2b 2a−2b 2a+b 0	
ASSOCIATIONS (ab) 3ab ABSORPTIONS (a) 4a REPLACEMENTS (c) 2c 8a : 7ab 12a		8ab	3ab ab b	5ab 2b		5ab 2ab 2a		
*	*	*	*	*	*	*	*	*
		INCLUDE ITEM WRITTEN DOWN UNCHANGED (with or without brackets) OR NOT COMPLETELY SIMPLIFIED ab+a aba		ab+b a−bb		2a+4−4.		
	'BLANK' IF ALL OTHERS BLANK (other than 2a+5a=7a)							

14 $r=s+t$, $r+s+t=30$	15 SIDES→ DIAGONALS 57→	k→	16 $c+d=10$ c IS LESS THAN d	17 WAGES	4 HOURS	18 $A+B+C = C+A+B$	$L+M+N = L+P+N$	19 $a=b+3$ $\Delta b=2$ $\Delta a=?$	$f=3g+1$ $\Delta g=2$ $\Delta f=?$	20 CAKES AND BUNS $4c+3b$	21 $x = \frac{6}{5}$	22 BLUE AND RED PENCILS	MACHINE	
$r=15$	54	$k-3$	$c<5$ $(c>5)$	$W=20+2h$ $w = 20 + 2\times h$	28	ALWAYS OR ALWAYS/SOMETIMES WHEN 'GENERAL' "POSITIVE"	SOMETIMES (ALWAYS) WHEN $M=P$	$\Delta a=2$	$\Delta f=6$	EXPLICIT "TOTAL COST" "COST OF 4 CKS AND 3 BUNS"	$\frac{6}{5}$ 1·2	$5b+6r=90$	X5 +50	1 CORRECT
		WRONG BUT CONSISTENT WITH ($k-2$ ONLY)	$c\le5$ > $c\le4$ $c<4$ $c<6$ SYSTEMATIC LIST (4 VALUES) 1–4 1234 0.1234(5)	TWO NUMERICAL PAIRS OF VALUES (CORRECT) $W=28$ $h=4$ $W=30$, $h=5$ etc OR 28,4 30,5 etc				STATIC– $a+2=b+5$ $a+2=b+3$ $a=b+1$ $a+2$.	"CORRECT" $f+6$ = [illegible] $f=3(g+1)+1$ $f=3g+7$ $f=3g-5$ $f+6$.	LITERAL "4 CAKES AT c PENCE AND 3 BUNS AT b PENCE"		TWO NUMERICAL PAIRS OF VALUES (CORRECT) FROM 6,10 12,5 18,0 0,15 OR $b\le18$, $r\le15$		2 CORRECT WEAK (SEVERAL CORRECT NUMERICAL ANSWERS)
RE-WRITE $r=30-s-t$ $r=30-r$ "$r<30$"		$-3k$ (−2) $3-k$	$C=10-d$	"$W+2h$ = wages" $W+2h$ $20+2h$ EXPRESSIONS INVOLVING "n_w, n_h" (CORRECT). AMBIGUOUS PUNCT. eg $W=20+h\times2$	.	ALWAYS/SOMETIMES WHEN 'SPECIFIC' eg $A=1$ $B=2$ $C=3$ SOMETIMES	SOMETIMES 'ALWAYS' (+ [illegible]) SOMETIMES WHEN M = NUMBER	"NO CHANGE" (NOT AMBIGUOUS eg	"ACTUALLY") PUNCTUATION $f=3g+2+1$ [illegible]	IMPLICIT "COST" "AMOUNT"	$1<x<2$ $x=5/6$ (NOT $x=1$ $x=2$)	NO CORRECT ANSWERS BUT CALCULATION SHOWN $b+r$ = NUMBER OF PENCILS EXPRESSIONS INVOLVING n_b, n_r (CORRECT)	"NO"	3 AMBIGUOUS
	.	k3 (k2) 3k (2k) $k+3$ ($k+2$) $k\times3$ ($k\times2$)	UNSYSTEMATIC 2,1,3,4, etc OR LESS THAN 4 VALUES (MORE THAN ONE)	$W=20+h$ $20+h$ w = wage + h TOTAL WAGE = $W+h$ $h\to W$ $W=W+h$				"INCREASES" OR "INCREASES BY..." eg "f increases by 2" $f+2$		"NUMBER OF CAKES & BUNS" (GENERAL)		$b+r$ = number of pencils $b+r=90$ or $=90$ $b+r\le90$		4 LETTER AS OBJECT GENERAL
		.	.	$W+h$. wh $w\cap h$				STATIC–INCORRECT $a=b+5$ $a=2b+3$ $a=3b+3$ ALL "OTHER" EXPRESSIONS IN b	$f=3g+3$ $f=3g-1$ $f=5g+\ldots$ $f=6g\ldots$			$b\cdot r$		5 LETTER AS OBJECT / ASSOCIATIVE GENERAL
		-3 (−2) 3 (2)	$C=4$ 5 3 2 1 0	ONE PAIR OF VALUES (CORRECT) $W=28$ $h=4$ etc OR 28,4 etc				$a=2$	$f=6$	(1 bun, 1 cake) 4p+3p 7p		ONE PAIR OF VALUES (CORRECT) $b=6$, $r=10$ etc [EXCEPT "6, 10"]		6 ONE CORRECT NUMERICAL ANSWER
		$k\to h$ (9) i (h) ~ 8 (7) 9 (8)		$2h=24W$ $4h=28W$, etc OR $24=W+2h$ $28=W+4h$ etc (NUMBERS CORRECT)				$(a\to c)$ etc	$(f\to l)$ etc	25p (NOT ×18p)		$6b+10r$ etc (NUMBERS CORRECT) $b=30$ $r=60$ etc (6 BLUE + 10 RED)	CONSISTENT 10 NOT 5 NOT ANY 5 NUMBER	7 LETTER AS OBJECT, SPECIFIC/[illegible] CONSISTENT
10	*	*	* (ALL ARITH ERRORS OTHER THAN C=5)	$W=20$, $h=2$ OR $20W+2h$ OR $W=10h$, etc (IMPLIES $W=20$, $h=2$) £22 $22=W+h$	*	.		$a=5$	$f=7$	4 CAKES, 3 BUNS 7 cakes	30	$5b+6r$ $b=5$, $r=6$ $b<r$ $b+r=11$ $b+r<r$ etc	5 10 OR 10 5	8 LETTER AS OBJECT, SPECIFIC/[illegible] OBVIOUS
		k	INCLUDE $c<d$	INCLUDE ALL ARITH ERRORS WITH $w=2h$ eg $W=28$ $h=8$ $W+h>20$ $28=W+h$ $2h=4W$ $28W+8h$ [illegible]		NEVER	NEVER	"CHANGES" $a+b+3$ $a<$ expression	$f+3g+1$ $f<$ expression	INCLUDE 7cb 7c,b 7c+b etc 4xc + 3xb		(LINE + PENCILS [illegible]) INCLUDE ALL ARITHMETICAL ERRORS eg $b=10$ $r=6$	INCLUDE 5 NOT 50 5 NOT 10	9 OTHERS
														0 BLANK

APPENDIX 2

Extract from Bell, Costello and Küchemann (1983, p. 135) illustrating the approximate Piagetian sub-stages relative to types of items and interpretation of the algebraic letter.

Piagetian Sub-Stage	Item	Question Number on test	% Correct	EVALUATED	IGNORED	OBJECT	SPECIFIC UNK	GNRLSD NMBR	VARIABLE	QUESTION	Common Wrong Answers	%
Early Concrete	A	5i	97		x					If $a + b = 43$, $a + b + 2 =$		
	B	9i	94			x				 $p =$		
	C	6i	92	x						If $a + 5 = 8$, $a =$		
	D	7i	91							 $A =$		
	E	7ii	89							 $A =$		
	F	1i	88							If $x \rightarrow x + 2$, $6 \rightarrow$		
Late Concrete	G	5ii	74		x					If $n - 246 = 762$, $n - 247 =$	763	13
	H	2	72		x					smallest, largest of: $n + 1$, $n + 4$, $n - 3$, n, $n - 7$		
	I	4i	68		x					Add 4 onto $n + 5$	9	20
	J	7iii	68			x				 $A =$		
	K	9ii	68			x				 $p =$	$p = 4ht$ or $hhhht$	20
	L	9iii	64			x				 $p =$	$p = 2u16$ or $uu556$	16
	M	11ii	62	x						If $m = 3n + 1$ and $n = 4$, $m =$		
	N	11i	61	x						If $u = v + 3$ and $v = 1$, $u =$	2	14
Early Formal	O	5iii	41				x			If $e + f = 8$, $e + f + g =$	12	26
	P	14	41				x			If $r = s + t$ and $r + s + t = 30$, $r =$	10	21
	Q	9iv	38				x			n-sided polygon, each side of length 2; $p =$	36, 38, etc.	18
	R	4ii	36				x			Add 4 onto $3n$	$7n$	31
	S	16	30					x		What can you say about c if $c + d = 10$ and c is less than d	4 only	39
	T	18ii	25					x		Is $L + M + N = L + P + N$ always, sometimes or never true?	never	51
	U	20	22				x			Cakes cost c pence each and buns cost b pence each. If I buy 4 cakes and 3 buns, what does $4c + 3b$ stand for?	4 cakes and 3 buns	39
Late Formal	V	4iii	17				x			Multiply $n + 5$ by 4.	$n + 20$	31
	W	7iv	12				x			 $A =$	$e + 10$, $10e$, $7e$	28
	X	22	11				x			"blue pencils and red pencils" (see text).	$b + r = 90$	17
	Y	17i	5				x			Mary's basic wage is £20 per week. She is also paid another £2 for each hour of overtime that she works. If h stands for the number of hours of overtime that she works, and if W stands for her *total* wage (in £'s) write down an equation connecting W and h:................	$W + h$ or $W = 20 + h$	27
	Z	3	6						x	Which is larger, $2n$ or $n + 2$? Explain.	$2n$	71

APPENDIX 3

CODE FREQUENCY TABLES

ALGEBRA - ANALYSIS FOR STANDARD 6

NUMBER OF PUPILS = 218

VAR	1	2	3	4	5	6	7	8	9	0
1	69.7	1.8	0.5	0.0	0.0	0.0	0.0	6.4	17.0	4.6
2	70.2	0.0	2.8	1.4	0.0	2.3	0.0	3.2	11.5	8.7
3	59.6	0.0	0.5	0.0	0.0	0.5	0.0	12.8	21.6	5.0
4	63.8	0.0	1.4	0.5	0.0	0.0	0.0	0.5	33.5	0.5
5	0.5	0.5	0.9	65.6	0.0	0.0	0.0	28.0	0.5	4.1
6	60.6	0.5	20.6	0.9	0.0	6.9	0.0	3.2	6.4	0.9
7	17.9	0.0	7.8	0.0	60.6	5.5	0.0	1.4	5.0	1.8
8	9.2	0.0	12.4	0.0	56.9	4.6	0.0	2.3	12.8	1.8
9	66.1	0.9	12.4	0.0	2.8	4.6	0.0	0.9	10.6	1.8
10	93.1	0.0	0.0	0.0	0.0	0.0	0.0	5.5	0.5	0.9
11	70.2	0.0	0.0	0.0	0.0	0.0	0.0	11.9	15.6	2.3
12	35.3	0.0	10.6	0.0	0.0	17.9	2.8	21.6	7.3	4.6
13	93.6	0.5	0.9	0.0	0.0	0.0	0.0	2.3	1.8	0.9
14	10.6	0.0	5.0	0.0	0.0	0.0	0.0	11.0	49.1	24.3
15	78.4	0.0	0.0	0.0	0.0	0.0	0.0	19.7	0.0	1.8
16	77.1	0.0	0.0	0.0	0.0	0.0	0.0	20.6	0.0	2.3
17	74.3	0.0	0.5	3.7	0.0	0.0	0.0	0.0	19.3	2.3
18	5.0	0.5	11.0	1.4	57.3	6.0	0.9	2.8	7.3	7.8
19	95.4	0.9	0.0	0.0	0.0	0.0	0.0	2.8	0.0	0.9
20	95.9	0.0	0.5	0.9	0.0	0.0	0.0	0.5	0.5	1.8
21	53.7	0.0	33.9	5.5	2.3	0.0	0.0	0.5	2.3	1.8
22	23.9	27.1	27.1	3.7	6.9	0.0	0.0	0.9	6.9	3.7
23	22.5	6.4	0.9	0.9	10.6	0.0	0.9	40.4	4.1	13.3
24	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
25	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
26	77.1	1.4	0.0	0.0	0.0	0.5	0.0	6.4	9.6	5.0
27	59.6	0.9	0.0	0.0	0.0	0.0	0.0	21.6	8.3	9.6
28	60.6	0.0	28.0	1.8	0.0	0.0	0.5	0.5	3.2	5.5
29	75.7	0.0	0.0	15.1	6.9	0.0	0.0	0.9	0.9	0.5
30	41.3	0.0	0.9	2.8	53.7	0.0	0.0	0.0	0.0	1.4
31	30.3	0.0	51.4	2.3	7.3	0.5	0.0	0.0	5.0	3.2
32	50.0	0.0	7.8	15.6	25.2	0.0	0.0	0.0	0.0	1.4
33	14.2	0.0	15.6	24.3	25.2	0.0	0.0	1.4	8.3	11.0
34	6.4	0.0	14.7	32.6	23.9	0.0	0.0	3.7	10.1	8.7
35	34.4	0.0	26.1	22.0	5.0	0.0	0.0	7.3	2.8	2.3
36	27.1	0.5	12.8	17.4	22.5	0.5	0.0	1.8	7.3	10.1
37	4.1	0.5	9.6	41.7	17.4	0.0	0.0	0.9	15.6	10.1
38	23.4	0.0	1.8	0.0	0.0	0.0	0.0	19.7	33.0	22.0
39	61.9	0.0	0.0	0.0	0.0	0.0	0.0	31.2	0.5	6.4
40	34.9	1.4	3.2	2.8	0.0	0.0	11.9	5.0	31.7	9.2
41	2.3	19.7	0.5	0.0	0.0	52.8	0.0	4.6	6.9	13.3
42	2.3	0.0	1.8	9.6	28.4	0.0	0.5	18.3	17.4	21.6
43	69.7	0.0	0.0	0.0	0.0	0.0	0.0	19.7	5.0	5.5
44	86.2	0.0	5.5	0.0	0.0	0.0	0.0	0.0	4.1	4.1
45	5.0	0.0	8.7	0.0	0.0	0.0	0.0	0.0	83.0	3.2
46	17.4	2.8	3.2	9.2	11.9	0.5	1.8	14.7	28.9	9.6
47	1.4	0.5	3.2	27.1	15.1	2.8	1.8	8.3	26.1	13.8
48	8.7	4.1	9.6	3.7	0.0	1.4	0.5	27.1	35.3	9.6
49	0.5	0.0	2.3	0.0	0.0	0.0	0.0	1.8	58.7	36.7
50	1.4	0.9	1.8	18.3	6.0	3.7	2.3	5.0	38.1	22.5

CSMS

TEST 1 YEAR 2 NUMBER OF CHILDREN = 1128 ALGEBRA 1976

VAR	1	2	3	4	5	6	7	8	9	0
1	83.2	0	0	0	0	0	0	5.2	5.8	5.8
2	64.7	0	3.9	.3	0	7.2	2.5	5.4	5.7	10.4
3	74.4	0	.1	0	0	0	0	7.9	9.8	7.8
4	66.3	0	1.6	0	0	0	0	3.6	26.1	2.4
5	2.1	1.9	1.5	62.1	0	0	.1	24.1	1.0	7.3
6	61.3	0	5.9	.2	1.2	20.0	.4	2.8	4.0	4.2
7	21.7	0	3.9	.2	40.8	17.3	0	5.3	5.2	5.6
8	7.6	0	11.5	.1	38.9	16.4	0	6.7	8.5	10.2
9	39.2	0	6.2	.5	11.3	16.0	0	6.9	8.5	11.3
10	92.1	0	0	0	0	0	0	3.6	.5	3.7
11	58.9	0	.1	0	0	0	0	22.3	9.7	9.0
12	24.7	0	5.7	0	0	10.2	4.1	28.7	13.7	12.9
13	85.5	0	.1	0	0	0	0	1.6	1.6	11.3
14	29.1	0	2.7	0	0	0	0	10.7	27.0	30.6
15	80.8	0	.4	0	0	0	0	12.1	.4	6.4
16	78.9	0	.4	0	0	0	0	13.8	.4	6.5
17	51.3	3.0	1.0	4.8	.1	0	.3	6.4	13.9	19.2
18	7.0	.1	14.5	3.6	27.2	9.5	1.1	6.0	12.1	18.9
19	95.3	.1	0	0	0	0	0	1.7	0	2.9
20	90.3	.3	.2	0	0	0	.9	2.2	1.9	4.3
21	57.4	.4	22.3	1.0	3.5	0	0	2.7	7.4	5.1
22	27.7	25.9	20.4	.8	2.0	0	0	4.3	11.0	8.0
23	9.3	14.5	.8	.4	3.3	.1	2.7	25.4	20.3	23.1
24	0	1.2	2.5	1.5	0	5.0	20.1	48.6	11.9	9.3
25	2.2	0	.4	0	0	6.1	9.1	64.7	7.9	9.5
26	49.3	.1	.4	0	0	0	0	15.2	13.6	21.5
27	43.5	.1	0	0	0	0	0	22.3	7.0	27.0
28	47.3	0	17.8	1.7	0	0	5.7	6.7	6.2	14.6
29	77.4	0	.1	.7	9.5	0	0	4.7	.4	7.2
30	29.3	0	6.6	1.1	45.3	.1	0	5.5	.7	11.3
31	37.7	0	9.0	1.1	18.7	0	0	7.4	4.7	21.5
32	40.2	0	3.2	1.3	26.4	0	0	7.0	2.0	19.9
33	15.2	0	3.4	25.4	19.7	0	0	6.8	4.3	25.3
34	10.6	0	3.1	19.9	21.9	0	0	7.6	5.0	31.8
35	29.6	.4	2.6	16.3	11.2	0	0	9.4	5.4	25.2
36	26.5	0	1.7	6.4	23.5	0	0	8.4	2.5	31.0
37	11.4	1.3	2.8	28.5	9.9	0	0	6.9	5.0	34.0
38	27.4	0	2.2	0	0	0	0	24.1	17.7	28.5
39	62.6	0	0	0	0	0	0	22.6	0	14.8
40	33.2	1.0	2.0	.8	.1	2.6	14.2	10.5	13.7	22.1
41	7.1	13.9	1.2	1.2	0	42.6	0	2.0	9.1	23.0
42	2.0	0	1.1	5.5	12.5	.2	.8	13.6	21.1	43.3
43	65.0	0	0	0	0	0	0	15.6	3.6	15.8
44	58.4	0	14.3	0	0	0	0	0	10.5	16.8
45	10.5	0	16.2	0	0	0	0	0	55.5	17.7
46	9.4	10.6	.9	4.9	22.1	.7	1.9	8.1	15.1	26.4
47	3.0	1.4	.5	9.7	29.4	2.2	1.3	1.7	14.3	36.4
48	3.5	3.8	6.3	2.3	0	2.1	1.2	35.5	19.0	26.4
49	4.3	.2	1.1	0	0	0	0	1.7	33.5	59.3
50	2.2	.2	.6	10.6	2.0	3.3	5.6	4.4	27.0	44.1

ALGEBRA - ANALYSIS FOR STANDARD 6

NUMBER OF BOYS = 100

VAR	1	2	3	4	5	6	7	8	9	0
1	67.0	1.0	0.0	0.0	0.0	0.0	0.0	6.0	19.0	7.0
2	69.0	0.0	2.0	2.0	0.0	0.0	0.0	4.0	10.0	13.0
3	64.0	0.0	0.0	0.0	0.0	1.0	0.0	12.0	15.0	8.0
4	74.0	0.0	3.0	1.0	0.0	0.0	0.0	0.0	22.0	0.0
5	1.0	0.0	2.0	66.0	0.0	0.0	0.0	29.0	0.0	2.0
6	60.0	0.0	21.0	1.0	0.0	10.0	0.0	2.0	5.0	1.0
7	18.0	0.0	13.0	0.0	56.0	6.0	0.0	1.0	5.0	1.0
8	13.0	0.0	9.0	0.0	50.0	7.0	0.0	3.0	17.0	1.0
9	63.0	2.0	11.0	0.0	4.0	6.0	0.0	2.0	11.0	1.0
10	94.0	0.0	0.0	0.0	0.0	0.0	0.0	4.0	1.0	1.0
11	78.0	0.0	0.0	0.0	0.0	0.0	0.0	7.0	12.0	3.0
12	40.0	0.0	7.0	0.0	0.0	21.0	6.0	14.0	9.0	3.0
13	97.0	0.0	0.0	0.0	0.0	0.0	0.0	2.0	0.0	1.0
14	14.0	0.0	4.0	0.0	0.0	0.0	0.0	14.0	43.0	25.0
15	85.0	0.0	0.0	0.0	0.0	0.0	0.0	14.0	0.0	1.0
16	82.0	0.0	0.0	0.0	0.0	0.0	0.0	16.0	0.0	2.0
17	77.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	21.0	1.0
18	6.0	0.0	12.0	2.0	58.0	7.0	2.0	3.0	6.0	4.0
19	93.0	0.0	0.0	0.0	0.0	0.0	0.0	6.0	0.0	1.0
20	96.0	0.0	0.0	1.0	0.0	0.0	0.0	1.0	0.0	2.0
21	59.0	0.0	28.0	7.0	2.0	0.0	0.0	1.0	2.0	1.0
22	31.0	30.0	21.0	5.0	4.0	0.0	0.0	1.0	6.0	2.0
23	23.0	5.0	1.0	1.0	10.0	0.0	1.0	41.0	5.0	13.0
24	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
25	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
26	82.0	0.0	0.0	0.0	0.0	1.0	0.0	6.0	6.0	5.0
27	67.0	0.0	0.0	0.0	0.0	0.0	0.0	20.0	4.0	9.0
28	60.0	0.0	25.0	2.0	0.0	0.0	0.0	1.0	5.0	7.0
29	75.0	0.0	0.0	13.0	9.0	0.0	0.0	2.0	0.0	1.0
30	42.0	0.0	1.0	2.0	53.0	0.0	0.0	0.0	0.0	2.0
31	41.0	0.0	45.0	0.0	8.0	0.0	0.0	0.0	3.0	3.0
32	52.0	0.0	6.0	9.0	30.0	0.0	0.0	0.0	0.0	3.0
33	20.0	0.0	12.0	32.0	22.0	0.0	0.0	2.0	4.0	8.0
34	3.0	0.0	12.0	40.0	22.0	0.0	0.0	6.0	9.0	8.0
35	33.0	0.0	19.0	31.0	4.0	0.0	0.0	4.0	5.0	4.0
36	32.0	1.0	11.0	21.0	18.0	0.0	0.0	3.0	6.0	8.0
37	8.0	1.0	6.0	47.0	16.0	0.0	0.0	2.0	12.0	8.0
38	35.0	0.0	1.0	0.0	0.0	0.0	0.0	17.0	30.0	17.0
39	60.0	0.0	0.0	0.0	0.0	0.0	0.0	35.0	0.0	5.0
40	34.0	1.0	3.0	1.0	0.0	0.0	16.0	5.0	32.0	8.0
41	4.0	31.0	0.0	0.0	0.0	49.0	0.0	3.0	5.0	8.0
42	3.0	0.0	2.0	9.0	31.0	0.0	0.0	20.0	16.0	19.0
43	76.0	0.0	0.0	0.0	0.0	0.0	0.0	16.0	3.0	5.0
44	90.0	0.0	4.0	0.0	0.0	0.0	0.0	0.0	2.0	4.0
45	5.0	0.0	9.0	0.0	0.0	0.0	0.0	0.0	84.0	2.0
46	19.0	4.0	3.0	12.0	11.0	1.0	3.0	12.0	28.0	7.0
47	2.0	1.0	4.0	31.0	13.0	0.0	2.0	7.0	27.0	13.0
48	6.0	5.0	12.0	3.0	0.0	1.0	1.0	33.0	34.0	5.0
49	1.0	0.0	4.0	0.0	0.0	0.0	0.0	2.0	61.0	32.0
50	2.0	1.0	0.0	17.0	4.0	6.0	4.0	5.0	41.0	20.0

ALGEBRA - ANALYSIS FOR STANDARD 6

NUMBER OF GIRLS = 104

VAR	1	2	3	4	5	6	7	8	9	0
1	69.2	2.9	1.0	0.0	0.0	0.0	0.0	6.7	17.3	2.9
2	70.2	0.0	2.9	1.0	0.0	3.8	0.0	1.9	14.4	5.8
3	52.9	0.0	1.0	0.0	0.0	0.0	0.0	13.5	30.8	1.9
4	50.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	48.1	1.0
5	0.0	1.0	0.0	65.4	0.0	0.0	0.0	26.9	1.0	5.8
6	62.5	1.0	18.3	1.0	0.0	4.8	0.0	3.8	7.7	1.0
7	17.3	0.0	2.9	0.0	64.4	4.8	0.0	1.9	5.8	2.9
8	6.7	0.0	15.4	0.0	62.5	2.9	0.0	1.0	9.6	1.9
9	69.2	0.0	12.5	0.0	1.0	2.9	0.0	0.0	11.5	2.9
10	91.3	0.0	0.0	0.0	0.0	0.0	0.0	7.7	0.0	1.0
11	64.4	0.0	0.0	0.0	0.0	0.0	0.0	16.3	17.3	1.9
12	30.8	0.0	13.5	0.0	0.0	16.3	0.0	27.9	5.8	5.8
13	90.4	1.0	1.9	0.0	0.0	0.0	0.0	2.9	2.9	1.0
14	8.7	0.0	6.7	0.0	0.0	0.0	0.0	8.7	51.0	25.0
15	75.0	0.0	0.0	0.0	0.0	0.0	0.0	22.1	0.0	2.9
16	75.0	0.0	0.0	0.0	0.0	0.0	0.0	22.1	0.0	2.9
17	73.1	0.0	1.0	6.7	0.0	0.0	0.0	0.0	15.4	3.8
18	4.8	1.0	7.7	1.0	58.7	4.8	0.0	2.9	6.7	12.5
19	97.1	1.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0
20	95.2	0.0	1.0	1.0	0.0	0.0	0.0	0.0	1.0	1.9
21	50.0	0.0	39.4	4.8	1.9	0.0	0.0	0.0	1.0	2.9
22	20.2	21.2	33.7	2.9	7.7	0.0	0.0	1.0	8.7	4.8
23	23.1	7.7	1.0	1.0	10.6	0.0	1.0	39.4	2.9	13.5
24	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
25	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
26	73.1	2.9	0.0	0.0	0.0	0.0	0.0	6.7	12.5	4.8
27	52.9	1.9	0.0	0.0	0.0	0.0	0.0	23.1	11.5	10.6
28	60.6	0.0	31.7	1.9	0.0	0.0	1.0	0.0	1.0	3.8
29	75.0	0.0	0.0	19.2	4.8	0.0	0.0	0.0	1.0	0.0
30	42.3	0.0	1.0	2.9	52.9	0.0	0.0	0.0	0.0	1.0
31	23.1	0.0	54.8	4.8	5.8	1.0	0.0	0.0	6.7	3.8
32	50.0	0.0	9.6	22.1	18.3	0.0	0.0	0.0	0.0	0.0
33	8.7	0.0	20.2	19.2	25.0	0.0	0.0	1.0	11.5	14.4
34	8.7	0.0	16.3	27.9	23.1	0.0	0.0	1.9	12.5	9.6
35	29.8	0.0	34.6	16.3	6.7	0.0	0.0	10.6	1.0	1.0
36	23.1	0.0	15.4	16.3	23.1	1.0	0.0	1.0	8.7	11.5
37	0.0	0.0	13.5	35.6	19.2	0.0	0.0	0.0	20.2	11.5
38	15.4	0.0	2.9	0.0	0.0	0.0	0.0	24.0	31.7	26.0
39	61.5	0.0	0.0	0.0	0.0	0.0	0.0	28.8	1.0	8.7
40	31.7	1.9	3.8	4.8	0.0	0.0	9.6	4.8	31.7	11.5
41	1.0	11.5	0.0	0.0	0.0	55.8	0.0	6.7	9.6	15.4
42	1.9	0.0	1.9	8.7	27.9	0.0	1.0	17.3	18.3	23.1
43	61.5	0.0	0.0	0.0	0.0	0.0	0.0	25.0	7.7	5.8
44	81.7	0.0	7.7	0.0	0.0	0.0	0.0	0.0	5.8	4.8
45	4.8	0.0	8.7	0.0	0.0	0.0	0.0	0.0	81.7	4.8
46	16.3	1.0	2.9	4.8	9.6	0.0	1.0	18.3	32.7	13.5
47	1.0	0.0	1.9	21.2	15.4	5.8	1.9	10.6	26.9	15.4
48	12.5	2.9	6.7	3.8	0.0	1.9	0.0	22.1	37.5	12.5
49	0.0	0.0	1.0	0.0	0.0	0.0	0.0	1.0	57.7	40.4
50	0.0	1.0	1.0	20.2	7.7	1.9	1.0	4.8	39.4	23.1

ALGEBRA - ANALYSIS FOR STANDARD 7

NUMBER OF PUPILS = 205

VAR	1	2	3	4	5	6	7	8	9	0
1	72.7	2.4	0.0	0.0	0.0	0.0	0.0	8.3	13.2	3.4
2	76.1	0.0	2.9	0.5	0.0	4.9	0.5	2.0	9.8	3.4
3	67.8	1.0	0.5	0.0	0.0	0.0	0.0	12.2	14.1	4.4
4	80.0	0.0	1.5	0.5	0.0	0.0	0.0	0.0	17.1	1.0
5	2.9	2.0	1.0	66.3	0.0	0.0	0.0	23.9	2.0	2.0
6	68.8	0.0	15.1	0.5	0.0	6.8	0.0	2.0	6.3	0.5
7	33.7	0.0	4.9	0.0	50.2	6.3	0.0	1.5	2.9	0.5
8	25.9	0.0	11.2	0.0	48.8	6.8	0.0	1.5	4.4	1.5
9	71.7	0.0	7.3	2.0	3.4	6.3	0.0	0.5	6.8	2.0
10	95.1	0.0	0.0	0.0	0.0	0.0	0.0	4.4	0.0	0.5
11	68.3	0.0	0.0	0.0	0.0	0.0	0.0	17.6	13.2	1.0
12	44.9	0.0	10.2	0.0	0.0	10.7	0.0	23.4	7.8	2.9
13	96.1	0.0	0.0	0.0	0.0	0.0	0.0	1.5	1.0	1.5
14	22.4	0.0	6.3	0.0	0.0	0.0	0.0	9.8	40.0	21.5
15	77.6	0.0	0.0	0.0	0.0	0.0	0.0	21.4	0.0	1.0
16	79.5	0.0	0.0	0.0	0.0	0.0	0.0	20.5	0.0	0.0
17	81.0	1.0	0.5	2.4	0.0	0.0	0.0	0.5	14.6	0.0
18	14.1	0.0	4.4	2.4	60.5	3.4	0.0	2.0	10.2	2.9
19	98.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	1.0
20	99.0	0.0	0.0	0.0	0.0	0.0	0.0	0.5	0.5	0.0
21	65.9	0.0	24.4	5.9	1.0	0.0	0.0	0.0	2.9	0.0
22	46.3	16.1	16.6	4.4	7.3	0.0	0.0	1.5	7.3	0.5
23	33.2	6.8	0.5	2.0	4.4	0.0	0.5	38.5	8.8	5.4
24	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
25	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
26	82.0	1.0	0.0	0.0	0.0	0.0	0.0	6.8	6.8	3.4
27	66.3	0.0	0.0	0.0	0.0	0.0	0.0	18.5	9.3	5.9
28	59.0	0.0	30.7	2.9	0.0	0.0	0.0	1.0	2.4	3.9
29	90.2	0.0	0.5	5.4	3.9	0.0	0.0	0.0	0.0	0.0
30	65.4	0.0	2.0	0.5	32.2	0.0	0.0	0.0	0.0	0.0
31	48.8	0.0	39.5	2.9	2.4	0.0	0.0	0.0	5.9	0.5
32	77.1	0.0	5.4	6.3	10.7	0.0	0.0	0.0	0.5	0.0
33	34.6	0.0	20.5	23.9	10.7	0.0	0.0	0.0	6.8	3.4
34	30.2	0.0	18.5	31.2	8.3	0.0	0.0	1.0	5.9	4.9
35	62.9	0.5	12.2	17.1	2.0	0.0	0.0	2.0	2.4	1.0
36	68.8	0.0	6.8	12.2	4.9	0.0	0.0	0.0	5.9	1.5
37	30.7	1.0	20.5	33.7	4.9	0.0	0.0	0.5	6.3	2.4
38	38.0	0.5	6.8	0.0	0.0	0.0	0.0	20.5	24.4	9.8
39	74.1	0.0	0.0	0.0	0.0	0.0	0.0	21.5	1.0	3.4
40	51.7	2.4	5.9	2.9	0.0	1.5	6.3	1.5	23.4	4.4
41	10.7	22.0	2.9	0.5	0.0	41.5	0.0	2.4	11.7	8.3
42	6.8	0.0	2.4	19.5	18.5	0.0	1.5	16.6	23.4	11.2
43	76.6	0.0	0.0	0.0	0.0	0.0	0.0	18.5	3.4	1.5
44	93.2	0.5	3.9	0.0	0.0	0.0	0.0	0.0	2.4	0.0
45	15.6	0.5	13.7	0.0	0.0	0.0	0.0	0.0	70.2	0.0
46	29.8	6.3	1.5	10.2	13.2	0.5	0.0	14.1	21.0	3.4
47	4.4	1.0	1.0	31.7	15.6	2.0	0.5	10.7	25.4	7.8
48	19.0	3.9	12.7	6.8	0.0	1.0	0.0	23.9	26.8	5.9
49	9.8	0.0	2.0	0.0	0.0	0.0	0.0	3.4	53.7	31.2
50	7.3	1.0	0.5	32.7	0.5	3.4	2.0	4.9	33.7	14.1

CSMS

TEST 1 YEAR 3 NUMBER OF CHILDREN = 961 ALGEBRA 1976

VAR	1	2	3	4	5	6	7	8	9	0
1	87.9	.1	0	0	0	0	0	3.9	5.0	3.1
2	75.0	0	1.0	.2	0	8.1	1.0	4.2	4.3	6.1
3	83.7	0	0	0	.1	.2	0	4.4	7.7	4.0
4	72.4	0	1.8	0	0	0	0	4.6	19.5	1.8
5	3.4	2.7	2.4	70.7	0	0	.1	15.7	.7	4.3
6	67.7	0	3.5	0	.7	20.0	0	2.2	3.1	2.7
7	36.0	0	3.2	.2	31.4	16.0	0	4.8	5.4	2.9
8	16.8	.1	19.1	.2	31.1	15.2	.1	4.7	6.3	6.3
9	44.5	0	5.4	.3	13.8	16.2	0	6.8	7.1	5.8
10	96.9	0	0	0	0	0	0	1.8	0	1.4
11	73.9	0	0	0	0	0	0	13.4	7.5	5.2
12	40.8	0	3.2	0	0	6.3	2.0	26.0	11.4	10.2
13	92.3	0	0	0	0	0	0	1.1	1.7	4.9
14	41.1	.1	1.6	0	0	0	0	11.6	23.5	22.2
15	91.4	0	0	0	0	0	0	5.4	.7	2.5
16	88.6	.1	0	0	0	0	0	8.1	.6	2.6
17	67.1	1.0	.4	5.3	0	0	0	5.2	9.5	11.4
18	11.8	.2	17.5	2.9	28.0	12.6	.4	5.0	8.6	13.0
19	97.4	0	0	0	0	0	0	1.1	.2	1.2
20	93.8	.2	0	0	0	0	.1	2.1	1.5	2.4
21	67.6	.3	20.0	1.4	.9	0	0	2.1	4.9	2.8
22	36.9	27.0	16.1	1.7	3.3	.1	0	2.8	7.4	4.7
23	23.0	15.1	.7	.6	4.8	0	.4	17.5	15.6	22.3
24	.2	.9	4.0	2.7	0	5.3	23.3	52.1	6.9	4.6
25	3.6	0	.4	0	0	5.2	6.7	72.7	5.6	5.7
26	61.0	0	.5	.1	0	.1	0	14.2	10.0	14.2
27	61.8	0	.1	0	0	0	0	14.3	4.8	19.0
28	63.3	0	14.0	1.4	0	0	2.2	2.9	5.3	10.9
29	86.4	0	.3	.4	7.5	0	0	1.9	.4	3.1
30	44.7	0	9.6	.2	34.2	0	0	4.2	.5	6.6
31	53.3	0	9.5	.7	14.6	0	0	5.1	3.7	13.1
32	59.7	0	2.4	.8	20.3	0	0	4.6	.8	11.3
33	22.8	.2	4.1	27.9	14.5	0	0	4.8	5.4	20.4
34	14.8	0	4.2	24.1	19.4	0	0	4.3	6.6	26.7
35	43.6	.8	2.4	13.3	11.2	0	0	6.0	6.9	15.7
36	46.7	0	1.4	5.1	20.2	0	0	4.6	2.0	20.1
37	17.5	1.7	4.9	28.6	7.6	0	0	2.8	5.4	31.5
38	35.2	0	5.7	0	0	0	0	21.3	15.5	22.3
39	74.6	.1	0	0	0	0	0	16.2	.2	8.8
40	51.1	.9	4.0	.9	0	.9	8.2	8.8	10.3	14.6
41	11.3	18.5	4.0	1.1	0	39.3	0	1.2	8.3	16.1
42	5.3	.1	2.7	12.9	13.9	.1	2.2	11.2	19.1	32.4
43	76.6	0	0	0	0	0	0	11.0	3.1	9.3
44	71.9	0	10.9	0	0	0	0	0	6.8	10.4
45	24.5	0	13.7	0	0	0	0	.1	50.7	11.0
46	21.0	9.5	2.4	4.7	20.2	.4	1.1	8.1	14.7	17.9
47	6.5	2.2	1.6	18.0	26.8	2.5	.2	2.1	15.3	24.9
48	10.7	2.9	8.0	2.8	0	1.8	.6	38.9	15.6	18.6
49	11.8	0	2.4	0	0	0	0	3.6	35.3	46.9
50	10.0	.7	1.1	17.0	.8	2.4	6.2	3.9	22.6	35.3

ALGEBRA - ANALYSIS FOR STANDARD 7

NUMBER OF BOYS = 105

VAR	1	2	3	4	5	6	7	8	9	0
1	70.5	2.9	0.0	0.0	0.0	0.0	0.0	8.6	14.3	3.8
2	78.1	0.0	4.8	1.0	0.0	2.9	1.0	1.0	6.7	4.8
3	68.6	1.0	0.0	0.0	0.0	0.0	0.0	11.4	14.3	4.8
4	85.7	0.0	1.0	0.0	0.0	0.0	0.0	0.0	12.4	1.0
5	4.8	1.9	1.9	63.8	0.0	0.0	0.0	21.0	3.8	2.9
6	66.7	0.0	18.1	1.0	0.0	3.8	0.0	2.9	7.6	0.0
7	38.1	0.0	7.6	0.0	44.8	4.8	0.0	1.9	2.9	0.0
8	32.4	0.0	11.4	0.0	43.8	3.8	0.0	1.9	5.7	1.0
9	64.8	0.0	9.5	2.9	5.7	4.8	0.0	1.0	10.5	1.0
10	95.2	0.0	0.0	0.0	0.0	0.0	0.0	3.8	0.0	1.0
11	76.2	0.0	0.0	0.0	0.0	0.0	0.0	11.4	12.4	0.0
12	53.3	0.0	6.7	0.0	0.0	9.5	0.0	20.0	7.6	2.9
13	95.2	0.0	0.0	0.0	0.0	0.0	0.0	1.0	1.0	2.9
14	31.4	0.0	3.8	0.0	0.0	0.0	0.0	7.6	38.1	19.0
15	80.0	0.0	0.0	0.0	0.0	0.0	0.0	19.0	0.0	1.0
16	81.9	0.0	0.0	0.0	0.0	0.0	0.0	18.1	0.0	0.0
17	81.9	1.0	1.0	1.9	0.0	0.0	0.0	0.0	14.3	0.0
18	23.8	0.0	4.8	2.9	52.4	1.0	0.0	2.9	9.5	2.9
19	98.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.9
20	98.1	0.0	0.0	0.0	0.0	0.0	0.0	1.0	1.0	0.0
21	68.6	0.0	22.9	4.8	1.0	0.0	0.0	0.0	2.9	0.0
22	50.5	17.1	14.3	4.8	4.8	0.0	0.0	2.9	4.8	1.0
23	36.2	6.7	0.0	2.9	2.9	0.0	1.0	37.1	7.6	5.7
24	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
25	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
26	79.0	0.0	0.0	0.0	0.0	0.0	0.0	6.7	8.6	5.7
27	68.6	0.0	0.0	0.0	0.0	0.0	0.0	13.3	7.6	10.5
28	63.8	0.0	25.7	2.9	0.0	0.0	0.0	1.0	3.8	2.9
29	93.3	0.0	1.0	3.8	1.9	0.0	0.0	0.0	0.0	0.0
30	68.6	0.0	1.9	0.0	29.5	0.0	0.0	0.0	0.0	0.0
31	51.4	0.0	34.3	3.8	3.8	0.0	0.0	0.0	5.7	1.0
32	82.9	0.0	1.9	4.8	9.5	0.0	0.0	0.0	1.0	0.0
33	36.2	0.0	20.0	19.0	13.3	0.0	0.0	0.0	7.6	3.8
34	33.3	0.0	16.2	26.7	9.5	0.0	0.0	1.9	5.7	6.7
35	69.5	0.0	8.6	14.3	1.9	0.0	0.0	2.9	2.9	0.0
36	74.3	0.0	5.7	9.5	4.8	0.0	0.0	0.0	4.8	1.0
37	36.2	1.0	20.0	27.6	5.7	0.0	0.0	1.0	6.7	1.9
38	44.8	0.0	8.6	0.0	0.0	0.0	0.0	18.1	19.0	9.5
39	72.4	0.0	0.0	0.0	0.0	0.0	0.0	24.8	0.0	2.9
40	50.5	3.8	7.6	1.0	0.0	0.0	9.5	0.0	22.9	4.8
41	16.2	28.6	2.9	0.0	0.0	31.4	0.0	1.9	10.5	8.6
42	7.6	0.0	1.0	21.0	22.9	0.0	1.0	9.5	25.7	11.4
43	81.9	0.0	0.0	0.0	0.0	0.0	0.0	12.4	4.8	1.0
44	91.4	0.0	4.8	0.0	0.0	0.0	0.0	0.0	3.8	0.0
45	23.8	0.0	12.4	0.0	0.0	0.0	0.0	0.0	63.8	0.0
46	34.3	7.6	1.9	8.6	8.6	0.0	0.0	10.5	23.8	4.8
47	7.6	1.0	1.0	29.5	14.3	0.0	1.0	5.7	27.6	12.4
48	23.8	0.0	14.3	6.7	0.0	1.0	0.0	26.7	21.9	5.7
49	17.1	0.0	1.9	0.0	0.0	0.0	0.0	2.9	45.7	32.4
50	7.6	1.9	1.0	27.6	1.0	6.7	1.9	3.8	34.3	14.3

ALGEBRA - ANALYSIS FOR STANDARD 7

NUMBER OF GIRLS = 94

VAR	1	2	3	4	5	6	7	8	9	0
1	73.4	2.1	0.0	0.0	0.0	0.0	0.0	8.5	12.8	3.2
2	73.4	0.0	1.1	0.0	0.0	7.4	0.0	3.2	12.8	2.1
3	68.1	1.1	1.1	0.0	0.0	0.0	0.0	10.6	14.9	4.3
4	74.5	0.0	2.1	1.1	0.0	0.0	0.0	0.0	21.3	1.1
5	1.1	2.1	0.0	70.2	0.0	0.0	0.0	25.5	0.0	1.1
6	71.3	0.0	11.7	0.0	0.0	10.6	0.0	1.1	4.3	1.1
7	30.9	0.0	2.1	0.0	53.2	8.5	0.0	1.1	3.2	1.1
8	19.1	0.0	10.6	0.0	53.2	10.6	0.0	1.1	3.2	2.1
9	77.7	0.0	5.3	1.1	1.1	8.5	0.0	0.0	3.2	3.2
10	94.7	0.0	0.0	0.0	0.0	0.0	0.0	5.3	0.0	0.0
11	61.7	0.0	0.0	0.0	0.0	0.0	0.0	21.3	14.9	2.1
12	38.3	0.0	13.8	0.0	0.0	10.6	0.0	26.6	7.4	3.2
13	96.8	0.0	0.0	0.0	0.0	0.0	0.0	2.1	1.1	0.0
14	13.8	0.0	9.6	0.0	0.0	0.0	0.0	11.7	40.4	24.5
15	75.5	0.0	0.0	0.0	0.0	0.0	0.0	23.4	0.0	1.1
16	77.7	0.0	0.0	0.0	0.0	0.0	0.0	22.3	0.0	0.0
17	80.9	1.1	0.0	3.2	0.0	0.0	0.0	1.1	13.8	0.0
18	4.3	0.0	4.3	2.1	69.1	6.4	0.0	1.1	10.6	2.1
19	97.9	0.0	0.0	0.0	0.0	0.0	0.0	2.1	0.0	0.0
20	100.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
21	63.8	0.0	24.5	7.4	1.1	0.0	0.0	0.0	3.2	0.0
22	43.6	16.0	17.0	4.3	10.6	0.0	0.0	0.0	8.5	0.0
23	30.9	6.4	1.1	1.1	6.4	0.0	0.0	39.4	9.6	5.3
24	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
25	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
26	86.2	2.1	0.0	0.0	0.0	0.0	0.0	7.4	3.2	1.1
27	64.9	0.0	0.0	0.0	0.0	0.0	0.0	24.5	9.6	1.1
28	55.3	0.0	34.0	3.2	0.0	0.0	0.0	1.1	1.1	5.3
29	87.2	0.0	0.0	7.4	5.3	0.0	0.0	0.0	0.0	0.0
30	63.8	0.0	1.1	1.1	34.0	0.0	0.0	0.0	0.0	0.0
31	48.9	0.0	44.7	2.1	1.1	0.0	0.0	0.0	3.2	0.0
32	71.3	0.0	9.6	7.4	11.7	0.0	0.0	0.0	0.0	0.0
33	35.1	0.0	19.1	29.8	7.4	0.0	0.0	0.0	5.3	3.2
34	27.7	0.0	20.2	36.2	7.4	0.0	0.0	0.0	5.3	3.2
35	57.4	1.1	14.9	19.1	2.1	0.0	0.0	1.1	2.1	2.1
36	63.8	0.0	7.4	13.8	5.3	0.0	0.0	0.0	7.4	2.1
37	26.6	0.0	21.3	40.4	3.2	0.0	0.0	0.0	5.3	3.2
38	29.8	1.1	5.3	0.0	0.0	0.0	0.0	24.5	28.7	10.6
39	75.5	0.0	0.0	0.0	0.0	0.0	0.0	19.1	1.1	4.3
40	53.2	1.1	3.2	5.3	0.0	3.2	3.2	3.2	23.4	4.3
41	5.3	13.8	3.2	1.1	0.0	54.3	0.0	3.2	10.6	8.5
42	6.4	0.0	4.3	17.0	14.9	0.0	2.1	23.4	21.3	10.6
43	70.2	0.0	0.0	0.0	0.0	0.0	0.0	25.5	2.1	2.1
44	94.7	1.1	3.2	0.0	0.0	0.0	0.0	0.0	1.1	0.0
45	7.4	1.1	16.0	0.0	0.0	0.0	0.0	0.0	75.5	0.0
46	24.5	5.3	1.1	12.8	16.0	1.1	0.0	19.1	18.1	2.1
47	1.1	1.1	1.1	36.2	16.0	3.2	0.0	17.0	21.3	3.2
48	14.9	8.5	9.6	7.4	0.0	1.1	0.0	20.2	33.0	5.3
49	2.1	0.0	2.1	0.0	0.0	0.0	0.0	4.3	60.6	30.9
50	7.4	0.0	0.0	39.4	0.0	0.0	2.1	6.4	29.8	14.9

ALGEBRA – ANALYSIS FOR STANDARD 8

NUMBER OF PUPILS = 131

VAR	1	2	3	4	5	6	7	8	9	0
1	91.6	3.8	0.0	0.0	0.0	0.0	0.0	3.1	1.5	0.0
2	93.1	0.0	0.8	0.0	0.0	3.1	0.0	0.8	2.3	0.0
3	89.3	0.8	0.0	0.0	0.0	0.0	0.0	4.6	5.3	0.0
4	84.7	0.0	2.3	0.0	0.0	0.0	0.0	0.0	12.2	0.8
5	3.1	3.1	1.5	80.2	0.0	0.0	0.0	8.4	2.3	1.5
6	76.3	0.0	9.9	0.8	0.8	5.3	0.0	1.5	4.6	0.8
7	54.2	0.0	6.1	0.0	32.8	3.8	0.0	0.8	2.3	0.0
8	48.1	0.8	16.8	0.8	26.7	3.1	0.0	0.8	1.5	1.5
9	73.3	3.8	6.9	0.0	5.3	6.9	0.0	0.0	2.3	1.5
10	98.5	0.0	0.0	0.0	0.0	0.0	0.0	1.5	0.0	0.0
11	80.2	0.0	0.0	0.0	0.0	0.0	0.0	9.9	8.4	1.5
12	66.4	0.0	9.2	0.0	0.0	6.1	2.3	10.7	3.8	1.5
13	98.5	0.0	0.8	0.0	0.0	0.0	0.0	0.8	0.0	0.0
14	25.2	0.0	9.2	0.0	0.0	0.0	0.0	9.9	31.3	24.4
15	87.0	0.0	0.0	0.0	0.0	0.0	0.0	12.2	0.0	0.8
16	86.3	0.0	0.0	0.0	0.0	0.0	0.0	13.7	0.0	0.0
17	86.3	0.8	0.0	0.8	0.0	0.0	0.0	0.0	12.2	0.0
18	22.9	1.5	8.4	3.1	43.5	6.1	0.8	2.3	9.2	2.3
19	99.2	0.0	0.0	0.0	0.0	0.0	0.0	0.8	0.0	0.0
20	98.5	0.8	0.0	0.0	0.0	0.0	0.0	0.8	0.0	0.0
21	75.6	0.8	14.5	7.6	0.8	0.0	0.0	0.8	0.0	0.0
22	57.3	13.7	10.7	3.1	6.9	0.0	0.0	2.3	6.1	0.0
23	54.2	12.2	1.5	0.0	2.3	0.0	0.0	18.3	6.9	4.6
24	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
25	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
26	94.7	0.0	0.0	0.0	0.0	0.0	0.0	0.8	2.3	2.3
27	91.6	0.0	0.0	0.0	0.0	0.8	0.0	3.1	3.1	1.5
28	75.6	0.0	21.4	1.5	0.0	0.0	0.0	0.8	0.0	0.8
29	93.1	0.0	0.0	5.3	1.5	0.0	0.0	0.0	0.0	0.0
30	71.0	0.0	3.1	0.8	23.7	0.0	0.0	0.8	0.0	0.8
31	62.6	0.0	27.5	0.8	0.0	0.0	0.0	0.0	9.2	0.0
32	84.7	0.0	3.1	4.6	6.9	0.0	0.0	0.0	0.8	0.0
33	47.3	0.0	19.8	13.7	6.1	0.0	0.0	0.0	10.7	2.3
34	43.5	0.0	13.0	21.4	2.3	0.0	0.0	0.0	13.7	6.1
35	78.6	0.0	8.4	8.4	0.8	0.0	0.0	0.0	1.5	2.3
36	80.9	0.0	7.6	2.3	1.5	0.0	0.0	0.0	3.8	3.8
37	34.4	0.0	18.3	20.6	0.8	0.0	0.0	3.1	14.5	8.4
38	42.7	0.8	16.8	0.0	0.0	0.0	0.0	9.9	24.4	5.3
39	85.5	0.8	0.0	0.0	0.0	0.0	0.0	12.2	0.8	0.8
40	74.0	0.0	4.6	0.8	0.0	0.0	3.8	0.0	10.7	6.1
41	19.1	29.8	11.5	0.8	0.0	24.4	0.0	1.5	6.1	6.9
42	9.2	0.0	4.6	27.5	13.0	0.0	0.0	14.5	19.1	12.2
43	88.5	0.0	0.0	0.0	0.0	0.0	0.0	9.2	1.5	0.8
44	95.4	0.0	2.3	0.0	0.0	0.0	0.0	0.0	1.5	0.8
45	19.1	0.0	2.3	0.0	0.0	0.0	0.0	0.0	78.6	0.0
46	35.9	12.2	0.8	12.2	15.3	0.0	0.0	5.3	15.3	3.1
47	9.2	1.5	4.6	32.1	17.6	0.0	0.0	9.9	14.5	10.7
48	22.9	6.9	19.8	5.3	0.0	2.3	0.0	20.6	15.3	6.9
49	12.2	0.0	5.3	0.0	0.0	0.0	0.0	1.5	53.4	27.5
50	21.4	0.0	0.8	38.2	0.8	1.5	0.0	5.3	22.1	9.9

CSMS

TEST 1 YEAR 4 NUMBER OF CHILDREN = 731 ALGEBRA 1976

VAR	1	2	3	4	5	6	7	8	9	0
1	82.1	0	0	0	0	.1	0	4.4	5.7	7.7
2	68.4	0	3.7	.3	0	5.5	.1	2.9	6.2	13.0
3	78.4	0	0	0	0	.3	0	2.9	9.8	8.6
4	71.7	0	2.3	0	0	0	0	2.5	20.2	3.3
5	5.9	3.7	2.2	66.3	0	0	0	15.2	1.6	5.1
6	68.9	0	6.4	0	2.1	13.7	.1	1.9	3.3	3.6
7	41.0	0	4.1	0	30.4	12.0	0	2.9	4.9	4.7
8	24.5	0	17.6	0	29.4	11.1	0	3.1	5.2	9.0
9	50.2	0	8.1	.3	12.6	10.8	0	2.9	6.6	8.6
10	94.5	0	0	0	0	0	0	2.2	0	3.3
11	73.9	0	0	0	0	0	0	10.9	6.8	8.3
12	49.8	0	7.1	.1	.1	5.5	1.1	15.7	7.8	12.7
13	93.0	0	.1	0	0	0	0	1.1	1.0	4.8
14	40.2	0	2.6	0	0	0	0	12.9	20.7	23.7
15	90.3	0	0	0	0	0	0	5.3	.3	4.1
16	89.9	0	0	0	0	0	0	6.2	.3	3.7
17	74.7	1.1	.4	2.6	0	0	0	2.6	6.2	12.4
18	16.1	0	18.6	1.5	30.5	8.8	.1	5.2	7.0	12.2
19	95.6	.1	0	0	0	0	0	.8	0	3.4
20	92.9	0	.3	0	0	0	.3	.8	1.1	4.7
21	71.8	1.0	12.9	3.4	1.5	0	0	1.0	3.0	5.5
22	44.7	21.9	12.3	2.9	2.2	0	0	1.0	7.1	7.9
23	23.0	17.5	.5	1.0	3.8	.1	.3	16.7	15.3	21.8
24	1.5	1.1	4.8	1.8	0	1.6	36.8	38.2	7.9	6.3
25	5.3	0	.4	.1	0	3.0	7.7	72.4	3.7	7.4
26	69.9	0	.4	0	0	0	0	7.7	4.9	17.1
27	67.4	0	.1	0	0	0	0	8.6	3.3	20.5
28	63.5	0	15.6	2.9	0	0	2.5	2.5	2.5	10.7
29	87.4	0	.1	.3	6.0	0	0	1.6	.3	4.2
30	50.9	0	5.5	1.1	34.2	0	0	2.2	.3	5.9
31	54.7	0	18.2	1.1	8.3	0	0	3.0	2.6	12.0
32	65.9	0	3.6	.8	17.4	0	0	2.6	.1	9.6
33	32.0	.3	6.4	23.4	14.5	0	0	2.9	4.7	15.9
34	19.6	0	5.9	27.9	16.1	0	0	1.8	5.5	23.3
35	51.7	.7	4.0	12.2	8.8	0	0	3.0	3.7	16.0
36	55.7	0	1.9	6.4	13.5	0	0	1.4	1.2	19.8
37	25.2	1.1	9.3	23.7	7.7	0	0	1.4	6.6	25.2
38	31.3	0	7.9	0	0	0	0	20.8	16.0	23.9
39	72.0	0	0	0	0	.1	0	16.4	.5	10.9
40	52.9	1.4	3.3	.4	0	1.1	7.8	4.7	10.0	18.5
41	15.5	19.3	3.7	.8	0	35.0	0	.8	8.9	16.0
42	7.9	0	3.0	18.6	12.4	0	1.2	10.5	15.5	30.8
43	75.9	.1	0	0	0	0	0	10.7	2.5	10.8
44	73.2	0	11.9	.1	0	0	0	0	4.1	10.7
45	27.2	0	11.5	0	0	0	0	0	50.1	11.2
46	25.9	7.4	1.9	6.2	17.9	.7	.7	9.3	11.1	19.0
47	13.0	3.1	1.5	16.7	23.3	1.8	.3	4.0	13.1	23.3
48	17.0	4.1	9.3	2.2	0	1.8	.5	25.9	21.1	18.2
49	15.7	0	4.2	0	0	0	0	2.1	30.2	47.7
50	12.2	.7	1.0	17.1	2.5	2.9	4.5	4.4	22.7	32.1

ALGEBRA - ANALYSIS FOR STANDARD 8

NUMBER OF BOYS = 66

VAR	1	2	3	4	5	6	7	8	9	0
1	89.4	4.5	0.0	0.0	0.0	0.0	0.0	4.5	1.5	0.0
2	93.9	0.0	0.0	0.0	0.0	1.5	0.0	0.0	4.5	0.0
3	87.9	1.5	0.0	0.0	0.0	0.0	0.0	6.1	4.5	0.0
4	83.3	0.0	3.0	0.0	0.0	0.0	0.0	0.0	12.1	1.5
5	6.1	4.5	1.5	80.3	0.0	0.0	0.0	6.1	1.5	0.0
6	75.8	0.0	10.6	0.0	1.5	6.1	0.0	1.5	4.5	0.0
7	51.5	0.0	6.1	0.0	33.3	4.5	0.0	0.0	4.5	0.0
8	53.0	1.5	15.2	1.5	21.2	4.5	0.0	0.0	1.5	1.5
9	68.2	6.1	7.6	0.0	4.5	7.6	0.0	0.0	4.5	1.5
10	98.5	0.0	0.0	0.0	0.0	0.0	0.0	1.5	0.0	0.0
11	84.8	0.0	0.0	0.0	0.0	0.0	0.0	9.1	4.5	1.5
12	68.2	0.0	9.1	0.0	0.0	4.5	1.5	6.1	7.6	3.0
13	98.5	0.0	0.0	0.0	0.0	0.0	0.0	1.5	0.0	0.0
14	36.4	0.0	7.6	0.0	0.0	0.0	0.0	9.1	28.8	18.2
15	83.3	0.0	0.0	0.0	0.0	0.0	0.0	16.7	0.0	0.0
16	81.8	0.0	0.0	0.0	0.0	0.0	0.0	18.2	0.0	0.0
17	81.8	1.5	0.0	1.5	0.0	0.0	0.0	0.0	15.2	0.0
18	24.2	3.0	10.6	0.0	39.4	6.1	1.5	1.5	10.6	3.0
19	100.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
20	98.5	1.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
21	74.2	1.5	15.2	9.1	0.0	0.0	0.0	0.0	0.0	0.0
22	54.5	12.1	13.6	4.5	9.1	0.0	0.0	1.5	4.5	0.0
23	53.0	12.1	0.0	0.0	1.5	0.0	0.0	22.7	6.1	4.5
24	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
25	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
26	95.5	0.0	0.0	0.0	0.0	0.0	0.0	1.5	1.5	1.5
27	92.4	0.0	0.0	0.0	0.0	1.5	0.0	3.0	3.0	0.0
28	74.2	0.0	24.2	1.5	0.0	0.0	0.0	0.0	0.0	0.0
29	92.4	0.0	0.0	6.1	1.5	0.0	0.0	0.0	0.0	0.0
30	63.6	0.0	4.5	1.5	28.8	0.0	0.0	0.0	0.0	1.5
31	53.0	0.0	31.8	1.5	0.0	0.0	0.0	0.0	13.6	0.0
32	80.3	0.0	4.5	4.5	9.1	0.0	0.0	0.0	1.5	0.0
33	42.4	0.0	22.7	12.1	6.1	0.0	0.0	0.0	15.2	1.5
34	39.4	0.0	12.1	21.2	4.5	0.0	0.0	0.0	13.6	9.1
35	71.2	0.0	12.1	10.6	1.5	0.0	0.0	0.0	1.5	3.0
36	74.2	0.0	10.6	3.0	1.5	0.0	0.0	0.0	4.5	6.1
37	34.8	0.0	22.7	19.7	0.0	0.0	0.0	3.0	10.6	9.1
38	50.0	1.5	15.2	0.0	0.0	0.0	0.0	7.6	19.7	6.1
39	84.8	1.5	0.0	0.0	0.0	0.0	0.0	13.6	0.0	0.0
40	77.3	0.0	3.0	0.0	0.0	0.0	3.0	0.0	13.6	3.0
41	21.2	33.3	9.1	0.0	0.0	22.7	0.0	0.0	6.1	7.6
42	12.1	0.0	4.5	34.8	9.1	0.0	0.0	9.1	18.2	12.1
43	87.9	0.0	0.0	0.0	0.0	0.0	0.0	9.1	1.5	1.5
44	95.5	0.0	3.0	0.0	0.0	0.0	0.0	0.0	0.0	1.5
45	19.7	0.0	3.0	0.0	0.0	0.0	0.0	0.0	77.3	0.0
46	40.9	13.6	1.5	13.6	10.6	0.0	0.0	6.1	12.1	1.5
47	12.1	3.0	6.1	33.3	16.7	0.0	0.0	6.1	12.1	10.6
48	27.3	9.1	25.8	6.1	0.0	0.0	0.0	16.7	9.1	6.1
49	9.1	0.0	9.1	0.0	0.0	0.0	0.0	1.5	48.5	31.8
50	21.2	0.0	1.5	40.9	0.0	1.5	0.0	6.1	21.2	7.6

ALGEBRA - ANALYSIS FOR STANDARD 8

NUMBER OF GIRLS = 65

VAR	1	2	3	4	5	6	7	8	9	0
1	93.8	3.1	0.0	0.0	0.0	0.0	0.0	1.5	1.5	0.0
2	92.3	0.0	1.5	0.0	0.0	4.6	0.0	1.5	0.0	0.0
3	90.8	0.0	0.0	0.0	0.0	0.0	0.0	3.1	6.2	0.0
4	86.2	0.0	1.5	0.0	0.0	0.0	0.0	0.0	12.3	0.0
5	0.0	1.5	1.5	80.0	0.0	0.0	0.0	10.8	3.1	3.1
6	76.9	0.0	9.2	1.5	0.0	4.6	0.0	1.5	4.6	1.5
7	56.9	0.0	6.2	0.0	32.3	3.1	0.0	1.5	0.0	0.0
8	43.1	0.0	18.5	0.0	32.3	1.5	0.0	1.5	1.5	1.5
9	78.5	1.5	6.2	0.0	6.2	6.2	0.0	0.0	0.0	1.5
10	98.5	0.0	0.0	0.0	0.0	0.0	0.0	1.5	0.0	0.0
11	75.4	0.0	0.0	0.0	0.0	0.0	0.0	10.8	12.3	1.5
12	64.6	0.0	9.2	0.0	0.0	7.7	3.1	15.4	0.0	0.0
13	98.5	0.0	1.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0
14	13.8	0.0	10.8	0.0	0.0	0.0	0.0	10.8	33.8	30.8
15	90.8	0.0	0.0	0.0	0.0	0.0	0.0	7.7	0.0	1.5
16	90.8	0.0	0.0	0.0	0.0	0.0	0.0	9.2	0.0	0.0
17	90.8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	9.2	0.0
18	21.5	0.0	6.2	6.2	47.7	6.2	0.0	3.1	7.7	1.5
19	98.5	0.0	0.0	0.0	0.0	0.0	0.0	1.5	0.0	0.0
20	98.5	0.0	0.0	0.0	0.0	0.0	0.0	1.5	0.0	0.0
21	76.9	0.0	13.8	6.2	1.5	0.0	0.0	1.5	0.0	0.0
22	60.0	15.4	7.7	1.5	4.6	0.0	0.0	3.1	7.7	0.0
23	55.4	12.3	3.1	0.0	3.1	0.0	0.0	13.8	7.7	4.6
24	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
25	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
26	93.8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	3.1	3.1
27	90.8	0.0	0.0	0.0	0.0	0.0	0.0	3.1	3.1	3.1
28	76.9	0.0	18.5	1.5	0.0	0.0	0.0	1.5	0.0	1.5
29	93.8	0.0	0.0	4.6	1.5	0.0	0.0	0.0	0.0	0.0
30	78.5	0.0	1.5	0.0	18.5	0.0	0.0	1.5	0.0	0.0
31	72.3	0.0	23.1	0.0	0.0	0.0	0.0	0.0	4.6	0.0
32	89.2	0.0	1.5	4.6	4.6	0.0	0.0	0.0	0.0	0.0
33	52.3	0.0	16.9	15.4	6.2	0.0	0.0	0.0	6.2	3.1
34	47.7	0.0	13.8	21.5	0.0	0.0	0.0	0.0	13.8	3.1
35	86.2	0.0	4.6	6.2	0.0	0.0	0.0	0.0	1.5	1.5
36	87.7	0.0	4.6	1.5	1.5	0.0	0.0	0.0	3.1	1.5
37	33.8	0.0	13.8	21.5	1.5	0.0	0.0	3.1	18.5	7.7
38	35.4	0.0	18.5	0.0	0.0	0.0	0.0	12.3	29.2	4.6
39	86.2	0.0	0.0	0.0	0.0	0.0	0.0	10.8	1.5	1.5
40	70.8	0.0	6.2	1.5	0.0	0.0	4.6	0.0	7.7	9.2
41	16.9	26.2	13.8	1.5	0.0	26.2	0.0	3.1	6.2	6.2
42	6.2	0.0	4.6	20.0	16.9	0.0	0.0	20.0	20.0	12.3
43	89.2	0.0	0.0	0.0	0.0	0.0	0.0	9.2	1.5	0.0
44	95.4	0.0	1.5	0.0	0.0	0.0	0.0	0.0	3.1	0.0
45	18.5	0.0	1.5	0.0	0.0	0.0	0.0	0.0	80.0	0.0
46	30.8	10.8	0.0	10.8	20.0	0.0	0.0	4.6	18.5	4.6
47	6.2	0.0	3.1	30.8	18.5	0.0	0.0	13.8	16.9	10.8
48	18.5	4.6	13.8	4.6	0.0	4.6	0.0	24.6	21.5	7.7
49	15.4	0.0	1.5	0.0	0.0	0.0	0.0	1.5	58.5	23.1
50	21.5	0.0	0.0	35.4	1.5	1.5	0.0	4.6	23.1	12.3

ALGEBRA - ANALYSIS FOR STANDARD 8 HIGHER GRADE

NUMBER OF PUPILS = 55

VAR	1	2	3	4	5	6	7	8	9	0
1	89.1	1.8	0.0	0.0	0.0	0.0	0.0	5.5	3.6	0.0
2	92.7	0.0	0.0	0.0	0.0	5.5	0.0	0.0	1.8	0.0
3	89.1	0.0	0.0	0.0	0.0	0.0	0.0	3.6	7.3	0.0
4	78.2	0.0	1.8	0.0	0.0	0.0	0.0	0.0	18.2	1.8
5	5.5	0.0	3.6	78.2	0.0	0.0	0.0	9.1	0.0	3.6
6	81.8	0.0	10.9	0.0	1.8	1.8	0.0	0.0	1.8	1.8
7	65.5	0.0	7.3	0.0	21.8	1.8	0.0	0.0	3.6	0.0
8	63.6	1.8	14.5	1.8	10.9	1.8	0.0	0.0	1.8	3.6
9	74.5	3.6	9.1	0.0	5.5	1.8	0.0	0.0	1.8	3.6
10	98.2	0.0	0.0	0.0	0.0	0.0	0.0	1.8	0.0	0.0
11	83.6	0.0	0.0	0.0	0.0	0.0	0.0	9.1	7.3	0.0
12	63.6	0.0	10.9	0.0	0.0	5.5	5.5	7.3	5.5	1.8
13	98.2	0.0	0.0	0.0	0.0	0.0	0.0	1.8	0.0	0.0
14	29.1	0.0	7.3	0.0	0.0	0.0	0.0	7.3	30.9	25.5
15	87.3	0.0	0.0	0.0	0.0	0.0	0.0	10.9	0.0	1.8
16	90.9	0.0	0.0	0.0	0.0	0.0	0.0	9.1	0.0	0.0
17	90.9	0.0	0.0	1.8	0.0	0.0	0.0	0.0	7.3	0.0
18	30.9	1.8	7.3	1.8	40.0	5.5	0.0	0.0	10.9	1.8
19	98.2	0.0	0.0	0.0	0.0	0.0	0.0	1.8	0.0	0.0
20	98.2	1.8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
21	72.7	0.0	18.2	9.1	0.0	0.0	0.0	0.0	0.0	0.0
22	65.5	9.1	10.9	3.6	9.1	0.0	0.0	1.8	0.0	0.0
23	54.5	10.9	1.8	0.0	3.6	0.0	0.0	10.9	9.1	9.1
24	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
25	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
26	96.4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	3.6	0.0
27	94.5	0.0	0.0	0.0	0.0	0.0	0.0	1.8	3.6	0.0
28	80.0	0.0	18.2	1.8	0.0	0.0	0.0	0.0	0.0	0.0
29	90.9	0.0	0.0	5.5	3.6	0.0	0.0	0.0	0.0	0.0
30	80.0	0.0	3.6	1.8	10.9	0.0	0.0	1.8	0.0	1.8
31	63.6	0.0	25.5	1.8	0.0	0.0	0.0	0.0	9.1	0.0
32	87.3	0.0	3.6	5.5	1.8	0.0	0.0	0.0	1.8	0.0
33	50.9	0.0	20.0	10.9	1.8	0.0	0.0	0.0	14.5	1.8
34	49.1	0.0	7.3	25.5	0.0	0.0	0.0	0.0	12.7	5.5
35	89.1	0.0	3.6	3.6	0.0	0.0	0.0	0.0	1.8	1.8
36	81.8	0.0	7.3	3.6	1.8	0.0	0.0	0.0	3.6	1.8
37	50.9	0.0	10.9	10.9	0.0	0.0	0.0	5.5	12.7	9.1
38	45.5	1.8	23.6	0.0	0.0	0.0	0.0	3.6	21.8	3.6
39	92.7	0.0	0.0	0.0	0.0	0.0	0.0	7.3	0.0	0.0
40	83.6	0.0	5.5	1.8	0.0	0.0	1.8	0.0	5.5	1.8
41	23.6	32.7	12.7	1.8	0.0	14.5	0.0	0.0	7.3	7.3
42	16.4	0.0	5.5	32.7	5.5	0.0	0.0	10.9	18.2	10.9
43	89.1	0.0	0.0	0.0	0.0	0.0	0.0	7.3	1.8	1.8
44	98.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.8
45	21.8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	78.2	0.0
46	41.8	7.3	1.8	10.9	12.7	0.0	0.0	7.3	14.5	3.6
47	16.4	3.6	3.6	23.6	20.0	0.0	0.0	5.5	14.5	12.7
48	29.1	10.9	21.8	1.8	0.0	0.0	0.0	16.4	10.9	9.1
49	14.5	0.0	9.1	0.0	0.0	0.0	0.0	0.0	40.0	36.4
50	29.1	0.0	0.0	34.5	1.8	3.6	0.0	3.6	16.4	10.9

ALGEBRA - ANALYSIS FOR STANDARD 8 STANDARD GRADE

NUMBER OF PUPILS = 61

VAR	1	2	3	4	5	6	7	8	9	0
1	95.1	3.3	0.0	0.0	0.0	0.0	0.0	1.6	0.0	0.0
2	93.4	0.0	1.6	0.0	0.0	1.6	0.0	1.6	1.6	0.0
3	88.5	0.0	0.0	0.0	0.0	0.0	0.0	6.6	4.9	0.0
4	88.5	0.0	3.3	0.0	0.0	0.0	0.0	0.0	8.2	0.0
5	1.6	4.9	0.0	80.3	0.0	0.0	0.0	9.8	3.3	0.0
6	68.9	0.0	8.2	1.6	0.0	9.8	0.0	3.3	8.2	0.0
7	41.0	0.0	6.6	0.0	44.3	6.6	0.0	1.6	0.0	0.0
8	36.1	0.0	21.3	0.0	36.1	4.9	0.0	1.6	0.0	0.0
9	68.9	4.9	4.9	0.0	6.6	13.1	0.0	0.0	1.6	0.0
10	100.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
11	75.4	0.0	0.0	0.0	0.0	0.0	0.0	9.8	11.5	3.3
12	65.6	0.0	9.8	0.0	0.0	8.2	0.0	13.1	3.3	0.0
13	100.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
14	14.8	0.0	13.1	0.0	0.0	0.0	0.0	9.8	34.4	27.9
15	86.9	0.0	0.0	0.0	0.0	0.0	0.0	13.1	0.0	0.0
16	83.6	0.0	0.0	0.0	0.0	0.0	0.0	16.4	0.0	0.0
17	83.6	1.6	0.0	0.0	0.0	0.0	0.0	0.0	14.8	0.0
18	11.5	1.6	9.8	4.9	49.2	8.2	1.6	4.9	6.6	1.6
19	100.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
20	98.4	0.0	0.0	0.0	0.0	0.0	0.0	1.6	0.0	0.0
21	75.4	1.6	11.5	8.2	1.6	0.0	0.0	1.6	0.0	0.0
22	45.9	19.7	11.5	3.3	4.9	0.0	0.0	3.3	11.5	0.0
23	50.8	13.1	1.6	0.0	1.6	0.0	0.0	27.9	3.3	1.6
24	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
25	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
26	93.4	0.0	0.0	0.0	0.0	0.0	0.0	1.6	0.0	4.9
27	90.2	0.0	0.0	0.0	0.0	1.6	0.0	3.3	1.6	3.3
28	70.5	0.0	26.2	1.6	0.0	0.0	0.0	1.6	0.0	0.0
29	93.4	0.0	0.0	6.6	0.0	0.0	0.0	0.0	0.0	0.0
30	60.7	0.0	3.3	0.0	36.1	0.0	0.0	0.0	0.0	0.0
31	54.1	0.0	34.4	0.0	0.0	0.0	0.0	0.0	11.5	0.0
32	78.7	0.0	3.3	4.9	13.1	0.0	0.0	0.0	0.0	0.0
33	41.0	0.0	23.0	13.1	11.5	0.0	0.0	0.0	8.2	3.3
34	32.8	0.0	18.0	21.3	4.9	0.0	0.0	0.0	14.8	8.2
35	65.6	0.0	14.8	14.8	1.6	0.0	0.0	0.0	0.0	3.3
36	78.7	0.0	9.8	1.6	1.6	0.0	0.0	0.0	3.3	4.9
37	14.8	0.0	26.2	32.8	1.6	0.0	0.0	0.0	14.8	9.8
38	37.7	0.0	8.2	0.0	0.0	0.0	0.0	16.4	29.5	8.2
39	80.3	0.0	0.0	0.0	0.0	0.0	0.0	16.4	1.6	1.6
40	62.3	0.0	4.9	0.0	0.0	0.0	6.6	0.0	16.4	9.8
41	14.8	23.0	9.8	0.0	0.0	37.7	0.0	1.6	6.6	6.6
42	3.3	0.0	1.6	23.0	21.3	0.0	0.0	16.4	21.3	13.1
43	86.9	0.0	0.0	0.0	0.0	0.0	0.0	13.1	0.0	0.0
44	93.4	0.0	4.9	0.0	0.0	0.0	0.0	0.0	1.6	0.0
45	9.8	0.0	3.3	0.0	0.0	0.0	0.0	0.0	86.9	0.0
46	29.5	14.8	0.0	8.2	21.3	0.0	0.0	3.3	19.7	3.3
47	3.3	0.0	4.9	36.1	19.7	0.0	0.0	13.1	16.4	6.6
48	9.8	3.3	21.3	8.2	0.0	4.9	0.0	27.9	18.0	6.6
49	11.5	0.0	0.0	0.0	0.0	0.0	0.0	3.3	67.2	18.0
50	11.5	0.0	1.6	41.0	0.0	0.0	0.0	8.2	27.9	9.8

ALGEBRA - ANALYSIS FOR STANDARD 9

NUMBER OF PUPILS = 33

VAR	1	2	3	4	5	6	7	8	9	0
1	81.8	6.1	0.0	0.0	0.0	0.0	0.0	0.0	12.1	0.0
2	90.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	9.1	0.0
3	81.8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	18.2	0.0
4	90.9	0.0	3.0	0.0	0.0	0.0	0.0	0.0	3.0	3.0
5	9.1	3.0	3.0	75.8	0.0	0.0	0.0	9.1	0.0	0.0
6	87.9	0.0	3.0	0.0	3.0	3.0	0.0	0.0	3.0	0.0
7	75.8	0.0	6.1	0.0	15.2	3.0	0.0	0.0	0.0	0.0
8	72.7	0.0	9.1	0.0	12.1	3.0	0.0	0.0	3.0	0.0
9	72.7	0.0	15.2	0.0	6.1	3.0	0.0	0.0	3.0	0.0
10	100.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
11	90.9	0.0	0.0	0.0	0.0	0.0	0.0	6.1	3.0	0.0
12	78.8	0.0	6.1	0.0	0.0	6.1	0.0	0.0	9.1	0.0
13	100.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
14	51.5	0.0	0.0	0.0	0.0	0.0	0.0	18.2	12.1	18.2
15	87.9	0.0	0.0	0.0	0.0	0.0	0.0	12.1	0.0	0.0
16	87.9	0.0	0.0	0.0	0.0	0.0	0.0	12.1	0.0	0.0
17	87.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	12.1	0.0
18	27.3	0.0	3.0	0.0	45.5	12.1	0.0	0.0	12.1	0.0
19	100.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
20	90.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	9.1	0.0
21	81.8	0.0	6.1	6.1	0.0	0.0	0.0	0.0	6.1	0.0
22	69.7	12.1	6.1	6.1	3.0	0.0	0.0	0.0	0.0	3.0
23	69.7	12.1	0.0	3.0	6.1	0.0	0.0	3.0	3.0	3.0
24	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
25	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
26	97.0	0.0	0.0	0.0	0.0	0.0	0.0	3.0	0.0	0.0
27	100.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
28	100.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
29	97.0	0.0	0.0	3.0	0.0	0.0	0.0	0.0	0.0	0.0
30	78.8	0.0	0.0	3.0	18.2	0.0	0.0	0.0	0.0	0.0
31	78.8	0.0	18.2	3.0	0.0	0.0	0.0	0.0	0.0	0.0
32	90.9	0.0	3.0	6.1	0.0	0.0	0.0	0.0	0.0	0.0
33	69.7	0.0	12.1	12.1	3.0	0.0	0.0	0.0	3.0	0.0
34	66.7	0.0	9.1	15.2	6.1	0.0	0.0	0.0	3.0	0.0
35	87.9	0.0	6.1	6.1	0.0	0.0	0.0	0.0	0.0	0.0
36	81.8	0.0	3.0	9.1	0.0	0.0	0.0	0.0	6.1	0.0
37	54.5	0.0	21.2	15.2	3.0	0.0	0.0	0.0	0.0	6.1
38	48.5	0.0	33.3	0.0	0.0	0.0	0.0	0.0	12.1	6.1
39	97.0	0.0	0.0	0.0	0.0	0.0	0.0	3.0	0.0	0.0
40	93.9	0.0	0.0	0.0	0.0	0.0	6.1	0.0	0.0	0.0
41	12.1	57.6	6.1	6.1	3.0	12.1	0.0	0.0	3.0	0.0
42	27.3	3.0	12.1	15.2	24.2	0.0	3.0	6.1	9.1	0.0
43	93.9	0.0	0.0	0.0	0.0	0.0	0.0	3.0	3.0	0.0
44	100.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
45	57.6	0.0	6.1	0.0	0.0	0.0	0.0	0.0	36.4	0.0
46	63.6	9.1	0.0	6.1	3.0	0.0	0.0	6.1	12.1	0.0
47	9.1	3.0	0.0	51.5	12.1	0.0	0.0	9.1	12.1	3.0
48	54.5	3.0	6.1	0.0	0.0	0.0	0.0	18.2	12.1	6.1
49	42.4	0.0	3.0	0.0	0.0	0.0	0.0	0.0	42.4	12.1
50	51.5	18.2	0.0	18.2	0.0	0.0	3.0	3.0	3.0	3.0

APPENDIX 4

TABLE OF CODES AND LEVELS

SCHOOL: ...... G .... CLASS: 8 ....

NAME	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
A	9	1	1	1	4	1	5	7	1	1	1	1	1	0	1	1	1	1	1	1	1	1	8
B	1	6	9	9	8	9	5	5	1	1	1	8	1	8	1	1	1	5	1	1	1	1	8
C	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D	1	1	1	1	4	1	5	5	1	1	1	1	1	1	1	1	1	5	1	1	4	3	1
E	1	1	1	1	9	1	5	9	1	1	1	8	1	9	1	1	1	5	1	1	1	5	8
F	1	1	1	1	2	1	1	1	1	1	8	1	1	1	1	1	1	5	1	1	1	9	0
↓																							
etc.																							

NAME	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	LEVEL 1	2	3	4	
A	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	5	1	1	9	2	4	9	0	1	✓	✓	✓	✗	
B	1	8	3	4	4	3	4	3	3	4	3	4	9	8	9	6	0	8	1	9	9	9	9	9	0	✓	✗	✗	✗	
C	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	4	1	1	3	9	✓	✓	✓	✓	
D	1	1	1	1	1	3	3	1	3	1	3	3	1	1	1	6	0	1	1	9	1	0	1	0	4	✓	✗	✓	✗	error type
E	1	1	1	1	1	1	1	4	1	1	4	1	0	1	3	9	3	1	1	9	9	9	8	9	4	✓	✓	✗	✗	
F	1	1	1	1	1	9	1	9	9	1	1	9	1	1	1	2	5	1	1	1	1	4	3	9	4	✓	✓	✓	✗	
↓																											↓			
etc.																														